# Lecture 17 Artin Rings

Recall a ring is Artinian if it satisfies the descending chain condition on ideals

every sequence

$I_1 \supseteq I_2 \supseteq \cdots$ stabilises.

Examples. 1) $K$ a field is Artinian.

$K \supseteq \{0\}$ are the only ideals.

2) $\mathbb{Z}$ is NOT Artinian

$(p) \supseteq (p^2) \supseteq \cdots$ does not stabilise

3) If $R$ has a finite # of elements it must be Noetherian + Artinian

**Lemma** If $R$ is Artinian it has a finite # of maximal ideals.

**Proof** If infinite consider the chain

$$m_1 \supseteq m_1 \cap m_2 \supseteq m_1 \cap m_2 \cap m_3 \supseteq \cdots$$

it must stabilise meaning

$$m_1 \cap \cdots \cap m_n = m_1 \cap \cdots \cap m_{n+1} \quad \text{for all}$$

$n \geq N$ for some $N$.

meaning $m_{n+1} \supseteq m_1 \cap \cdots \cap m_n$.

Since ideals are all distinct we can find $x_i \in m_i \setminus m_{n+1} \ \forall i$.

s.t. $x = \prod_{i=1}^{\uparrow} x_i \in m_1 \cap \cdots \cap m_n$ but

$x \notin m_{n+1}$ since $m_{n+1}$ maximal $\Rightarrow$ prime

contradiction. $\square$

meaning there is a minimal elt.

$m_1 \cap \cdots \cap m_n$. and any

max ideal $m$ satisfies

$$m \cap m_1 \cap \cdots \cap m_n = m_1 \cap \cdots \cap m_n$$

therefore $m \supseteq m_1 \cap \cdots \cap m_n$,

By (1.11) $m \supseteq m_i$ for some

Let $a_1, \ldots, a_n$ be ideals and

$P$ a prime ideal containing $\bigcap a_i$

then $P \supseteq a_i$ for some $i$.

1.11. Atiyah Macdonald

**Lemma** Let $R$ be an Artinian ring. If $R$ is an integral domain it is a field.

**Proof** consider $x \neq 0$ $x \in R$, the chain

$(x) \supseteq (x^2) \supseteq (x^3) \ldots$ must stabilize

hence $x^N = yx^{N+1}$ for some $N$. $y \in R$.

Since $R$ is an integral domain

$\Rightarrow$ $1 = yx$ for some $y \in R$ and

$x$ is a unit hence $R$ is a field.

**Lemma** In an Artinian ring, every prime ideal is maximal.

**Proof** $R$ Artinian $\Rightarrow$ $R/I$ is Artinian for any ideal. If $P \subseteq R$ is prime ideal, then $R/P$ is an integral domain and Artinian. By previous lemma $R/P$ is a field $\Rightarrow$ $P$ is maximal. $\square$.

**Def** Let $R$ be a ring The Krull dimension of $R$ is the supremum of lengths of chains of prime ideals in $R$. $P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n$

**Examples**

1) $\dim K = 0$

2) $\dim \mathbb{Z} = 1$

3) $\dim K[x] = 1 \quad (x) \supsetneq (0)$

4) $\dim K[x_1 \dots x_n] = n \mid (x_1 \dots x_n) \supsetneq (x_1 \dots x_{n-1}) \supsetneq \dots \supsetneq (x_1) \supsetneq (0)$

**Remark** Krull dimension is important in alg geometry. A prime ideal $P \subseteq K[x_1 \dots x_n]$ has a zero locus:

$$V(P) = \{(a_1 \dots a_n) \in K^n \mid f(a_1 \dots a_n) = 0 \mid f \in P\}$$

a chain of primes provides a chain

nested sequence of vanishing sets

If $k = \mathbb{C}$, these sets increase by dimension in $\mathbb{C}^n$.

Goal today Show a ring $R$ is Artinian $\iff$ it is Noetherian and $\dim R = 0$.

Proposition A ring $R$ has $\dim R = 0$ if and only if every prime ideal is maximal

Proof If $P \subseteq R$ is non-maximal, then $\exists\, m$ maximal such that $m \supseteq P$ so $\dim R \geq 1$.

If $\dim R \geq 1$ then $\exists\, P_0 \supseteq P_1$ and $P_1$ is prime not maximal $\square$.

Corollary An Artinian ring $R$ has $\dim R = 0$

**Prop** Every Artinian ring is Noetherian.

**Sketch** Let $m_1, \dots m_n$ be max ideals of $R$.

**Prop 8.4 AM** $\exists\ e$ s.t.

$$m_1^e \dots m_n^e = (0).$$

**Lemma 6.11 AM** Let $R$ be a ring in which $(0) = m_1 \dots m_k$ product of not necessarily distinct max ideals. Then $R$ Noetherian $\iff R$ is Artinian.

Consider the chain

$$A \supseteq m_1 \supseteq m_1 m_2 \supseteq \dots \supseteq (0)$$

Each factor $m_1 \dots m_{i-1} / m_1 \dots m_i$

is a vector space over the field $R/m_i$. Hence a.c.c $\Leftrightarrow$ d.c.c for each factor. Also a.c.c (or d.c.c) for each factor $\Leftrightarrow$ a.c.c (or d.c.c) for $R$. using short exact sequences recursively. $\square$

**Prop.** A Noetherian ring with $\dim R = 0$ is Artinian.

**Proof** Assume it isn't and consider $T = \{ I \subseteq R \mid R/I$ not Artinian$\}$. Since $R$ is Noetherian and $(0) \in T$ we can choose a max

element $I \in T$. Consider $B = R/I$

Then, • B is Noetherian and not Artinian.

• Every prime ideal of B is maximal. (same holds for R dim R=1)

• If $(0) \subsetneq J \subsetneq B$ is an ideal then $J = J'/I$ for some $I' \supseteq I$

hence $B/J = R/I \,/\, J'/I \cong R/J'$

so $B/J$ is Artinian.

<u>Claim</u> B is an integral domain.

assume claim: every prime ideal in B is max and integral domain

$\Rightarrow$ B is a field so B is Artinian

contradiction.

Proof of claim: Suppose $0 \neq x, y \in B$ and $xy = 0$

$$0 \to B/\mathrm{Ann}(x) \xrightarrow{\cdot x} B \to B/(x) \to 0.$$

We have $x \in (x)$, $y \in \mathrm{Ann}(x)$ so $(x), \mathrm{Ann}(x) \neq 0$. So both right & left modules in S.E.S are Artinian $\Rightarrow$ B would be Artinian $\Rightarrow$ contradiction.

Structure Thm for Artin rings

A Artinian ring $R$ is (uniquely) a finite direct product of Artinian local rings

$$R \cong \prod_{i=1}^{n} R/m_i^e$$ where $m_i$ max ideals $m_1^e \cdots m_n^e = 0$

**Proof** Recall we used AM 8.4, that

$$\prod_{i=1}^{n} m_i^e = 0$$

Then $m_i^e, m_j^e$ are comaximal $\forall\, i \neq j$

(if radicals are comaximal, then ideals are comaximal)

Prop 1.16.

Therefore $\bigcap m_i^e = \prod m_i^e$ and

$\phi : R \to \prod R/m_i^e$ is surjective since

pairwise comaximal and $\ker\phi = \prod m_i^e = 0$.

Each $R/m_i^e$ is Artinian, moreover its

max ideals are $m/m_i^e$ but since

$m_i^e \subseteq m \Rightarrow m = m_i$ there is a unique

max ideal $m_i/m_i^e$ hence it's local.

□.