

Lecture 14 Chain conditions Ch 6 AM
Lecture 9 E11.

Recall a chain $\mathcal{C} = \{M_i\}_{i \in I}$ in a partially ordered set Σ is a linearly ordered subset

$$\dots \subseteq M_i \subseteq M_{i+1} \subseteq M_{i+2} \subseteq \dots$$

An ascending chain is a sequence

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$$

A descending chain is a sequence

$$\dots \supseteq M_2 \supseteq M_1 \supseteq M_0$$

In either case the chain is stabilising if $\exists N$ s.t. $M_n = M_N$ for all $n \geq N$.

Lemma Let Σ be a partially ordered set TFAE:

- 1) Every ascending sequence stabilises
- 2) Every non-empty subset $T \subseteq \Sigma$ contains a maximal element.

$2 \Rightarrow 1$ easy.

$1 \Rightarrow 2$ Suppose there is no max elt of T . Therefore for every $t \in T$ we can choose $f(t) \in T$ with $f(t) \geq t$. Take $s_1 = t$ $s_2 = f(t)$
 $\dots$ $s_i = f(f(\dots(f(t))))$
 $i \rightarrow \infty$

to get a non-stabilising chain

Note that we can say the same for descending sequences & minimal elts.

Def Let R be a ring and M an R -module. Let Σ be submodule of M .

• M is Noetherian if every ascending sequence in Σ stabilises

• M is Artinian if every descending sequence in Σ stabilises.

"satisfies ascending chain condition"
descending

A ring R is Noetherian/Artinian if it is as a module over itself
(submodules of R are ideals)

Named after Emmy Noether (1882-1935 German Mathematician)
and Emil Artin (1898-1962 Austrian Mathematician)

Examples $\mathbb{Z}$ is Noetherian but not Artinian

$$(n_1) \subsetneq (n_2) \subsetneq \dots$$

$\Rightarrow n_i \mid n_{i+1}$ so it not stabilize.

However

$$(2) \supsetneq (4) \supsetneq (8) \supsetneq \dots$$

Does not stabilize

Example let $R = K$ a field. M ^{vector space}

M is Noetherian $\iff M$ is Artinian

$\iff M$ has finite dimension

dimension not drop (or increase)

$$\text{if } M_i \subsetneq M_{i+1} \quad M_i \supsetneq M_{i+1}$$

Example The ring of polynomials

$R[x]$ is not Artinian.

$$(1) \neq (x) \neq (x^2) \neq \dots \neq (x^k) \neq \dots$$

$R[x]$ is Noetherian $\iff R$ is Noetherian

$\Rightarrow$ a non-stabilizing chain in R would be non-stabilizing in $R[x]$ ($R[x]/(x) = R$)
quotient rings

To prove $\Leftarrow$ we will use another equivalent condition to be Noetherian

Example $C(\mathbb{R})$ smooth functions on $\mathbb{R}$

is neither A nor N. $(1) \neq (x) \neq \dots$

and it's not N since

$$(\sin(x)) \neq (\sin(\frac{x}{2})) \neq (\sin(\frac{x}{4})) \neq \dots$$

To see inclusion use double angle:

$$\sin\left(\frac{2x}{2}\right) = 2\cos x \sin x$$

Theorem (Noether 1921)

Let R be a ring and M a module over R TFAE

- 1) M is Noetherian
- 2) Every non-empty subset of ΣM has a maximal element.
- 3) Every submodule of M is finitely generated.

Proof $1 \Rightarrow 2$ if not 2 then construct an infinite chain (kind of like before)

$2 \Rightarrow 3$ let $\mathcal{F}_N \subseteq \Sigma M$ be the subset of finitely generated submodules of N .
let $N_{\max}$ be the max element of $\mathcal{F}_N$.
If $N_{\max} \neq N$ then take $x \in N \setminus N_{\max}$

$N_{\max}$ is generated by $x_1, \dots, x_n$ then $xR + N_{\max}$ is generated by $x, x_1, \dots, x_n$.

Therefore it is finitely generated
and so $xR + N_{\max} \in \mathcal{F}_N$ it must
be equal to $N_{\max}$ so it is
finitely generated.

If every submodule $M_i \subseteq M$ is finitely
generated let $M_1 \subseteq \dots$ be a
ascending chain and let

$$M' = \bigcup_{i=1}^{\infty} M_i \subseteq M \text{ must be}$$

finitely generated by $m_1, \dots, m_n$

There must be some finite $\#$ of M_i 's
containing all of these so $M' = \bigcup_{i=1}^N M_i$

and $M_n = M'$ for all $n \geq N$.

The chain stabilises.

Corollary A ring is Noetherian $\Leftrightarrow$ all ideals are finitely generated.

If K is a field $K[x]$ is Noetherian since $K[x]$ is a P.I.D.

Theorem Let

$$0 \rightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \rightarrow 0$$

be exact sequence of R modules

M Noetherian $\Leftrightarrow M'$ and M'' Noetherian

M Artinian $\Leftrightarrow M', M''$ Artinian.

Proof (Noetherian) If $M_1' \subseteq M_2' \subseteq \dots$

is a chain of submodules of M' then

$i(M_1') \subseteq i(M_2') \subseteq \dots$ is a chain in

M . If M is Noetherian, then $i(M_k')$ not stabilise. By injectivity of i ,

$M_1' \subseteq M_2' \subseteq \dots$ must stabilise.

If $M_1'' \subseteq M_2'' \subseteq \dots$ is a chain in M'' .

then $p^{-1}(M_1'') \subseteq p^{-1}(M_2'') \subseteq \dots$ is a chain in M which stabilises since M is Noetherian. By surjectivity of p the chain in M'' must stabilise.

$\Leftarrow$ Given a chain $M_1 \subseteq M_2 \subseteq \dots$ in M we obtain chains in M', M'' by

$$i^{-1}(M_1) \subseteq i^{-1}(M_2) \subseteq \dots$$

$$p(M_1) \subseteq p(M_2) \subseteq \dots$$

These chains must stabilise, so $\exists$ a Nst.

$$i^{-1}(M_n) = i^{-1}(M_{n+1}) \quad \forall n \geq N.$$

$$p(M_n) = p(M_{n+1})$$

Use the snake lemma:

$$\begin{array}{ccccccc}
0 \longrightarrow & i^{-1}(M_n) & \xrightarrow{i} & M_n & \xrightarrow{p} & p(M_n) & \longrightarrow 0 \\
& \textcolor{red}{112} \downarrow f' & & \downarrow f & & \textcolor{red}{112} \downarrow f'' & \\
0 \longrightarrow & i^{-1}(M_{n+1}) & \xrightarrow{i'} & M_{n+1} & \xrightarrow{p'} & p(M_{n+1}) & \longrightarrow 0
\end{array}$$

$$0 \rightarrow \ker f' \rightarrow \ker f \rightarrow \ker f'' \rightarrow \operatorname{coker} f' \rightarrow \operatorname{coker} f \rightarrow \operatorname{coker} f'' \rightarrow 0.$$

Since f', f'' are both isomorphisms we have $0 \rightarrow 0 \rightarrow \ker f \rightarrow 0 \rightarrow 0 \rightarrow \operatorname{coker} f \rightarrow 0 \rightarrow 0$ is an exact sequence. $\Rightarrow \ker f = \operatorname{coker} f = 0$ so f is an isomorphism for all $n \geq N$.
 $\Rightarrow M_1 \subseteq M_2 \subseteq \dots$ stabilises and M is Noetherian. $\square$

Corollary If $M_1, \dots, M_n$ are Noetherian, then $\bigoplus_{i=1}^n M_i$ is Noetherian.

Proof

There is a short exact sequence

$$0 \rightarrow M_j \rightarrow \bigoplus_{i=1}^j M_i \rightarrow \bigoplus_{i=1}^{j-1} M_i \rightarrow 0$$

$\forall j \in \{1, \dots, n\}$. Apply previous proposition to prove statement by induction $\square$.

Prop

If R is Noetherian and M a finitely generated R -module, then M is Noetherian, (same for Artinian)

Proof

R is Noetherian $\Rightarrow R^n$ is Noetherian

Since M is finitely generated we have

$$0 \rightarrow \ker \phi \rightarrow R^n \xrightarrow{\phi} M \rightarrow 0$$

is exact. Apply proposition and we obtain

$\ker \phi, M$ are both Noetherian. $\square$

Prop If R is Noetherian ring, I ideal of R , then R/I is Noetherian

Proof Use $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$,

Examples 1) Any field K is Noetherian.
No non-trivial ideals

2) The ring $K[x_1, \dots, x_n]$ is not Noetherian, but it is an integral domain.
 $K(x_1, \dots, x_n)$ field of fractions is Noetherian. Hence subrings of Noetherian subrings need not be Noetherian.