MAT4210—Algebraic geometry I: Notes 8

Non-singular curves

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Hot themes in Notes 8:
Super-Preliminary version 0.0 as of 12th March 2018 at 9:56am—Well, still not really a version at all, but better. Improvements will follow!
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Intoduction

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Overview

Rational and birational maps

Just like we spoke about rational functions on a variety being function defined and regular on a non-empty open subset, one may speak about rational maps from a variety X to another Y. Strictly speaking, this is a pair consisting of an open subset $U \subseteq X$ and a morphism $\phi\colon U \to Y$. Commonly a rational map is indicate by a broken arrow like $\phi\colon X \dashrightarrow Y$.

If V is another open subset of X containing U, an *extension* of ϕ to V is a morphism $\psi\colon V \to Y$ such that $\psi|_U = \phi$; it is common usage to say that ϕ is defined on V. An open subset $U \subseteq X$ is called a *maximal subset of definition* for ϕ if ϕ can not be extended to any open subset strictly containing U. The next proposition tells us that every rational map ϕ has unique maximal set of definition:

Proposition 8.1 *Let X and Y be two varities, and $U \subseteq X$ an open non-empty set. Suppose that $\phi\colon U \to Y$ a morphism. Then ϕ has a unique maximal set of definition.*

Proof: Since X is a Noetherian topological space, any non-empty collection of open subsets has a maximal element. Hence maximal sets of definition exists, and merely the unicity statement requires some work.

Assume that V_1 and V_2 open subsets of X cotaining U and both being maximal sets of definition for ϕ. Let the two extensions be ϕ_1 and ϕ_2. Both restrict to morphisms on the intersection $V_1 \cap V_2$, and the salient point is that these two restrictions coincide. Indeed, both ϕ_1 and ϕ_2 restrict to ϕ on U, and because Y is a variety (open subset of varieties are varieties) the Hausdorff axiom holds for $V_1 \cap V_2$.

Consequently, the subset of $V_1 \cap V_2$ where ϕ_1 and ϕ_2 coincide, is closed; and since they coincide on U, which is dense in $V_1 \cap V_2$, they coincide along the entire intersection $V_1 \cap V_2$. This means that ϕ_1 and ϕ_2 can be patched together to give a map defined on $V_1 \cup V_2$, which is a morphism (being a morphism is a local property). My maximality, it follows that $V_1 = V_2$. ❑

Dominant rational maps enjoy a weeker but similar functorial property as morphisms. "By composition" they induce in a contravariant way a k-algebra homomorphism, but merely between the function fields of the two involved varieties.

To be precise, assume that $\phi\colon X \dashrightarrow Y$ is the dominant, rational map, and that ϕ is defined on the open set U_ϕ. For any open $V \subseteq Y$, the inverse image $\phi^{-1}(V \cap \phi U_\phi)$ is non-empty since ϕ is dominating and of course it is open. A member f of the function field $K(Y)$ is a regular function defined on some open set V_f of Y and the composition $f \circ \phi$ is a regular function on $\phi^{-1}(V_f \cap \phi U_\phi)$, and hence defines an element in function field $K(X)$. In this way we obtain a homomorphism $\phi^*\colon K(Y) \to K(X)$.

An important proertry is that this construction is reversible:

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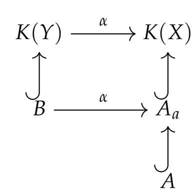
Theorem 8.1 *Given two varities X and Y and a k-algebra isomoprhism $\alpha\colon K(Y) \to K(X)$. Then there exists a unique dominant rational map $\phi\colon X \dashrightarrow Y$ such that $\phi^* = \alpha$.*

Notice that the α is a field isomorphism but it must act trivially on the constants k.

Proof: We begin by choosing an open and affine set in each of the varieties X and Y. Call them U and V with $U \subseteq X$ and $V \subseteq Y$. They have coordinate rings $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(V)$; then $A \subseteq K(X)$ and $B \subseteq K(Y)$. Furthermore, the function fields are the fractio fieldsa of A and B respectively. As U and V were randomly chose, there is no reason for α to send B into A; but we shall replace B with a localization for this to happen.

The k-algebra B is finitely generated over k; let $b_1, \ldots, b_s$ be generators. The images $\alpha(b_i)$ are form $\alpha(b_i) = a_i a^{-1}$ with the a_i's and a all belonging to A (the field $K(X)$ is the fraction field of A). But then α sends B into the localized ring A_a.



Translating this algebra into geometry will finish the proof. The localization A_a is the coordinate ring of the distinguished affine open subset U_a of U, and by the main theorem about morphisms between affine varieties, there is a morphism $\phi\colon U_a \to V$ with ϕ^* equal to $\alpha|_V$. Hence ϕ represents a rational and dominating map with the requested property that $\alpha = \phi^*$ ❑

A *birational map* is a rational map which has a rational inverse. To be precise, assume that X and Y are two varieties; To give a birational map from X to Y is to give open sets $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $\phi\colon U \to V$. When there is birational map between X and Y one sais that X and Y are *birationally equivalent*. Be aware that the open set U might be smaller than the maximal set of definition U_ϕ; like in example 8.1 below. The main theorem (theorem 8.1 above) tells us that two varities X and Y are birationally equivalent if and only if their faunctions fields are isomorphic as k-algebras.

Birational maps

Birationally equivalent varities

BIRATIONAL GEOMETRY did almost dominate algebraic geometry at a certain period. The classification of varities up to birational equivalence is a much courser classification than classification up to isomorphism, and hence it is *a priori* an easier task (but still, challenging enough). However, for non-singular projective curves, as we later shall see, the two are equivalent. Two such curves are isomorphic if and only if they are birationally equivalent.

Already for projective non-singular surfaces, the situation is completely different. There are infinitely many non-isomorphic surface in the same birational class (see example 8.3 below for a simple example of two), and they can form a very complicated hierarchy. For varities of higher dimension, the picture is even more complicated, but the so called *Mori Minimal Model Program* that as evolved during the last twenty years, shed some light on the situation.

EXAMPLE 8.1 Consider the map $\sigma(x;y;z) = (yz;xz;xy)$ which is a rational map from $\mathbb{P}^2$ to $\mathbb{P}^2$ (by lemma **??** above, it is morphism where it is defined). The map σ is certainly define away from the three points $e_z = (0;0;1)$, $e_y = (0;1;0)$ and $e_x = (1;0;0)$, but can not be extended beyond any of these. Let us check this for the point $e_z = (0;0;1)$. To that end, introduce the two lines $L_x = Z_+(x)$ and $L_y = Z_+(y)$. Now, the point is that σ maps $L_x \setminus e_z$ and $L_y \setminus e_z$ to two different points, namely to e_x and e_y repectively. And this, of course, excludes an extension of σ to a neighbourhood of e_z. For the two other points symmetric arguments hold, and we can conclude that the maximal set of definition for σ is the open set $U_\sigma = \mathbb{P}^2 \setminus \{e_x, e_y, e_z\}$. ☆

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EXAMPLE 8.2 Any rational map $\mathbb{P}^1 \to \mathbb{P}^n$ is defined everywhere; in other words, the maximal set of definition U_ϕ of ϕ is equal to the entire $\mathbb{P}^1$. Let $D = D_+(x_i)$ be one the basic open sets which meet the image of U_ϕ under ϕ. The variety D is an affine n-space with coordinates $\{x_j x_i^{-1}\}$.

The inverse image $V = \phi^{-1}(D \cap \phi(U_\pi))$ is an open set, and the n component functions of $\phi|_V$ are rational functions on $\mathbb{P}^1$. They may be brought on the form f_j/f_i with $0 \leq j \leq n$ and $j \neq i$, where the polynomial f_i is their common denominator and does not vanish on V; that is, at points in V the relation $x_j x_i^{-1} = f_j f_i^{-1}$ holds.

The idea is to use the f_k's (now including f_i) as the homogenous components of a morphism of $\mathbb{P}^1$ into $\mathbb{P}^n$. However, it could happen that the $n+1$ polynomials f_k have a common factor, but it can be discarded and hence we can assume that f_k's are without common zeros. This allows us to define a map $\Phi(x) = (f_0(x); \ldots ; f_n(x))$ which is easily checked to be a morphism that extends ϕ. ☆

EXAMPLE 8.3 The quadric $Q = Z_+(xz - yw) \subseteq \mathbb{P}^3$ is birationally equivalent to the projective plane $\mathbb{P}^2$, the two are not isomorphic. This is one of the simplest example of two non-isomorphic projective and non-singular surfaces being birationally equivalent.

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To begin with, the two are not isomorphic. They are not even homeomorphic since any two curves in $\mathbb{P}^2$ intersect, but on the quadric there families of disjoint lines. For example the two disjoint lines $x = y = 0$ and $x + z = y + w = 0$ both lie on Q.

Next we exhibit a birational map $\phi\colon Q \dashrightarrow \mathbb{P}^2$. It will be defined on the open set $U = D_+(x) \cap Q$. In $D_+(x) \simeq \mathbb{A}^3$ with coordinates y, z and w, the equation of Q becomes, $z = yw$. It is almost obvious that the projection $\mathbb{A}^3 \to \mathbb{A}^2$ sending (y, z, w) to (y, w) induces an isomorphism from $Q \cap D_+(x)$ to $\mathbb{A}^2$, but a rewarding exercise for the sudents to check all details. ☆

The case of curves

In this section X will denote a curve; that is, a variety of dimension one. The fundamental property of curves in this context is that any rational map from a curve into a projective varletry is defined at all non-singular points of the curve. This implies that binational maps between projective non-singular curves are isomorphisms, and consequently there is up to isomorphism only one non-singular and projective curve in a binational class.

Another consequence of the extension property is that every non-singular curve is isomorphic to an open set of a non-singular *projective* curve (it could of course be equal to the whole). In particular, any field of trancendence degree one over an algebraically closed field k is the function field of a projective and non-singular curve.

An easy algebraic preparation

If $P \in X$ is a non-singular point, the local ring $\mathcal{O}_{X,P}$ of X at P is regular of dimension one. It is also an integral domain, X being[1] a variety.

[1] It is a theorem that regular rings are domains, but we have not proven that.

That $\mathcal{O}_{X,P}$ is regular of dimension one implies that the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,P}$ requires just one generator. Let t be one. Then t is a rational function on X which is regular in a neighbourhood V of P, and if the neighbourhood is sufficiently small, t has no other zeros in V but P. Such a function t is often called a *uniformizing parameter* at P.

Uniformizing parameters

The following easy lemma from commutative algebra, tells us that any rational function f on X may be expressed as $f = \alpha t^\nu$ where α is a rational function, regular and non-vanishing at P, and where ν is an integer. It holds true that $\nu \geq 0$ precisely when f is regular at P, and $\nu = 0$ exactly when f is regular and non-vanishing at P.

Lemma 8.1 *In a local Noetherian domain whose maximal ideal is principal, all ideals are powers of the maximal ideal.*

PROOF: Let x a generator for the maximal ideal $\mathfrak{m}$ and let $\mathfrak{a} \subseteq A$ be a non-zero ideal. Let n be the largest integer such that $\mathfrak{a} \subseteq \mathfrak{m}^n$; an n like that exists by *e.g.,* Krull's intersection theorem. Since $\mathfrak{a} \not\subseteq \mathfrak{m}^{n+1}$, there is an $a \in \mathfrak{a}$, such that $a = \alpha x^n$ with $\alpha \notin \mathfrak{m}$, that is α is a unit since the ring is local. It follows that $(x^n) \subseteq \mathfrak{a}$, and we are done. ❏

The lemma shows that A is a *discrete valuation ring*; any element in its fraction field K can be written as αt^ν with α a unit in A and ν an integer.

The extension lemma

The main property of curves in this context is that any rational map from a curve into a projective varletry is defined at all non-singular points of the curve.

One may think about this as an advanced form of "l'Hôpital's" rule. The tactics of the proof is firts to realize the mapping in a neighbourhood of P as the composition $\pi \circ \Phi$ where $\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and where $\Phi = (g_0, \ldots, g_n)$ with the g_i's regular near P, and then cancel out common factors of the g_i's vanishing at P.

ExtensionLemma

Lemma 8.2 *Let U be a curve and $P \in U$ a non-singular point. Assume that $\phi \colon U \setminus \{P\} \to \mathbb{P}^n$ is a morphism. Then there exists a morphism $\psi \colon U \to \mathbb{P}^n$ extending ϕ.*

PROOF: The first observation is that it suffices to find an open $U_0 \subseteq U$ containing P over which ϕ extends. Indeed, if $\psi_0 \colon U_0 \to \mathbb{P}^n$ is such an extension, the two morphisms ψ_0 and ϕ coincide on $U_0 \setminus \{P\}$, and hence they patch together to a morphism on U. It follows that we may assume U to be affine.

Secondly, we may, possible after having renumbered the coordinates, assume that the image $\phi(U \setminus \{P\})$ meets the basic open set

$D = D_+(x_0)$; then the inverse image $V = \phi^{-1}D$ is a non-empty open subset of U, and by shrinking V if necessary, V will be an affine open subset being mapped into D The basic open set D is an affine n-space with coordinates $x_1x_0^{-1}, \ldots, x_nx_0^{-1}$, and the map $\phi|_V$ is therefore given by m component functions on V. They are all rational function on U, and may therefore be written as fractions $f_i = g_i/g_0$ of regular functions on U.

Consider the morphism $\Phi(x) = (g_0, g_1, \ldots, g_n)$ from U into $\mathbb{A}^{n+1}$. It is well defined at the point P, but of course, it might be that it maps P to the origin. However, if this is not the case, the composition $\pi \circ \phi$ is defined at P and extends ϕ to the neighbourhood of P where the g_i's do not vanish simultaneously, and we will be done.

Now, the salient point is that we have the liberty to alter the morphism Φ by cancelling common factors of the g_i's without changing the composition $\pi \circ \Phi$: after such a modification the composition $\pi \circ \Phi$ and the original morphism ϕ coinside where they both are defined. Indeed, it holds true that $(hg_0; \ldots; hg_n) = (g_0; \ldots; g_n)$ where both sets of homogeneous coordinates are legitimate.

To get rid of common zeros of g_i's might have at the point P, we introduce a uniformizing parameter t at p; that is, a regular function t on U which generates the maximal ideal of the local ring $\mathcal{O}_{U,P}$. One may then write $g_i = \alpha_i t^{\nu_i}$ with the α_i's being regular functions on U that do not vanish at P, and where the ν_i's are non-negative integers. Putting $\nu = \min_i \nu_i$, the differences $\mu_i = \nu_i - \nu$ will be non-negative and at least one will be zero. Hence replacing g_i by $g_i t^{-n} = \alpha_i t^{\nu_i - \nu}$ we arrive at the requested modification of Φ. ❑

The theorems

Most of the work is done in proving the lemma , and we can collect the fuits. Here comes the theorems:

ExtensionTheorem

Theorem 8.2 *Let X be a curve and $P \in X$ a non-singular point. Any rational map $\phi\colon X \dashrightarrow Y$ where Y is a projective variety, is defined at P.*

Proof: Assume the projective variety Y is a closed subvariety of $\mathbb{P}^m$; that is, $Y \subseteq \mathbb{P}^m$. Let U be a neighbourhood of P such that ϕ is defined on $U \setminus \{P\}$. By the extension lemma (lemma 8.2 above), the map ϕ composed with the inclusion Y into $\mathbb{P}^m$ extends to P, and the extension takes values in Y since Y is closed in $\mathbb{P}^m$. ❑

It is paramount that P be a non-singular point. If X has *e.g.,* two different branches passing theorug P, the "limit" of ϕ at P along the two branches may be different.

Theorem 8.3 *Assume that X and Y are two projective and non-singular curves that are birationally equivalent. Then they are isomorphic.*

PROOF: Let $U \subseteq X$ and $V \subseteq Y$ be two open sets such that there is an isomorphism $\phi\colon U \dashrightarrow V$. Since Y is projective and X is non-singular a repeated application of theorem 8.2 above gives a morphism $\Phi\colon X \to Y$ extending ϕ. Similarly, there is morphism $\Psi\colon Y \to X$ extending ϕ^{-1}. Finally, the Hausdorff axiom holds for both X and Y, and one infers that $\Phi \circ \Psi = \mathrm{id}_Y$ and $\Psi \circ \Phi = \mathrm{id}_X$ since they extend $\phi \circ \phi^{-1} = \mathrm{id}_V$ and $\phi^{-1} \circ \phi = \mathrm{id}_U$ respectively. ❑