

Rings of Fractions (Chap 2 AM)

If R integral domain, we can form

$K(R)$ - field of fractions

"smallest" ring containing R s.t. $\forall r \in R$ has $r^{-1} \in K(R)$.

Examples 1) $R = \mathbb{Z}$ $K(\mathbb{Z}) = \mathbb{Q}$. $\frac{n}{m} \neq 0$

$$\frac{n}{m} = \frac{n'}{m'} \quad \text{iff} \quad nm' - mn' = 0$$

2) $R = K[x]$ $K(K[x])$ rational func's

$$\frac{f(x)}{g(x)} = \frac{\tilde{f}(x)}{\tilde{g}(x)} \iff f(x)\tilde{g}(x) - \tilde{f}(x)g(x) = 0 \quad \begin{matrix} g(x) \neq 0 \\ \tilde{g}(x) \neq 0 \end{matrix}$$

In general, $K(R)$ is constructed from

1) $(a, b) \in R \times R \quad b \neq 0$

2) $(a, b) \sim (a', b')$ if $ab' - a'b = 0$.

need R to be an integral domain
for $\sim$ to be an equivalence rel'n.

4) $K(R) = \underbrace{\{(a, b) \mid a, b \in R \quad b \neq 0\}}_{\sim}$

5) Addition

Multiplication

$$(a, b) + (a', b') = (ab' + a'b, bb')$$

$$(a, b)(a', b') = (aa', bb')$$

3) If R not integral domain $\mathbb{Z}/(4)$

$$(2, 2) \sim (1, 1) \quad 2 \cdot 1 - 1 \cdot 2 = 0$$

$$(2, 2) \sim (0, 2) \quad 2 \cdot 2 - 0 \cdot 2 = 0$$

$$(1, 1) \sim (0, 2) \quad 2 \cdot 1 - 1 \cdot 0 = 2 \neq 0$$

So $\sim$ is not an equivalence relation

Goal Given R a ring and a subset $S \subseteq R$ construct a new ring $S^{-1}R$ where if $s \in S$ then $\exists s^{-1} \in S^{-1}R$.

Def $S \subseteq R$ is multiplicatively closed if

1) $1 \in S$

2) if $s_1, s_2 \in S$ then $s_1 s_2 \in S$.

Note 0 is not forbidden in S !!

Examples 1) $S =$ non-zero divisors mult. closed
if x, y are non-zero divisors and $(xy)z = 0$ then $x(yz) = 0$ so x, y are zero divisors $\Rightarrow$ '.

2) let P be a prime ideal then $S = R - P$ is multiplicatively closed
($S = R - I$ closed $\Rightarrow I$ prime)

(This will be a main example)

Def Ring of fractions R a ring and $S \subseteq R$ multiplicatively closed set

The ring of fractions with respect to

S 13

$$S^{-1}R = \frac{\{(a, s) \mid a \in R, s \in S\}}{\sim}$$

$$(a, s) \sim (b, t) \iff \exists u \in S \text{ s.t. } (at - sb)u = 0.$$

EQUIVALENCE RELATION NEXT PAGE.

with operations:

$$(a, s) + (b, t) = (at + bs, st)$$

$$(a, s)(b, t) = (ab, st).$$

Write $(a, s) = \frac{a}{s}$.

identities are $(0, 1) = 0_{S^{-1}R}$ $(1, 1) = 1_{S^{-1}R}$.

Example If $0 \in S$ then $S^{-1}R \cong 0$

since $(at - sb) \cdot 0 = 0 \quad \forall a, b \in R, s, t \in S$.

In fact, $S^{-1}R = 0 \iff 0 \in S$.

" $\sim$ " is an equivalence relation.

" $\sim$ " is symmetric + reflexive,

To see it is transitive consider

$$(a, s) \sim (b, t) \text{ and } (b, t) \sim (c, u).$$

$$\text{then } (at - bs) \cdot v = 0 \text{ for some } v \in S$$

$$(bu - ct) \cdot w = 0 \text{ for some } w \in S$$

eliminate b :

$$(at - bs) \cdot v \cdot (uw) = 0$$

$$(bu - ct) \cdot w \cdot (sv) = 0$$

Now add equations

$$(au - cs) \cdot \underbrace{tvw} = 0$$

$\in S$ since
mult. closed

$$\Rightarrow (a, s) \sim (c, u) \text{ and } \sim \text{ is transitive}$$

Prop The operations of addition + and multiplication are well defined.

$$\text{i.e., if } (a, s) \sim (a', s') \quad (b, t) \sim (b', t')$$

$$\text{then } (a, s) + (b, t) \sim (a', s') + (b', t')$$

$$\text{same } (a, s) \cdot (b, t) \sim (a', s') \cdot (b', t'),$$

and $S^{-1}R$ is a commutative ring with identity.

There is a ring homomorphism

$$f: R \rightarrow S^{-1}R \quad r \mapsto \frac{r}{1}.$$

Proof This week's exercises.



The homomorphism $f: R \rightarrow S^{-1}R$ may not be injective!

Example 1) When R is an integral domain and $S = R - 0$ then $S^{-1}R = K(R)$.

2) "localisation at P " P prime ideal

If P is prime ideal then $S = R - P$ is multiplicatively closed.

$1 \in S = R - P$ if $s_1, s_2 \notin P$ then

$s_1 s_2 \notin P$. Prove from contrapositive

If $s_1 s_2 \in P \Rightarrow$ either $s_1 \in P$ or $s_2 \in P$.
(definition of P prime)

Write $R_P := S^{-1}R$

Prop The ring R_P for P prime ideal is local.

Proof R_P is local $\iff$ non-units form an ideal

Claim

$\mathcal{M} = \left\{ \frac{a}{s} \mid a \in P, s \notin P \right\} \subseteq R_P$ is an ideal
Check.

$\frac{1}{s} \notin M$ since $\frac{1}{s} = \frac{a}{s} \iff (s-a) \cdot u = 0 \nexists u$
but $a \in P$ $s \notin P$ so $s-a \notin P$ so we can't
have $\frac{1}{s} \in M$ so the elements of M are non
units.

if $\frac{b}{s} \in R_P$ is not in M then $b \notin P$

and hence b is a unit in R_P and so
 $\frac{b}{s}$ is a unit. $\implies M$ is max ideal

this means any ideal $I \subseteq R_P$ not
contained in M contains a unit.

$\implies R_P$ is local and M is unique
max ideal. $\square$

3) $f \in R$ and let $S = \{f^n \mid n \geq 0\}$
write $R_f = S^{-1}R$.

4) If $R = \mathbb{Z}$ and $P = (p)$ p prime

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \mid \gcd(n, p) = 1 \right\} \subseteq \mathbb{Q}$$

This is a local ring with max ideal
 $\mathcal{M} = \left\{ \frac{p^n}{n} \mid p \nmid n \right\}$

Example $P = (x) \subseteq R[x]$ Then

$$R[x]_{(x)} = \left\{ \frac{f}{g} \mid g \notin (x) \right\}$$

rational functions without poles at 0.

Prop (Universality of $S^{-1}R$)

Let $\phi: R \rightarrow R'$ be a ring homomorphism
 s.t. $\forall s \in S \quad \phi(s)$ is a unit in R' .

Then there exists a unique homomorphism

$$\rho: S^{-1}R \rightarrow R' \quad \text{s.t.}$$

$$\begin{array}{ccc} R & \xrightarrow{\iota} & S^{-1}R \xrightarrow{\tau} \\ & \searrow \phi & \downarrow \rho \\ & & R' \end{array}$$

Proof Uniqueness. Let $r \in R$, then

$$\rho\left(\frac{r}{1}\right) = \rho(f(r)) = \phi(r)$$

Let $s \in S$ then

$$\begin{aligned}\rho\left(\frac{1}{s}\right) &= \rho(s^{-1}) = \rho(f(s)^{-1}) \\ &= (\rho(f(s)))^{-1} = \phi(s)^{-1}\end{aligned}$$

Since $\rho\left(\frac{r}{s}\right) = \rho(r) \rho\left(\frac{1}{s}\right)$, ρ is determined.

Existence Define $\rho\left(\frac{r}{s}\right) := \phi(r) \phi(s)^{-1}$

check it is well-defined $\square$.

Modules on $S^{-1}R$

M R -module, $S \subseteq R$ multiplicatively closed

Def The $S^{-1}R$ module $S^{-1}M$ is the

$$\underbrace{\{(m, s) \mid m \in M, s \in S\}}_{\sim}$$

where $(m, s) \sim (n, q) \iff \exists u \in S \mid (qm - sn)u = 0$

addition: $(m, s) + (n, q) = (mq + sn, sq)$

and for $(a, s) \in S^{-1}R$. module structure
 $(a, s) \cdot (m, q) = (am, sq)$.

Again we write $\frac{m}{s}$.

Example. $R = \mathbb{Z}$ $S = \mathbb{Z} \setminus 0$ $S^{-1}\mathbb{Z} = \mathbb{Q}$.

$M = \mathbb{Z}$ then $S^{-1}M = \mathbb{Q}$.

$M = \mathbb{Z}/n$ then $S^{-1}M = 0$ since

$$\begin{aligned} \frac{m}{s} &= \frac{nm}{ns} & \text{because } n(ms - ms) &= 0 \\ &= \frac{0}{s} & \text{in } \underline{M}. \end{aligned}$$

Notational examples

1) If $S = R - P$, P prime ideal we write R_P and $M_P = S^{-1}M$.

2) $S = \{f^n \mid n \geq 0\}$ $S^{-1}R = R_f$ write M_f for $S^{-1}M$.