MAT4210—Algebraic geometry I: Notes 3

Sheaves, morphisms and varieties

14th February 2018

Hot themes in notes 3: Sheaves of rings— regular function—regular functions on algebraic sets—affine varieties—general varieties—morphisms—morphisms into affine varieties.

Preliminary version 1.5 as of 14th February 2018 at 3:33pm—Prone to misprints and errors—new version will follow.
Added four exercises, rewritten parts, added an example of a "bad set".
Rewritten proof of proposition 3.3; corrected many errors
2018-01-31 10:24:55: Added two more examples, 3.5 and 3.6; corrected misprints.
2018-02-05 10:25:18. Added two exercises 3.2 and 3.3. Added a small paragraph about products on page 12. Corrected misprints and stupid errors.
2018-02-06 23:25:37 Expanded example 3.6 on page 7. Added proposition 3.7 on page 13.
2018-02-08 16:18:07 Have split exercise 3.3 on page 13 into several problems and given lot of hints (I was far too difficult as it stood). Have added a lemma 3.5 on page 14 that we shall need later.
2018-02-14 15:32:07: Corrected several misprint and smaller errors. This is (hopefully) the last version this semester.

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Introduction

A central feature of modern geometry is that a space of some geometric type comes equipped with a distinguished set of functions. For instance, topological spaces carry continuous functions and smooth manifolds carry C^∞-functions.

There is a common way of *axiomatically* introducing the different types of "functions" on a topological space, namely the so called *sheaves of rings*.

Jean Leray (1906–1998)
French mathematician

There are many variants of sheaves involving other structures than ring structures. They play an important role in geometry—they are omnipresent in modern algebraic geometry—and there is a vast theory about them. However, we confine ourselves to "sheaves of functions" in this introductory course. Our only reason to introduce sheaves is we need them to give a uniform and clear definition of varieties, which are after all our main objects of study. So we cut the story about sheaves to a bare minimum (those pursuing studies of algebraic geometry will certainly have the opportunity to be well acquainted with sheaves of all sorts).

Figure 1: A sheaf

Sheaves were invented by the french mathematician Jean Leray during his imprisonment as a prisoner of war during WWII. The (original) french name[1] is *faisceau*.

[1] In Norwegian one says "knippe" which is close to the meaning of the French word faisceau. There was a certain discussion among the mathematicians about the terminology, some proposed "feså"!!

Sheaves

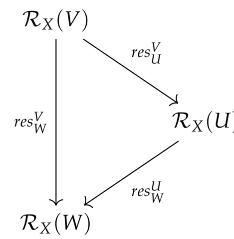
Let X be a topological space. A *presheaf* of rings has two constituents. Firstly, one associates to any open subset $U \subseteq X$ a ring $\mathcal{R}_X(U)$, and secondly, to any pair $U \subseteq V$ of open subsets a ring homomorphism

Presheaves

$$res^V_U \colon \mathcal{R}_X(V) \to \mathcal{R}_X(U),$$

subjected to the following two conditions:

- $res^V_V = id_{\mathcal{R}_X(V)}$,
- $res^U_W \circ res^V_U = res^V_W$,



where the last condition must be satisfied for any three open sets $W \subseteq U \subseteq V$.

The elements of $\mathcal{R}_X(U)$ are frequently called *sections* of $\mathcal{R}_X$ over U, although they often have particular names in specific contexts. It is also common usage to denote $\mathcal{R}_X(U)$ by $\Gamma(U, \mathcal{R}_X)$ or by $H^0(U, \mathcal{R}_X)$ (indicating that there are mysterious gadgets $H^i(U, \mathcal{R}_\mathcal{X})$ around), but we shall stick to $\mathcal{R}_X(U)$. The homomorphisms res^V_U are called *restriction maps*. An alternative notation for the restriction maps is the traditional $f|_U$.

sections of a sheaf

The two conditions above reflect familiar properties of functions (to fix the ideas, think of continuous functions on X). The first reflects the utterly trivial fact that restriction from V to V does not change functions, and the second the fact that restricting from V to W can be done by restricting via an intermediate open set U.

STUDENTS INITIATED in the vernacular of category theory will recognize a presheaf of rings as a contravariant functor from the category of open subsets of X with inclusions as maps to the category of rings. That absorbed, one easily imagines what a presheaf with values in any given category is; for instance, presheaves of abelian groups, which are commonly met in mathematics. To give such a sheaf, is to give an abelian group $\mathcal{A}(U)$ every open U of X and restrictions maps $\mathcal{A}(V) \to \mathcal{A}(U)$ that are group homomorphisms, and of course such that the two axioms are valid.

MANY CLASSES OF FUNCTIONS comply to a principle that can be subsumed in the phrase "Functions of the class are determined locally". There are two aspects of this principle. Given an open U in X and a covering $\{U_i\}$ of U of open sets. Two functions on U that agree on each U_i, are equal on U, and a collection of functions, one on each U_i, agreeing on the intersections $U_i \cap U_j$ can be patched together to a give a global function on U. The defining properties of function classes obeying to this principle, must be of local nature; like being continuous or differentiable. But being bounded, for instance, is not a local property, and te set of bounded functions do not in general form sheaves.

THIS LEADS to two new axioms. A presheaf $\mathcal{R}_X$ is called a *sheaf of rings* when the following conditions are fulfilled: For every open $U \subseteq X$ and every covering $\{U_i\}_{i \in I}$ of U by open subsets it holds true that:

Sheaf of rings

- Whenever $f, g \in \mathcal{R}_X(U)$ are two sections satisfying $res^U_{U_i} f = res^U_{U_i} g$ for every $i \in I$, it follows that $f = g$.
- Assume there are given sections $f_i \in \mathcal{R}_X(U_i)$, one for each $i \in I$, satisfying
$$res^{U_i}_{U_j \cap U_i} f_i = res^{U_j}_{U_i \cap U_j} f_j$$
for each pair of indices i, j. Then there exists an $f \in \mathcal{R}_X(U)$ such that $res^U_{U_i} = f_i$.

EXAMPLE 3.1 The simplest examples of sheaves of rings are the sheaves of continuous real or complex valued functions on a topological space X. ☆

EXAMPLE 3.2 In our algebraic world it is also natural to consider the *sheaf* $\mathcal{C}_X$ of continuous functions with values in $\mathbb{A}^1$ on a topological space X. Where, of course, $\mathbb{A}^1$ is equipped with the Zariski topology, so that a function into $\mathbb{A}^1$ is continuous if and only if the fibres are closed. Since $\mathbb{A}^1 = k$ we can consider $\mathbb{A}^1$ being a field, and ring operations in $\mathcal{C}_X$ can be defined pointwise. Notice that the space of global sections $C_X(X)$—that this, the set of continuous maps $X \to \mathbb{A}^1$—is a k-algebra equipped with pointwise addition and multiplication. ☆

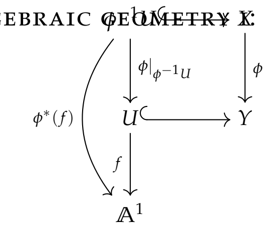
The sheaf $\mathcal{C}_X$ of continuous $\mathbb{A}^1$-valued functions.

ASSUME THAT $\phi\colon X \to Y$ is a continuous map between two topological spaces. The two spaces carry their sheaves of continuous functions into $\mathbb{A}^1$, respectively $\mathcal{C}_X$ and $\mathcal{C}_Y$. Composition with ϕ gives us some kind of "map" ϕ^* between the two sheaves $\mathcal{C}_Y$ and $\mathcal{C}_X$; or rather a well organized collection of maps, one would called it.

In precise terms, for any open $U \subseteq Y$ and any $f \in \mathcal{C}_Y(U)$ one forms the composition $\phi^*(f) = f \circ \phi|_{\phi^{-1}U}$, which is a section in $\mathcal{C}_X(\phi^{-1}U)$. Clearly ϕ^* is a k-algebra homomorphism (operations are defined pointwise), and it is compatible with the restriction maps: For any open $V \subseteq U$ it holds true that

$$\phi^*(f)|_{\phi^{-1}V} = \phi^*(f|_V).$$

In particular, one has a k-algebra homomorphism $\phi^*\colon \mathcal{C}_Y(Y) \to \mathcal{C}_X(X)$ between the k-algebras of global continuous functions to $\mathbb{A}^1$.



This "upper star operation" is *functorial* in the sense that if ϕ and ψ are composable continuous maps, it holds true that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. Notice the change of order of the two involved maps—the "upper star operation" is *contravariant*, as one says.

Subsheaves or rings

GIVEN A TOPOLOGICAL space X with a sheaf of rings $\mathcal{R}$ on it. It should be intuitively clear what is meant by a *subsheaf or rings* $\mathcal{R}' \subseteq \mathcal{R}$; namely, for every open subset U of X one is given a subring $\mathcal{R}'(U) \subseteq \mathcal{R}(U)$ that satisfies two conditions. First of all, the different subrings $\mathcal{R}'(U)$ must be *compatible with the restrictions*; that is, for every pair of open subsets $U \subseteq V$, the restriction maps res^V_U takes $\mathcal{R}'(V)$ into $\mathcal{R}'(U)$. This makes $\mathcal{R}'$ a presheaf, and the second condition requires $\mathcal{R}'$ to be a *sheaf*. The first sheaf axiom for $\mathcal{R}'$ is inherited from $\mathcal{R}$, but the second imposes a genuine condition on $\mathcal{R}'$. Patching data in $\mathcal{R}'$ gives rise to a section in $\mathcal{R}$, and for $\mathcal{R}'$ to be a sheaf, the resulting section must lie in $\mathcal{R}'$.

AT A CERTAIN POINT we shall be interested in subsheaves of rings of the sheaves $\mathcal{C}_X$ of continuous $\mathbb{A}^1$-functions on topological spaces and isomorphisms between such. So let X and Y be topological spaces with a *homeomorphism* $\phi\colon X \to Y$ given. Then the composition map ϕ^* maps $\mathcal{C}_Y$ isomorphically into $\mathcal{C}_X$. If $\mathcal{R}_Y$ and $\mathcal{R}_X$ are subsheaves of rings of respectively $\mathcal{C}_X$ and $\mathcal{C}_Y$, there is a very natural criterion for when ϕ^* induces and isomorphism between $\mathcal{R}_X$ and $\mathcal{R}_Y$—if it is true locally, it holds globally:

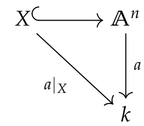
Lemma 3.1 *If there is a basis $\{U_i\}_{i\in I}$ for the topology of X such that ϕ^* takes $R_X(U_i)$ into $\mathcal{R}_Y(\phi^{-1}U_i)$, then ϕ^* maps $\mathcal{R}_X$ isomorphically into $\mathcal{R}_Y$.*

PROOF: It suffices to see that ϕ^* maps $\mathcal{R}_X(U)$ into $\mathcal{R}_Y(\phi^{-1}U)$ for every open subset $U \subseteq X$.

So, let $U \subseteq X$ be open, and take a section s in $\mathcal{R}_X(U) \subseteq \mathcal{C}_X(U)$. It is sent to a section $\phi^*(s) \in C_Y(\phi^{-1}U)$. Now, there is subset J of the index set I so that $\{U_j\}_{j\in J}$ is an open covering of U, and by assumption, $\phi^*(s|_{U_J})$ lies in $\mathcal{R}_Y(\phi^{-1}U_j)$. Since $\phi^*(s|_{U_J}) = \phi^*(s)|_{\phi^{-1}U_j}$, these sections coincide on the intersections $\phi^{-1}U_j \cap \phi^{-1}U_{j'}$, and consequently they patch together to a section in $\mathcal{R}_{\mathcal{Y}}(\phi^{-1}U)$. ❑

Functions on irreducible algebraic sets

In this section we shall work with an irreducible closed algebraic set $X \subseteq \mathbb{A}^n$. It has a coordinate ring $A(X) = k[x_1, \ldots, x_n]/I(X)$, which is an integral domain. The coordinate ring can easily be identified with the so-called *polynomial functions* on X which we met in Notes 2;



that is, the functions on X that are restrictions of polynomials in $k[x_1, \ldots, x_n]$. Indeed, two polynomials a and b restrict to the same function on X precisely when their difference vanishes along X; in other words, if and only if $a - b \in I(X)$; and hence, if and only if a and b have the same image in $A(X)$.

For points $p \in X$ we denote by $\mathfrak{m}_p$ the ideal in $A(X)$ of polynomial functions vanishing at p. It is a maximal ideal, and the Nullstellensatz tells us it is generated by the elements $x_i - p_i$ for $1 \leq i \leq m$ where the p_i's are the coordinates of the point p.

Rational and regular functions on irreducible algebraic sets

The field of rational functions

We shall denote the fraction field of $A(X)$ by $K(X)$. It is called the *rational function field*, or for short *the function field*, of X, and the elements of $K(X)$ are called *rational functions*. The name stems from the case of the affine line $\mathbb{A}^1$ whose coordinate ring is the polynomial ring $k[x]$, and therefore whose function field equals $k(x)$; the field of rational functions in one variable; familiar function we know from earlier days. Similarly, the function field of $\mathbb{A}^n$ is the field $k(x_1, \ldots, x_n)$ of rational functions in n variables; the elements are quotients of polynomials in the x_i's.

PROPERLY SPEAKING rational functions are not functions on X; they are only defined on open subsets of X. However, a statement like that requires a precise definition of what is meant by a function being defined at a point[2].

[2] Remember those endless problems in calculus courses with L'Hôpital's rule?

Regular functions

So let $p \in X$ be a point. One says that a rational function $f \in K(X)$ is *defined* at x, or is *regular* at p, if f can be represented as a fraction $f = a/b$ of two elements in $A(X)$ where the denominator b does not vanish at p; that is, it holds that $b \notin \mathfrak{m}_p$ where, as usual, $\mathfrak{m}_p$ denotes the maximal ideal of functions vanishing at p. The subring of $K(X)$ consisting of functions regular at p is just the localization $A(X)_{\mathfrak{m}_p}$ of $A(X)$ at the maximal ideal $\mathfrak{m}_p$. This ring is commonly denoted by $\mathcal{O}_{X,p}$ and called *the local ring* at p.

The local ring $\mathcal{O}_{X,p}$ at a point

EXAMPLE 3.3 An element $f = a/b$ with $a, b \in A(X)$ is certainly defined at all points in the distinguished open set X_b where the denominator b does not vanish. Be aware, however, that it can be defined on a bigger set; the stupid example being $a = b$. For a less stupid example see example 3.4 below. ☆

The ideal of denominators

TO A RATIONAL FUNCTION $f \in K(X)$ one associates the ideal $\mathfrak{a}_f$ defined by $\mathfrak{a}_f = \{ b \in A(X) \mid bf \in A(X) \}$; that is, the *ideal of denominators* for f. One has

Lemma 3.2 *The maximal open set where the rational function f is defined, is the complement of $Z(\mathfrak{a}_f)$. A function f on X is regular if and only if it belongs to $A(X)$.*

The last statement in the lemma says that the regular functions on X are precisely the polynomial functions.

PROOF: Let $p \in X$ be a point. If $\mathfrak{a}_f \not\subseteq \mathfrak{m}_p$, there is an element $b \in \mathfrak{a}$ not vanishing at p with $f = a/b$ for some a, hence f is regular at p. If f is regular at p, one can write $f = a/b$ with $b \notin \mathfrak{m}_p$, hence $\mathfrak{a}_f \not\subseteq \mathfrak{m}_p$. For the second statement, the Nullstellensatz tells us that $Z(\mathfrak{a}_f)$ is empty if and only if $1 \in \mathfrak{a}_f$, which is equivalent to f lying in $A(X)$. ❑

Lemma 3.3 *Let $b \in A(X)$. The regular functions on X_b equals $A(X)_b$.*

The ring $A(X)_b$ is the localization of $A(X)$ in the element b; *i.e.*, in the multiplicative system $S = \{ b^i \mid i \in \mathbb{N} \}$.

PROOF: Clearly functions of the form a/b^r are regular on the distinguished open set X_b where b does not vanish. For the other way around: Assume that f is regular on X_b. Because $X_b^c = Z(b)$, this means that $Z(\mathfrak{a}_f) \subseteq Z(b)$. We infer by the Nullstellensatz that $b \in \sqrt{\mathfrak{a}_f}$, so $f = ab^r$ for some r and some $a \in A(X)$, which is precisely to say that $f \in A(X)_b$. ❑

EXAMPLE 3.4 Consider the algebraic set $X = Z(xw - yz)$ in $\mathbb{A}^4$. It is irreducible, and one has the equality $f = x/y = z/w$ in the fraction field $K(X)$. The rational function f is defined on the open set $X_y \cup X_w$ which is strictly larger than both X_y and X_w, but the maximal open set where f is defined, is not a distinguished open set.

Indeed, assume it is, say it equals X_b for some $b \in A(X)$ (then there is an inclusion $X_y \cup X_w \subseteq X_b$). By lemma 3.2 above, it follows that $\mathfrak{a}_f = (b)$ and hence $(y, w) \subseteq (b)$. Now letting $A = k[x, y, z, w]/(xw - yz)$, one has $A/(y, w) = k[x, z]$. Hence the prime ideal (z, w) is of height two, contradicting Krull's Hauptidealsatz[3] which says that the principal ideal (b) is of height at most one. ☆

[3] We haven't spoken about Krull's Hauptidealsatz yet, but we'll do in due course

EXAMPLE 3.5 The coordinate ring $A(X)$ from the previous example is not a UFD—in fact, it is in some sense the arche-type of a k-algebra that is not a UFD—and this is the reason behind f not being defined on a distinguished open subset. One has

Proposition 3.1 *Let $X \subseteq \mathbb{A}^n$ be a irreducible closed algebraic set, and assume that the coordinate ring is a UFD. Then the maximal open subset where a rational function f is defined, is of the form X_b.*

PROOF: Let $f \in K(X)$ be a rational function and assume let $b', b \in \mathfrak{a}_f$ be two elements. That is, it holds true that $f = a/b = a'/b'$ so that $ab' = a'b$, and we may well cancel common factors and assume that a and b (respectively a' and b') are without common factors (remember, $A(X)$ is a UFD). Now, we can write $b = cg$ and $b' = c'g$ with c and c' without common factors. It follows that $ac' = a'c$ and hence c is a factor in a and c' one in a'. It follows that c and c' are units, and b and b' are both equal to g up to a unit. Hence $\mathfrak{a}_f = (g)$. ❑

☆

EXAMPLE 3.6 When $n \geq 2$, any regular function on $\mathbb{A}^n \setminus \{0\}$ extends to $\mathbb{A}^n$ and is thus a polynomial function. Indeed, the coordinate ring of $\mathbb{A}^n$ is the polynomial algebra $k[x_1, \ldots, x_n]$ which is UFD, hence the maximal set where a regular function is defined is of the form $\mathbb{A}^n_f$, but when $n \geq 2$, $\mathbb{A}^n \setminus \{0\}$ is not of this form (the ideal $(x_1, \ldots, x_n)$ is not a principal ideal). ☆

The sheaf of regular functions on affine varieties

The sheaf $\mathcal{O}_X$ of regular functions

Time has come to define *the sheaf $\mathcal{O}_X$ of regular functions* on X—remember, the assumption that X be an irreducible closed algebraic set in $\mathbb{A}^n$ is still in force. The ring $\mathcal{O}_X(U)$ associated to an open subset $U \subseteq X$ is simply defined by

$$\mathcal{O}_X(U) = \{ f \in K(X) \mid f \text{ is regular on } U \} = \bigcap_{x \in U} \mathcal{O}_{X,x},$$

and the restriction maps are just, well, the usual restrictions. The rings $\mathcal{O}_X(U)$ are thus subrings of the function field $K(X)$, and when $U \subseteq V$ are two open subsets, the restriction from V to U is just the inclusion $\bigcap_{x \in V} \mathcal{O}_{X,x} \subseteq \bigcap_{x \in U} \mathcal{O}_{X,x}$. This gives us a *presheaf* of rings—the two presheaf axioms are trivially verified— and it turns out to be a sheaf.

When working with the sheaf $\mathcal{O}_X$, one should have in mind that all sections of $\mathcal{O}_X$ are elements of $K(X)$ and all restriction maps are identities. For a given open subset U, the presheaf merely picks out which rational functions in $K(X)$ are regular in U. This simplifies matters considerably and makes the following proposition almost trivial:

Proposition 3.2 *Let X be an irreducible closed algebraic set. The presheaf $\mathcal{O}_X$ is a sheaf.*

PROOF: There are two axioms to verify. The first one is trivial: If $U \subseteq V$ are two opens, the restriction map, just being the inclusion $\bigcap_{x \in V} \mathcal{O}_{X,x} \subseteq \bigcap_{x \in U} \mathcal{O}_{X,x}$, is injective.

As to the second requirement, assume first that f and g are regular on opens subsets U and V respectively, and that $f|_{U \cap V} = g|_{U \cap V}$.

Then $f = g$ as elements in $K(X)$. Next, let $\{U_i\}$ be a covering of U and assume given sections f_i of $\mathcal{O}_X$ over U_i coinciding on the pairwise intersections. Since all the intersections $U_j \cap U_i$ are non-empty, the f_i's all correspond to the same element $f \in K(X)$, and since the U_i's cover U, that element is regular in U. ❑

Notice that lemma 3.2 on page 5 interpreted in the context of sheaves, says that the global sections of the structure sheaf $\mathcal{O}_X$ is the coordinate ring $A(X)$; in other words, one has $\mathcal{O}_X(X) = A(X)$. In particular, when $X = \mathbb{A}^m$, one has $\mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \ldots, x_n]$.

The definition of a variety

In this section we introduce the main objects of study in this course, namely the varieties. We begin by telling what an affine variety is, and subsequently the affine varieties will serve as building blocks for general varieties. The general definition may appear rather theoretical, but soon, when we come to projective varities, there will be many examples illustrating its necessity and how it functions in practice.

As alluded to in the introduction, varieties will be topological spaces endowed with sheaves of rings of regular functions.

Affine varities

The definition of an *affine variety* which we are about to give, can appear unnecessarily complicated. Of course, the model affine variety is an irreducible closed algebraic set X endowed with the sheaf $\mathcal{O}_X$ of regular functions, but the theory requires a slightly wider and more technical definition. We must accept gadgets that are "isomorphic" to X in a certain sense. An *affine variety* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ a subsheaf of rings of the sheaf $\mathcal{C}_X$ of continuous functions on X with values in $\mathbb{A}^1$. The pair is subjected to the following condition: There is an irreducible algebraic set X_0 and a homeomorphism $\phi\colon X \to X_0$, so that the map $\phi^*\colon \mathcal{C}_{X_0} \to \mathcal{C}_X$ induces an isomorphism between $\mathcal{O}_X$ and $\mathcal{O}_{X_0}$. This means that for all open subsets $U \subseteq X_0$ the map ϕ^* takes $\mathcal{O}_{X_0}(U)$ isomorphically into $\mathcal{O}_X(\phi^{-1}(U))$.

Affine varities

$$\begin{array}{ccc} \mathcal{C}_{X_0}(U) & \xrightarrow[\simeq]{\phi^*} & \mathcal{C}_X(\phi^{-1}(U)) \\ \uparrow & & \uparrow \\ \mathcal{O}_{X_0}(U) & \xrightarrow[\simeq]{} & \mathcal{O}_X(\phi^{-1}(U)) \end{array}$$

THE DISTINGUISHED OPEN sets X_f of an algebraic set X we met in Notes 2 illustrate well the reason for this somehow cumbersome definition of an affine variety. *Per se*—as open subsets of X—they are not closed algebraic sets, but endowed with the restriction $\mathcal{O}_X|_{X_f}$ of the sheaf of regular functions as sheaf of rings, they turn out to be affine varieties:

Proposition 3.3 *Let X be an irreducible closed algebraic set and let $f \in A(X)$. Then the pair $(X_f, \mathcal{O}_X|_{X_f})$ is an affine variety.*

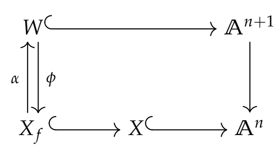
Proof: We need to exhibit a closed algebraic set W and a homeomorphism $\phi \colon W \to X_f$ inducing an isomorphism between the sheaf of rings.

To this end, assume that $X = Z(\mathfrak{a})$ for an ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$. The requested W will be the closed algebraic subset $W \subseteq \mathbb{A}^n \times \mathbb{A}^1 = \mathbb{A}^{n+1}$ that is the zero locus of the following ideal[4]:

$$\mathfrak{b} = \mathfrak{a}k[x_1, \ldots, x_{n+1}] + (1 - f \cdot x_{n+1}).$$

[4] We already came across this ideal when performing the Rabinowitsch trick, but contrary to then, in the present situation f does not belong to $\mathfrak{a}$

The subset W is contained in inverse image $X \times \mathbb{A}^1$ of X under the projection onto $\mathbb{A}^n$, and consists of those points there where $x_{n+1} = 1/f(x_1, \ldots, x_n)$. We let ϕ denote the restriction of the projection to W; it is bijective onto X_f with the the map α sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, 1/f(x_1, \ldots, x_n))$ as inverse.



The salient point is that ϕ^* and α^* are mutually inverse homomorphism between $A(W)$ and $A(X_f)$. As α and ϕ are mutually inverse, the only ting to verify is that $A(W)$ and $A(X_f)$ are mapped into each other.

A regular function g on W is a polynomial in the coordinates $x_1, \ldots, x_{n+1}$, and substituting $1/f(x_1, \ldots, x_n)$ for x_{n+1}, gives a regular function on X_f since $A(X_f) = A(X)_f$ (this is lemma 3.3 on page 6). So α^* takes $A(W)$ into $A(X_f)$.

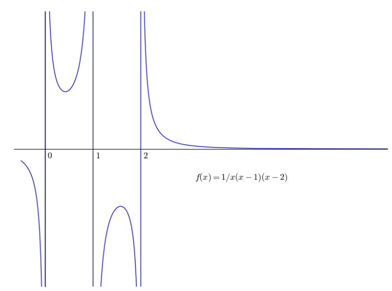


Figure 2: The set W is the graph of the function $1/f$ with $f(x) = x(x-1)(x-2)$.

Similarly, if g is regular on X_f, it is expressible in the form a/f^r where a is a polynomial in $x_1, \ldots, x_n$, and therefore $g \circ \phi$ is regular on W; indeed, it holds true that

$$a(\phi(x_1, \ldots, x_{n+1}))/f(\phi(x_1, \ldots, x_{n+1}))^r = a(x_1, \ldots, x_n)x_{n+1}^r.$$

To finish the proof, we have to show that ϕ^* takes the sheaf of rings $\mathcal{O}_{X_f}$ into the sheaf of regular functions $\mathcal{O}_W$, but because of lemma 3.1 on page 4, it suffices to show that for any distinguished open set $X_g \subseteq X_f$ and any regular function h on X_g, the composite $g \circ \phi$ is regular on $\phi^{-1}X_g = W_{\phi^* g}$; but this is now obvious since ϕ^* is an isomorphism and $A(X_g) = A(X_f)_g$ and $A(X_{\phi^* g}) = A(W)_{\phi^*(g)}$.

❑

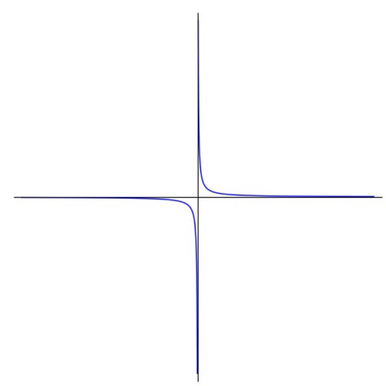


Figure 3: Projections onto the x-axis, makes the hyperbola $xy = 1$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$.

The set W in the proof is nothing but the *graph* of the function $1/f$ embedded in $\mathbb{A}^{n+1}$. Two simple examples of the situation are depicted in the margin (Figure 2 and 3) in both cases $X = \mathbb{A}^1$. In the first figure the function f is given as $f(x) = x(x-1)(x-2)$, and in the second f is the coordinate x. In the latter case $X_f = \mathbb{A}^1 \setminus \{0\}$, and W is the hyperbola $xy = 1$.

General prevarieties

To begin with we define a version of geometric gadgets called *prevarieties*, which at least for us is provisional. One of the axioms that varieties must fulfil—the so called Hausdorff axiom—is momentary lacking since its formulation requires the concept of a morphism. Therefore the outline is first to introduce *prevarieties*, then *morphisms* between such and finally define what is meant by a *variety*.

PREVARIETIES ARE DEFINED as follows: A *prevariety* is a topological space X endowed with a subsheaf of rings $\mathcal{O}_X$ of the sheaf $\mathcal{C}_X$ such that *Prevarieties*

- X is an irreducible topological space;
- There is an open covering $\{X_i\}$ of X such that each $(X_i, \mathcal{O}_X|_{X_i})$ is an affine variety.

The sheaf $\mathcal{O}_X$ is called *the structure sheaf* of X, and the sections of $\mathcal{O}_X$ over an open subset U are called *regular functions* on U. *The structure sheaf* *Regular functions*

The first axiom can be weakened to requiring that X be *connected*, since connectedness in the presence of the second axiom implies that X irreducible.

PROBLEM 3.1 Show that a connected space having an open covering of irreducible open sets is irreducible. ★

EXAMPLE 3.7 Assume $U \subseteq X$ is an open subset of a prevariety X. We may endow U with the restriction of the structure sheaf $\mathcal{O}_X$ to U; that is, we put $\mathcal{O}_U = \mathcal{O}_X|_U$. Then $(U, \mathcal{O}_U)$ will be a prevariety. In fact, this follows from the slightly more general statement:

Proposition 3.4 *A prevariety X has a basis for the topology consisting of open affine subsets.*

PROOF: Let $\{X_i\}$ be an open affine covering of X as in the second axiom. If $U \subseteq X$ is an open subsets of X, the sets $U_i = U \cap X_i$ form an open covering of U. The U_i's will not necessarily be affine, but we know that the distinguished open sets in X_i form a basis for its topology, and by proposition 3.3 on page 8 above they are affine varieties. Hence we can cover each of the U_i's, and thereby U, by affine opens. ❑

☆

Morphisms between prevarieties

On the fly, we also define what is meant by a *morphism* between pre- *Morphisms of varieties*

varieties (the usage will soon degenerate into the more practical term "maps of prevarieties"). Morphisms are always maps that conserve structures; in our present context this means they are continuous maps that conserve the sheaves of regular functions.

ASSUME THAT X and Y are two prevarieties. A continuous map from $\phi\colon X \to Y$ is called a *morphism* if for all open subsets $U \subseteq Y$ and all regular functions f on U, the function $f \circ \phi|_{\phi^{-1}(U)}$ is regular on $\phi^{-1}(U)$. This is equivalent to requiring that the map ϕ^* from $\mathcal{C}_Y$ to $\mathcal{C}_X$ sends the structure sheaf $\mathcal{O}_Y$ of Y into the structure sheaf $\mathcal{O}_X$ of X.

Morphisms

If X and Y are varieties, an *isomorphism* from X to Y is a morphism $\phi\colon X \to Y$ which has an inverse morphism; that is, there is a morphism $\psi\colon Y \to X$ such that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$.

Isomorphisms

BEING A MORPHISM is a *local property* of a continuous map $\phi\colon X \to Y$ between two prevarieties; that is, one can check it being a morphism on appropriate open coverings. One has:

Lemma 3.4 *Let X and Y be two prevarieties and let $\phi\colon X \to Y$ be a continuous map between them. Suppose one can find open coverings $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ of respectively X and Y such that ϕ maps U_i into V_i's, and such that $\phi|_{U_i}$ is a morphism between U_i and V_i, then ϕ is a morphism.*

PROOF: Let f be a regular function on some open $V \subseteq Y$ and let $U \subseteq X$ be an open subset with $\phi(U) \subseteq V$. To see that $f \circ \phi|_U$ is regular in U, it suffices by the patching property of sheaves, to show that its restriction to each $U_i \cap U$ is regular. But $U_i \cap U$ maps into V_i, and by hypothesis $f \circ \phi|_{U_i}$ is regular, and because restrictions of regular functions are regular, it follows that $f \circ \phi|_{U_i \cap U}$ is regular. ❑

PROBLEM 3.2 Show that the composition of two composable morphisms is a morphism. Show that morphisms to $\mathbb{A}^1$ are just regular functions. ★

Maps into affine space

Given a prevariety X and a set $f_1, \ldots, f_m$ of regular functions on X. Letting the f_i's serve as component functions one builds a mapping $\phi\colon X \to \mathbb{A}^m$ by putting $\phi(x) = (f_1(x), \ldots, f_m(x))$. It is obviously continuous and as would be expected, ϕ is a *morphism*.

Indeed, since being a morphism is a local property (Lemma 3.4 above), it suffices to check the defining property on the distinguished open subsets of $\mathbb{A}^m$. So let $U = \mathbb{A}^m_b$ be one. Regular functions on U are (Lemma 3.3 on page 6) of the form $g = a/b^r$ where a is a polynomial and r a non-negative integer. For points x in the inverse image $\phi^{-1}(U)$, it holds true that $b(f_1(x), \ldots, f_m(x)) \neq 0$, hence

$$g \circ \phi(x) = a(f_1(x), \ldots, f_m(x))/b(f_1(x), \ldots, f_m(x))^r$$

is regular in $\phi^{-1}(U)$.

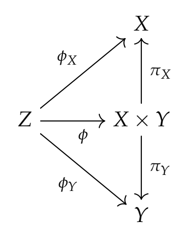
On the other hand, if $\phi\colon X \to \mathbb{A}^m$ is a morphism, the component functions f_i of ϕ being the compositions $f_i = x_i \circ \phi$ of the morphism ϕ with the coordinate functions, are morphisms. Hence we have proven the following proposition whose content is the quite natural property that morphisms from a prevariety X to the affine m-space $\mathbb{A}^m$ are given by giving regular component functions:

Proposition 3.5 *Assume that X is a prevariety. Sending ϕ to ϕ^* sets up a one-to-one correspondence between morphisms $\phi\colon X \to \mathbb{A}^m$ and k-algebra homomorphisms $\phi^*\colon k[x_1, \ldots, x_m] \to \mathcal{O}_X(X)$.*

THIS PROPERTY MAY be interpreted in yet another way. There is standard way of defining what is meant by a product in general category which generalizes the Cartesian product of two sets. In our context, it goes as follows: If X and Y are two prevarieties, a *product* of X and Y is *triple* consisting of a prevariety $X \times Y$ and two morphisms π_X and π_Y from $X \times Y$ to respectively X and Y, with the property that given two morphism $\phi_X\colon Z \to X$ and $\phi_Y\colon Z \to Y$, there is a unique morphism $\phi\colon Z \to X \times Y$ such that $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. In other words, giving ϕ is the same as giving the component functions.

The product of two varieties

What we checked above is, when $m = 2$, equivalent to saying that $\mathbb{A}^2$ is the product of $\mathbb{A}^1$ in the category of prevarieties. The definition of a product of two prevarieties generalizes *mutatis mutandis* to a product of a finite number of prevarieties; and what we proved above is that $\mathbb{A}^m$ is the m-fold product of $\mathbb{A}^1$.



THIS PROPOSITION has an immediate generalization. One may replace the affine n-space with any affine variety. It holds true

Theorem 3.1 *Assume that X is a prevariety and Y an affine variety. The the assignment $\phi \mapsto \phi^*$ sets up a one-to-one correspondence between morphisms $\phi\colon X \to Y$ and k-algebra homomorphisms $\phi^*\colon A(Y) \to \mathcal{O}_X(X)$.*

PROOF: Suppose that $Y \subseteq \mathbb{A}^n$. Giving a morphism $\phi\colon X \to Y$ is the same as giving a morphism $\phi\colon X \to \mathbb{A}^n$ that factors through Y. By proposition 3.5 above, giving a $\phi\colon X \to \mathbb{A}^n$ is the same as giving the algebra homomorphisms $\phi^*\colon k[x_1, \ldots, x_n] \to \mathcal{O}_X(X)$, and ϕ takes values in Y if and only if $f(\phi(x)) =$ for all $f \in I(Y)$;

that is, the composition map ϕ^* vanishes on the ideal $I(Y)$. Hence ϕ takes values in Y if and only if ϕ^* factors through the quotient $A(Y) = k[x_1, \ldots, x_n]/I(X)$. ❑

Specializing the prevariety X to be affine as well, we get the following corollary

Proposition 3.6 *Assume that X and Y are two affine varieties. Then $\phi \mapsto \phi^*$ is a one-to-one correspondence between morphisms $\phi\colon X \to Y$ and k-algebra homomorphisms $\phi^*\colon A(Y) \to A(X)$.*

An immediate corollary is the following:

Proposition 3.7 *Let X and Y be two affine varieties and $\phi\colon X \to Y$ a morphism. Then ϕ is an isomorphism if and only if ϕ^* is an isomorphism. In particular, X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as k-algebras.*

EXAMPLE 3.8 The variety $\mathbb{A}^n \setminus \{0\}$ is not affine if $n \geq 2$. Indeed, by example 3.6 on page 7 any regular function on the inclusion $\iota\colon \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n$ induces an isomorphism i^* between the spaces of global sections of the structure sheaves. If $\mathbb{A}^n \setminus \{0\}$ were affine, the inclusion would therefore have been an isomorphism after proposition 3.7 above, and this is of course not the case. ☆

IN A CATEGORICAL language Proposition 3.1 above says in view of exercise **??** on page **??** in Notes 2, that the category of affine varieties is equivalent to the category of finitely generated k-algebras that are integral domains. So the study of the affine varieties is in some sense equivalent to the study of k-algebras; but luckily, the world is more versatile than that! There is the vast host of projective varieties—beautiful, challenging and intricate and sometimes even untouchable!

PROBLEM 3.3 *(The product of affine varieties)* Assume that $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are two affine varieties. Let $\mathfrak{a}$ be the ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by $I(X)$ and $I(Y)$.

a) Show that $Z(\mathfrak{a}) = X \times Y$ by checking the universal property above.

b) Show that $X \times Y$ is irreducible. HINT: Assume that $X \times Y = Z_1 \cup Z_2$ and show that the sets $X_i = \{\, x \in X \mid \{x\} \times Y \subseteq Z_i \,\}$ are closed in X and cover X.

c) Show that $Z(\mathfrak{a}) \subseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ is the product of X and Y in the category of prevarieties. ★

PROBLEM 3.4 Let A and B be two k-algebras finitely generated over the algebraically closed field k. Assume that both are reduced. Show that $A \otimes_k B$ is reduced. HINT: Show that any $d \in A \otimes_k B$ may be represented as $d = \sum a_i \otimes b_i$ with the b_i's linearly independent over k. Show then that if d is nilpotent, the a_i's lie in every maximal ideal in A. ★

PROBLEM 3.5 With reference to problem 3.3 above, show that $A(X \times Y) = A(X) \otimes_k A(Y)$. ★

The Hausdorff axiom

THE HAUSDORFF AXIOM is the third axiom required of varieties. Our Zariski topologies are as we have seen far from being Hausdorff. Some properties[5] of Hausdorff spaces can be salvaged by this third axiom, so in some sense it is a substitute for the topologies being Hausdorff. A prevariety $(X, \mathcal{O}_X)$ is called a *variety* if the following condition is fulfilled

[5] That is properties expressed in terms of morphisms not in terms continuous maps.

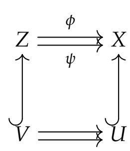
Varieties

- Given any two morphisms $\phi, \psi \colon U \to X$ where U is a prevariety, the set of points in U where ϕ and ψ coincide is closed; that is, the subset $\{ x \in U \mid \phi(x) = \psi(x) \}$ is closed.

Proposition 3.8 *Any affine variety is a variety.*

PROOF: To begin with, observe that if f and g are two regular functions on a prevariety U, the set where they coincide is closed. Indeed, the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ being the zero locus of $x_1 - x_2$ is closed and the map $U \to \mathbb{A}^2$ given as $x \mapsto (f(x), g(x))$ is continuos. Since preimages of closed sets by continuous maps are closed, it follows that $\{ x \mid f(x) = g(x) \}$ is closed.

Now, let $X \subseteq \mathbb{A}^n$ be affine, and assume that ϕ and ψ map U into X. If the coordinate functions on $\mathbb{A}^n$ are y_i, the compositions $y_i \circ \psi$ and $y_i \circ \phi$ are regular functions on U. The set where ϕ and ψ coincide is the intersection of the subsets where each pair $y_i \circ \psi$ and $y_i \circ \phi$ coincide. By the initial observations each of these subsets is closed, hence their intersection is closed. ❑

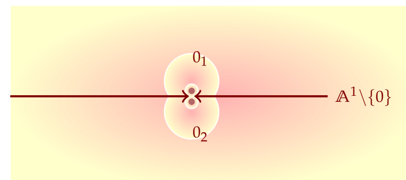


Lemma 3.5 *Assume that X is a prevariety such any two different points are contained in an open affine subset. Then X is a variety.*

PROOF: Let Z be a prevariety and ϕ and ψ two maps from Z to X. Let $x \in Z$ be a point such that $\phi(x) \neq \psi(x)$. Then by assumption there is an open affine set U in X containing both $\phi(x)$ and $\psi(x)$, and $V = \phi^{-1}U \cap \psi^{-1}U$ is an open set in Z where x lie. Now U is a variety by proposition 3.8 above, hence the set $W \subseteq V$ where the two maps ψ and ϕ coincide is closed; but this means that $V \setminus W$ is an open set in Z containing x entirely contained in the complement of $\{ z \in Z \mid \phi(z) = \psi(z) \}$, and the complement is open since x was an arbitrary member. ❑

EXAMPLE 3.9 — A BAD GUY This is an example of a prevariety X for which the Hausdorff axiom is not satisfied. These "*non separated prevarieties*", as they often are called, exist on the fringe of the algebro-geometric world, you very seldom meet them—although now and then they materialize from the darkness and serve a useful purpose. Anyhow, this is the only place such a creature will appear in this course, and the only reason to include it is to convince you that the Hausdorff axiom is needed.

The intuitive way to think about X is as an affine line with "two origins. It does not carry enough functions that the two origins can be separated—if a function is regular in one, it is regular in both and takes the same value there.



The underlying topological space is the set $X = (\mathbb{A}^1 \setminus \{0\}) \cup \{0_1, 0_2\}$ endowed with the topology of finite complements. It has two copies of the affine line $\mathbb{A}^1$ lying within it; either with one of the twin origins as origin; that is $A_1 = (\mathbb{A}^1 \setminus \{0\}) \cup \{0_1\}$ and $A_2 = (\mathbb{A}^1 \setminus \{0\}) \cup \{0_2\}$. Both these sets are open sets and their intersection A is given as $A = A^1 \cap A^2 = \mathbb{A}^1 \setminus \{0\}$. Obviously, the Hausdorff axiom is not satisfied, because the two inclusions of $\mathbb{A}^1$ in X are equal on $\mathbb{A}^1 \setminus \{0\}$ which is not closed in X.

To tell what regular functions X carries, let $U \subseteq X$ be any open subset. There are two cases:

- The complement $X \setminus U$ contains both the twin origins: Then the ring $\mathcal{O}_X(U)$ of regular functions in U is the set of rational functions $a(x)/b(x)$ in one variable with $b(x) \neq 0$ for $x \in U$—so the sheaf $\mathcal{O}_X|A$ equals $\mathcal{O}_{\mathbb{A}^1}|_A$.
- One or both the twin origins belongs to U: Then $\mathcal{O}_X(U)$ will be the set of $a(x)/b(x)$ of rational functions in one variable such that $b(x) \neq 0$ for $x \in U \cap A$ and additionally $b(0) \neq 0$.

So the point, is that when U is an open subset containing both 0_1 and 0_2 the subsets U, $U \setminus \{0_1\}$ and $U \setminus \{0_2\}$ all carry the *same* regular functions.

We leave it as an exercise to students interested in the dark corners at the fringes of the universe to fill in details and check that the axioms for a prevariety are fulfilled. ☆

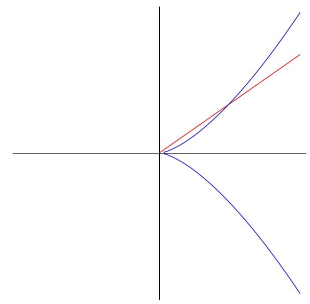
AS AN EPILOGUE, we remind you that a variety has two ingredients: a topological space X and the structure sheaf $\mathcal{O}_X$. Among the two the structure sheaf is the main player, the Zariski topology having a more supportive role. For instance, if X is an irreducible and Noetherian space whose only closed irreducible sets are the points, the closed sets, apart form the entire space, are precisely the finite subsets. This means that all such spaces are homeomorphic as longs as their cardinality is the same. So for instance, the affine lines $\mathbb{A}^1$ over different countable[6] fields are homeomorphic, and they are even homeomorphic to the bad guys we just constructed.

[6] The algebraic closure of finite fields $\mathbb{F}_q$ and of the rationals $\mathbb{Q}$ are all countable.

Later on, after having introduced the concept of dimension, we shall see that any irreducible space of dimension one fall in this category, so they are all homeomorphic. But there is an extremely rich fauna of such varieties!

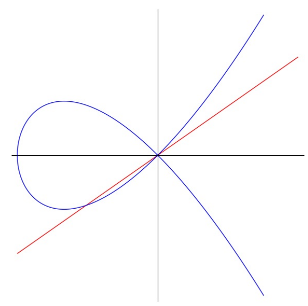
In higher dimensions the Zariski topologies play a more decisive role, but still they do not distinguish varieties very well.

Problems



The rational cusp $y^2 = x^3$.

3.6 *(Rational cusp)* Consider the curve C in $\mathbb{A}^2$ whose equation is $y^2 - x^3$. Show that C can be parametrized by the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ defined as $\phi(t) = (t^2, t^3)$. Describe the map $\phi^*\colon A(C) \to A(\mathbb{A}^1)$. Show that ϕ is bijective but not an isomorphism. Show that the function field of C equals $k(t)$.

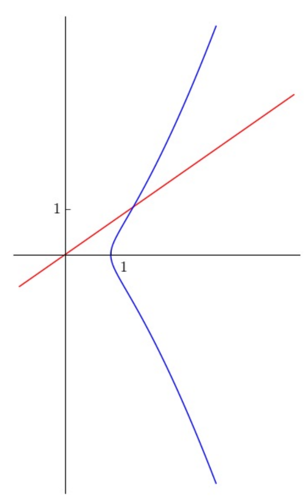


The rational node $y^2 = x^2(x+1)$.

3.7 *(Rational node)* In this exercise we let C be the curve in $\mathbb{A}^2$ whose equation is $y^2 - x^2(x+1)$. Define a map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ by $\phi(t) = (t^2 - 1, t(t^2-1))$. Show that $\phi(\mathbb{A}^1) = C$, and describe the map $\phi^*\colon A(C) \to A(\mathbb{A}^1)$. Show that ϕ is not an isomorphism, but induces an isomorphism $\mathbb{A}^1 \setminus \{\pm 1\} \to C \setminus \{0\}$. Show that the function field of C equals $k(t)$.

3.8 Let C be one of the curves from the two previous exercises. Show that, except for finitely many, every line through the origin intersects C in exactly one other point. What are the exceptional lines in the two cases? Use this to give a geometric interpretation of the parametrizations in the previous exercises.

3.9 *(An acnode)* Consider the curve D given by $y^2 = x^2(x-1)$ in $\mathbb{A}^2$. Make a sketch of the real points of D (see the figure in the margin); notice that the origin is isolated among the real points—such a point is called an *acnode*. Show that $(t^2+1, t(t^2+1))$ is a parametrization of D. Exhibit a complex linear change of coordinates in $\mathbb{A}^2$ that brings D on the form in problem 3.7 above.



An acnode at the origin $y^2 = x^2(x+1)$.

3.10 Let R be a UFD. Show that any prime ideal of height one (that is a prime ideal properly containing no other prime ideals than the zero ideal) is principal.

3.11 Let X be a variety and let $Y \subseteq X$ be a closed irreducible subset. For any open $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the subset of regular functions on U that vanish on $Y \cap U$. Show that $\mathcal{I}_Y(U)$ is an *ideal* in $\mathcal{O}_X(U)$.

Show that if $V \subseteq U$ are two open sets, then res^U_V takes $\mathcal{I}_Y(U)$ into $\mathcal{I}_Y(V)$. Show that $\mathcal{I}_Y$ is a sheaf (of abelian groups, in fact of rings if one ignores the unit element).

3.12 Let X be a prevariety and assume that $Y \subseteq X$ is a closed irreducible subset. Show that Y can be given the structure of a prevariety in a unique way so that the inclusion $Y \to X$ is a morphism.

3.13 Let $\mathcal{B}$ be the presheaf of bounded continuous real valued functions on $\mathbb{R}$. Show that $\mathcal{B}$ is not a sheaf. HINT: It does not satisfy the second sheaf axiom.

3.14 Let X be a topological space and let $\mathbf{A}$ be a ring equipped with the discrete topology. For any open set $U \subseteq X$ let $\mathbf{A}(U)$ be the set of continuous functions $U \to A$. Show that $\mathbf{A}(U)$ is a sheaf.

3.15 *(For fringy people)* Let X be any closed algebraic set and let $Y \subseteq X$ be a proper closed subset. Construct a prevariety $X_{\text{♊}}$ containing unseparable twin copies of Y and two different open subsets both isomorphic to X that intersect along $X \setminus Y$.

3.16 Show that the algebraic closure of a countable field is countable.

★