Mandatory assignment MAT4210 – Spring 2021

The assignment must be submitted via Canvas by **14:30, Thursday February 18th**.

You need to solve 3 problems to pass. If you have any questions or comments about the problems, feel free to email me at `johnco@math.uio.no`.

All varieties are over the field $k = \mathbb{C}$.

Problem 1. Assume X and Y are two affine varieties and that $\phi : X \to Y$ is a morphism. Show that ϕ is a closed embedding if and only if the map $\phi^* : A(Y) \to A(X)$ between the coordinate rings is surjective.

Problem 2. Consider the algebraic set $X = Z(I) \subset \mathbb{A}^3$ given by the ideal

$$I = \left(y - x^2, yz^2, xz^2\right) \subset \mathbb{C}[x, y, z]$$

Find a decomposition of X into irreducible components and compute its dimension.

Problem 3. Find all the singular points on the curve

$$C = Z(x^4 + y^3 z - x^2 yz) \subset \mathbb{P}^2$$

and show that C is rational (i.e., birational to $\mathbb{P}^1$).

Problem 4. Consider $V \subset \mathbb{A}^2 \times \mathbb{P}^1$ given by the equation

$$u_0 x^2 - u_1 y = 0$$

where $(u_0 : u_1)$ are homogeneous coordinates on $\mathbb{P}^1$ and x, y are affine coordinates on $\mathbb{A}^2$.

(i) Show that V is irreducible and compute its dimension.

(ii) Describe the fibers of the morphism $\pi = p_1 : V \to \mathbb{A}^2$ and show that V is rational.

(iii) Describe the fibers of the morphism $p = p_2 : V \to \mathbb{P}^1$. Which fibers are singular?

(iv) Find all sections of p, i.e., morphisms $\sigma : \mathbb{P}^1 \to V$ so that $p \circ \sigma = \mathrm{id}_{\mathbb{P}^1}$.