

Lecture II : Ideals

* Ideal quotient correspondence

* prime + maximal ideals

* existence of maximal ideals

Recall: $I \subseteq R$ an ideal we have
 $\phi: R \rightarrow R/I$ ring homomorphism.

Thm There is a one to one order preserving correspondence:

$$\{\text{Ideals of } R \text{ containing } I\} \longleftrightarrow \{\text{Ideals of } R/I\}$$

$$I \subseteq J \mapsto \phi(J) = J'_I$$

$$\phi^{-1}(J') \longleftarrow J'$$

where $\phi: R \rightarrow R/I$ is the quotient homomorphism

Lemma Given a homomorphism $f: S \rightarrow R$

If $I \subseteq R$ ideal then

$f^{-1}(I) \subseteq S$ is an ideal.

"pullback/contraction"

Proof
preimage is an additive group:

$$\phi(0_S) = 0_R \quad 0_S \in \phi^{-1}(I)$$

if $s \in \phi^{-1}(I)$ then $-s \in \phi^{-1}(I)$

if $s_1, s_2 \in \phi^{-1}(I)$ then

$$\phi(s_1) + \phi(s_2) \in I \text{ and } s_1 + s_2 \in \phi^{-1}(I)$$

if $\phi(s_1) \in I$ then $-\phi(s_1) \in I \Rightarrow -s_1 \in \phi^{-1}(I)$

If $s \in \phi^{-1}(I)$ and $x \in S$ then

$$\phi(xs) = \phi(x)\phi(s) \in I$$

So $\phi^{-1}(I)$ is an ideal. $\square$.

Lemma If $f: S \rightarrow R$ is a surjective homomorphism and I an ideal of S , then $f(I)$ is an ideal of R .

Proof exercises.

Ex $f: \mathbb{Z} \rightarrow \mathbb{Q}$ inclusion mapping
 $I = (n) \subseteq \mathbb{Z}$ $f(I) = \{kn \mid k \in \mathbb{Z}\}$
is not an ideal of $\mathbb{Q}$.

Ideals of $\mathbb{Q}$ are $\{0\}, \mathbb{Q}$.

In general $f(I)$ is **NOT** an ideal of R . However $Rf(I)$ is an ideal of R for any homomorphism

$Rf(I)$ is the "Extension" / "Pushout" of I by f .

Back to correspondence

Order preserving means:

$$J_1 \subseteq J_2 \iff \phi(J_1) \subseteq \phi(J_2)$$

This is clear. Thm is proved $\square$.

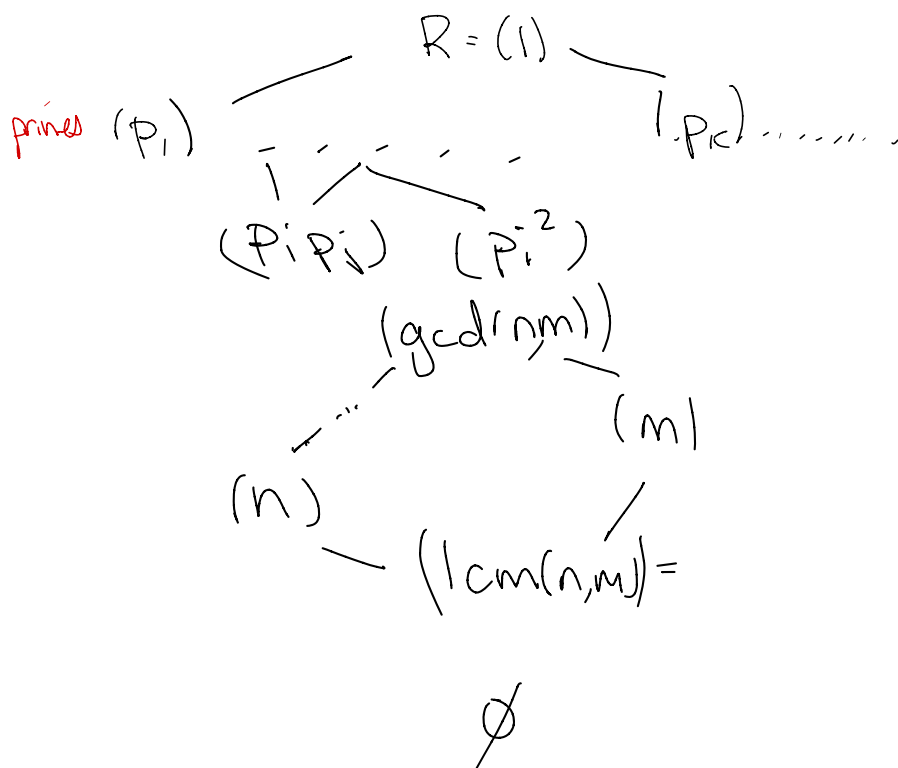
Inclusion defines a partial order on ideals in R , so

$$\mathcal{J}(R) = \{ I \subseteq R \mid I \text{ ideal} \}$$

forms a poset (partially ordered set).

Example $R = \mathbb{Z}$ all ideals are principal
 $(m) \subseteq (n) \iff n \mid m$.

Diagram of $\mathcal{I}(R)$:



Given any pair of ideals $I, J \subseteq \mathcal{I}(R)$ in the poset there is:

- least upper bound $I + J$.
- greatest lower bound $I \cap J$.

The poset $\mathcal{I}(R)$ is a "lattice" (operations are called "join" + "meet" resp.)

Ideal sum $I + J = \{a+b \mid a \in I, b \in J\}$

• $0 = 0 + 0 \in I + J$

• $r_1, r_2 \in I + J$ then $r_i = a_i + b_i$

$r_1 + r_2 = a_1 + a_2 + b_1 + b_2 \in I + J.$

• if $r \in I + J$ $-r = -(a+b) = -a-b$
 $-a \in I, -b \in J \Rightarrow -r \in I + J. \checkmark$

If $r \in I + J$, $r = a + b$ then
 $c \cdot r = c(a+b) = ca + cb \in I + J.$
 $\square$

Ideal intersection: $I \cap J$ is an ideal

abelian group $\checkmark$

Suppose $a \in I \cap J$ then $ra \in I$

$ra \in J \Rightarrow ra \in I \cap J.$

$\checkmark.$

Zero divisors, nilpotents + units

Zero divisor is $x \in R$ s.t. $xy = 0$
for some $0 \neq y \in R$

If R has no zero divisors it is
an integral domain

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$\mathbb{Z}/(n)$ for n composite is NOT

$x \in R$ is nilpotent if $x^n = 0$
for some $n > 0$

A unit is $x \in R$ s.t. $xy = 1$
for some $y \in R$

Every $a \neq 0$ is a unit in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Units of $\mathbb{Z}$ are $+1, -1$.

x is a unit $\Leftrightarrow (x) = R$.

A field is a ring with $1 \neq 0$
and $\forall x \neq 0$ is a unit.

Theorem $R \neq 0$ TFAE

- 1) R is a field
- 2) The only ideals are R and 0
- 3) Every homomorphism of R into $S \neq 0$ is injective

1) $\Rightarrow$ 2) If I ideal $\neq 0$, then
 $x \in I$ $x \neq 0$ but x unit $\Rightarrow I = R$.

2) $\Rightarrow$ 3) if $f: R \rightarrow S$ then $\ker f = 0, R$
it can't be R if $S \neq 0$ since $1 \neq 0$.
so $\ker f = 0$ injective.

3) $\Rightarrow$ 1) contrapositive. suppose $0 \neq x \in R$ is

Not a unit. Then $(x) \neq R$ so $R/(x)$ is not zero ring and $\phi: R \rightarrow R/(x)$ has $\ker(\phi) = (x)$ not injective $\square$.

Prime + Maximal ideals

Def An ideal $I \subseteq R$ is prime if $I \neq (1)$ and $ab \in I \Rightarrow a \in I$ or $b \in I$

I prime $\Leftrightarrow R/I$ integral domain

I is maximal if $I \neq (1)$ and $\exists J$ s.t. $I \subsetneq J \subsetneq (1)$

I maximal $\Leftrightarrow R/I$ field

(use correspondence of ideals)

Fields are integral domains $\Rightarrow$
maximal ideals are prime

Extension/contraction of prime/Maximal

Proposition If $f: S \rightarrow R$ homomorphism
and $I \subseteq R$ is prime then $f^{-1}(I)$
is prime.

Proof. $f: S/f^{-1}(I) \rightarrow R/I$ is an
injective homomorphism $\Rightarrow S/f^{-1}(I)$ is
isomorphic to a subring of R/I
 $\Rightarrow$ integral domain. $\square$

 $Rf(S)$ is not necessarily prime!

$I = (p) \subseteq \mathbb{Z}$ $i: \mathbb{Z} \rightarrow \mathbb{Q}$ $\mathbb{Q}I = \mathbb{Q}$
not prime!

⚠ If $I \subseteq R$ is maximal $f^{-1}(I)$ is not necessarily maximal,

Example If R is a field max ideal is $I = (0)$

$f: \mathbb{Z} \rightarrow \mathbb{Q}$ injection but

$f^{-1}((0)) = (0)$ not maximal!

Theorem Every $R \neq 0$ contains at least one maximal ideal.

Proof. Zorn's lemma: A poset $\mathcal{P}$ in which every chain has an upper bound contains a maximal element

chain: $\{I_\alpha\}_{\alpha \in A} \subseteq \mathcal{I}(R)$ s.t.

$\forall \alpha, \beta \in A \quad I_\alpha \subseteq I_\beta \text{ or } I_\beta \subseteq I_\alpha$

Given a chain $\{I_\alpha\}_{\alpha \in A}$ of ideals $I_\alpha \neq (1)$ claim upper bound is $X = \bigcup_{\alpha \in A} I_\alpha$. (must prove it is an ideal)

Abelian group:

1) $0 \in X$

2) If $a, b \in X$ then $a \in I_\alpha, b \in I_\beta$
since chain $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$
 $\Rightarrow a, b \in I_\alpha$ or $I_\beta \Rightarrow a+b \in X$.

3) if $a \in X$ then $a \in I_\alpha$ for some $\alpha \Rightarrow -a \in I_\alpha \Rightarrow -a \in X$

4) If $x \in X$ then $x \in I_\alpha$ so $rx \in I_\alpha$
 $rx \in X$.

X is an ideal and an upper bound.

Moreover $X \neq (1)$ since $1 \notin I_\alpha \forall \alpha$.

Apply Zorn's lemma to poset $\mathcal{I}(R) \setminus \{(1)\}$

Corollary If $I \subsetneq R$ is an ideal
there is a maximal ideal of R
containing I .

Proof Apply ideal correspondence
under $R \rightarrow R/I$ $\square$

Corollary every non-unit $x \in R$
is contained in a max ideal.

Proof Apply above corollary to $(x) = I$.