

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4200 — Commutative Algebra

Day of examination: Thursday 11 December 2025

Examination hours: 09:00–13:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 subproblems, each weighted equally. You may answer in English or Norwegian.

Problem 1

What is the length of the $\mathbb{Z}$ -module $\mathbb{Z}/(16)$?

Problem 2

Let k be a field, let $I = (z^3 - x^2 + y^2) \subseteq k[x, y, z]$, and let $A = k[x, y, z]/I$. Write $\bar{x}, \bar{y}, \bar{z}$ for the residue classes of x, y, z in A .

2a

Prove that $\bar{x}$ and $\bar{y}$ are algebraically independent over k .

2b

Prove that $k[\bar{x}, \bar{y}] \subset A$ is an integral ring extension. Determine the dimension of A .

Problem 3

Determine the number of maximal ideals in the ring $\mathbb{C}[x, y]/(3y^2 - x^3, y + x)$.

Problem 4

Let k be a field. Compute the Hilbert polynomial of the graded ring

$$k[x, y, z]/(x^4 - y^4).$$

(Continued on page 2.)

Problem 5

Let A be a ring. Recall that for an A -module M , we have the annihilator ideal

$$\text{Ann}_A(M) = \{a \in A \mid am = 0 \ \forall m \in M\} \subseteq A.$$

5a

Let M be an A -module and $S \subseteq A$ a multiplicatively closed subset. Prove that

$$S^{-1}(\text{Ann}_A(M)) \subseteq \text{Ann}_{S^{-1}A}(S^{-1}M),$$

5b

Let M be an A -module such that $\text{Ann}_{A_P}(M_P) = 0$ for all prime ideals P in A . Prove that $\text{Ann}_A(M) = 0$.

5c

Let

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

be an exact sequence of A -modules. Prove that

$$\text{Ann}_A(M_1) \text{Ann}_A(M_3) \subseteq \text{Ann}_A(M_2) \subseteq \text{Ann}_A(M_1) \cap \text{Ann}_A(M_3).$$

5d

With the M_i as above, for $i = 1, 2, 3$, let $\phi_i: M_i \rightarrow S^{-1}M_i$ be the standard homomorphism of A -modules. Prove that if ϕ_1 and ϕ_3 are injective, then ϕ_2 is injective.

Problem 6

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Assume that there is a $k \geq 1$ such that $\mathfrak{m}^k = \mathfrak{m}^{k+1}$.

Prove that $\mathfrak{m}^k = 0$ and that $\mathfrak{m}$ is the unique prime ideal of A .

END