

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4200 — Commutative algebra

Day of examination: Friday, December 15, 2017

Examination hours: 14.30–18.30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

EXERCISE 1 Let R be a ring, and M and N R -modules. Recall that the length $l(M)$ of a module is the infimum of the lengths of all composition series of submodules of M .

- a) Explain what we mean by *additivity of length*.
- b) Suppose $\phi : M \rightarrow N$ is an injective map of R -modules of finite length, and that $l(M) = l(N)$. Show that ϕ is an isomorphism.

EXERCISE 2 Let R be a ring, $S \subset R$ a multiplicatively closed set with $1 \in S$, $\mathfrak{a}$ an ideal. The *saturation* of the ideal $\mathfrak{a}$ with respect to S is denoted $\mathfrak{a}^S$ and defined by

$$\mathfrak{a}^S := \{a \in R \mid \exists s \in S \text{ such that } as \in \mathfrak{a}\}$$

- a) Show that $\mathfrak{a}^S$ is an ideal, containing $\mathfrak{a}$.
- b) Let $\mathfrak{p}$ be a prime ideal of R , and suppose $\mathfrak{p} \cap S = \emptyset$. Show that $\mathfrak{p}^S = \mathfrak{p}$.
- c) Let $\phi_S : R \rightarrow S^{-1}R$ be the map sending $r \in R$ to $\frac{r}{1}$. Show that $\phi_S^{-1}(\mathfrak{a}S^{-1}R) = \mathfrak{a}^S$.

EXERCISE 3 Let k be a field, and R the graded ring given by

$$R = k[x, y, z]/(x^2 - y^2, y^2 - z^2)$$

- a) Compute the Hilbert polynomial of R . What is the dimension of R ?
- b) Let $\mathfrak{m} = (x, y, z)$ be the maximal graded ideal of R , and (x) the principal ideal generated by x . Show that $\mathfrak{m}^3 \subseteq (x)$.
- c) Show that (x) is a $\mathfrak{m}$ -primary ideal in R with the least possible number of generators.

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EXERCISE 4 Let R be a Noetherian ring, $\mathfrak{p} \subset R$ a prime ideal and M a finitely generated R -module. The prime ideal $\mathfrak{p}$ is associated to M if there exists $x \in M$, $x \neq 0$ such that $\text{Ann}(x) = \mathfrak{p}$, and we write $\mathfrak{p} \in \text{Ass}(M)$. The set of prime ideals $\mathfrak{p} \subset R$ such that $M_{\mathfrak{p}} \neq 0$ is called the **support** of M , denoted $\text{Supp}(M)$.

- a) Show that $\text{Ass}(M) \subseteq \text{Supp}(M)$.

Let R be a ring. An R -module M is said to be *simple* if for any submodule $N \subseteq M$ either $N = 0$ or $N = M$.

- b) Show that if M is simple, then

$$M \cong R/\mathfrak{m}$$

for some maximal ideal $\mathfrak{m} \subset R$.

- c) Let M be a simple module. Show that $\text{Supp}(M)$ consists of only one element.
- d) Recall that $\mathfrak{p} \in \text{Supp}(M)$ if and only if there is a prime ideal $\mathfrak{p}' \in \text{Ass}(M)$, such that $\mathfrak{p}' \subseteq \mathfrak{p}$.

Prove that if $\dim(R) = 0$, then $\text{Ass}(M) = \text{Supp}(M)$.

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