

Exact sequences

A sequence of R -modules + R -module homomorphisms

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is exact if $\ker f_{i+1} = \operatorname{Im} f_i \quad \forall i$

Example

1) $0 \rightarrow M' \xrightarrow{f} M$ exact $\Leftrightarrow f$ injective

2) $M' \xrightarrow{g} M \rightarrow 0$ exact $\Leftrightarrow g$ surjective

3) $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact

$\Leftrightarrow f$ is injective, g is surjective

and
$$\begin{array}{ccc} M / \operatorname{Im} f(M') & \xrightarrow{\cong} & M'' \\ \text{"} & \text{"} & \\ \operatorname{Coker}(f) & \xrightarrow{g} & \end{array}$$

3) is called a short exact sequence

Any long exact sequence

$$\dots \rightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \rightarrow \dots$$

can be broken up into short exacts:

$$0 \rightarrow \ker f_{i+1} \rightarrow M_i \rightarrow \operatorname{Im} f_{i+1} \rightarrow 0$$

exact at M_i by homomorphism ψ_m :

$$M_i / \ker f_{i+1} \cong \operatorname{Im} f_{i+1}$$

Example M' submodule of M then

$$0 \rightarrow M' \rightarrow M \rightarrow M/M' \rightarrow 0 \text{ is } \underline{\text{exact}}.$$

Up to isomorphism every short exact seq. is of this form. $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$
 $\Rightarrow M'$ is iso to submodule of M , M'' is iso to quotient

Example For any modules M, N

$$0 \rightarrow M \rightarrow M \oplus N \rightarrow N \rightarrow 0$$
$$m \mapsto (m, 0)$$
$$(m, n) \mapsto n$$

is exact.

Def

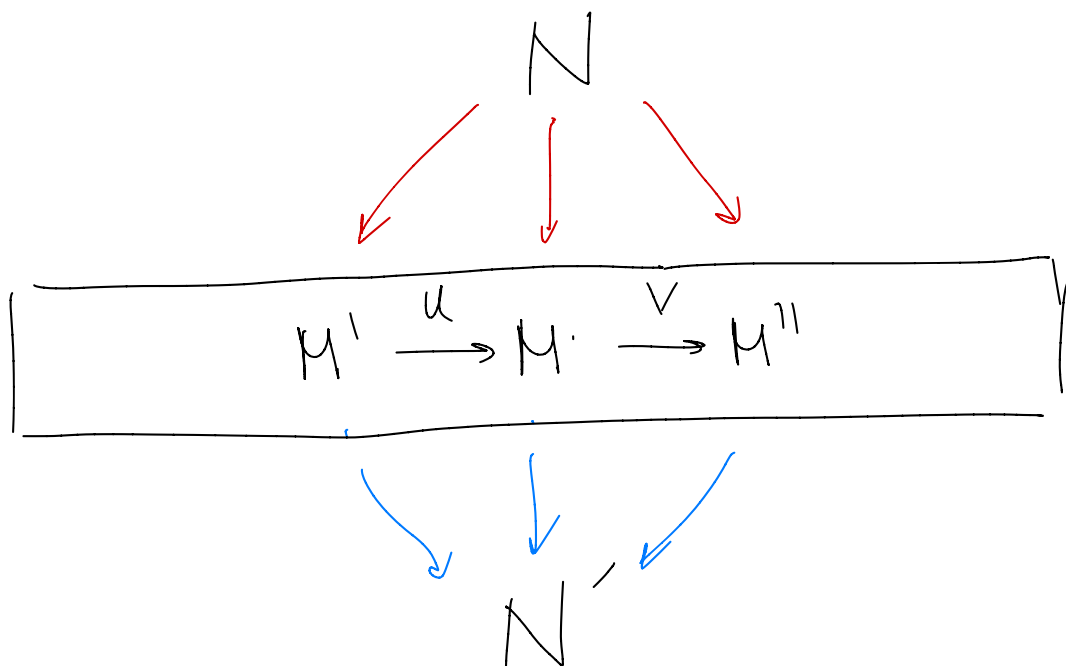
A short exact sequence is split if it is isomorphic to a sequence of the above form.

Example

$$0 \rightarrow \mathbb{Z}/(2) \rightarrow \mathbb{Z}/(4) \rightarrow \mathbb{Z}/(2) \rightarrow 0$$

NOT split. (Warm up)

$\text{Hom}(-, -)$ and exactness



Given morphism u and v above we get

$$\begin{array}{ccccc} \text{Hom}(N, M') & \xrightarrow{u_*} & \text{Hom}(N, M) & \xrightarrow{v_*} & \text{Hom}(N, M'') \\ f & \mapsto & u \circ f & & \\ & & g & \mapsto & v \circ g \end{array}$$

$$\begin{array}{ccccc} \text{Hom}(M'', N') & \xrightarrow{v^*} & \text{Hom}(M, N') & \xrightarrow{u^*} & \text{Hom}(M', N') \\ f & \mapsto & f \circ v & & \\ & & g & \mapsto & g \circ u \end{array}$$

What happens with exactness?

Proposition

$$1) \quad 0 \rightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \text{ is exact}$$
$$\quad \quad \quad \downarrow \quad \quad \quad \downarrow$$
$$\quad \quad \quad g \quad \quad \quad h \mapsto 0$$

$0 \rightarrow \text{Hom}(N, M') \rightarrow \text{Hom}(N, M) \rightarrow \text{Hom}(N, M'')$ is exact for all R -modules N .

$$2) \quad M' \rightarrow M \rightarrow M'' \rightarrow 0 \text{ is exact}$$
$$\quad \quad \quad \updownarrow$$

$0 \rightarrow \text{Hom}(M'', N') \rightarrow \text{Hom}(M, N') \rightarrow \text{Hom}(M', N')$ is exact for all R modules N'

Both are left exact

$\text{Hom}(-, N)$ covariant $\text{Hom}(N, -)$ is contravariant

For 1) $\Rightarrow$ if $u \circ f \equiv 0$ then $f \equiv 0$ since u is injective so u_* injective.

$$\text{Im } u = \text{Ker } v \text{ so } v \circ u \circ f \equiv 0$$

$$\Rightarrow \text{Im } u_* \subseteq \text{Ker } v_*$$

$$\text{If } g \in \text{Ker } v_* \Rightarrow v \circ g \equiv 0 \quad g(N) \subseteq \text{Ker } v = \text{Im } u$$

$$\text{Since } u \text{ is injective} \quad M' \xrightarrow[\cong]{u} \text{Im } u \subseteq M.$$

meaning $\exists u^{-1}: \text{Im } u \rightarrow M'$. We can

$$\text{define } f = u^{-1} \circ g \in \text{Hom}(N, M').$$

$$\text{So } \text{Im } u_* = \text{Ker } v_* \text{ exact.}$$



Example If $R = K$ a field, Hom is exact on both sides,

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0 \text{ exact}$$

$$\Updownarrow$$

$$0 \rightarrow \text{Hom}(W, V') \rightarrow \text{Hom}(W, V) \rightarrow \text{Hom}(W, V'') \rightarrow 0$$

is exact $\forall W$.

$$\Updownarrow$$

$$0 \rightarrow \text{Hom}(V'', W) \rightarrow \text{Hom}(V, W) \rightarrow \text{Hom}(V', W) \rightarrow 0$$

is exact $\forall W$.

Example In general $\text{Hom}(M, -)$ is not right exact.

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}/(n) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(\mathbb{Z}/(n), \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}/(n), \mathbb{Z}) \rightarrow 0$$

$\downarrow$ $\downarrow$ $\downarrow$
 0 0 $\mathbb{Z}/(n)$

Snake Lemma Let

$$\begin{array}{ccccccc}
 0 & \rightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' \rightarrow 0 \\
 & & f' \downarrow & & f \downarrow & & f'' \downarrow \\
 0 & \rightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' \rightarrow 0
 \end{array}$$

be a commutative diagram of
 $\mathcal{S} \in \mathcal{S}$. (Rows are exact and
 $f'' \circ v = v' \circ f$ + $f \circ u = u' \circ f'$)

then there is a long exact seq

$$\begin{array}{ccccccc}
 0 \rightarrow \ker f' & \xrightarrow{u} & \ker f & \xrightarrow{v} & \ker f'' & \xrightarrow{d} & \operatorname{cok} f' \\
 \xrightarrow{\bar{u}'} & \operatorname{cok} f' & \xrightarrow{\bar{v}'} & \operatorname{cok} f'' & \rightarrow & 0
 \end{array}$$

Proof Define the map d . For $x'' \in \ker f''$
 $x'' \in M''$ and v is surjective so $\exists x \in M$
s.t. $v(x) = x''$ $v'(f(x)) = f''(v(x)) = 0$
 $f(x) \in \ker v' \Rightarrow f(x) \in \operatorname{Im} u'$ so

$f(x) = u'(y')$ for $y' \in N'$. Define
 $dx'' := y' + \text{Im } f' \in \text{Cok}(f')$.

Must prove d is well-defined
and exact Diagram chase.

Def Let C be some set of R -modules and $\lambda: C \rightarrow G$ some abelian group G . (ex $\mathbb{Z}$).

λ is additive $\forall \forall$ SES

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \quad M, M', M'' \in C$$

we have $\lambda(M') - \lambda(M) + \lambda(M'') = 0$.

Ex $R = K$ $C = \text{finite dim'l vector space}$

$$\lambda = \dim$$

Proposition $0 \xrightarrow{f_0} M_0 \xrightarrow{f_1} \dots \rightarrow M_n \xrightarrow{f_{n+1}} 0$
exact with M_i and $\ker f_i \in C^i$

Then $\sum (-1)^i \lambda(M_i) = 0$

for additive function $\lambda: C \rightarrow G$.

Proof Split into SES

$$0 \rightarrow \underbrace{\ker f_i}_{N_i} \rightarrow M_i \rightarrow \underbrace{\ker f_{i+1}}_{N_{i+1}} \rightarrow 0$$

$$\lambda(M_i) = \lambda(N_i) + \lambda(N_{i+1}) \quad \text{and } N_0,$$

$N_{n+1} = 0$. Upon taking alternating
sums we get 0.