

MAT4200 - 2026 EXERCISES WEEK 2

WARM UPS

- (1) Let $a_1, \dots, a_n, b \in A$, and let $1 \leq i, j \leq n$ with $i \neq j$. Prove that

$$(a_1, \dots, a_n) = (a_1, \dots, a_{j-1}, a_j + ba_i, a_{j+1}, \dots, a_n)$$
- (2) Let $a_1, \dots, a_n \in A$, let $1 \leq i \leq n$, and let $c \in A$ be a unit. Prove that

$$(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, ca_i, a_{i+1}, \dots, a_n).$$
- (3) Let A be a ring, and let $a \in A$. Prove that $A[x]/(x - a) \cong A$.
- (4) Describe the quotient of ideals $((144) : (24))$ in $\mathbb{Z}$.
- (5) Determine the nilradical of $\mathbb{Z}/(30)$.

SUGGESTED PROBLEMS WEEK 2

[AM]: Exercise 1.12 and 1.13 (pages 8,9)

[E]: 2.2, 2.5, 2.26, 2.28, 2.37 (“coprime” is “comaximal” here)

- (1) Let A be a ring, let $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$. Prove that

$$(a_1, \dots, a_n)(b_1, \dots, b_m) = (a_1b_1, a_1b_2, \dots, a_nb_m),$$

where the sequence in the rightmost brackets contains every product a_ib_j with $1 \leq i \leq n$ and $1 \leq j \leq m$.

- (2) Let A be a ring, let $a, b \in A$. Prove that $A[x]/(x - a, b) \cong A/(b)$.
- (3) Let A be a ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals. Prove that $\mathfrak{a}, \mathfrak{b}$ are coprime if and only if there is no maximal ideal $\mathfrak{m}$ with $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$.
- (4) Let A be a ring and $\mathfrak{m}$ a maximal ideal. When are $\mathfrak{m}$ and $\mathfrak{a}$ coprime for any ideal $\mathfrak{a}$ of A .
- (5) Let k be a field, and let $f, g \in k[x]$, show that there exists a polynomial $h \in k[x]$ such that $(f, g) = (h)$.
- (6) Let k be a field, and let $f, g \in k[x]$. Prove that if (f) and (g) are coprime if and only if f and g have no common irreducible factor.¹
- (7) Let k be a field, and $0 \neq f \in k[x]$. Prove that $k[x]/(f)$ is isomorphic to a direct product of rings of the form $k[x]/(g^n)$, where g is irreducible.
- (8) Let $(a_1, b_1) \neq (a_2, b_2) \in \mathbb{C}^2$. Prove that the ideals $(x - a_1, y - b_1)$ and $(x - a_2, y - b_2)$ in $\mathbb{C}[x, y]$ are coprime.
- (9) Show that the ideals $(x - 2)$ and $(2x^2 - 2)$ in $\mathbb{Z}[x]$ are not coprime.
- (10) Let $\mathfrak{a}, \mathfrak{p} \subseteq A$ be ideals with $\mathfrak{p}$ prime such that $\mathfrak{a}$ is not contained in $\mathfrak{p}$. Prove that $(\mathfrak{p} : \mathfrak{a}) = \mathfrak{p}$.
- (11) Let $p_1, \dots, p_k$ be distinct prime numbers, let $m = \prod_{i=1}^k p_i^{e_i}$ and $n = \prod_{i=1}^k p_i^{f_i}$ with $e_i, f_i \geq 0$. Prove that $((m) : (n)) = (r)$ with $r = \prod_{i=1}^k p_i^{\max(e_i - f_i, 0)}$.

¹Recall that in $k[x]$ every ideal is principal, every polynomial can be factored into irreducible polynomials, and $h \in k[x] \setminus \{0\}$ is irreducible if and only if (h) is prime.

EXTRA CURRICULAR

- (12) (*) Let $a_1, \dots, a_n \in A$, and let $(b_{ij})_{1 \leq i, j \leq n}$ be an invertible matrix with coefficients in A .² Prove that

$$(a_1, \dots, a_n) = \left(\sum_{j=1}^n b_{1j} a_j, \sum_{j=1}^n b_{2j} a_j, \dots, \sum_{j=1}^n b_{nj} a_j \right).$$

- (13) (*) Let $A = \mathbb{Z}[x]/(x^2 + 1)$, and let $p \in \mathbb{Z}$ be a prime. Let $\phi: \mathbb{Z} \rightarrow A$ be the natural map, and show that $\phi((p))A \subseteq A$ is a prime ideal if and only if -1 is a quadratic residue modulo p , that is if and only if the equation $x^2 + 1$ admits solutions in $\mathbb{Z}/(p)$. *Hint:* Show that $A/\phi((p))A \cong \mathbb{Z}/(p)[x]/(x^2 + 1)$ and consider what it means for this ring to be an integral domain.
- (14) (*) This exercise proves the implication “ $f \in A$ nilpotent” $\Rightarrow$ f is contained in every prime ideal of A .

We will assume that f is not nilpotent, and construct a prime ideal not containing f . Let $N = \{f^n \mid n \geq 0\}$, and let S be the set of ideals $\mathfrak{a}$ of A such that $\mathfrak{a} \cap N = \emptyset$.

- Prove that S is non-empty.
- Use a Zorn’s lemma argument to prove that S has a maximal element under partial ordering.
- Let $\mathfrak{p} \in S$ be a maximal element, and let $g \notin \mathfrak{p}$. Prove that we can find $n \geq 0, x \in A, y \in \mathfrak{p}$ such that

$$f^n = y + xg.$$

- Prove that $\mathfrak{p}$ is a prime ideal not containing f .

²This means that there exist $(c_{ij})_{1 \leq i, j \leq n}$ with $c_{ij} \in A$ such that

$$\sum_{k=1}^n b_{ik} c_{kj} = \sum_{k=1}^n c_{ik} b_{kj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$