

MAT4200 - 2026 EXERCISES WEEK 3

MODULES

WARM UPS

- (1) Let $R[x]_d$ be the polynomials with coefficients in R of degree less than or equal to d . Exhibit $R[x]_d$ as a R -module. Is it finitely generated? If so, give generators. Is $R[x]_d$ a subring of $R[x]$?
- (2) Can you make $\mathbb{Z}/(3)$ a $\mathbb{Z}/(2)$ -module?
- (3) What is $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2))$ as a set? As a $\mathbb{Z}$ -module? Can you generalise this for any prime p ? What about $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(3), \mathbb{Z}/(2))$?

SUGGESTED PROBLEMS WEEK 3

From [E]: 4.2, 4.11, 4.12, 4.17, 4.18, 4.20

- (1) Let G be an abelian group and $n \geq 1$. Show that G admits a structure of $\mathbb{Z}/(n)$ -module if and only if $\overbrace{g + \cdots + g}^n = 0$ for all $g \in G$. Show that if this condition holds, then the structure of $\mathbb{Z}/(n)$ -module on G is unique.
- (2) Let G be an abelian group, and let $\phi: G \rightarrow G$ be a group homomorphism. Prove that there is a unique structure of $\mathbb{Z}[x]$ -module on G such that $xg = \phi(g)$ for all $g \in G$.
- (3) Let A be an integral domain and $0 \neq a \in A$. Show that A and the ideal (a) are isomorphic as A -modules. Give an example when A is not an integral domain and the statement fails.
- (4) Let A be an integral domain and $I \neq 0$ an ideal. Show that $\text{Hom}_A(A/I, A) = 0$. Give an example where this is not true if you drop the assumption that A is an integral domain.
- (5) Show that $\text{Ann}(M + N) = \text{Ann}(M) \cap \text{Ann}(N)$
- (6) Let M be an A -module. Show that the map $F: \text{Hom}_A(A, M) \rightarrow M$ given by

$$F(\phi) = \phi(1)$$

is an isomorphism of A -modules.

- (7) Let $M_1, \dots, M_n$ and N be A -modules. Prove that $\text{Hom}(M_1 \oplus \cdots \oplus M_n, N)$ is isomorphic as an A -module to $\text{Hom}(M_1, N) \oplus \cdots \oplus \text{Hom}(M_n, N)$.
- (8) Let A be a local ring and M a finitely generated A module. Show that all minimal generating sets of M have the same size. Recall a generating set is minimal if no subset generates M .
- (9) Let $A \subset \mathbb{R}(x)$ be the ring of rational functions defined at 0 as in the lecture notes, with maximal ideal $\mathfrak{m} \subset A$. Thinking of $\mathbb{R}(x)$ as an A -module, prove that $\mathbb{R}(x) = \mathfrak{m}\mathbb{R}(x)$. Why does this not contradict Nakayama's lemma?

EXTRA CURRICULAR

(1) Ellingsrud 4.12

(2) Let $\{M_i\}_{i \in S}$ and N be A -modules. Prove that

$$\operatorname{Hom}\left(\bigoplus_{i \in S} M_i, N\right) \cong \prod_{i \in S} \operatorname{Hom}(M_i, N)$$

and

$$\operatorname{Hom}\left(N, \prod_{i \in S} M_i\right) \cong \prod_{i \in S} \operatorname{Hom}(N, M_i)$$

as A -modules.