

MAT4200 - 2026 EXERCISES WEEK 7

SUGGESTED PROBLEMS WEEK 7

- (1) Let A be a ring and $I \subset A$ an ideal. If P a prime ideal such that I is not contained in P prove that $(A/I)_P = 0$ and that $I_P \cong A_P$ as A_P -modules.
- (2) Let M be an A -module, and let $f \in A$ be such that $fm = 0$ for all $m \in M$. Prove that $M_P = 0$ for all prime ideals $P \subset A$ with $f \notin P$.
- (3) Let M be a *finitely generated* A -module, and let $P \subset A$ be a prime ideal such that $M_P = 0$. Prove that there exists an $f \in A \setminus P$ such that $fm = 0$ for all $m \in M$.
- (4) Show that the condition “finitely generated” in the previous problem is necessary as follows. Let M be the $\mathbb{Z}$ -module

$$M = \bigoplus_{n \geq 1} \mathbb{Z}/(n).$$

Prove that $M_{(0)} = 0$ and that there is no $f \in \mathbb{Z}$ such that $fm = 0$ for all $m \in M$.

- (5) Atiyah-Macdonald Exercise 3.8.

EXTRA CURRICULAR

- (1) Show that “being finitely generated” is not a local property of A -modules, i.e. give an example of a ring A and a module M such that all localisations $M_{\mathfrak{p}}$ are finitely generated $A_{\mathfrak{p}}$ -modules, but M is not a finitely generated A -module.
- (2) Prove that the functor $M \mapsto S^{-1}M$ does not commute with infinite intersections of submodules; that is, give an example of a ring A , a multiplicatively closed subset $S \subseteq A$, an A -module M , and an infinite set of submodules $\{M_i\}_{i \in T}$ of M such that

$$\bigcap_{i \in T} S^{-1}M_i \neq S^{-1}\left(\bigcap_{i \in T} M_i\right).$$

Hint: There is an example with $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$.