

Lecture 5 Modules. Definitions, examples, homomorphisms.

R a ring. An R -module is an abelian group $(M, +)$ together with a map.

$$R \times M \rightarrow M$$
$$(r, m) \mapsto r \cdot m \quad \text{"scalar multiplication"}$$

Satisfying:

$$1) \quad 1 \cdot m = m \quad \forall m \in M$$

$$2) \quad r_1(r_2 m) = (r_1 r_2) m \quad \forall r_1, r_2 \in R, m \in M$$

$$3) \quad (r_1 + r_2) m = r_1 m + r_2 m \quad \forall$$

$$4) \quad r(m_1 + m_2) = r m_1 + r m_2 \quad \forall r \in R, m_1, m_2 \in M.$$

Examples 1) $\{0\}$ is an R -module $\forall R$
The zero-module O -module.

2) R is an R -module

3) If $S \subseteq R$ subring then
 R is an S module]

4) I ideal of R is a submodule

5) If $R = K$ a field, then K -modules
are exactly vector spaces.
(compare with vector space axioms).

6) $R = K[x]$ K -field then R -module
is a K vector space with a
linear transformation

$$K[x] \times M \rightarrow M$$

$\cup$

$$K \times M \rightarrow M \Rightarrow M \text{ must be a vector space.}$$

$$M = V \quad (x, v) \rightarrow x \cdot v$$

defines a linear map

$$\phi: V \rightarrow V$$

$$\phi(v) = x \cdot v$$

$$\phi(\lambda v) = \lambda \phi(v)$$

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2).$$

5) $R = \mathbb{Z}$ then $\mathbb{Z}$ -module M is no more than an abelian group G .

(ie the map $\mathbb{Z} \times M \rightarrow M$ has no choice). The axioms force:

$$n \cdot g = \begin{cases} (1 + \dots + 1)g = g + \dots + g & n > 0 \\ 0 & n = 0 \\ -(1 + \dots + 1)g = -g + (-g) + \dots + (-g) & n < 0 \end{cases}$$

6) G finite group, K a field

The group ring is:

$$K[G] = \left\{ \sum_{\text{finite sums}} \lambda_i g_i \mid \lambda_i \in K, g_i \in G \right\}$$

addition is "formal" (think of g_i 's as variables)

$$\lambda_1 g_i + \lambda_2 g_i = (\lambda_1 + \lambda_2) g_i$$

$g_i + g_j$ is what it is.

$K[G]$ commutative $\iff G$ abelian

$K[G]$ behaves like $K\langle g_1, \dots, g_k, \dots \rangle$

In particular, $K[G]$ module is a K -vector space V together with linear maps $\phi_{g_i} : V \rightarrow V \quad \forall g_i \in G$

satisfying $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$

This is exactly the definition of a group representation MAT 4270.

7) If $f: R \rightarrow S$ is a ring homom.
Then S is an R -module.
 S abelian group $\checkmark$.

$$\begin{aligned} R \times S &\rightarrow S \\ (a, b) &\mapsto f(a) \cdot b \end{aligned}$$

Recall If $f: K \rightarrow K'$ is a homomorphism of fields then K' is a K -vector space.

Def. M, N R -modules a homomorphism is a map $f: M \rightarrow N$ s.t.

$$f(m + m') = f(m) + f(m') \quad m, m' \in M$$

$$f(am) = a f(m) \quad \forall m \in M \\ a \in R$$

f is an isomorphism if bijective.

Examples 1) $0: M \rightarrow N$ $0(m) = 0 \forall m \in M$

2) $R = K$ the K -module homomorphisms are exactly linear maps of vector spaces, M, N finite dimensional F is given by a matrix $(n \times m)$ -matrix

3) $R = \mathbb{Z}$ and M, N $\mathbb{Z}$ -modules

$f: M \rightarrow N$ homom is homom. of abelian groups. (condition $f(rx) = rx$ is automatically satisfied by additive condition).

4) $\phi: R \rightarrow (a)$ $a \in R$
 $\phi(r) = r \cdot a$ is a homomorphism

it is surjective and $\text{Ker } \phi = \text{Ann}(a)$

Def M, N R -modules:

The set of homomorphisms $\text{Hom}_R(M, N)$ from M to N is an R module:

$$f, g \in \text{Hom}_R(M, N) \Rightarrow f+g, rf \in \text{Hom}_R(M, N)$$

$$(f+g)(m) = f(m) + g(m)$$

$$(rf)(m) = r f(m)$$

Module constructions

Direct sums. $M_1, \dots, M_n$ R -modules

$$\bigoplus_{i=1}^n M_i = \{ (m_1, \dots, m_n) \mid m_i \in M_i \}.$$

$$\begin{aligned} \text{addition} : (m_1, \dots, m_n) + (m'_1, \dots, m'_n) \\ = (m_1 + m'_1, m_2 + m'_2, \dots, m_n + m'_n) \end{aligned}$$

$$\text{scalar mult} : a(m_1, \dots, m_n) = (am_1, \dots, am_n)$$

For any set $\{M_i\}_{i \in S}$ of R -modules

Direct sum

$$\bigoplus_{i \in S} M_i = \{ (m_i)_{i \in S} \mid m_i \in M_i, m_i \neq 0 \text{ for any finitely many } i \}$$

Direct product

$$\prod_{i \in S} M_i = \{ (m_i)_{i \in S} \mid m_i \in M_i \}$$

$$\text{If } |S| < \infty \text{ then } \bigoplus_{i \in S} M_i = \prod_{i \in S} M_i$$

Example $R[x]$ and $R[[x]]$ are K -modules. As modules:

$$R[x] \cong \bigoplus_{i \in \mathbb{N}} R$$

$$R[[x]] \cong \prod_{i \in \mathbb{N}} R$$



Notice ring structures are different!

$$f = \sum_{i=0}^n a_i x^i \quad g = \sum_{i=0}^{\infty} b_i x^i$$

$f \cdot g$ does not have coefficients $a_i b_i$

Example Every vector space V over K has a basis $\{v_i\} \in S$.

$$\phi: \bigoplus_{i \in S} K \longrightarrow V$$

$$\phi((a_i)_{i \in S}) \mapsto \sum_{i \in S} a_i v_i$$

Submodules M an R -module then

$M' \subseteq M$ is a submodule if $M' \subseteq M$ is

a subgroup and $\forall a \in R \ m \in M'$

$$am \in M'.$$

Examples The submodules of R as an R -module are precisely the ideals of R .

A submodule of a $\mathbb{Z}$ module M (recall abelian group) is simply a subgroup $M' \subseteq M$. Since in M' is a subgroup it is closed under addition and

$$n \cdot m = m + \dots + m \quad \forall m \in M' \quad n \in \mathbb{Z} \\ \in M'.$$

Given submodules $(M_i)_{i \in S}$ $M_i \subseteq N$ their sum is a submodule of N .

$$\sum_{i \in S} M_i = \left\{ \sum_{\text{finite sums}} m_i \mid m_i \in M_i \right\} \subseteq N.$$

This generalises sum of ideals.

their intersection is also a submodule

$$\bigcap_{i \in S} M_i = \{ m \mid m \in M_i \quad \forall i \in S \}.$$

Poset of submodules is a complete lattice

Quotient modules $M' \subseteq M$ submodule
of R -module M then

$$M/M' = \{ m + M' \mid m \in M \}$$

quotient group with $R \times M/M' \rightarrow M/M'$

defined by $a(m + M') = am + M'$.

In particular, for any ideal $I \subseteq R$
the quotient ring R/I is an
 R -module.

Kernels, images, quotients

For $\phi: M \rightarrow N$ homom. of R -modules
submodule

$$\text{Ker } \phi = \{ m \in M \mid \phi(m) = 0 \} \subseteq M$$

$$\text{Im } \phi = \{ n \in N \mid \exists m \in M, \phi(m) = n \}$$

$\text{submodule} \subseteq N$

Cok $\phi = N / \text{Im } M.$ quotient module.
"co kernel"

Example Fix $a_1, \dots, a_n \in R$ and define

$$\phi: \bigoplus_{i=1}^n R \rightarrow R$$

$$\phi(r_1, \dots, r_n) \mapsto r_1 a_1 + \dots + r_n a_n$$

$$\text{Im } (\phi) = (a_1, \dots, a_n) \text{ ideal of } R.$$

Module isomorphism theorem

(i) Let $\phi: M \rightarrow N$ be a homom. of R modules then,

$$M / \text{Ker } \phi \cong \text{Im } \phi.$$

2) Let $M'' \subseteq M' \subseteq M$ be R -modules
 then $M/M' \cong (M/M'')/(M'/M'')$.

3) Let $M, N \subseteq P$ be submodules so
 that $M+N \subseteq P$ and $M \cap N \subseteq P$.

Then $(M+N)/N \cong M/(M \cap N)$.

Proof

For 2) and 3) define homomorphisms:

2) $\phi: M/M'' \rightarrow M/M'$
 $m + M'' \mapsto m + M'$ well defined

$\ker \phi = M'/M''$.

3) $\phi: M_2 \rightarrow M_1 + M_2 \rightarrow (M_1 + M_2)/M_1$
 $m_2 \mapsto 0 + m_2 \mapsto$
 $m_1 + m_2 \mapsto m_1 + m_2 + M_1$

$\ker \phi = M_1 \cap M_2$.