

Restriction + Extension of Scalars

$f: R \rightarrow S$ a rng homomorphism.

restriction of scalars: N an S -module
can be made an R -module

$$(r, n) \mapsto f(r) \cdot n$$

Recall we saw that f gives an
 R -module structure to $S (= N)$.

Extension of scalars: M R -module

then form $M_S := S \otimes_R M$

and give S -module structure via

$$(s, s' \otimes m) \mapsto ss' \otimes m. \quad (\text{well-defined})$$

Prop 1) If N is finitely generated as an S module and S is finitely generated as an R module, then N is finitely generated as a R module.

Restriction of scalar

Proof If $x_1, \dots, x_n$ generate N over S and $y_1, \dots, y_m$ generate S over R then $\{x_i y_j\}$ generate N over R .

2) If M is finitely generated as an R module, then M_S is finitely generated as an S module.

Proof $1 \otimes x_i$ generate M_S over S if x_i generate M over R .

Def An R -algebra B is a ring with a ring homomorphism $f: R \rightarrow B$. Hence a pair (B, f) .

B an R -algebra $\Rightarrow B$ is a ring and an R -module:

$$r \cdot b := f(r) \cdot b$$

An R -algebra homomorphism is a homomorphism $\phi: B_1 \rightarrow B_2$ with $f_2 = \phi \circ f_1$.

$$\begin{array}{ccc} R & \xrightarrow{f_1} & B_1 \xrightarrow{\phi} B_2 \\ & \searrow f_2 & \nearrow \end{array}$$

Examples Every ring R is a $\mathbb{Z}$ -algebra in a unique way. $f: \mathbb{Z} \rightarrow R$
 $f(n) = n1_R$.

Every homomorphism of rings is a homomorphism of $\mathbb{Z}$ -algebras.

Example For any R $f: R \rightarrow R[x_1, \dots, x_n]$
 $r \mapsto r$

makes $R[x_1, \dots, x_n]$ an R algebra.

Also, $R[x_1, \dots, x_n]/I$ is an R -algebra with

$$\bar{f}: R \rightarrow R[x_1, \dots, x_n]/I$$
$$r \mapsto r + I$$

Def Let B be an R -algebra

• B is finite R -algebra if finitely gen
as an R -algebra ($\exists b_1, \dots, b_n \in B$ s.t.

$$\forall b \in B \quad b = \sum a_i b_i$$

• B is a finite type R -algebra if
 $\exists$ a finite set of elements $b_1, \dots, b_n \in B$
s.t. $\forall b \in B \quad b = \sum a_{i_1, \dots, i_n} b_1^{i_1} \dots b_n^{i_n}$.

Notice finite $\Rightarrow$ finite type.

Not the other way around.

Example $R[x_1, \dots, x_n]$ is finite type but not finite as R -module.

Take $b_i = x_i$

Proposition Let B be an R -algebra. TFAE:

- 1) B is of finite type
- 2) $\exists$ a surjective R -algebra homomorphism

$$\phi: R[x_1, \dots, x_n] \rightarrow B$$

- 3) $\exists$ an R -algebra isomorphism

$$R[x_1, \dots, x_n] / \mathcal{I} \rightarrow B.$$

Proof 1 $\rightarrow$ 2 the map sending $x_i \mapsto b_i$ is surjective

2 $\rightarrow$ 3 Fundamental homom. thm $\Rightarrow$

$\frac{R[x_1, \dots, x_n]}{\ker \phi} \cong B$ as rngs. Check alg. homom.

3 $\rightarrow$ 1 let $\overline{x_i} = x_i + I$ then every elt
of $\frac{R[x_1, \dots, x_n]}{I} \cong B$ can be written as

$$\sum a_i \dots i_n \overline{x_1}^{i_1} \dots \overline{x_n}^{i_n} \rightarrow \text{finite type.}$$

Prop If B, C are R -algebras then
 $B \otimes_R C$ is an R -algebra with multiplication
defined by

$$(b \otimes c)(b' \otimes c') = bb' \otimes cc' \quad \leftarrow \text{elementary tensors}$$

$$\left(\sum b_i \otimes c_i \right) \left(\sum b'_j \otimes c'_j \right) = \sum_{i,j} (b_i b'_j \otimes c_i c'_j)$$

Proof The homom. $R \rightarrow B \otimes_R C$ is
 $\varphi \otimes \varphi$
 $R \cong R \otimes_R R \xrightarrow{\varphi \otimes \varphi} B \otimes_R C$

must check that multiplication is
well defined. Exercise

Example Let $B = R[x]$ $C = R[y]$ then
as R -modules (not as rings!)

$$B = \bigoplus R x^i \quad C = \bigoplus R y^j$$

$$B \otimes_R C \cong (\bigoplus R x^i) \otimes (\bigoplus R y^j)$$

$$\cong \bigoplus R x^i \otimes y^j$$

$$\text{product } (x^{i_1} \otimes y^{j_1})(x^{i_2} \otimes y^{j_2}) = x^{i_1+i_2} \otimes y^{j_1+j_2}$$

There is an isomorphism of R -algebras

$$B \otimes_R C \longrightarrow R[x, y]$$

Example $\mathbb{C}$ is an $\mathbb{R}$ -algebra

$$\mathbb{C} \cong \mathbb{R}1 \oplus \mathbb{R}i \quad \mathbb{R}\text{-vector space, 2 dim!}$$

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{R}$ vector space with

$$\text{basis } \begin{array}{cccc} 1 \otimes 1 & 1 \otimes i & i \otimes 1 & i \otimes i \\ 1 & x & y & z \end{array}$$

$$x^2 = (1 \otimes i)(1 \otimes i) = (1 \otimes i^2) = -(1 \otimes 1) = -1$$

$$y^2 = -1$$

$$z^2 = (i \otimes i)(i \otimes i) = 1 \otimes 1 = 1$$

$$xy = z$$

$$yz = -x$$

$$zx = -y$$

As a ring $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$.



$$\phi(u, v) = \frac{1}{2} (u+v) \otimes 1 + \frac{1}{2} \frac{(u-v)}{i} 1 \otimes i$$

$$\text{ie } \phi(e_1) = \frac{1}{2} (1 \otimes 1) - \frac{1}{2} (i \otimes i)$$

$$\phi(e_2) = \frac{1}{2} (1 \otimes 1) + \frac{1}{2} (i \otimes i)$$



This is a consequence that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

$$\cong \mathbb{C} \times \mathbb{C}.$$

Rings of Fractions (Chap 2 AM)

If R integral domain, we can form

$K(R)$ - field of fractions

"smallest" ring containing R s.t. $\forall$
 $r \in R$ has $r^{-1} \in K(R)$.

Examples 1) $R = \mathbb{Z}$ $K(\mathbb{Z}) = \mathbb{Q}$. $\frac{n}{m} \neq 0$

$$\frac{n}{m} = \frac{n'}{m'} \quad \text{iff} \quad nm' - mn' = 0$$

2) $R = K[x]$ $K(K[x])$ rational func's

$$\frac{f(x)}{g(x)} = \frac{\tilde{f}(x)}{\tilde{g}(x)} \iff f(x)\tilde{g}(x) - \tilde{f}(x)g(x) = 0 \quad \begin{matrix} g(x) \neq 0 \\ \tilde{g}(x) \neq 0 \end{matrix}$$

In general, $K(R)$ is constructed from

1) $(a, b) \in R \times R \quad b \neq 0$

2) $(a, b) \sim (a', b')$ if $ab' - a'b = 0$.

need R to be an integral domain
for $\sim$ to be an equivalence rel'n.

4) $K(R) = \underbrace{\{(a, b) \mid a, b \in R \quad b \neq 0\}}_{\sim}$

5) Addition

Multiplication

$$(a, b) + (a', b') = (ab' + a'b, bb')$$

$$(a, b)(a', b') = (aa', bb')$$

3) If R not integral domain $\mathbb{Z}/(4)$

$$(2, 2) \sim (1, 1) \quad 2 \cdot 1 - 1 \cdot 2 = 0$$

$$(2, 2) \sim (0, 2) \quad 2 \cdot 2 - 0 \cdot 2 = 0$$

$$(1, 1) \sim (0, 2) \quad 2 \cdot 1 - 1 \cdot 0 = 2 \neq 0$$

So $\sim$ is not an equivalence relation

Goal Given R a ring and a subset $S \subseteq R$ construct a new ring $S^{-1}R$ where if $s \in S$ then $\exists s^{-1} \in S^{-1}R$.

Def $S \subseteq R$ is multiplicatively closed if

1) $1 \in S$

2) if $s_1, s_2 \in S$ then $s_1 s_2 \in S$.

Note 0 is not forbidden in S !!

Example let P be a prime ideal
then $S = R - P$ is multiplicatively closed
($S = R - I$ closed $\Rightarrow I$ prime)

(This will be a main example)

Def Ring of fractions R a ring and $S \subseteq R$ multiplicatively closed set
The ring of fractions with respect to

S is

$$S^{-1}R = \frac{\{(a, s) \mid a \in R, s \in S\}}{\sim}$$

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ s.t. } (at - sb)u = 0.$$

with operations:

$$(a, s) + (b, t) = (at + bs, st)$$

$$(a, s)(b, t) = (ab, st).$$

$$\text{write } (a, s) = \frac{a}{s}.$$

$$\text{identities are } (0, 1) = 0_{S^{-1}R} \quad (1, 1) = 1_{S^{-1}R}.$$

There is a ring homomorphism

$$\phi: R \rightarrow S^{-1}R \quad r \mapsto \frac{r}{1}.$$

Example If $0 \in S$ then $S^{-1}R \cong 0$