

Lecture 1 Rings Review

Goals: Definitions + Examples

Rings

Ideals

Homomorphisms

Def A (commutative) ring is a set R together with binary operations

- addition $a + b \in R \quad \forall a, b \in R$
- multiplication $ab \in R \quad \forall a, b \in R$

Satisfying the following axioms:

1) $(\mathbb{R}, +)$ is an abelian group with "additive" identity element $0 \in \mathbb{R}$

2) $(ab)c = a(bc)$ associativity of multiplication

3) $ab = ba$ commutativity of multiplication

4) $(a+b)c = ac + bc$ distributivity

5) $\exists$ multiplicative identity $1 \in \mathbb{R}$:

$$\forall a \in \mathbb{R} \quad 1 \cdot a = a \cdot 1 = a$$

From the axioms:

$$(-a)(-b) = ab, \quad -a(b) = -ab$$

$$0a = 0$$

Examples $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
are all rings.

Trivial example: 0 ring is
 $R = \{0\}$, In particular $0 = 1$.

Exercise: If $0 = 1 \in R$ then
 $R = \{0\}$

Non-examples 1) $\mathbb{N} = \{0, 1, \dots\}$
is closed under addition and
multiplication but is not a
ring. $\nexists$ additive inverses

2) $2\mathbb{Z} = \{-2k \mid k \in \mathbb{Z}\}$ has
no multiplicative identity!

Example. $R = \{ f: \mathbb{R} \rightarrow \mathbb{R} \}$ forms
a ring

More generally for any set A and
ring R

$R^A := \{ f: A \rightarrow R \}$ is a
ring. Exercise.

Polynomial rings: R a ring
polynomials in one variable over R
form a ring $R[x]$

$R[x] = \{ a_n x^n + \dots + a_1 x + a_0 \mid a_i \in R \}$
addition + multiplication of polynomials

In n -variables: $R[x_1, \dots, x_n] =$

$$\left\{ \sum_{i_1, \dots, i_n \geq 0} a_{i_1, i_2, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} \mid \text{finite sum} \right\}$$

$a_{i_1, \dots, i_n} \in R$

Exercise $(R[x])[y] = R[x, y]$.

Polynomial rings $\Rightarrow$ algebraic geometry.

Homomorphisms

S, R rings $\phi: S \rightarrow R$ is a
homomorphism if

- 1) $\phi(1_S) = 1_R$
- 2) $\phi(aa') = \phi(a) \cdot \phi(a') \quad \forall a, a' \in S$
- 3) $\phi(a+a') = \phi(a) + \phi(a') \quad \forall a, a' \in S$

If ϕ is bijective call it an isomorphism ($\Rightarrow \phi^{-1}(0) = 0$ and $\phi(S) = R$).

A subring $S \subseteq R$ is a subset closed under addition + multiplication and $1_R \in S$.

The inclusion map $i: S \rightarrow R$ is a homomorphism.

Example $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R} \rightarrow \mathbb{C}$ are ring inclusions

Example evaluation homomorphism

$$\begin{aligned} \phi_a: R[x] &\rightarrow R \\ f(x) &\mapsto f(a) \end{aligned}$$

also works for R^A A set
R ring.

The kernel of $\phi: S \rightarrow R$

$$\text{Ker } \phi = \{ a \in S \mid \phi(a) = 0_R \}.$$

Example. $\phi_a: R[x] \rightarrow R$

$$\text{Ker } \phi_a = \{ f \mid a \text{ is a root of } f \}.$$

Exercise: ?

Ideals.

Def A subset I of a ring R is an ideal if

1) I is a subgroup of $(R, +)$

$\left(\begin{array}{l} 0_R \in I, \\ I \text{ closed under addition and inverses} \end{array} \right)$

2) $\forall r \in R$ and $a \in I$ $ra \in I$

Note: If I ideal then $1 \in I \iff$
 $I = R$. $I \neq R$ is called proper.

Trivial ideal: $\{0\}$

Example $\phi: S \rightarrow R$ homom. then

$\text{Ker } \phi \subseteq S$ is an ideal.

Proof. $(\text{Ker}(\phi), +) \subseteq (\mathcal{S}, +)$
subgroup

$$\phi(0_S) = 0_R \Rightarrow 0_S \in \text{Ker} \phi$$

$$\begin{aligned} \text{if } a, b \in \text{Ker} \phi &\Rightarrow \phi(a+b) \\ &= \phi(a) + \phi(b) \\ &= 0 + 0 = 0 \end{aligned}$$

$$\Rightarrow a+b \in \text{Ker} \phi$$

$$\text{and } \phi(-a) = -\phi(a) = -0 = 0$$

$$\Rightarrow -a \in \text{Ker} \phi$$

Moreover, if $a \in \text{Ker} \phi$ then

$$\phi(sa) = \phi(s)\phi(a) = \phi(s) \cdot 0 = 0$$

$$\text{for } \forall s \in \mathcal{S} \Rightarrow sa \in \text{Ker} \phi$$

$\Rightarrow \text{Ker} \phi$ is an ideal.

Def. For $x \in R$ the principal ideal generated by x is
 $(x) := \{ ax \mid a \in R \} \subseteq R.$

Example All ideals of $\mathbb{Z}$ are principal

Suppose $I \subseteq \mathbb{Z}$ is an ideal

consider $\min \{ a \mid a > 0, a \in I \} = n$

Claim $I = (n)$

Certainly $(n) \subseteq I.$

If $I \neq (n)$ then $\exists k \in I$ s.t.

$n \nmid k$ so $k = nx + r$ $0 < r < n$

but $r = nx - k \in I. \Rightarrow \Leftarrow$

$\mathbb{Z}$ is a principal ideal domain (PID).

Quotient Rings If $I \subset R$ is an ideal the quotient ring

R/I has elements:

$$\{x + I \mid x \in R\}.$$

$$\left(\begin{array}{l} \text{Notice } x + I = x' + I \iff \\ x - x' \in I \end{array} \right)$$

with operations:

$$\left. \begin{array}{l} (x + I) + (y + I) = x + y + I \\ (x + I)(y + I) = xy + I \end{array} \right\} \text{well defined.}$$

$$0_{R/I} = 0 + I = I \quad 1_{R/I} = 1_R + I.$$

The quotient homomorphism

$$\phi: R \rightarrow R/I$$

$$r \mapsto r + I.$$

$$\text{Ker } \phi = I.$$

Examples 1) $R/R \cong \{0\}$ $R/(0) \cong R.$

$$2) \mathbb{Z}/(n) \cong \{0 + (n), 1 + (n), \dots, n-1 + (n)\}.$$

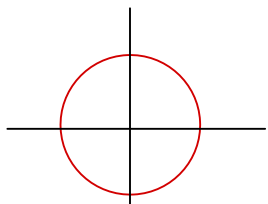
$$\mathbb{Z} \rightarrow \mathbb{Z}/(n)$$

$m \mapsto r + (n)$ where r is remainder
when dividing by n .

The fundamental homomorphism theorem For $\phi: S \rightarrow R$ a ring homomorphism, the image $\phi(S)$ is a subring of R and we have $S/\text{Ker } \phi \cong \phi(S)$

ϕ is injective if and only if $\text{Ker } \phi = \{0\}$

Examples $f \in \mathbb{C}[x, y]$ $f = x^2 + y^2 - 1$.



Unit circle = $\{(x, y) \mid f(x, y) = 0\}$

$\mathbb{C}[x, y] \rightarrow \frac{\mathbb{C}[x, y]}{(f)} \leftarrow$ polynomial functions on the unit circle