

Lecture 8

M, N, P be R -modules

Def $f: M \times N \rightarrow P$ is R -bilinear if

for $\forall x \in M$ the map $f_x: N \rightarrow P$
 $y \mapsto f(x, y)$

is R -linear (aka module homomorphism)

and $\forall y \in N$ $f_y: M \rightarrow P$ is
 $x \mapsto f(x, y)$

R -linear (aka module homomorphism).

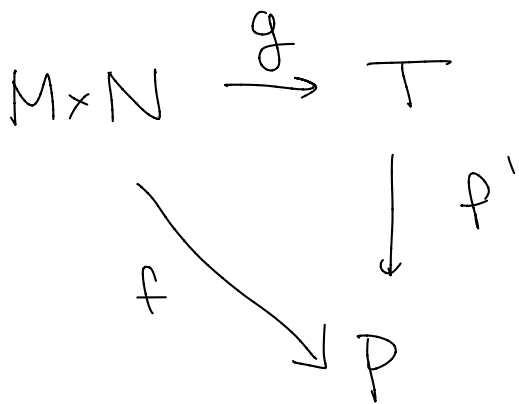
Tensor product M, N R -modules

$\exists (T, g)$ with T R -module

$g: M \times N \rightarrow T$ R bilinear s.t.

1) (universality) $\forall P$ R -module and

$f: M \times N \rightarrow P$ R bilinear there
exists $f': T \rightarrow P$ module homomorphism



$$f = f' \circ g$$

every bilinear map
"factors" "steps by"
 T .

(uniqueness) $(T, g), (T', g')$ both have
the above property $\exists$ unique
 $j: T \rightarrow T'$ isomorphism with
 $g' = j \circ g$.

Proof existence let $C = R^{M \times N}$ i.e.

$$C = \left\{ \sum_{i=1}^n a_i (x_i, y_i) \mid a_i \in R, \begin{matrix} x_i \in M \\ y_i \in N \end{matrix} \right\}$$

$\cup$

$$D = \left\langle \begin{array}{l} (x, y + y') - (x, y) - (x, y'), \\ (x + x', y) - (x, y) - (x', y), \\ (ax, y) - a(x, y), (x, ay) - a(x, y) \end{array} \right\rangle$$

Then $T = C/D$. Tensor Product

T is generated by elts $\frac{x \otimes y}{\text{notation}}$.

where $\phi: C \rightarrow C/D$

$$(x, y) \mapsto x \otimes y.$$

Write
 $T = M \otimes N$

uniqueness as exercise.

Prop If M, N are finitely generated then so is $M \otimes N$.

Proof If $x_1, \dots, x_m$ generate M and $y_1, \dots, y_n$ generate N then

$\{x_i \otimes y_j\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ generate $M \otimes N$.

Find surjective homom: $A^{m \times n} \rightarrow M \otimes N$. □

⚠ The notation $x \otimes y$ is ambiguous
It really depends in which tensor
product you are considering it

Example $M' = 2\mathbb{Z} \subseteq \mathbb{Z} = M$ $\mathbb{Z}$ -modules
 $N' = N = \mathbb{Z}/2\mathbb{Z}$.

The element $2 \otimes x \in M \otimes N$ is
zero since where $x \neq 0 \in N$.

$$2 \otimes r = 2 \cdot (1 \otimes r) = 1 \otimes 2r = 1 \otimes 0 = 0$$

But $2 \otimes x \in M' \otimes N'$ is not.

cannot "factor" $2 \in 2\mathbb{Z}$ as
 $r \cdot a$ with $r \in \mathbb{Z}$ and $a \in M = 2\mathbb{Z}$
unless $r = \pm 1$.

Check at Prop 2.12 and 2.14 of
Atiyah-Macdonald.

Example 2 $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$

$$\begin{aligned} x \otimes y &= 1(x \otimes y) = (3-2)(x \otimes y) \\ &= 3(x \otimes y) - 2(x \otimes y) = (3x \otimes y) - (x \otimes 2y) \\ &= 0 \otimes y - x \otimes 0 = 0. \end{aligned}$$

Question Formulate a general statement

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong ?$$

Example 3 $R \otimes_R M \cong M.$

$$\begin{aligned} \varphi: M &\rightarrow R \otimes_R M & \varphi^{-1}: R \otimes_R M &\rightarrow M \\ x &\mapsto 1 \otimes x & r \otimes x &\mapsto rx. \end{aligned}$$

must verify φ^{-1} is a module homomorphism
(well-defined)

Prop M, N, P R -modules :

$$1) M \otimes_R N \cong N \otimes_R M$$

Exercises

$$2) M \otimes_R (N \oplus P) \cong M \otimes_R N \oplus M \otimes_R P \quad \downarrow$$

$$M \otimes \left(\bigoplus_{i \in I} N_i \right) \cong \bigoplus_{i \in I} M \otimes N_i; \quad M \otimes \left(\prod_{i \in I} N_i \right) \not\cong \prod_{i \in I} M \otimes N_i$$

$$3) M \otimes_R (N \otimes_R P) \cong (M \otimes_R N) \otimes_R P$$

Example Tensor of free modules

$$\begin{aligned} R^{\oplus m} \otimes R^{\oplus n} &= (R \oplus \dots \oplus R) \otimes (R \oplus \dots \oplus R) \\ &= \bigoplus R^{mn}. \end{aligned}$$

If $R = K$ field V, W vector spaces

$V \otimes W$ has basis $\{e_i \otimes f_j\}$ $\{e_i\}$ basis of V
 $\{f_j\}$ basis of W

Example $R[x] \cong \bigoplus_{i=0}^{\infty} R \leftarrow \text{free.}$

$$M \otimes_R R[x] \cong M \otimes \left(\bigoplus_{i=0}^{\infty} R \right) \cong \bigoplus_{i=0}^{\infty} M \otimes R \cong M[x]$$

Useful Properties

$$1) R/I \otimes_R M \cong M/IM = \left\{ \sum_{finite} a_i m_i \mid a_i \in I, m_i \in M \right\}$$

$$2) M/IM \otimes_R N/IN \cong M \otimes_R N \otimes_R R/I$$

$$3) R/I \otimes_R R/J \cong R/(I+J).$$

return to $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$

Proposition M, N, P R -modules

$$\text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \text{Hom}(N, P))$$

$$g: M \otimes N \rightarrow P \mapsto f \quad \text{s.t.}$$

$$f(m) \in \text{Hom}(N, P)$$

$$f(m)(n) := g(m \otimes n).$$

$$g(m \otimes n) = f(m)(n). \longleftarrow f: M \rightarrow \text{Hom}(N, P).$$

In fact, both sides are canonically iso.

$$\{ f : M \times N \rightarrow P \mid R\text{-bilinear maps} \}.$$

Prop Right exactness of tensor product

Let $M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ be exact
sequence of R -modules and N any
 R -module then

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} \underline{M'' \otimes N} \rightarrow 0$$

is exact. right side.

Tensor product of maps Given homomorphism

$$f : M \rightarrow M' \quad g : N \rightarrow N' \quad \text{then}$$

$$f \otimes g : M \otimes N \rightarrow M' \otimes N'$$

$$m \otimes n \mapsto f(m) \otimes g(n)$$

is a R -module homomorphism.

Proof right exactness

Suppose $M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact
then $\forall$ R -module Q we have

$$0 \rightarrow \text{Hom}(M'', Q) \rightarrow \text{Hom}(M, Q) \rightarrow \text{Hom}(M', Q)$$

is exact. let $Q = \text{Hom}(N, P) \forall P$.

Using isomorphism

$$\text{Hom}(M, \text{Hom}(N, P)) \cong \text{Hom}(M \otimes N, P)$$

we get :

$$0 \rightarrow \text{Hom}(M'' \otimes N, P) \rightarrow \text{Hom}(M \otimes N, P) \rightarrow \text{Hom}(M' \otimes N, P)$$

is exact for all P .

$$\Rightarrow M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0$$

is exact for any N . □

Example $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$

is exact. Tensor with M

$$I \otimes_R M \xrightarrow{\phi} R \otimes_R M \xrightarrow{\psi} (R/I) \otimes M \rightarrow 0$$

$\Downarrow$
 M

is exact.

$$\phi(a \otimes m) = a \otimes m = am \in M.$$

$$\psi(1 \otimes m) = (1 + I) \otimes m. \quad \leftarrow \text{surjective easy to see}$$

$\Downarrow$
 m

$$\ker \psi = \operatorname{Im} \phi = IM$$

$$R/I \otimes_R M \cong M/IM \quad \text{by homomorphism Theorem.}$$

Example explicit failure of left exactness

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}(2) \rightarrow 0$$

$$\mathbb{Z} \otimes \mathbb{Z}(2) \rightarrow \mathbb{Z} \otimes \mathbb{Z}/(2) \rightarrow \mathbb{Z}_2 \otimes \mathbb{Z}_2 \rightarrow 0$$

this map becomes 0. $1 \otimes 1 \mapsto 2 \otimes 1 = 2(1 \otimes 1) = (1 \otimes 2) = 0$