

MAT4200 - 2026 EXERCISES WEEK 5

WARM UPS

- (1) Consider $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as the tensor product of algebras over $\mathbb{R}$. Write

$$((2+i) \otimes (i))((1-i) \otimes (1+i))$$

in the basis $1 \otimes 1, i \otimes 1, 1 \otimes i, i \otimes i$.

- (2) Consider the tensor product of $\mathbb{R}$ algebras $\mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y]$. Write the product

$$((x^2 + 1) \times (y + 2))((x + 1) \otimes (y^2))$$

as a sum $\sum a_{ij} x^i \otimes y^j$.

- (3) Is an ideal a multiplicatively closed subset of a ring?
 (4) Show that the intersection of multiplicatively closed subsets is multiplicatively closed.

SUGGESTED PROBLEMS WEEK 6

- (1) Let B be an A -algebra, and let $I \subset B$ be an ideal. Prove that if B is a finite A -algebra, then so is B/I .
 (2) Let B be an A -algebra, and let $I \subset B$ be an ideal. Prove that if B is a finite type A -algebra, then so is B/I .
 (3) Let B be an A -algebra. Prove that

$$A[x] \otimes_A B \cong B[x]$$

- (4) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}$ as a ring.
 (5) Let R be a ring, let B and C be R -algebras, with R -algebra structure given by

$$\psi_B: R \rightarrow B \text{ and } \psi_C: R \rightarrow C.$$

Let $\phi_B: B \rightarrow B \otimes_R C$ and $\phi_C: C \rightarrow B \otimes_R C$ be given by $\phi_B(b) = b \otimes 1$ and $\phi_C(c) = 1 \otimes c$. Verify that the following diagram of ring homomorphisms is commutative.

$$\begin{array}{ccc} R & \xrightarrow{\psi_B} & B \\ \downarrow \psi_C & & \downarrow \phi_B \\ C & \xrightarrow{\phi_C} & B \otimes_R C, \end{array}$$

that is, that $\phi_B \circ \phi_B = \phi_C \circ \psi_C$.

- (6) Let I be an ideal of R . Show that $1 + I$ is a multiplicatively closed subset.
 (7) Suppose that R is a local ring with maximal ideal $\mathfrak{m}$. Show that $R_{\mathfrak{m}} \cong R$, where $R_{\mathfrak{m}}$ is the localization at the maximal (hence prime) ideal $\mathfrak{m}$.
 (8) Let $S \subseteq R$ be a multiplicatively closed subset. Show that the kernel of the homomorphism $\phi: R \rightarrow S^{-1}R$ is the set

$$\bigcup_{s \in S} \text{Ann}(s) \subseteq R$$

- (9) Let $n \geq 1$ be an integer with prime factorisation $n = p_1^{e_1} \dots p_k^{e_k}$, let $l \in \mathbb{Z}$, and let S be the set of i such that p_i does not divide l . Thinking of l as an element of $\mathbb{Z}/n$, prove that

$$(\mathbb{Z}/n)_l \cong \mathbb{Z}/m$$

where $m = \prod_{i \in S} p_i^{e_i}$.

- (10) Let R be a ring and $f \in R$. Prove that $R[x]/(fx - 1) \cong R_f$.
 (11) Let R be a ring and let $S \subseteq R$ be a multiplicatively closed subset. Prove that if $S^{-1}R = 0$, then $0 \in S$.
 (12) Prove that any intermediate ring $\mathbb{Z} \subset R \subset \mathbb{Q}$ isomorphic to the localization of $\mathbb{Z}$ in a multiplicatively closed set S .
 (13) If S and T are two multiplicatively closed subsets of R . Let U be the image of T in $f : R \rightarrow S^{-1}R$. Show that the rings $(ST)^{-1}R \cong U^{-1}(S^{-1}R)$.
 (14) Let I be an ideal of R and $S = 1 + I$. If M is a finitely generated R module. Show that $S^{-1}M = 0$ if and only if there is a $s \in S$ such that $sM = 0$.

EXTRA CURRICULAR

- (1) Exercise 7 of Atiyah-MacDonald on saturated multiplicatively closed subsets.
 (2) With notation as in Problem (??) let D be a ring, and let $\rho_B : B \rightarrow D$ and $\rho_C : C \rightarrow D$ be ring homomorphisms such that $\rho_B \circ \psi_B = \rho_C \circ \psi_C$. Prove that there is a unique ring homomorphism

$$\rho : B \otimes_A C \rightarrow D$$

such that $\rho_B = \rho \circ \phi_B$ and $\rho_C = \rho \circ \phi_C$.

In diagrams: Prove that the commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\psi_B} & B \\ \downarrow \psi_C & & \downarrow \phi_B \\ C & \xrightarrow{\phi_C} & B \otimes_R C \end{array} \quad \begin{array}{c} \nearrow \rho_B \\ \searrow \rho_C \end{array} \quad \begin{array}{c} \\ \\ D \end{array}$$

can be extended uniquely to a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{\psi_B} & B \\ \downarrow \psi_C & & \downarrow \phi_B \\ C & \xrightarrow{\phi_C} & B \otimes_R C \end{array} \quad \begin{array}{c} \nearrow \rho_B \\ \searrow \rho_C \end{array} \quad \begin{array}{c} \\ \nearrow \rho \\ D \end{array}$$