

Lecture 16 Noetherian rings + Hilbert basis theorem.

Recall a ring is Noetherian if it satisfies the ascending chain condition on ideals.

Noether's thm

R Noetherian $\iff \forall$ ideal finitely generated
 $\iff$ every non-empty collection of ideals has a maximal element.

Prop If R is Noetherian and $\phi: R \rightarrow R'$ is a surjective homomorphism then R' is Noetherian.

Proof $R' \cong R / \ker \phi$

Prop Let $R \subseteq R'$ be a subring.
Suppose R is Noetherian and R'
is finitely generated as an R -module.
then R' is a Noetherian ring.

Proof Every R' -module is also an
 R -module

Example $\mathbb{Z}[i]$ Gaussian integers are
finitely generated over $\mathbb{Z}$. hence Noetherian.

Prop If R is Noetherian and $S \subseteq R$ mlt.
closed then $S^{-1}R$ is Noetherian.

Recall the ideals of $S^{-1}R$ are in correspond.

with contracted ideals of R via
 $\phi: R \rightarrow S^{-1}R$.

Hilbert's Basis Theorem

If R is Noetherian, then $R[x]$ is Noetherian.

Proof Let $I \subseteq R[x]$ be an ideal.

We will find a finite generating set "basis" (old fashioned terminology).

The leading coefficients of elts of I form an ideal $I_{\text{lead}} \subseteq R$.

Why? let $r_1, r_2 \in I_{\text{lead}}$ then $r_1 + r_2 \in I_{\text{lead}}$

Since $\exists f_1, f_2$ with leading coefficients r_1, r_2
build a polynomial $f_1 + x^{\deg f_1 - \deg f_2} f_2$
with leading coefficient $r_1 + r_2$.

If $r_i \in I_{\text{lead}}$ then $-r_i \in I_{\text{lead}}$ since
 $f_i \in I \Rightarrow -f_i \in I$.

If $r_i \in I_{\text{lead}}$ and $r \in R$ then
 $r_i r \in I_{\text{lead}}$ since $rf_i \in I$.

Since R is Noetherian I_{lead} is fin.
generated $a_1, \dots, a_n$.

For each $i=1, \dots, n$ $\exists f_i = a_i x^{r_i} + \text{lower terms}$
 $f_i \in I$. set $r = \max r_i$ and let
 $I' = \langle f_1, \dots, f_n \rangle \subseteq I \subseteq R[x]$.

let $f = ax^m + (\text{lower terms}) \in I$ then
 $a \in I_{\text{lead}}$.

If $m \geq r$ we have:

$a = \sum_{i=1}^n u_i a_i$ $u_i \in R$ and then

$f - \underbrace{\sum u_i f_i x^{m-r_i}}_{\in I'} \in I$ and has

degree $< m$. Continuing this way
we can arrive at

$$f = g + \underbrace{h}_{\in I'} \quad \deg g < r.$$

Now let M be the R -module
generated by $1, x, \dots, x^{r-1}$.

Then $I = I \cap M + I'$. M is finitely
generated, hence Noetherian, so $I \cap M$ is
finitely generated as an R -module.

If $g_1, \dots, g_m$ generate $I \cap M$ and $f_1, \dots, f_n$
generate I' , then $g_1, \dots, g_m, f_1, \dots, f_n$ generate
 I . So $R[x]$ is Noetherian $\square$.

Corollary If R is Noetherian, then

$R[x_1, \dots, x_n]$ is Noetherian (finit # of variables)

Proof When $R = K$ a field

Define the lexicographical order on monomials

$$x_1^{a_1} \dots x_n^{a_n} \geq x_1^{b_1} \dots x_n^{b_n} \iff$$

$$(a_1, \dots, a_n) \geq_{\text{lex}} (b_1, \dots, b_n).$$

For $f \in R[x_1, \dots, x_n]$ define $LT(f)$ to be the highest term of f w.r.t. $\leq_{\text{lex}}$.

Define $LT(I) = \langle LT(f) \mid f \in I \rangle$.

leading term ideal. This is generated by monomials.

Claim every monomial ideal in
 $K[x_1, \dots, x_n]$ is finitely generated.

$$\Rightarrow LT(I) = \langle LT(g_1), \dots, LT(g_m) \rangle$$

This is a consequence of Dickson's
lemma for semi-groups:

$S \subseteq \mathbb{Z}_{\geq 0}^n$ closed under
addition is finitely generated.

It can be proved by induction
on n .

Claim $I = \langle g_1, \dots, g_m \rangle$

$$\langle g_1, \dots, g_m \rangle \subseteq I \quad \checkmark$$

for $f \in I$ can write

$$f = q_1 g_1 + \dots + q_m g_m + r \quad \text{so no}$$

monomial in r is divisible by $LT(g_i)$

$$\text{Now } r = f - \sum q_i g_i \in I$$

If $r \neq 0$ then $LT(r) \in LT(I)$

which means $LT(r)$ must be
divisible by $LT(g_i)$ for some g_i

$\Rightarrow \Leftarrow$. So $r = 0$ and

$$f = \sum q_i g_i$$

□

I is finitely generated and $K[x_1, \dots, x_n]$
is Noetherian

Definition $I \subseteq K[x_1, \dots, x_n]$ an ideal.

A Groebner basis of I is $g_1, \dots, g_m$
s.t. $\langle g_1, \dots, g_m \rangle = I$ and

$$\langle LT(g_1), \dots, LT(g_m) \rangle = LT(I)$$

Corollary Let A be a finitely generated
 R -algebra, if R is Noetherian,
then so is A .

Proof A finitely generated R -algebra
(finite type R -algebra) means

$$A \cong R[x_1, \dots, x_n] / I.$$

Hence, A is Noetherian as a ring.

Hilbert's (weak) Nullstellensatz

Recall that a field extension K' of K is $K \subseteq K'$ is finite if K' is a finite dimensional vector space over K .

Thm

Let $\mathfrak{m} \subset K[x_1, \dots, x_n]$ be a maximal ideal. Then $K[x_1, \dots, x_n]/\mathfrak{m}$ is a finite field extension over K .

If K is algebraically closed then

$K[x_1, \dots, x_n]/\mathfrak{m} \cong K$ in particular

$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_i \in K$.

Proof of weak Nullstellensatz

i) $K[x_1, \dots, x_n]/\mathfrak{m}$ is a field so Zariski's lemma applies.

ii) If K is algebraically closed, then for $K \subseteq K'$ finite we must have

$K = K'$. Thus $f: K \rightarrow K[x_1, \dots, x_n]/\mathfrak{m}$

must be an isomorphism and for

each $i=1, \dots, n$ we must have $x_i = a_i$

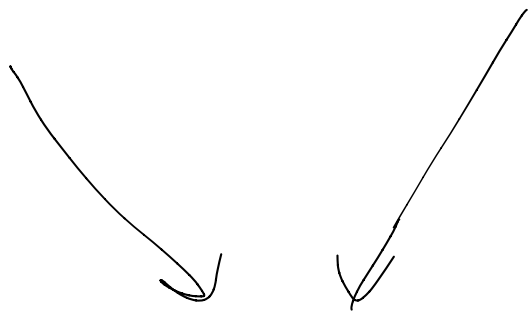
this means $x_i - a_i \in \mathfrak{m}$

$$(x_1 - a_1, \dots, x_n - a_n) \subseteq \mathfrak{m}$$

the ideal on the left side is maximal

so it is equal to $\mathfrak{m}$ $\square$

$$(M \otimes_2 N) \times P \longrightarrow (M \otimes_2 N) \otimes P$$



$$M \otimes_2 (S \otimes_S P). \text{ module.}$$