

Lecture 15 Chain conditions + composition series

Def A module is simple if it has no non-trivial submodules

You may remember this from group theory!

Example p prime $\mathbb{Z}/(p)$ is a simple $\mathbb{Z}$ -module
no non-trivial subgroups. $\mathbb{Z}/(n)$ for
 n composite is not simple $\mathbb{Z}/(k) \subseteq \mathbb{Z}/(n)$
if $k \mid n$.

Example $K[x] / (x-a)$, $\frac{K[x_1, \dots, x_n]}{(x_1-a_1, \dots, x_n-a_n)} \cong K$

are both simple $K[x]$ modules.

Prop An R -module is simple $\iff$ it is isomorphic to $R/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$.

Proof Suppose M is simple and consider.

$$\phi_m: R \rightarrow M \quad r \mapsto mr \quad \text{for some } m \neq 0.$$

Since $m \neq 0$ and M simple $\text{Im}(\phi_m) = M$

therefore $M \cong R/\mathfrak{I}$ for some $\mathfrak{I}$.

$\mathfrak{I} = \text{Ann}(M)$. We claim $\text{Ann}(M)$ is maximal.

If not there is a larger ideal $\mathfrak{M}$ containing $\mathfrak{I}$ and then $0 \neq M/\mathfrak{I} \subsetneq R/\mathfrak{I}$

$\cong M$ but M is simple $\Rightarrow \Leftarrow$.

For $\mathfrak{m}$ a max ideal of R , suppose

$0 \subsetneq M \subseteq R/\mathfrak{m}$ for some module M

Then let $\phi: R \rightarrow R/\mathfrak{m}$ be the projection

$\phi^{-1}(M) \subseteq R$ is an ideal $\mathfrak{m} \subseteq \phi^{-1}(M) \subseteq R$.

Since $\mathfrak{M}$ is maximal $p^{-1}(\mathfrak{M}) = \mathfrak{M}$

or $p^{-1}(\mathfrak{M}) = R$ either $\mathfrak{M} = 0$ or $\mathfrak{M} = R_{\mathfrak{M}}$

Def A composition series of $\mathfrak{M}$ is a finite chain:

$$\mathfrak{M} = \mathfrak{M}_0 \supsetneq \mathfrak{M}_1 \supsetneq \dots \supsetneq \mathfrak{M}_{n-1} \supsetneq \mathfrak{M}_n = 0$$

which is maximal i.e. cannot be extended by adding some $\mathfrak{M}'$:

$$\mathfrak{M}_i \supsetneq \mathfrak{M}' \supsetneq \mathfrak{M}_{i+1}$$

A chain is a composition series $\Leftrightarrow$

$\mathfrak{M}_i / \mathfrak{M}_{i+1}$ is simple for every i .

Example. $n = pq$ p, q prime $l(\mathbb{Z}/(pq)) = 2$

$$\mathbb{Z}/(n) \supsetneq \begin{smallmatrix} (p) \\ 1 \end{smallmatrix} / \begin{smallmatrix} (pq) \\ 1 \end{smallmatrix} \supsetneq \begin{smallmatrix} (pq) \\ 2 \end{smallmatrix} / \begin{smallmatrix} (pq) \\ 2 \end{smallmatrix} = 0.$$

$$n = p_1^{e_1} \cdots p_k^{e_k}$$

$$\mathbb{Z}/(n) \supsetneq (p_1)/(n) \supsetneq (p_1^2)/(n) \supsetneq \cdots \supsetneq (p_1^{e_1})/(n)$$

$$(p_1^{e_1} p_2)/(n) \cdots \supsetneq (p_1^{e_1} \cdots p_{k-1} p_k)/(n)$$

$$\supsetneq (n)/(n) = 0.$$

$$\# \text{ of modules in chain} = \sum e_i.$$

$$l(\mathbb{Z}/(n)) = \sum e_i$$

Definition The length of a module M is the minimal length of a composition series of M . $l(M) = \infty$ if no composition series exists.

$$M \text{ is simple} \iff l(M) = 1.$$

Lemma Let M have finite length,
then $N \subsetneq M \Rightarrow l(N) < l(M)$.

Proof let $M = M_0 \supsetneq \dots \supsetneq M_n = 0$.

be a composition series for M .

set $N_i = M_i \cap N \forall i$. Then,

$$\phi_i: M_i \cap N \hookrightarrow M_i \rightarrow M_i/M_{i+1}.$$

$$\text{Ker } \phi_i = M_{i+1} \cap N. \quad \text{either } \text{Im } \phi_i = \begin{cases} M_i/M_{i+1} \\ 0 \end{cases}$$

since M_i/M_{i+1} is simple.

Case 1 $\text{Im } \phi_i = 0 \Rightarrow M_i \cap N \cong M_{i+1} \cap N$.

Case 2 $\text{Im } \phi_i = M_i/M_{i+1} \Rightarrow \frac{M_i \cap N}{M_{i+1} \cap N} \cong \frac{M_i}{M_{i+1}}$

in particular N_i/N_{i+1} is simple.

Removing the i 's which are in case 1
we obtain a composition series.

$$\Rightarrow l(N) \leq l(M).$$

Since $N \not\cong M$ we have $N = N \cap M_0 \neq M_0$.

while $0 = N \cap M_{l(M)} = M_{l(M)} = 0$.

$$\text{let } i = \min_i \{i \mid N \cap M_i = M_i\}$$

Then $M_{i-1} \not\cong N \cap M_{i-1} \supset N \cap M_i = M_i$

but $N \cap M_{i-1} = N \cap M_i$, so we must remove this i from the composition series for N . $\Rightarrow l(N) < l(M)$. $\square$.

Prop Let M be a module with $l(M) = n$

1) every composition series has length n .

2) for every chain $M_0 = M \supsetneq M_1 \supsetneq \dots \supsetneq M_k = 0$ has $k \leq n$ and can be extended to a composition series

Proof 1) If $M_0 = M \supsetneq \dots \supsetneq M_{n'} = 0$ is a composition series we must have

$$l(M_{n'}) < l(M_{n'-1}) < \dots < l(M_0)$$

$\Rightarrow l(M) \geq n'$ by Lemma.

However $l(M) \leq n'$ by definition

$$l(M) = n' = n.$$

2) If $k < n$ it is not a composition series hence M_i/M_{i+1} is not simple

and we can increase the chain:

$$M_i \supsetneq M' \supsetneq M_{i+1}.$$

Remark You may have seen composition series for groups. Simple groups have no normal subgroups. Jordan-Hölder theorem says length is invariant & so are intermediate quotients up to isomorphism.

Prop $\ell(M)$ finite $\iff M$ is Artinian and Noetherian

Def $\Rightarrow$ Any increasing/decreasing sequence must have finite length hence stabilising hence Noetherian/Artinian.

$\Leftarrow$ Define a composition series:

$M_0 = M$, M_1 is a maximal submodule which exists since M is Noetherian. Proceed by induction M_1 is Noetherian and M_2 is max submodule. Now we get a descending chain, since M is Artinian it stabilises and $M_n = 0$ for some n .

□.

Prop The function l is additive.

Given a short exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of R -modules, we have

$$l(M) = l(M') + l(M'').$$

Proof If one of the lengths is infinite it is handled by previous proposition and behavior of Noetherian Artinian under S.E.S.'s

Assume all lengths $< \infty$.

Consider composition series for M' and M''

$$\begin{aligned} M = p^{-1}(M_0'') &\supsetneq p^{-1}(M_1'') \supsetneq \dots \supsetneq \underbrace{p^{-1}(M_{l(M'')}'')}_{l(M')} \supsetneq p^{-1}(0) \\ &= i'(M) = i(M_0') \supsetneq \dots \supsetneq i(M_{l(M')}') = 0 \end{aligned}$$

Since $p^{-1}(M_i'')/p^{-1}(M_{i+1}'') \cong M_i''/M_{i+1}''$

and $i(M_i')/i(M_{i+1}') \cong M_i'/M_{i+1}'$

we have a composition series for M
 and $l(M) = l(M') + l(M'')$. $\square$