

MAT4200 - 2026 EXERCISES WEEK 5

WARM UPS

- (1) Show that $R \times R \rightarrow R$ given by $(r, s) = rs$ is a bilinear map but it is not a homomorphism of R -modules. On the other hand, recall or reprove that the map $R \otimes_R R \rightarrow R$ given by $r \otimes s \rightarrow rs$ is an isomorphism.
- (2) Convince yourself that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

are the zero module.

- (3) For two R -modules M and N show that $M \otimes_R N$ we have

$$(m_1 + m_2) \otimes (n_1 + n_2) = m_1 \otimes n_1 + m_1 \otimes n_2 + m_2 \otimes n_1 + m_2 \otimes n_2.$$
- (4) Consider $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ and let e_1, e_2 be a standard basis for $\mathbb{R}^2$. Write down a basis for $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ and express $(2, 3) \otimes (-1, 1)$ in this basis.
- (5) In the same set up as above, show that

$$2e_1 \otimes e_1 + 4e_1 \otimes e_2 + e_2 \otimes e_1 + 2e_2 \otimes e_2$$

can be written as a simple tensor, i.e. in the form $v \otimes w$ for vectors $v, w \in \mathbb{R}^2$.

- (6) Use the above to notice that the map $g : M \times N \rightarrow T$ sending $(m, n) \mapsto m \otimes n$ is NOT a R -module homomorphism.
- (7) Verify for yourself using the relations of the tensor product that g above is R -bilinear.

SUGGESTED PROBLEMS WEEK 5

- (1) Let $R = \mathbb{Z}[x]$ and consider the ideal $I = (2, x)$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ cannot be written as a simple tensor.
- (2) Let R and I be as above. Show that the element $2 \otimes x - x \otimes 2$ is annihilated by 2 and also by x .
- (3) Prove that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/c\mathbb{Z},$$

where $c = \gcd(m, n)$. Begin by proving that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is generated by a single element $1 \otimes 1$.

- (4) More generally, if R is any ring and I and J ideals then

$$R/I \otimes_R R/J \cong R/(I + J).$$

Hint: First prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$. Then show that there is an R -module homomorphism sending

$$(r + I) \otimes (r' + J) \mapsto rr' + (I + J).$$

- (5) Ellingsrud Exercise 6.1 on interchanging tensor products with arbitrary direct sums.
- (6) Prove that $M_1 \oplus M_2$ is flat if and only if both M_1 and M_2 are flat. Extra curricular: generalise this to arbitrary direct sums.

- (7) Prove that if M and N are both flat R -modules then $M \otimes N$ is a flat R module.

EXTRA CURRICULAR

- (1) The following two examples show that the tensor product does not commute with infinite direct products.

Show that there is a canonical module homomorphism

$$M \otimes \left(\prod_{i \in I} N_i \right) \rightarrow \prod_{i \in I} (M \otimes N_i),$$

given on elementary tensors by $x \otimes (y_i) \mapsto (x \otimes y_i)$.

This homomorphism may be neither injective nor surjective.

- Consider $R = \mathbb{Z}$ and $M = \mathbb{Q}$ and $N_n = \mathbb{Z}/n\mathbb{Z}$ for all $n \in \mathbb{N}$. Show that the canonical map above is not injective. Hint the right hand side is zero so it suffices to find a non-zero element on the left.
 - Consider $R = \mathbb{Z}$ and $M = \mathbb{Q}$ and $N_n = \mathbb{Z}$ for all n . Show that the map above is not surjective.
- (2) Prove directly that $- \otimes_R M$ is a right exact functor (i.e. without using the Hom trickery).
- (3) Show that $R[x]$ is a flat R module. Hint: Prove the extra curricular part of 6.