

MAT4200 - 2026 EXERCISES WEEK 4

WARM UPS

- (1) Convince yourself that

$$0 \rightarrow \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/(n) \rightarrow 0$$

is exact where $\phi(k) = kn$.

- (2) Find maps that make the following sequence exact

$$0 \rightarrow \mathbb{Z}/(n) \rightarrow \mathbb{Z}/(nm) \rightarrow \mathbb{Z}/(m) \rightarrow 0.$$

Is the above sequence split?

- (3) Write down the long exact sequence from the snake lemma for the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/60\mathbb{Z} & \longrightarrow & \mathbb{Z}/30\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/20\mathbb{Z} & \longrightarrow & \mathbb{Z}/5\mathbb{Z} \longrightarrow 0, \end{array}$$

where the rows are the maps you found in part 2) and the vertical maps are non-zero.

SUGGESTED PROBLEMS WEEK 4

- (1) Let A be a ring, let $M = A \oplus A$, and let $\phi: M \rightarrow M$ be the homomorphism given by $\phi(x, y) = (0, x)$. Prove that the sequence

$$\dots \xrightarrow{\phi} M \xrightarrow{\phi} M \xrightarrow{\phi} M \xrightarrow{\phi} \dots$$

is exact.

- (2) Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules. Prove that if M' and M'' are finitely generated then so is M .
- (3) Prove the five lemma: Given the following commutative diagram with exact rows, if f' and f'' are isomorphisms then so is f .

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' \longrightarrow 0. \end{array}$$

How you weaken the hypothesis if you only want injectivity (or surjectivity) of f ?

- (4) For R -modules M', M, M'' , prove that $M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact if and only if for all R -modules N

$$0 \rightarrow \text{Hom}_R(M'', N) \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M', N)$$

is exact

- (5) Let k be a field, let $n \geq 3$, and let

$$0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_n \rightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Prove that the homomorphism $V_{n-2} \rightarrow V_{n-1}$ is surjective if and only if $\sum_{i=1}^{n-1} (-1)^i \dim V_i = 0$.

EXTRA CURRICULAR

- (1) Let A be a local ring with maximal ideal $\mathfrak{m}$, let $\phi: N \rightarrow M$ be a homomorphism of A -modules with M finitely generated. Use the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{m}N & \longrightarrow & N & \longrightarrow & N/\mathfrak{m}N \longrightarrow 0 \\ & & & & \downarrow \phi & & \downarrow \\ 0 & \longrightarrow & \mathfrak{m}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{m}M \longrightarrow 0, \end{array}$$

to show that if $N/\mathfrak{m}N \rightarrow M/\mathfrak{m}M$ is an isomorphism, then $\mathfrak{m}M \subseteq \phi(\mathfrak{m}N)$.