

## MAT4200 - 2026 EXERCISES WEEKS 8 AND 9

### WARM UPS

- (1) Is the ring  $\mathbb{Z}$  Noetherian? Is the ring  $\mathbb{Z}$  Artinian?
- (2) Is the ring  $k[x]$  Noetherian for  $k$  a field? Is the ring  $k[x_1, x_2, \dots]$  for a field  $k$  Noetherian?
- (3) What is the length of the  $\mathbb{Z}$ -module  $\mathbb{Z}/(18)$ ?
- (4) Let  $R = \mathbb{Q}[x]$  and  $M = R/(x^2)$ . Is the  $R$ -module  $M$  simple? What is its length?

### SUGGESTED PROBLEMS WEEKS 8 AND 9

- (1) Let  $M$  be an  $R$ -module and  $u : M \rightarrow M$  a module homomorphism. Prove that if  $M$  is Noetherian and  $u$  is surjective then  $u$  is an isomorphism.  
 Prove that if  $M$  is Artinian and  $u$  is injective, then  $u$  is an isomorphism.  
 Hint: consider the kernels and cokernels of  $u^n$ .
- (2) Let  $M$  be an  $R$ -module. Prove that if every non-empty set of finitely generated submodules of  $M$  has a maximal element, then  $M$  is Noetherian.
- (3) Let  $M$  be an  $R$ -module and let  $N_1, N_2$  be submodules of  $M$ . Prove that if  $M/N_1$  and  $M/N_2$  are Noetherian, then so is  $M/(N_1 \cap N_2)$ . Prove the analogous statement with Noetherian replaced with Artinian.
- (4) Prove that if a direct sum of Noetherian modules is Noetherian, then the sum is finite and all the summands are Noetherian.
- (5) Show that a direct sum of finitely many simple modules is both Noetherian and Artinian.
- (6) Let  $M$  be a Noetherian  $R$ -module. Prove that  $R/\text{Ann}(M)$  is a Noetherian ring.

### EXTRA CURRICULAR

- (1) Let  $k$  be a field, and consider

$$k[x]_x = \left\{ \frac{f}{x^n} \mid n \geq 0, f \in k[x] \right\} \subseteq k(x)$$

as a  $k[x]$ -module, along with the submodule  $k[x] \subseteq k[x]_x$ . Prove that (as  $k[x]$ -modules)

- $k[x]_x$  is not Noetherian
- $k[x]_x$  is not Artinian
- $k[x]_x/k[x]$  is not Noetherian
- $k[x]_x/k[x]$  is Artinian (\*)

For the final point, let  $M \subseteq k[x]_x/k[x]$  be a  $k[x]$ -submodule and prove that

$$\frac{x^n + \cdots + a_0}{x^m} \in M \Leftrightarrow x^{n-m} \in M.$$

Use this to prove that if  $0 \subsetneq M \subsetneq k[x]_x/k[x]$ , then there is some  $n$  such that

$$M = \left\{ \frac{f}{x^n} \mid f \in k[x] \right\}.$$