UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4200 — Commutative algebra

Day of examination: Wednesday 16 December 2015

Examination hours: 14:30 – 18:30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider the polynomial ring $A := \mathbb{Z}[X, Y]$ in two variables with integer coefficients.

1a

Set $B := A/(3, X^2)$ and let $\overline{X}$ denote the class of X in B. Show that the only zero-divisors in B are the elements in the ideal $(\overline{X})$.

1b

Define what a primary ideal is, and what it means that a primary ideal is $\mathfrak{p}$-primary for some prime ideal $\mathfrak{p}$. Show that $(3, X^2) \subset A$ is a $(3, X)$-primary ideal.

1c

Show that the radical of the ideal $\mathfrak{a} := (3X, 3Y, 9, X^2) \subset A$ is equal to $(3, X)$.

1d

Find a minimal primary decomposition of $\mathfrak{a} = (3X, 3Y, 9, X^2)$ and the prime ideals belonging to it. Which of these are minimal and which are embedded?

Problem 2

Let k be a field and $A := k[X, Y, Z]$ the polynomial ring in three variables over k, considered as a graded ring. For a finitely generated graded A-module M, the Poincaré series of M is $P(M, t) := \sum_{n \geq 0} \dim_k M_n t^n$.

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2a

Assume

$$0 \to K \to L \to M \to 0$$

is an exact sequence of finitely generated graded A-modules, where the maps preserve degrees. Express the Poincaré series of M in terms of the Poincaré series of K and L.

2b

Let $f \in A_r$ be a homogeneous polynomial of degree r, and consider the graded A-module $M := A/(f)$. Express the Poincaré series of M in terms of the Poincaré series of A.

2c

Show that the Hilbert polynomials $h_M(n)$ of M and $h_A(n)$ of A are related by the equation

$$h_M(n) = h_A(n) - h_A(n-r),$$

and deduce an explicit formula (in terms of r) for $h_M(n)$.

Problem 3

Let A be a Noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field $k := A/\mathfrak{m}$. Let M be a finitely generated A-module. State Nakayama's lemma and use it to show that if $x_1, \ldots, x_n \in M$ are elements such that their images in $M/\mathfrak{m}M$ form a basis for this k-vector space, then $x_1, \ldots, x_n$ generate the A-module M.

Problem 4

Let A be a ring, and let $\mathfrak{n} := \sqrt{(0)}$ denote its nilradical.

4a

Let $S \subseteq A$ be a multiplicative subset of A. Prove that the nilradical of $S^{-1}A$ is equal to $S^{-1}\mathfrak{n}$.

4b

We say that a ring is *reduced* if its nilradical is equal to the zero ideal. Prove that the following statements are equivalent:

(i) A is reduced;

(ii) $S^{-1}A$ is reduced for every multiplicative subset $S \subseteq A$;

(iii) $A_\mathfrak{m}$ is reduced for every maximal ideal $\mathfrak{m}$ in A;

by showing that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i).

THE END