

MAT4200 FALL 2016

SUGGESTED SOLUTIONS TO THE EXAM PROBLEMS

Problem 1

a): Since the equation $X^3 + X + 1 = 0$ has no solutions in the field $\mathbb{F}_2 = \{0, 1\}$, the polynomial $X^3 + X + 1$ is irreducible, and hence the ideal $(X^3 + X + 1)$ is prime. Since it is not zero, it is maximal and hence the ring $\mathbb{F}_2[X]/(X^3 + X + 1)$ is a field.

The polynomial $(X^3 + X + 1) \in \mathbb{F}_3[X]$ has the root $X = 1$, hence is not irreducible, so the ideal it generates is not prime. Hence $\mathbb{F}_3[X]/(X^3 + X + 1)$ is not an integral domain, in particular not a field.

b): The local ring $\mathbb{F}_2[X]_{\mathfrak{p}}$ is an integral domain of dimension 1, and its maximal ideal is principal (generated by $(X^3 + X + 1)$). Therefore it is a discrete valuation ring (Prop. 9.2).

The element $\frac{1}{X^3 + X + 1} \in \mathbb{F}_2(X)$ has valuation -1 .

c): We can factor $(X^3 + X + 1) = (X - 1)(X^2 + X + 2) \in \mathbb{F}_3[X]$, where both factors are irreducible. It follows that the maximal ideals of $\mathbb{F}_3[X]/(X^3 + X + 1)$ are the two principal ideals generated by the images of $X - 1$ and $X^2 + X + 2$. This gives the isomorphism

$$\mathbb{F}_3[X]/(X^3 + X + 1) \cong \mathbb{F}_3[X]/(X - 1) \times \mathbb{F}_3[X]/(X^2 + X + 2) \cong \mathbb{F}_3 \times \mathbb{F}_3[X]/(X^2 + X + 2).$$

(Note that the Artinian ring $\mathbb{F}_3[X]/(X^2 + X + 2)$ is a finite field with $3^2 = 9$ elements.)

Problem 2

a): Multiplication with $f := x_1x_3 - x_2^2$ induces an exact sequence of graded A -modules

$$0 \rightarrow A(-2) \xrightarrow{f} A \rightarrow M \rightarrow 0,$$

where $A(-2)$ is the graded A -module given by $A(-2)_n = A_{n-2}$. It follows that

$$h_M(n) = h_A(n) - h_{A(-2)}(n) = h_A(n) - h_A(n-2) = \binom{n+3}{3} - \binom{n+1}{3}.$$

For $n \geq 2$, we have an exact sequence of vector spaces

$$0 \rightarrow A_{n-2} \xrightarrow{f} A_n \rightarrow M_n \rightarrow 0,$$

hence $\dim_k M_n = \dim_k A_n - \dim_k A_{n-2} = \binom{n+3}{3} - \binom{n+1}{3} = (n+1)^2$ for $n \geq 2$. But we see that $M_0 = k$, so $\dim_k M_0 = 1 = h_M(0)$, and $M_1 = kx_0 \oplus kx_1 \oplus kx_2 \oplus kx_3$, so $\dim_k M_1 = 4 = h_M(1)$, hence we have equality for all $n \geq 0$.

b): Since $g := x_0x_2 - x_1^2$ and $f = x_1x_3 - x_2^2$ have no common factors (both are irreducible), g is a non-zero divisor in M . Hence we get an exact sequence of graded A -modules

$$0 \rightarrow M(-2) \xrightarrow{g} M \rightarrow N \rightarrow 0,$$

and we conclude that

$$h_N(n) = h_M(n) - h_M(n-2) = (n+1)^2 - (n-1)^2 = 4n.$$

Problem 3

a): Since $\mathfrak{b} \subset (x, z)$, we have $\sqrt{\mathfrak{b}} \subseteq \sqrt{(x, z)} = (x, z)$. But $x \in \sqrt{\mathfrak{b}}$, since $x^3 \in \mathfrak{b}$, so we have equality. The ideal (x, z) is a prime ideal since $k[x, y, z]/(x, z) \cong k[y]$ is an integral domain. In the ring $k[x, y, z]/\mathfrak{b} \cong k[x, y]/(x^3)$, any non-zero zero-divisor must contain x as a factor, hence is nilpotent. Therefore $\mathfrak{b}$ is a primary ideal.

b): We have that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\mathfrak{a} \subseteq (x, y, z^2)$, so $\mathfrak{a}$ is contained in the intersection of these two ideals. Now suppose $f \in (x, y, z^2) \cap (x^3, z)$. Write $f = gx^3 + hz$. Since f and gx^3 are in the ideal (x, y, z^2) , we must have $hz \in (x, y, z^2)$, but this implies $h \in (x, y, z)$, and thus $hz \in \mathfrak{a}$. Since $gx^3 \in \mathfrak{a}$, we conclude that $f \in \mathfrak{a}$. It follows that $\mathfrak{a} = (x, y, z^2) \cap (x^3, z)$.

Since $\sqrt{(x, y, z^2)} = (x, y, z)$ is a maximal ideal, (x, y, z^2) is primary. Hence $\mathfrak{a} = (x, y, z^2) \cap (x^3, z)$ is a primary decomposition of $\mathfrak{a}$, and it is minimal since the two radicals (x, z) and (x, y, z) are distinct prime ideals. These are the two prime ideals belonging to $\mathfrak{a}$, the first is minimal and the second is embedded.

Another minimal primary decomposition is $\mathfrak{a} = (x, y, z)^2 \cap (x^3, z)$.

Problem 4

a): There is a natural homomorphism of A -modules

$$\text{Hom}_A(B, M) \rightarrow \text{Hom}_A(C, M_{[\varphi]})$$

given by $f \mapsto f \circ \varphi$. If $D \in \text{Der}_A(B, M)$, then $D \circ \varphi$ satisfies $(D \circ \varphi)(cc') = D(\varphi(c)\varphi(c')) = \varphi(c)D(\varphi(c')) + \varphi(c')D(\varphi(c))$ (in the C -module $M_{[\varphi]}$), hence $D \circ \varphi \in \text{Der}_A(C, M_{[\varphi]})$.

b): Assume φ surjective and $D \circ \varphi = 0$. Given $b \in B$, there is a $c \in C$ such that $b = \varphi(c)$. Hence $D(b) = D(\varphi(c)) = (D \circ \varphi)(c) = 0$. Hence $D = 0$.

If $b \in B$ and $D \in \text{Der}_A(B, M)$, then define bD by $(bD)(b') = bD(b')$. Clearly, $b(D + D') = bD + bD'$, $(b + b')D = bD + b'D$, $(bb')D = b(b'D)$, and $1 \cdot D = D$ all hold. Moreover, $bD \in \text{Hom}_A(B, M)$. Finally, we have $bD \in \text{Der}_A(B, M)$, since $(bD)(b'b'') = b(D(b'b'')) = b(b'D(b'')) + b''D(b') = b(b'D(b'')) + b(b''D(b')) = b'(bD)(b'') + b''(bD)(b')$.

c): If $f = \sum_{i,j} a_{ij}x^i y^j \in k[x, y]$ and $D \in \text{Der}_k(k[x, y], k[x, y])$, then since D is k -linear, $D(f) = \sum_{i,j} a_{ij}D(x^i y^j)$. Since D is a derivation, $D(x^m y^n) = x^m D(y^n) + y^n D(x^m)$. By induction on m (resp. n), we obtain $D(x^m) = m x^{m-1} D(x)$, and $D(y^n) = n y^{n-1} D(y)$. Hence D is determined by its value on x and y . If we set $D_1(x) = 1, D_1(y) = 0$, and $D_2(x) = 0, D_2(y) = 1$, then clearly D_1 and D_2 generate $\text{Der}_k(k[x, y], k[x, y])$ as a $k[x, y]$ -module.