

Lecture 13 Local properties

Recall properties of $S^{-1}M$.

- $S^{-1}R$ is flat module over R 
- S^{-1} commutes with quotients
- $M', M'' \subseteq M$ S^{-1} commutes with $M' + M''$ $M' \cap M''$
- $S^{-1}M \otimes_{S^{-1}R} S^{-1}N \cong S^{-1}(M \otimes_R N)$.
- $S^{-1}M \cong S^{-1}R \otimes_R M$. $\left(\frac{m}{s} \mapsto \frac{1}{s} \otimes m \right)$

Def A ring (alt. R -module) property P is local if :

A (altern. M) has $P \iff$

A_P (altern. M_P) has P for all

localisations M_P (P is prime ideal

$$M_P = S^{-1}M \quad S = M \setminus P).$$

Prop The operation $S^{-1}(-)$ is exact.
 i.e. If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M
 then $S^{-1}M' \rightarrow S^{-1}M \rightarrow S^{-1}M''$ is exact at M

Proof exact $\Rightarrow g \circ f = 0 \Rightarrow S^{-1}g \circ S^{-1}f = 0$
 we have $\text{Im } S^{-1}f \subseteq \text{Ker } S^{-1}g$

If $\frac{m}{s} \in \text{Ker } S^{-1}g$ then $\frac{g(m)}{s} = \frac{0}{1} \Rightarrow$

$(g(m) - s \cdot 0) \cdot u = 0$ for some $u \in S \subset R$
 $ug(m) = g(um)$ since g is a R -mod hom.

$\Rightarrow um \in \text{Ker } g = \text{Im } f \Rightarrow f(m') = um$

for some $m' \in M' \Rightarrow \frac{m}{s} = \frac{f(m')}{s \cdot u} = S^{-1}f\left(\frac{m'}{s \cdot u}\right)$
 $\in \text{Im } S^{-1}f.$

Exactness is proven.

Corollary $S^{-1}R$ is flat over R .

$S^{-1}R \otimes M \cong S^{-1}M$ (tensoring with $S^{-1}R$ is same as module of fractions)

Examples. ① Being 0 is a local property

$$M = 0 \Rightarrow M_P = 0 \quad \forall P \text{ prime ideals}$$

$$\Rightarrow M_m = 0 \quad \forall m \text{ max ideals.}$$

Now if $m \in M, m \neq 0$. Then

$$\text{Ann}(m) = \{ r \mid r \cdot m = 0 \} \subseteq_{\text{ideal}} R.$$

then $\exists m \supseteq \text{Ann}(m)$ m max ideal.

$$\text{but then } M_m = 0 \text{ and } \frac{m}{1} = \frac{0}{1} \quad \forall m$$

by definition $(m - 0) \cdot u = 0$ for $u \in R - m$

but $u \in \text{Ann}(m)$ so $u \in m \Rightarrow$

$$\Rightarrow M_m = 0 \quad \forall m \text{ max} \Rightarrow M = 0.$$

② Being flat is a local property.

(recall $\otimes$ is right exact

M flat if $M \otimes -$ is exact functor

Proposition Let $\phi: M \rightarrow N$ be an R -mod homom. then TFAE:

1) $\phi: M \rightarrow N$ is injective (surjective)

2) $\phi_P: M_P \rightarrow N_P$ is injective (surjective)
 $\forall$ prime ideals $P \subseteq R$

3) $\phi_{\mathfrak{m}}: M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is injective (surjective)
 $\forall$ max ideals $\mathfrak{m} \subseteq R$.

Proof recall $S^{-1}R$ is flat over R .

$0 \rightarrow \ker \phi \rightarrow M \rightarrow N$ is injective.

$\Downarrow$

$0 \rightarrow \ker \phi_P \rightarrow M_P \rightarrow N_P$ is injective.

$\Rightarrow \ker \phi_P = (\ker \phi)_P \Rightarrow \ker \phi = 0$

$\Updownarrow$

$\ker \phi_P = 0 \forall P$.

$\Updownarrow$

$\ker \phi_{\mathfrak{m}} = 0 \forall \mathfrak{m}$

1 $\Leftrightarrow$ 2 $\Leftrightarrow$ 3.

Prop Flatness is a local property.

For any R -module M TFAE.

1) M is flat over R

2) M_P is flat over $R_P \forall P$ prime ideal

3) M_m is flat over $R_m \forall m$ prime ideal.

$$\underline{1 \Rightarrow 2} \quad M_P \cong R_P \otimes_R M$$

Exercise. If $f: R \rightarrow S$ is a ring homomorphism and M a flat R -module then $S \otimes_R M$ is a flat S -module.

$$\underline{2 \Rightarrow 3} \quad \text{max} \Rightarrow \text{prime}$$

3 $\Rightarrow$ 1 let $N \rightarrow N'$ be a homom.

$N \rightarrow N'$ injective $\Rightarrow N_m \rightarrow N'_m$ injective

$N_m \otimes M_m \rightarrow N'_m \otimes M_m$ by assumption M_m flat

$$\Rightarrow (N \otimes_R M)_M \rightarrow (N' \otimes_R M)_M \text{ injective} \\ (\text{loc. commutes with } \otimes)$$

$$\Rightarrow N \otimes_R M \rightarrow N' \otimes_R M \text{ injective by prop}$$

so M is flat. $\square$

Recall extended contracted ideals.

$$f: R \rightarrow R'$$

contraction

$$J \subseteq R' \text{ ideal then } f^{-1}(J) \subseteq R \text{ ideal}$$

extension

$$I \subseteq R \text{ ideal then } f(I) \cdot R' = \left\{ \sum f(r_i) s_i \right\}$$

When f is $\phi: R \rightarrow S^{-1}R$ we have I^e

Prop $I \subseteq R$ $J \subseteq S^{-1}R$ ideal

$$I^e = \left\{ \frac{a}{s} \mid a \in I, s \in S \right\}$$

$$J^c = \left\{ a \mid \frac{a}{1} \in J \right\}$$

Proof. J^c is by definition

$$I^e = \left\{ \sum_{\text{finite}} \frac{a_i}{s_i} \mid a_i \in I, s_i \in S \right\}$$

write over common denominator

$$= \left\{ \sum \frac{s_1 \dots \hat{s}_i \dots s_n a_i}{s_1 \dots s_n} \in I \mid s_1 \dots s_n \in S \right\} \quad \square.$$

Prop For $J \in S^{-1}R$ we have

$$J^{ce} = J.$$

Corollary 1 Every $J \in S^{-1}R$ ideal is the extension of some $I \subseteq R$.

2) The map $J \mapsto J^c$ gives an inclusion $\mathcal{L}(S^{-1}R) \rightarrow \mathcal{L}(R)$.

Proof of prop If $\frac{a}{s} \in J$ then $\frac{s}{1} \cdot \frac{a}{s} = \frac{a}{1} \in J \Rightarrow a \in J^c \Rightarrow \frac{a}{s} \in J^{ce}$

If $\frac{a}{s} \in J^c$ then $\frac{a}{s} = \frac{b}{s'}$ for

some $b \in J^c \Rightarrow \frac{b}{1} \in J \Rightarrow$

$$\frac{b}{1} \cdot \frac{1}{s'} = \frac{b}{s'} = \frac{a}{s} \in J. \quad \square$$

Example Consider $\mathbb{Z}(p)$ for a prime p . Every $J \subseteq \mathbb{Z}(p)$ is the extension of an $I = (n)$ in $\mathbb{Z}$. $\rightarrow$

$$J = \left\{ \frac{n}{k} \mid p \nmid k \right\} \quad n = p^l \cdot q \quad p \nmid q$$

$$= \{m \cdot p^l\} \Rightarrow J = (p^l)^e.$$

The ideals are:

$$(1) \supsetneq (p) \supsetneq (p^2) \supsetneq \dots \supsetneq (p^k) \dots (0).$$

Which ideals are prime in $\mathbb{Z}(p)$.

Prop Prime ideals in $S^{-1}R$.

There is a 1-1 correspondence

$$\left\{ \begin{array}{l} \text{prime ideals} \\ \text{of } S^{-1}R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{prime ideals} \\ \text{of } R \text{ with} \\ S \cap P = \emptyset \end{array} \right\}$$

$$J \longmapsto \phi^{-1}(J) \text{ contraction}$$

$$\phi(I)S^{-1}R \longleftarrow I \text{ extension.}$$

Again here: $\phi: R \rightarrow S^{-1}R$

Compare this to prime ideals in R/I

Proof. If J is prime then $\phi^{-1}(J)$ is prime, moreover $\phi^{-1}(J) \cap S = \emptyset$, since otherwise $a \in \phi^{-1}(J)$ and $a \in S \Rightarrow \frac{a}{1} \in J$ and $\frac{a}{1} \cdot \frac{1}{a} \in J \Rightarrow 1 \in J$ not prime.

If $I \subseteq R$ is prime and $S \cap I = \emptyset$ then

We want to show $I \cdot S^{-1}R$ is prime:

Assume not prime, then $\exists$

$$\frac{a}{s} \cdot \frac{b}{s'} \in I \cdot S^{-1}R \quad \text{but} \quad a, b \notin I$$
$$\frac{c}{t} \quad \text{where} \quad c \in I \quad t \in S.$$

$$\Rightarrow (t \cdot a \cdot b - s s' \cdot c) \cdot u = 0 \quad \exists u \in S.$$

$$t \cdot a \cdot b \cdot u = \underline{s \cdot s' \cdot c \cdot u} \quad \& \text{ this is in } I.$$

$$\text{but } t, a, b, u \notin I$$

Finally, $I \subseteq \phi^{-1}(I \cdot S^{-1}R)$ (this is always true)

$$\text{If } a \in \phi^{-1}(I \cdot S^{-1}R) \text{ then } \frac{a}{1} = \frac{b}{s} \quad \begin{matrix} b \in I \\ s \in S \end{matrix}$$

$$(as - b \cdot 1) \cdot u = 0 \quad u \notin I.$$

$$asu = bu \in I \quad \text{but since } s, u \notin I$$

$$\text{and } I \text{ prime } asu \in I \Rightarrow a \in I \quad \square$$

Corollary The prime ideals of R_P where P is prime are all $\mathbb{Q}R_P$ where $\mathbb{Q} \subseteq P$ $\mathbb{Q}$ prime.

$\rightarrow S = R - P$ so $\mathbb{Q} \cap S = \emptyset \Leftrightarrow \mathbb{Q} \subseteq P. \quad \square$

Corollary The prime ideals of R_f are $\mathbb{Q}R_f$ where $\mathbb{Q}$ is prime and $f \notin \mathbb{Q}$.

$S = \{1, f, f^2, \dots\}$ so $\mathbb{Q} \cap S = \emptyset$
 $\Rightarrow f^k \notin \mathbb{Q}$ for any k but $\mathbb{Q}$ prime
so $f^k \in \mathbb{Q} \Leftrightarrow f \in \mathbb{Q}. \quad \square.$

Corollary The ring R_P is local.

Proof The max ideal is PR_P .

Corollary The nilradical of R is the intersection of all primes

Proof f nilpotent $\iff f \in$ every prime.

$$f^k = 0 \implies f \in P \text{ every prime.}$$

If f in every prime, then

$$\{P \text{ prime} \mid f \notin P\} = \emptyset$$

$\implies R_f$ has no prime ideals

$\implies R_f$ is 0 ring ($0=1$)

$$\implies \frac{1}{1} = \frac{0}{1} \in R_f \implies (1 - 1 \cdot 0)f^k = 0$$

in $R \implies f^k = 0$ nilpotent. $\square$

Examples The prime ideals of $\mathbb{Z}_{(2)}$

are $(2) \cdot \mathbb{Z}_{(2)} = \left(\frac{2}{1}\right)$

$$(0) \cdot \mathbb{Z}_{(2)} = \left(\frac{0}{1}\right)$$

The prime ideals of $\mathbb{Z}_2$ i.e. $\mathbb{Z}_f$

are $\left(\frac{3}{1}\right), \left(\frac{5}{1}\right), \dots, \left(\frac{p}{1}\right)$ any

odd prime p