Lecture 19 Primary ideals + decomposition

Def An ideal $Q \subsetneq R$ is primary if when $xy \in Q$ then either $x \in Q$ or $y^n \in Q$.

Examples 1) P prime $\Rightarrow P$ primary

2) $(x, y^2) \subseteq k[x,y]$ is primary but not prime nor P^n for any n.

3) primary ideals of $\mathbb{Z}$ are (0) and (p^k).

4) P^k is not necessarily primary for P prime

Consider $(\bar{x}, \bar{z})^2 \subseteq k[x,y,z]/(xy - z^2)$

$\bar{x}\bar{y} - \bar{z}^2 \in P^2$ but $\bar{x} \notin P^2$ $\bar{y} \in \sqrt{P^2} = P$.

Prop Q is primary $\Leftrightarrow R/Q \neq 0$ and all zero divisors of R/Q are nilpotent

Pf Exercise

Prop If Q is primary then $\sqrt{Q}$ is prime.

Proof $xy \in \sqrt{Q}$ then $(xy)^n \in Q$ for some $n \Rightarrow x^n$ or $y^n \in Q$ for some n, m. $\Rightarrow x$ or $y \in \sqrt{Q}$ $\square$.

Def A primary ideal Q is P-primary if

$$\sqrt{Q} = P$$

⚠ $\sqrt{I} = P$ prime does not mean it is primary. Consider $I = (xy, y^2) \subseteq K[x,y]$

$\sqrt{I} = (y, xy) = (y)$. So $\sqrt{I} = (y)$ is prime however it is not primary.

Since $xy \in I$ but $x^m \notin I$.

Also $K[x,y]/(xy, y^2)$ has x, y zero divisors $x^m \notin I$.

Prop If $\sqrt{Q} = m$ maximal, then Q is primary.

Pf In R/Q we have $\sqrt{(0)} = m/Q$.

yet $\sqrt{(0)} = \bigcap_{\text{prime ideals}} P \Rightarrow m/Q$ is the only prime ideal of R/Q. $\Rightarrow$ only max ideal $\Rightarrow$ every one out side m/Q is a unit so every zero divisor lies in $\sqrt{(0)}$ ie every zero divisor is nilpotent. $\square$.

Def A primary decomposition of an ideal I is $I = \bigcap_{i=1}^{n} Q_i$ Q_i primary

It is minimal if .

1) The primes $\sqrt{Q_i} = P_i$ are distinct

2) There is no i s.t.

$$Q_i \subseteq Q_1 \cap \dots \cap Q_{i-1} \cap Q_{i+1} \cap \dots \cap Q_n$$

Example $(n) \subseteq \mathbb{Z}$ then

$$(n) = (p_1^{e_1}) \cap \dots \cap (p_k^{e_k}) \text{ where}$$

$n = p_1^{e_1} \dots p_k^{e_k}$ is a prime factorization.

Minimal $\iff$ p_i are distinct.

Lemma. If Q_1, Q_2 are P-primary then $Q_1 \cap Q_2$ is P-primary.

Proof Exercise Hint: $\sqrt{Q_1 \cap Q_2} = \sqrt{Q_1} \cap \sqrt{Q_2}$.

Must show intersection is primary.

Corollary If a primary decomposition of I exists then exists a minimal one.

Proof Apply the previous lemma to satisfy condition 1. Toss out unneeded Q_i's to satisfy condition 2.

Example 1) $(xy) \subset k[x,y]$ has $(x,y) = (x) \cap (y)$. both ideals are prime!

2) $(xy, y^2) \subseteq k[x,y]$ has minimal decompositions:

$$(y) \cap (x,y)^2 = (y) \cap (y^2, x)$$

NOT ALWAYS UNIQUE!

Theorem (Lasker-Noether)

If R is Noetherian and $I \subseteq R$ an ideal then I admits a primary decomposition

$$I = \bigcap_{i=1}^{n} Q_i \qquad Q_i \text{ primary.}$$

Def An ideal $I \subseteq R$ is irreducible if

$$I = \bigcap_{i=1}^{k} I_i \Rightarrow I_j = I \text{ for some } i.$$

Lemma R Noetherian, then $\forall\, I \subseteq R$ can be decomposed into irreducibles J_i

$$I = J_1 \cap \dots \cap J_k$$

Proof Suppose such decomposition does not always exist and let I_{max} be the maximal such ideal in this set (I_{max} exists since Noetherian).

Then I_{max} cannot be irreducible hence

$$I_{max} = I_1 \cap I_2 \text{ where } I_1, I_2 \supsetneq I_{max}$$

hence I_1, I_2 can be decomposed into irreducibles therefore so can I_{max}. ☐

Lemma If R is Noetherian, then an irreducible ideal I is primary.

Proof Pass to R/I and we'll show (0) is primary.

Notice (0) is irreducible.

Suppose otherwise meaning $xy \in (0)$ and $x, y \neq 0$ $y^n \neq 0$ for any n.

Consider $\text{Ann}(y) \subseteq \text{Ann}(y^2) \subseteq \ldots\ldots$ chain of ideals. R/I Noetherian, so it must stabilise, meaning $\text{Ann}(y^n) = \text{Ann}(y^N)$ for all $n \geq N$. Since y is not nilpotent $(y^N) \neq 0$ and since $x \in \text{Ann}(y) \subseteq \text{Ann}(y^N)$ $\text{Ann}(y^N) \neq 0$

Claim $(y^N) \cap \text{Ann}(y^N) = (0)$ which contradicts (0) irreducible

Proof of claim: if $\neq 10)$ then $0 \neq ay^N \in (y^N)$ and if $ay^N \in \mathrm{Ann}(y^N)$ then $ay^{2N} = 0 \Rightarrow a \in \mathrm{Ann}(y^{2N}) = \mathrm{Ann}(y^N)$ $\Rightarrow ay^N = 0$. □

Uniqueness* Theorem

Recall $(I : x) := \{r \in R \mid rx \in I\}$

If $I \subseteq R$ has minimal primary decomposition $I = \bigcap_{i=1}^{k} Q_i$ then the primes $\sqrt{Q_i}$ are exactly the primes in $\{\sqrt{(I : x)} \mid x \in R\}$.

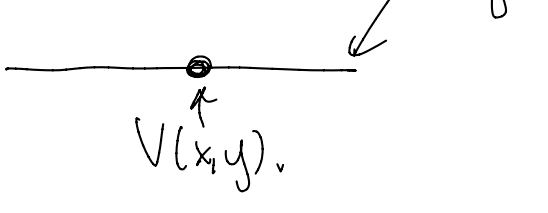
Def The prime ideals $\sqrt{Q_i}$ in the decomposition are called associated primes

The associated primes form a poset with inclusion

An associated prime is isolated/minimal if it is minimal. It is embedded otherwise

Example. $(y) \cap (x,y)^2$ we have (y) is isolated and (x,y) is embedded

$V((x,y)) \subseteq V((y))$



Proposition The isolated primes are exactly those minimal over I.