Summary of the course

Here is a list of the topics covered through the course. A more detailed list and exercises are in the schedule section.

Important: examples and counter-examples to the several results (and definitions) we met are important for the oral exam; so going through the exercise sheets is a good idea to be well prepared.

1 Affine varieties

- Definition of the algebraic set associated to an ideal.
- Zariski topology: definition, basis and examples.
- Irreducible topological spaces. Characterisation and consequences (e.g. not Hausdorff, any non-empty open subset is dense, etc...).
- Ideal associated to an algebraic set and properties.
- Hilbert Nullstellensatz (we only proved the equivalence between the three versions, always assuming the field algebraically closed).
- An algebraic set is irreducible iff its ideal is prime.
- Definition of affine and quasi-affine variety.
- Definition of coordinate ring associated to an algebraic set, properties and discussion about the maximal spectrum of a k-algebra.
- Bijection between algebraic sets (resp. affine varieties), radical ideals (resp. radical prime ideals) and finitely generated k-algebras (resp. finitely generated k-algebras that are integral domains).
- Noethersian topological spaces; example of Noethersian spaces (algebraic sets) and examples of non-Noetherian spaces.
- Characterisation: quasicompactness and existence of mininum in any non-empty family of closed subsets.
- Decomposition in irreducible components of a Noetherian space.

- Relation with primary decomposition of an ideal.
- Dimension of a (irreducible) topological space. Examples.
- Dimension of a closed subset and of an open subset.
- Recall on Krull dimension and the dimension formula (i.e. $ht(P) + \dim A/P = \dim A$).
- The dimension of an affine variety is the Krull dimension of its coordinate ring.
- Dimension of the closure of a quasi-affine variety.
- Dimension of $\mathbb{A}^n$.
- Recall: Krull dimension as transcendence degree; Krull principal ideal theorem (and geometric version of it).
- A Noetherian integral domain R is UFD iff any prime of height 1 is principal.
- An affine variety in $\mathbb{A}^n$ has codimension 1 iff it is the zero set of a single irreducible polynomial. (Remark: more generally this is true if we replace $\mathbb{A}^n$ by any variety X such that $A(X)$ is a UFD.)
- Definition of morphism of affine varieties as a polynomial map.
- Morphisms of affine varieties are continuous for the Zariski topology.
- Regular functions; global regular functions as morphisms to $\mathbb{A}^1$.
- Homomorphism of k-algebras induced by a morphism of affine varieties.
- Equivalence between the category of affine shemes and the category of finitely generated k-algebras that are integral domains.
- Germs of regular functions at a point; they form a local ring isomorphic to the localisation of the coordinate ring to the maximal ideal of the point.
- Rational functions; they form a field isomorphic to the field of fractions of the coordinate ring.
- Finite morphisms and geometric version of Noether normalisation lemma. Finite morphisms are closed and surjective.
- Regular functions of a quasi-affine variety and morphism between them.
- (Example of a quasi-affine variety that is not affine.)

2 Projective varieties

- Recall about the definition of projective space $\mathbb{P}^n$, and about graded rings and homogeneous ideals.
- Projective algebraic set associated to a homogeneous ideal and properties.
- Zariski topology on $\mathbb{P}^n$.
- The natural projection $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is continuous. In particular $\mathbb{P}^n$ is an irreducible topological space.
- Definition of projective and quasi-projective variety.
- Definition of the homogeneous ideal and the homogeneous coordinate ring of a projective algebraic set.
- $\mathbb{P}^n$ has an open cover composed by open subsets homeomorphic to $\mathbb{A}^n$.
- Projective closure of an affine variety; properties.
- Affine (and projective) cone over a projective variety; properties.
- Bijection between closed and non-empty subsets of $\mathbb{P}^n$ and closed and non-trivial cones in $\mathbb{A}^{n+1}$.
- Projective Nullstellensatz.
- Regular functions and morphisms of projective varieties.
- $\mathbb{P}^n$ has an open cover composed by open subset isomorphic to $\mathbb{A}^n$.
- Any (quasi-)projective variety is covered by affine varieties.
- Definition of homogeneous localisation of a graded ring.
- Germs of functions at a point form a local ring isomorphic to the homogeneous localisation of the homogeneous coordinate ring at the homogeneous linear ideal defining the point.
- Rational functions form a field isomorphic to the homogeneous localisation of the homogeneous coordinate ring at the zero ideal.
- A projective variety has no non-constant regular functions.
- Closed embedding; characterisation. Examples: Veronese and Segre embeddings.
- Rational normal curves; quadric surface.

3 Sheaves

- Definition of pre-sheaf and examples.
- Definition of morphism of pre-sheaves.
- Definition of sheaf. Examples and counter-examples.
- A sheaf is uniquely determined by its values on the open subsets of a basis.
- Regular functions of an affine variety form a sheaf.
- Stalks of a sheaf.
- Morphism between the stalks induced by a morphism of sheaves (i.e. a morphism of pre-sheaves).
- A morphism if sheaves is an isomorphism iff the morphism induced on the stalks is an isomorphism at every stalk.
- Definition of the kernel sheaf.
- Sheafification of a pre-sheaf.
- Definition of image of a sheaf.
- Examples: restriction sheaf and direct image sheaf.
- A morphism of affine varieties is a continuous map plus a morphism of sheaves.
- Definition of ringed space and morphism between ringed spaces.

4 Algebraic pre-varieties and varieties

- (Re-)definition of affine variety as a ringed space.
- definition of algebraic pre-variety.
- Any pre-variety is an irreducible Noetherian topological space.
- A (quasi-)affine or (quasi-)projective variety is a pre-variety.
- Gluing process: gluing sheaves and gluing affine varieties.
- Examples: $\mathbb{P}^1$ and the line with double origin.
- Definition of product object in a category.
- The category of pre-varieties has products.

- Definition of algebraic variety.
- Characterisation of the Hausdorff property.
- The line with the double origin is not a variety.
- A sub-pre-variety of a variety is a variety; product of varieties is a variety; affine varieties are varieties.
- If for any two points of a pre-variety there exists an open affine subset containing them, then the pre-variety is a variety.
- Any (quasi-)projective variety is a variety.
- Characterisation of varieties as those pre-varieties for which the intersection of two open affine subsets are still affine and their coordinate rings have are described as product (in the rational field of the variety) of the coordinate rings of the open affines.
- Morphisms from any variety to an affine variety are uniquely determined by their induced homomorphism on regular global functions.

5 Rational maps

- Morphisms of varieties are determined by their restriction to open non-empty dense subsets.
- Definition of rational, dominant and birational maps. Definition of the category of varieties with dominant maps.
- Homomorphism of field extensions of k associated to a dominant map.
- Equivalence between the category of varieties with dominant maps and the category of finitely generated field extensions of k with homomorphisms of field extensions.
- Characterisation of birational maps.
- Definition of blow up of $\mathbb{A}^n$ at the origin; properties and definition of strict transform of an affine variety.
- Interpretation of the blow up as incidence variety and as resolution of indeterminacy of a rational map.
- Blow up of $\mathbb{A}^3$ at a line (throught the origin); properties and strict transform of an affine surface.

6 Non-singularity

- Jacobian criterion for non-singularity of affine varieties.
- Definition of regular local ring.
- Intrinsic characterisation of the Jacobian criterion via regularity of the local rings of an affine variety.
- Definition of non-singular variety.
- Upper bound of the dimension of the variety (i.e. $\dim X \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$).
- The singular locus of a variety is a proper and closed subset.
- Non-singularity of affine plane curves: characterisation of non-singularity via multiplicities. Tangent directions at a singular point. Definition of node (i.e. multiplicity 2 with two distinct tangent directions). Blowing up a node in an affine plane curve resolves the singularity.
- Tangent cones and tangent spaces for affine varieties.
- Intrinsic characterisation of tangent cones and tangent spaces, and definition of both of them for any variety.
- Bijection between vector spaces over k and affine spaces over k; the tangent space as a vector space.
- The tangent cone is embedded in the tangent space and there is equality iff the variety is non-singular at the corresponding point iff the dimension of the variety is equal to the dimension of the tangent space at the point.

7 Intersection theory

- Long and detailed motivation (via examples) for why Bézout theorem should hold and for why we should define intersection multiplicities in that way.
- Dimension theory: lower bound for the dimension of the intersection of two affine varieties; lower bound for the dimension of the intersection of two projective varieties.
- Regular sequences. Intersection of r projective hypersurfaces has codimension exactly r.
- Varieties of dimension 0: recall on Artinian rings and their main features. Intersection multiplicities are well defined.

- Transversal intersections at a point; the intersection multiplicity at a point is 1 iff the intersection is transversal at that point.
- Recall about Hilbert polynomials; Hilbert-Serre theorem and quick sketch of its proof.
- Explicit computation of Hilbert polynomials of: projective space; projective hypersurfaces; intersection of projective hypersurfaces (associated to a regular sequence).
- Definition of degree of a projective variety via the Hilbert polynomial of the variety.
- Hilbert polynomial of a module supported on a line (through the origin).
- Proof of Bézout theorem.