Lecture 18 Ideals of $K[x_1,\dots x_n]$ and subsets of K^n.

Let K be algebraically closed

meaning every non-constant polynomial $f \in K[x]$ has a root in K. $\alpha \in K$ s.t. $f(\alpha) = 0$.

Def The vanishing locus of $f \in K[x_1,\dots x_n]$ is $V(f) := \{\alpha \in K^n \mid f(\alpha) = 0\}$

$\alpha = (a_1,\dots a_n)$.

for $f_1,\dots f_m$:

$$V(f_1,\dots f_m) := \{\alpha \in K^n \mid f_i(\alpha) = 0 \ \forall i\} = V(f_1) \cap \dots \cap V(f_m).$$

Ex. $V(f)$ = roots of f $f \in K[x]$.

- $V(x_1 - a_1, \dots, x_n - a_n) = (a_1,\dots a_n)$.
- $V(x_1^2 + x_2^2 - 1) \subseteq \mathbb{C}^2$ $V(x_1^2 + x_2^2 - 1) \cap \mathbb{R}^2 =$ 

Hilbert's weak Nullstellensatz

Thm

Let $m \subset K[x_1 \dots x_n]$ be a maximal ideal. Then $K[x_1 \dots x_n]/m$ is a finite field extension over K.

If K is algebraically closed then

$$K[x_1 \dots x_n]/m \cong K$$ in particular

$m = (x_1 - a_1, \dots x_n - a_n)$ for some $a_i \in K$.

* proof of many variants will come in alg geo 1.

Definition For $I \subseteq K[x_1 \dots x_n]$ an ideal define $V(I) = \{\alpha \in K^n \mid f(\alpha) = 0 \ \forall f \in I\} \subseteq K^n$.

A subset X of K^n is algebraic if $X = V(I)$ for some ideal I.

Example. $V((f)) = \{\alpha \mid g(\alpha) f(\alpha) = 0 \ \forall g\}$.

$= \{\alpha \mid f(\alpha) = 0\}$.

$= V(f)$

Lemma $\alpha \in V(I) \iff I \subseteq \mathfrak{m}_\alpha$.

Proof $\Leftarrow$ If $I \subseteq \mathfrak{m}_\alpha$ then $\forall f \in I$ we have $f \in \mathfrak{m}_\alpha$ so $f(\alpha) = 0$. and $\alpha \in V(I)$.

$\Rightarrow$ If $I \not\subseteq \mathfrak{m}_\alpha$ there is some $f \notin \mathfrak{m}_\alpha$ so $f(\alpha) \neq 0$. $\alpha \notin V(I)$.

Prop Let I, J be ideals

1) $V((0)) = K^n$
2) $V((1)) = \emptyset$ *
3) If $I \subseteq J$, then $V(J) \subseteq V(I)$
4) $V(I + J) = V(I) \cap V(J)$
5) $V(I \cdot J) = V(I) \cup V(J)$

6) $V(I) = V(\sqrt{I})$

Proof 1) By def. also $(0) \subseteq m_\alpha \ \forall \alpha$.

2) $(1) \not\subseteq m_\alpha$ for any α. $\alpha \notin V((1))$.

3) If $\alpha \in V(J)$ then $I \subseteq J \subseteq m_\alpha$

so $\alpha \in V(I)$

4) $\alpha \in V(I+J) \iff I+J \subseteq m_\alpha$

$\iff I \subseteq m_\alpha$ and $J \subseteq m_\alpha \iff \alpha \in V(I) \cap V(J)$

5) $IJ \subseteq I \cap J \subseteq I, J$. apply (3).

6) $I \subseteq \sqrt{I}$ so $V(\sqrt{I}) \subseteq V(I)$.

If $\alpha \in V(I)$ then $I \subseteq m_\alpha$ and

$\sqrt{I} \subseteq \sqrt{m_\alpha} = m_\alpha$ so $\alpha \in V(\sqrt{\alpha})$.

Notice also: $f \in \sqrt{I} \iff f^n \in I$.

and $f^n(\alpha) = 0 \iff f(\alpha) = 0$.

Recall $k[x_1, \dots, x_n]$ is Noetherian, i.e. every $I = (f_1, \dots, f_m)$ finitely generated.

So $V(I) = V((f_1) + \dots + (f_m)) = V(f_1) \cap \dots \cap V(f_m)$

$= V(f_1, \dots, f_m)$ a set of polynomials.

Hilbert basis theorem "gave" us generators

Lemma $V(I) \cap V(J) = \emptyset \iff I, J$ coprime.

Proof. $V(I) \cap V(J) = V(I + J)$.

$I + J = (1) \iff$ no max ideal containing I and J.

$\iff V(I + J) = \emptyset$. $\square$

Definition For $S \subseteq k^n$ define

$I(S) = \{ f \in k[x_1, \dots, x_n] \mid f(\alpha) = 0 \ \forall \alpha \in S \}$

$= \bigcap_{\alpha \in S} m_\alpha$ ← ideal.

Prop

1) $I(\emptyset) = (1)$
2) $I(K^n) = (0)$ *
3) $R \subseteq S \subseteq K^n \quad I(S) \subseteq I(R)$
4) $I(R \cup S) = I(R) \cap I(S)$

Proof

Weak Nullstellensatz (III)

K algebraically closed and $I \subseteq K[X_1 \dots X_n]$

then $Z(I) = \emptyset \iff I = (1)$.

Strong Nullstellensatz (equalent to weak)

K algebraically closed:

$J \subseteq K[x_1 \dots x_n]$ then $I(V(J)) = \sqrt{J}$.

Proof $f \in \sqrt{J}$ then $V(f) \supseteq V(\sqrt{J}) = V(J)$

so $f \in I(V(J))$.

The other direction is tricky. Rabinoff trick. see Ehrigsnd Lemma 14.17.

Def An ideal is radical if $\sqrt{I} = I$.

Corollary. There is a bijection

$$\left\{ \begin{array}{l} \text{radical ideals} \\ \text{of } K[x_1 \dots x_n] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{algebraic subsets} \\ \text{of } K^n \end{array} \right\}.$$

Notice the quotient ring $K[x_1 \dots x_n]/I$ for I radical is the ring of

polynomial functions on $V(I)$.

$f, g \in K[x_1 \dots x_n]$ then $f = g$ on $V(I)$

$\iff f \in g + I$ i.e. represent the same class in $K[x_1 \dots x_n]/I$.

Def An algebraic set $V(I)$ is **irreducible** if it cannot be written as $V(I) = V(I_1) \cup V(I_2)$ for $V(I_i) \neq \emptyset$.

Prop For I radical then $V(I)$ is irreducible $\iff$ I is prime.