MAT4200 - 2026 EXERCISES WEEK 11

Warm ups

(1) Give the primary decomposition of the ideal $(32) \subset \mathbb{Z}$.
(2) Give the primary decomposition of $(x^n, y) \subset k[x, y]$ for k a field.
(3) Let $h(x) = \frac{(x^2-1)}{x-1} \in \mathbb{C}(x)$. Compute $\nu_1(h(x))$.
(4) Let $p = 2$ and compute $\nu_2(2/5 + 2/3)$.

Suggested problems week 11

(1) Show that Q is a primary ideal of R if and only if R/Q is not 0 and all zero divisors of R/Q are nilpotent.
(2) Let R be a ring and P a prime ideal. If Q_1, Q_2 are P-primary then $Q_1 \cap Q_2$ is P-primary.
(3) Let I be an ideal of a ring R and consider $\phi : R \to R/I$. Show that if the ideal J if R/I is primary, then $\phi^{-1}(J)$ is a primary ideal of R.
(4) Let R be an integral domain of dimension 1. Prove that an ideal $I \subseteq R$ is primary if and only if $\sqrt{I}$ is prime.
(5) Let A be a ring such that every ideal is principal. Prove that $\dim A \leq 1$.
(6) Let v be a discrete valuation on a field K. Prove that if $x, y \in K$ are such that $v(x) < v(y)$, then $v(x + y) = v(x)$.
(7) Let v be a discrete valuation on a field K. Prove that if $x \in K$ is such that $x^n = 1$ for some $n \geq 1$, then $v(x) = 0$.
(8) Let k be a field. Prove that there is a unique discrete valuation v on $k(x)$ such that
$$v(f) = -n$$
if $f \in k[x]$ is a polynomial of degree n. Describe the associated discrete valuation ring and find a generator of its maximal ideal.
(9) Let R be a local domain which is not a field, in which the maximal ideal $\mathfrak{m}$ is principal, and for which $\cap_{i=1}^{\infty} \mathfrak{m}^n = 0$. Prove that R is a DVR.

Extra curricular

(1) For an example of an ideal which does not have a primary decomposition take a look at this quite readable arxiv preprint: https://arxiv.org/abs/1707.07783
(2) Let k be a field. Prove that there exists a discrete valuation v on $k(x, y)$ such that
$$v(\sum a_{i,j} x^i y^j) = n,$$
where $n = \min\{i + j \mid a_{i,j} \neq 0\}$. Describe the associated discrete valuation ring and find a generator of its maximal ideal.
(3) Let A be an integral domain. Let $v \colon A \to \mathbb{Z} \cup \{\infty\}$ be a function satisfying the three conditions of a valuation and that the elements $\{v(x) \mid x \in A\} \subset \mathbb{Z}$ generate $\mathbb{Z}$ as a group. Prove that v can be extended uniquely to give a discrete valuation on the fraction field of A.