

Recall from last time:

2) Let $M'' \subseteq M' \subseteq M$ be R -modules
then $M/M' \cong (M/M'')/(M'/M'')$.

3) Let $M, N \subseteq P$ be submodules so
Then $(M+N)/N \cong M/(M \cap N)$.

Proof

For 2) and 3) define homomorphisms:

2) $\phi: M/M'' \rightarrow M/M'$
 $m + M'' \mapsto m + M'$ well defined

$$\ker \phi = M'/M''.$$

3) $\phi: M_2 \rightarrow M_1 + M_2 \rightarrow (M_1 + M_2)/M_1$
 $m_2 \mapsto 0 + m_2 \mapsto$
 $m_1 + m_2 \mapsto m_1 + m_2 + M_1$

$$\ker \phi = M_1 \cap M_2.$$

Some more module operations

If N, M submodules of P , then

$$(N:M) = \{ r \in R \mid rM \subseteq N \}$$

is an ideal of R .

$$\text{Ann}(M) = (0:M) = \{ r \mid rM = 0 \}.$$

If $I \subseteq \text{Ann}(M)$, then M is also a R/I module. ($I \subseteq \text{Ann}(M)$ is what makes the multiplication map well defined).

Exercise

$$\text{Ann}(M+N) = \text{Ann}(M) \cap \text{Ann}(N)$$

$$(N:P) = \text{Ann}((N+P)/_N)$$

Lecture 6 Finitely generated modules

Definition A module M is finitely generated if either of the following equivalent conditions hold:

- $\exists m_1, \dots, m_n \in M$ s.t. $\forall m \in M$ is $m = \underbrace{r_1 m_1 + \dots + r_n m_n}_{\text{not necessarily unique } r_i\text{'s}}$ for $r_i \in R$

- $\exists$ a surjective homomorphism

$$\phi: \bigoplus_{i=1}^n R \rightarrow M$$

Examples • An ideal $I \subseteq R$ is finitely generated as a module $\Leftrightarrow$ f.g. as an ideal.

- K a field then a finitely generated K -module is a finite dimensional vector space $M \cong M' \Leftrightarrow \dim M = \dim M'$

- An abelian gp is finitely generated as a $\mathbb{Z}$ module $\iff$ it is a f.g. abelian gp.

Classification thm $\Rightarrow$

$$M \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \mathbb{Z}/(p_1^{e_1}) \oplus \dots \oplus \mathbb{Z}/(p_n^{e_n})$$

Therefore, $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$ -module.

• $R[x]$ is not finitely generated as an R -module. We would need generators $(1, x, x^2, \dots, x^n, \dots)$ NOT FINITE.

Def An R -module M is free if

$$M \cong \bigoplus_{i \in I} R \quad \text{for some } I.$$

A finitely generated free R -module

$$\text{is } M \cong \bigoplus_{i=1}^n R \quad \text{for some } n.$$

written: R^n .

Prop M is a finitely generated R module $\Leftrightarrow M$ is isomorphic to a quotient of R^n .

Proof. $\Rightarrow$ Use existence of surjective $\phi: R^n \rightarrow M$ homomorphism + module isomorphism theorem

$\Leftarrow M \cong R^n / N$ then $\exists$ a homom.

$\phi: R^n \rightarrow M$ surjective. Since

$e_i = (0, \dots, \underset{i}{1}, \dots, 0) \in R^n$ for $1 \leq i \leq n$

generate R^n , $\phi(e_i)$ generate M .

Nakayama's Lemma

Very useful in algebraic geometry

Nakayama " $IM = M \Rightarrow im = m$ ".

If I is an ideal of R and M a finitely generated R -module s.t.
 $IM \subseteq M$ then there exists $i \in I$
s.t. $im = m \quad \forall m \in M$.

Proof. Let $m_1, \dots, m_n$ be generators
of M . $m_i \in IM$ means

$$\forall i \quad m_i = b_{i1}m_1 + \dots + b_{in}m_n \quad (*)$$

with $b_{ij} \in I \quad \forall i, j$. Define a
matrix $T = I_n - (b_{ij})$ (entries in R)

Then $(*)$ translates to $T \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$
 $(**)$

There is a matrix $\text{adj}(T)$ s.t.
 $\text{adj}(T)T = T \cdot \text{adj}(T) = \det(T)I_n$

the adjugate/adjoint matrix

$$\text{adj}(T) = (c_{ij})$$

$$c_{ij} = (-1)^{i+j} \det(T_{ji}) \quad T_{ji} \text{ delete } i^{\text{th}} \text{ row } j^{\text{th}} \text{ column}$$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{adj } T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\text{adj } T$ and $\text{adj } T T = \det(T) I_n$ works over any ring!

Multiplying both sides of ** by $\text{adj}(T)$ we obtain

$$\det(T) I_n \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \det(T) m_i = 0 \quad \forall i$$

Moreover $\det(T) \in 1 + \mathcal{I}$

To see this perform cofactor expansion

$$T = \begin{pmatrix} 1-b_{11} & & -b_{1j} \\ & \ddots & \\ -b_{ij} & & 1-b_{nn} \end{pmatrix}$$

$$\begin{aligned} \det(T) &= (1-\underline{b_{11}}) \det(T_{11}) + \sum_{j=2}^n (-1) \underline{b_{1j}} \det(T_{1j}) \\ &= 1 + B \quad \text{where } B \in I. \end{aligned}$$

$$\text{Now } i = -B = 1 - \det(T) \in I$$

$$\begin{aligned} \text{and } i \cdot m &= (1 - \det(T))m \\ &= m - 0, \quad \forall m \in M. \end{aligned}$$

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Example Finite generation is necessary.

$$M = \mathbb{Q} \quad R = \mathbb{Z} \quad I = (2). \quad \text{then}$$

$$I\mathbb{Q} = \mathbb{Q} \quad \text{but there is no even}$$

$$K \text{ such that } Kq = q \quad \forall q \in \mathbb{Q}.$$

Corollary If M is finitely generated and $f: M \rightarrow M$ is a surjective endomorphism then f is injective.

Counter-example $M = \mathbb{Q}[x]$.

$$\frac{d}{dx}: \mathbb{Q}[x] \rightarrow \mathbb{Q}[x]$$
$$\sum a_i x^i \mapsto \sum a_i i x^{i-1} \quad \text{surjective}$$

but not injective!

Proof. Use f to make M an $R[x]$

Module $R[x] \times M \rightarrow M$

defined by $x \cdot m = f(m)$.

Since f is surjective $(x)M = M$

$\Rightarrow \exists x \cdot p(x) \in (x) \subseteq R[x] \text{ s.t.}$

$$m = x \cdot p(x) \cdot m \quad \forall m$$

$$\text{If } f(m) = 0 \Rightarrow m = p(x)x \cdot m =$$

$$= p(x) \cdot (f(m)) = p(x) \cdot 0 = 0$$

so $f(m) = 0 \Rightarrow m = 0$ f injective $\square$.

Local Nakayama's lemma If R is a local ring with max ideal $\mathfrak{m}$ and M a finitely generated module

$$\mathfrak{m}M = M \Rightarrow M = 0.$$

Proof. From previous version we found $\det(T) \cdot m_i = 0$ and $\det T = 1 + B$
 $B \in \mathfrak{m} \Rightarrow \det T \notin \mathfrak{m}$ so it must be a unit (every non-unit is contained in a max ideal). Since $\det T$ is a unit $\det T m_i = 0 \Rightarrow m_i = 0 \forall i \Rightarrow M = 0 \quad \square$.

Corollary R local M fin. gen. Suppose $m_i \in M$ are s.t. $\phi(m_i)$ are a basis of $M/\mathfrak{m}M$, then m_i generate M .

Proof. let $N \subseteq M$ be submodule generated by $m_1, \dots, m_n$. Then $N \rightarrow M \rightarrow M/\eta M$ is surjective

$$\text{hence } N + \eta M = M$$

quotienting by N we get.

$$M/N = \eta M/N \Rightarrow M/N = 0$$

Nakayama

$$\Rightarrow N = M.$$

Corollary R local M finitely generated

Let $m_1, \dots, m_n$ be a minimal generating set ($\{m_{i_1}, \dots, m_{i_k}\} \neq \{m_1, \dots, m_n\}$ does not generate)

$$\text{then } n = \dim M/\eta M.$$

All minimal generating sets have Exercise same cardinality.

Exact sequences

A sequence of R -modules + R -module homomorphisms

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is exact if $\ker f_{i+1} = \operatorname{Im} f_i \quad \forall i$

Example

1) $0 \rightarrow M' \xrightarrow{f} M$ exact $\Leftrightarrow f$ injective

2) $M' \xrightarrow{g} M \rightarrow 0$ exact $\Leftrightarrow g$ surjective

3) $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact

$\Leftrightarrow f$ is injective, g is surjective

and
$$\begin{array}{ccc} M / \operatorname{Im} f(M') & \xrightarrow{\cong} & M'' \\ \text{"} & \uparrow g & \\ \operatorname{Coker}(f) & & \end{array}$$

3) is called a short exact sequence

Any long exact sequence

$$\dots \rightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \rightarrow \dots$$

can be broken up into short exacts:

$$0 \rightarrow \ker f_{i+1} \rightarrow M_i \rightarrow \operatorname{Im} f_{i+1} \rightarrow 0$$

exact at M_i by homomorphism ψ_m :

$$M_i / \ker f_{i+1} \cong \operatorname{Im} f_{i+1}$$

Example M' submodule of M then

$$0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0 \text{ is } \underline{\text{exact}}.$$

Up to isomorphism every short exact seq. is of this form. $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$
 $\Rightarrow M'$ is iso to submodule of M , M'' is iso to quotient

Example For any modules M, N

$$0 \rightarrow M \rightarrow M \oplus N \rightarrow N \rightarrow 0$$
$$m \mapsto (m, 0)$$
$$(m, n) \mapsto n$$

is exact.

A short exact sequence is split if it is isomorphic to a sequence of the above form.

$$0 \rightarrow \mathbb{Z}/(2) \rightarrow \mathbb{Z}/(4) \rightarrow \mathbb{Z}/(2) \rightarrow 0$$

NOT split. (Warm up)

Snake Lemma Let

$$\begin{array}{ccccccc}
 0 & \rightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' \rightarrow 0 \\
 & & f' \downarrow & & f \downarrow & & f'' \downarrow \\
 0 & \rightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' \rightarrow 0
 \end{array}$$

be a commutative diagram of
 $\mathcal{S} \in \mathcal{S}$. (Rows are exact and
 $f'' \circ v = v' \circ f$ + $f \circ u = u' \circ f'$)

then there is a long exact seq

$$\begin{array}{ccccccc}
 0 \rightarrow \ker f' & \xrightarrow{u} & \ker f & \xrightarrow{v} & \ker f'' & \xrightarrow{d} & \operatorname{cok} f' \\
 \xrightarrow{\bar{u}'} & \operatorname{cok} f' & \xrightarrow{\bar{v}'} & \operatorname{cok} f'' & \rightarrow & 0
 \end{array}$$

Proof Define the map d . For $x'' \in \ker f''$
 $x'' \in M''$ and v is surjective so $\exists x \in M$
s.t. $v(x) = x''$ $v'(f(x)) = f''(v(x)) = 0$
 $f(x) \in \ker v' \Rightarrow f(x) \in \operatorname{Im} u'$ so

$f(x) = u'(y')$ for $y' \in N'$. Define
 $dx'' := y' + \text{Im } f' \in \text{Cok}(f')$.

Must prove d is well-defined
and exact Diagram chase.

Def Let C be some set of R -modules and $\lambda: C \rightarrow G$ some abelian group G . (ex $\mathbb{Z}$).

λ is additive $\forall \forall$ SES

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \quad M, M', M'' \in C$$

we have $\lambda(M') - \lambda(M) + \lambda(M'') = 0$.

Ex $R = K$ $C = \text{finite dim'l vector space}$
 $\lambda = \dim$

Proposition $0 \xrightarrow{f_0} M_0 \xrightarrow{f_1} \dots \rightarrow M_n \xrightarrow{f_{n+1}} 0$
exact with M_i and $\ker f_i \in C^i$

Then $\sum (-1)^i \lambda(M_i) = 0$

for additive function $\lambda: C \rightarrow G$.

Proof Split into SES

$$0 \rightarrow \underbrace{\ker f_i}_{N_i} \rightarrow M_i \rightarrow \underbrace{\ker f_{i+1}}_{N_{i+1}} \rightarrow 0$$

$$\lambda(M_i) = \lambda(N_i) + \lambda(N_{i+1}) \quad \text{and } N_0,$$

$N_{n+1} = 0$. Upon taking alternating sums we get 0.