

Continued from last time:

Example $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$

is exact. Tensor with M

$$I \otimes_R M \xrightarrow{\phi} R \otimes_R M \xrightarrow{\psi} (R/I) \otimes_R M \rightarrow 0$$

$\searrow \scriptstyle M$

is exact.

$$\phi(a \otimes m) = a \otimes m = am \in M.$$

$$\psi(1 \otimes m) = (1 + I) \otimes m. \quad \leftarrow \text{surjective easy to see}$$

$\scriptstyle m$

$$\ker \psi = \operatorname{Im} \phi = IM$$

$$R/I \otimes_R M \cong M/IM \text{ by homomorphism Theorem.}$$

Example explicit failure of left exactness

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}(2) \rightarrow 0$$

$$\mathbb{Z} \otimes \mathbb{Z}(2) \rightarrow \mathbb{Z} \otimes \mathbb{Z}/(2) \rightarrow \mathbb{Z}_2 \otimes \mathbb{Z}_2 \rightarrow 0$$

$\uparrow$

this map becomes 0. $1 \otimes 1 \mapsto 2 \otimes 1 = 2(1 \otimes 1) = (1 \otimes 2) = 0$

Lecture 9 Exactness, flatness, and a bit of category theory.

Def An R -module N is flat if

$\forall$ SES. of R -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \text{ we have}$$

$$0 \rightarrow M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0$$

is exact.

Prop The following are equivalent.

- 1) N is flat
- 2) Tensoring any exact sequence with N preserves exactness.
- 3) If $f: M' \rightarrow M$ is injective, then $f \otimes 1: M' \otimes N \rightarrow M \otimes N$ is injective

iv) If M', M are finitely generated and $f: M' \rightarrow M$ is injective then $f \otimes 1: M' \otimes N \rightarrow M \otimes N$ is injective.

Examples: 1) $\mathbb{Z}/(n)$ is not flat as a $\mathbb{Z}$ -module.

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}/(n) \rightarrow 0 \quad \text{but}$$

$$\mathbb{Z} \otimes \mathbb{Z}/(n) \xrightarrow{\cdot n} \mathbb{Z} \otimes \mathbb{Z}/(n) \rightarrow \mathbb{Z}/(n) \otimes \mathbb{Z}/(n) \rightarrow 0$$

2) $\mathbb{Q}, \mathbb{Z}$ are flat $\mathbb{Z}$ modules

A bit of category theory

A category $\mathcal{C}$ consists of

① $\text{Ob } \mathcal{C}$ a collection of objects

② $\forall x, y \in \text{Ob } \mathcal{C}$ a set of morphisms
 $\text{Mor}(x, y)$

③ For $x, y, z \in \text{Ob } \mathcal{C}$ a composition operation
 $\circ : \text{Mor}(x, y) \times \text{Mor}(y, z) \rightarrow \text{Mor}(x, z)$.

Satisfying 1) $f \circ (g \circ h) = (f \circ g) \circ h$

2) $\forall x \in \text{Ob } \mathcal{C} \quad \exists \text{id}_x \in \text{Mor}(x, x)$

s.t. $\text{id}_x \circ f = f \quad g \circ \text{id}_x = g$

$\forall f \in \text{Mor}(y, x) \quad g \in \text{Mor}(x, z)$

Examples

	$\text{Ob } \mathcal{C}$	Mor
Set	Sets	Maps of sets
Grp	Groups	homomorphisms
Ring	(Comm.) Rings	"
Mod_R	R-Modules	"
Top	Topological spaces	continuous maps

Def $\mathcal{C}, \mathcal{D}$ categories - A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a map of objects $F: \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{D}$ and $\forall x, y \in \text{Ob } \mathcal{C}$ $F: \text{Mor}_{\mathcal{C}}(x, y) \rightarrow \text{Mor}_{\mathcal{D}}(F(x), F(y))$ s.t.

- 1) $F(\text{id}_x) = \text{id}_{F(x)}$
- 2) $F(f \circ g) = F(f) \circ F(g) \quad \forall f, g \text{ composable}$

A contravariant functor is a map

$$F: \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{D} \quad \text{and} \quad \forall x, y \in \text{Ob } \mathcal{C}$$

$$\text{a map } F: \text{Mor}_{\mathcal{C}}(x, y) \rightarrow \text{Mor}_{\mathcal{D}}(F(y), F(x))$$

$$\text{satisfying 1) } \forall x \quad F(\text{id}_x) = \text{id}_{F(x)}$$

$$2) \quad F(f \circ g) = F(g) \circ F(f). \quad (\text{note reversal!})$$

Examples 1) $F: \text{Sets} \rightarrow \text{Rings}$ functor.

$$F(X) = \mathbb{R}^X = \{ \varphi: X \rightarrow \mathbb{R} \}$$

pointwise
addition +
multiplication

$$X \xrightarrow{f} Y \quad \text{map of sets}$$

$$F(Y) \xrightarrow{F(f)} F(X) \quad \text{contravariant}$$

Prop M an R -module then the following are functors $\text{Mod}_R \rightarrow \text{Mod}_R$

$$\textcircled{1} \text{Hom}_R(M, -) \quad N \in \text{Ob Mod}_R$$

$$N \mapsto \text{Hom}_R(M, N)$$

$$\downarrow \qquad \qquad \downarrow$$

$$P \mapsto \text{Hom}_R(M, P)$$

covariant

RIGHT EXACT

$$\textcircled{2} \text{Hom}_R(-, M) \quad N \quad \text{Hom}_R(N, M)$$

$$\downarrow \qquad \qquad \uparrow$$

$$P \quad \text{Hom}_R(P, M).$$

contravariant

RIGHT EXACT

$$\textcircled{3} M \otimes_R - \quad N \xrightarrow{f} P$$

$$M \otimes_R N \xrightarrow{1 \otimes f} M \otimes P.$$

Covariant.

LEFT EXACT

Two functors $F: \mathcal{D} \rightarrow \mathcal{C}$ (various)
 $G: \mathcal{C} \rightarrow \mathcal{D}$
are adjoint if $\forall X \in \mathcal{D}, Y \in \mathcal{C}$

$$\text{Mor}_{\mathcal{C}}(F(X), Y) = \text{Mor}_{\mathcal{D}}(X, G(Y))$$

We've seen that:

$$F = - \otimes_R N \quad G = \text{Hom}(N, -)$$

are adjoint functors $\text{Mod}_R \rightarrow \text{Mod}_R$.

$$\text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \text{Hom}(N, P))$$