Introduction to Schemes

Geir Ellingsrud and John Christian Ottem

September 29, 2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Contents

Acknowledgements 1
Notation and conventions 2

1 Algebraic sets 3
1.1 Algebraic sets 3
1.2 Hilbert's Nullstellensatz 5
1.3 The structure of algebraic sets 8
1.4 Regular and rational functions 10
1.5 Morphisms of affine varieties 12
1.6 Exercises 16

2 The spectrum of a ring 19
2.1 The spectrum of a ring 20
2.2 Affine spaces 24
2.3 Distinguished open sets 26
2.4 Maps between prime spectra 27
2.5 Finitely generated k-algebras 30
2.6 Fibers 31
2.7 Exercises 36

3 The structure sheaf 38
3.1 Presheaves and sheaves 38
3.2 Stalks 41
3.3 Sheaves defined on a basis 43
3.4 The pushforward of a sheaf 44
3.5 The structure sheaf on the spectrum of a ring 45
3.6 Application: Connectedness of $\operatorname{Spec} A$ 50
3.7 Exercises 51

4 Schemes 54
4.1 Locally ringed spaces 54
4.2 Schemes 57
4.3 Morphisms into an affine scheme 58
4.4 Schemes over a ring 61
4.5 Open embeddings and open subschemes 62
4.6 Closed embeddings and closed subschemes 63
4.7 Reduced schemes 65

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

4.8 Integral schemes, dominant morphisms, and function fields 66
4.9 Residue fields and ring-valued points 69
4.10 Affine varieties as schemes 71
4.11 Exercises 71

5 Gluing 75
5.1 Gluing two schemes together 75
5.2 Gluing sheaves 76
5.3 Gluing schemes 79
5.4 First examples 81
5.5 Exercises 86

6 Projective schemes 88
6.1 Projective space 88
6.2 Projective schemes 93
6.3 An important family of sheaves on $\mathbb{P}^n_A$ 94
6.4 The Proj construction 96
6.5 Exercises 104

7 Fiber products and the functor of points 107
7.1 Relative schemes 107
7.2 Fiber products 108
7.3 Fiber products of schemes 109
7.4 Examples 114
7.5 Base change 115
7.6 Scheme-theoretic fibers 117
7.7 Yoneda's Lemma 118
7.8 The Segre embedding 122
7.9 Exercises 123

8 Finiteness properties and dimension 129
8.1 Covering intersections by distinguished opens 129
8.2 Schemes of finite type 130
8.3 Affine morphisms and finite morphisms 132
8.4 Noetherian schemes 134
8.5 Dimension 136
8.6 Products of finite type schemes 141
8.7 Codimension 142
8.8 Products of finite type schemes 146
8.9 Dimensions of intersections 147
8.10 Dimensions of the fibers of a morphism 148
8.11 Exercises 149

9 More on sheaves 161
9.1 Kernels and images 161
9.2 Injective and surjective maps of sheaves 163
9.3 Exact sequences 165
9.4 The sheaf associated to a presheaf 166

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

9.5 Cokernels and quotients 169
9.6 The inverse image sheaf 169
9.7 Exercises 173

10 Quasi-coherent sheaves 175
10.1 Sheaves of modules 175
10.2 The sheaf associated to a module 176
10.3 Quasi-coherent sheaves 178
10.4 Kernels, images and cokernels 181
10.5 Direct sums and products 182
10.6 Tensor products 183
10.7 The Hom-sheaf 184
10.8 Finiteness conditions for $\mathcal{O}_X$-modules 185
10.9 Pushforwards 187
10.10 Pullbacks 190
10.11 Relative Spec 194
10.12 Closed subschemes and ideal sheaves 197
10.13 Exercises 199

11 Locally free sheaves 208
11.1 Locally free sheaves 208
11.2 Transition functions 210
11.3 Invertible sheaves and the Picard group 211
11.4 Fibers and local freeness 213
11.5 Vector bundles 219
11.6 Exercises 220

12 Sheaves on projective schemes 225
12.1 The sheaf associated to a graded module 225
12.2 Closed subschemes of projective space 233
12.3 Coherent sheaves on projective space 235
12.4 Remarks about non-standard gradings 236
12.5 Pullbacks of invertible sheaves 237
12.6 Morphisms to projective space 238
12.7 Application: Automorphisms of projective space 241
12.8 Exercises 241

13 Separated morphisms 245
13.1 Separated morphisms 245
13.2 Intersections of affine opens 248
13.3 Graphs and Equalizers 250
13.4 Rational maps 252
13.5 Exercises 252

14 Varieties and rational maps 256
14.1 Varieties 256
14.2 Rational maps 257
14.3 Generically finite maps and degree 260

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

14.4 Rationality 261

15 Proper morphisms 263
15.1 Proper morphisms 263
15.2 Projective space is proper 265
15.3 Finite morphisms and properness 268
15.4 Chow's Lemma 269
15.5 Exercises 271

16 Tangent spaces and regular schemes 274
16.1 Tangent spaces and cotangent spaces 274
16.2 Regular schemes 276
16.3 Regular embeddings 280
16.4 Zero sets of sections 283
16.5 Exercises 284

17 Differentials 289
17.1 Differentials 289
17.2 The sheaf of differentials 294
17.3 Differentials and cotangent spaces 296
17.4 Smoothness over a field 297
17.5 The tangent sheaf 301
17.6 Smooth closed subschemes 303
17.7 The Euler sequence and differentials of $\mathbb{P}^n_k$ 304
17.8 The sheaf of p-forms 305
17.9 Exercises 306

18 Normal schemes 310
18.1 Normal schemes 310
18.2 Normalization 314
18.3 Rational functions on normal schemes 316
18.4 Normal schemes and birational morphisms 317
18.5 Curves 319
18.6 Exercises 322

19 Divisors 326
19.1 Weil divisors 326
19.2 The sheaf associated to a Weil divisor 331
19.3 Effective divisors and linear systems 339
19.4 Pullbacks of divisors 340
19.5 Extended examples 341
19.6 Exercises 345

20 First steps in sheaf cohomology 350
20.1 Some homological algebra 351
20.2 Čech cohomology 352
20.3 Examples 356
20.4 Čech cohomology on schemes 358

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20.5 Cohomology of sheaves on affine schemes 360
20.6 Independence of the cover 361
20.7 Pushforwards and cohomology 362
20.8 Cohomology and dimension 363
20.9 Cohomology of the sheaves $\mathcal{O}(m)$ 364
20.10 Cohomology of sheaves on projective schemes 366
20.11 Example: Plane curves 370
20.12 Example: The twisted cubic 371
20.13 Example: Hyperelliptic curves 372
20.14 Example: Complete intersections 373
20.15 Coherence of pushforwards 375
20.16 Hodge numbers 376
20.17 Application: irrationality of hypersurfaces 377
20.18 Exercises 378

21 Projective embeddings and ampleness 383
21.1 Very ample and ample invertible sheaves 383
21.2 A criterion for being a closed embedding 388
21.3 Extended example: The blow-up of the projective plane 390
21.4 Exercises 391

22 Curve theory 393
22.1 Sheaves on curves 393
22.2 Splitting of locally free sheaves on $\mathbb{P}^1_k$ 394
22.3 Divisors on curves 395
22.4 The canonical divisor 398
22.5 Exercises 404

23 The Riemann–Roch theorem and applications 406
23.1 The Riemann–Roch formula 406
23.2 Base-point freeness and very ampleness on curves 410
23.3 Curves of genus 0 412
23.4 Curves of genus 1 414
23.5 Curves of genus 2 416
23.6 Curves of genus 3 416
23.7 Curves of genus 4 418
23.8 Proof of Serre duality 419
23.9 Exercises 425

Appendix A **Further constructions and examples** 429

Appendix B **Some category theory** 469

Appendix C **Some results from Commutative Algebra** 473
Bibliography 498

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Acknowledgements

Thanks to Nick Addington, Georges Elencwajg, Frank Gounelas, Johannes Nicaise, Dan Petersen, Kristian Ranestad, Jørgen Vold Rennemo, Stefan Schreieder and Burt Totaro for comments and suggestions. Also, thanks to Edvard Aksnes, Shamil Asgarli, Valentine Blanpain, Alex Brodbelt, Anne Brugård, Anakin Dey, Tim Duff, Søren Gammelgaard, Elias Giraud-Audine, Samit Ghosh, Kai Komori, Timo Kränzle, Simen Westbye Moe, Anirban Mondal, Torger Olsson, Nikolai Thode Opdan, Erik Paemurru, Sabrina Pauli, Gabriel Ribeiro, Renè Rubenbauer, Arne Olav Vik, Magnus Vodrup, Yang Yihan, Xiangzhuo Zeng and Qi Zhu for numerous corrections to the text. A special thanks goes to Aleksander Leraand and Leandros Emmanuel de Jonge for their careful reading and helpful comments on the manuscript.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Notation and conventions

$\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers.

$\subset$ means 'is subset of', i.e., the same thing as $\subseteq$.

If X and Y are two sets, then $X - Y$ denotes the set of elements in X which are not in Y.

All rings are commutative with 1.

Ring maps are required to send 1 to 1.

The zero ring is not an integral domain (and hence not a field).

For a ring A, we write $A_{\mathfrak{p}}$ and A_f for the localizations in the multiplicative sets $S = A - \mathfrak{p}$ and $S = \{1, f, f^2, \dots\}$ respectively.

For a ring A, $A^\times$ denotes the group of invertible elements of A.

The fraction field of an integral domain A is denoted by $K(A)$.

We use the words 'map' and 'morphism' interchangably. This means a morphism in the relevant category, e.g., a 'map of rings' is a ring homomorphism.

We will occasionally write $A = B$ if there is a *canonical* isomorphism $A \simeq B$. So for instance, $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$.

If $\mathcal{C}$ is a category, we denote its *opposite category* by $\mathcal{C}^{\text{op}}$. This category has the same objects as $\mathcal{C}$, but for any pair of objects, the morphisms from X to Y in $\mathcal{C}^{\text{op}}$ correspond to the morphisms from Y to X in $\mathcal{C}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

1

Algebraic sets

In this chapter, we discuss algebraic sets and affine varieties, which will serve as the main motivating examples in the theory of schemes. We confine ourselves to the basic definitions, examine a few key examples, and establish notation for later chapters. As we progress through the book, we will develop the theory of varieties in greater depth.

1.1 Algebraic sets

Algebraic sets and affine varieties are defined over a fixed ground field k. In this chapter we will assume that k is algebraically closed. It is useful to keep some specific fields in mind, e.g. the field of complex numbers $\mathbb{C}$, the field of algebraic numbers $\bar{\mathbb{Q}}$, or perhaps the algebraic closure of a finite field, $\bar{\mathbb{F}}_p$.

We will denote the set k^n by $\mathbb{A}^n(k)$, and refer to it as the *affine n-space*. The change in notation is meant to emphasize that there is more to $\mathbb{A}^n(k)$ than its underlying set of elements; it will soon be equipped with a topology, and ultimately, it will be a scheme. The elements of $\mathbb{A}^n(k)$ are called *points*.

Definition 1.1. For a subset S of $k[x_1, \dots, x_n]$, we define its *zero set* as

$$Z(S) = \{\, P \in \mathbb{A}^n(k) \mid f(P) = 0 \text{ for all } f \in S \,\}.$$

An *algebraic set* is a subset of $\mathbb{A}^n(k)$ of this form.

If $f_1, \dots, f_r \in S$, then every polynomial of the form $\sum_{i=1}^r b_i f_i$ where $b_i \in k[x_1, \dots, x_n]$, also vanishes at the points of $Z(S)$. This means that the zero set of the ideal $\mathfrak{a}$ generated by the elements of S is the same as $Z(S)$; that is, $Z(S) = Z(\mathfrak{a})$. We will therefore work almost exclusively with ideals and tacitly replace a set of polynomials by the ideal they generate. Hilbert's basis theorem (Theorem C.24) tells us that any ideal in $k[x_1, \dots, x_n]$ is finitely generated, so an algebraic subset is always described as the set of common zeros of *finitely* many polynomials.

The more constraints we impose, the smaller the solution set will be, so if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals with $\mathfrak{a} \subset \mathfrak{b}$, one has $Z(\mathfrak{a}) \supset Z(\mathfrak{b})$. The assignment $\mathfrak{a} \mapsto Z(\mathfrak{a})$ therefore defines an inclusion-reversing map from the set of ideals in $k[x_1, \dots, x_n]$ to the set of algebraic subsets of $\mathbb{A}^n(k)$.

Under this correspondence, different ideals can define the same algebraic set. For instance, the ideals (x) and (x^2) in $k[x]$, both have the origin of $\mathbb{A}^1(k)$ as their zero set. To take care

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

of this ambiguity, we recall the *radical* $\sqrt{\mathfrak{a}}$, which is defined by

$$\sqrt{\mathfrak{a}} = \{\, f \mid f^r \in \mathfrak{a} \text{ for some } r > 0 \,\}.$$

For a point $P \in \mathbb{A}^n(k)$ and a polynomial f, we have $f^r(P) = 0$ for some $r > 0$ if and only if $f(P) = 0$, so we obtain $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$. Two ideals with the same radical therefore have the same zero set. In fact, Hilbert's Nullstellensatz, which we will state below, tells us that the converse also holds.

Proposition 1.2. The zero sets in $\mathbb{A}^n(k)$ satisfy the following properties:

(i) (Arbitrary intersections) For a collection of ideals $\{\mathfrak{a}_i\}_{i\in I}$ in $k[x_1, \ldots, x_n]$,

$$\bigcap_{i\in I} Z(\mathfrak{a}_i) = Z\Big(\sum_{i\in I} \mathfrak{a}_i\Big). \tag{1.1}$$

(ii) (Finite unions) For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $k[x_1, \ldots, x_n]$,

$$Z(\mathfrak{a}) \cup Z(\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a} \cdot \mathfrak{b}). \tag{1.2}$$

Additionally, $Z(1) = \varnothing$ and $Z(0) = \mathbb{A}^n(k)$.

Proof (i): Note that $P \in \bigcap_{i\in I} Z(\mathfrak{a}_i)$ if and only if $f(P) = 0$ for all $f \in \mathfrak{a}_i$ and all $i \in I$. This is equivalent to $f(P) = 0$ for all $f \in \bigcup_{i\in I} \mathfrak{a}_i$. Since $\sum_{i\in I} \mathfrak{a}_i$ is the ideal generated by $\bigcup_{i\in I} \mathfrak{a}_i$, the first item follows.

(ii): We have $\mathfrak{a} \cdot \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ is contained in both $\mathfrak{a}$ and $\mathfrak{b}$. This shows that $Z(\mathfrak{a}) \cup Z(\mathfrak{b}) \subset Z(\mathfrak{a} \cap \mathfrak{b}) \subset Z(\mathfrak{a} \cdot \mathfrak{b})$. Conversely, if $P \notin Z(\mathfrak{a}) \cup Z(\mathfrak{b})$, then there exist $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ such that $f(P) \neq 0$ and $g(P) \neq 0$. But then $(fg)(P) \neq 0$, and so $P \notin Z(\mathfrak{a} \cdot \mathfrak{b})$. The identities $Z(1) = \varnothing$ and $Z(0) = \mathbb{A}^n(k)$ follow from the definitions. □

The identities in Proposition 1.2 tell us that the algebraic sets form the closed sets of a topology on the affine space $\mathbb{A}^n(k)$. This topology is called the *Zariski topology*.

If $X \subset \mathbb{A}^n(k)$ is any subset, we get an induced Zariski topology on X by declaring that the closed sets of X are those of the form $X \cap W$, where W is closed in $\mathbb{A}^n(k)$.

Examples

Example 1.3. The affine space $\mathbb{A}^1(k)$ is called the *affine line*. Every nonzero proper ideal in $k[t]$ is principal, generated by some polynomial f. Since k is algebraically closed, f factors as a product $(t - a_1) \cdots (t - a_r)$, and its zero set $Z(f)$ is precisely the finite set of roots $\{a_1, \ldots, a_r\}$. Conversely, any finite set of points arises this way. Hence the closed subsets of $\mathbb{A}^1(k)$ are precisely the finite sets and the whole space.

This means that the Zariski topology on $\mathbb{A}^1(\mathbb{C})$ behaves quite differently from the usual topology on $\mathbb{C}$. For instance, $\mathbb{A}^1(\mathbb{C})$ is not Hausdorff, as any two non-empty open sets are dense, so they must intersect. △

Example 1.4 (Plane conics)**.** Plane conics are classical examples of algebraic sets. Each is defined by a single quadratic equation in $\mathbb{A}^2(k)$. Three familiar examples are the circle $x^2 + y^2 - 1 = 0$, the parabola $y - x^2 = 0$, and the hyperbola $xy - 1 = 0$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Note also the conic $x^2 + y^2 + 1 = 0$, which has no solutions, for example when $k = \mathbb{R}$. Over $\mathbb{C}$, however, it is equivalent to $xy - 1 = 0$ by the linear change of variables $u = x + iy$, $v = -x + iy$. More generally, if k is algebraically closed of characteristic $\neq 2$, then after an affine linear change of coordinates any conic can be brought to one of the forms $xy - 1 = 0$, $y - x^2 = 0$, $x^2 = 0$, $xy = 0$, or $x^2 - 1 = 0$. △

Example 1.5. A more interesting example is the *Clebsch cubic surface*; the algebraic set in $\mathbb{A}^3(\mathbb{C})$ defined by the equation

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3.$$

The real points of the surface, i.e. the points in $\mathbb{A}^3(\mathbb{R})$ satisfying the equation, are shown below. This surface contains 27 lines, each defined over the real numbers. △



Figure 1.1 The Clebsch cubic surface

1.2 Hilbert's Nullstellensatz

There is a kind of converse to the 'take zero sets' operation. For a subset $V \subset \mathbb{A}^n(k)$, we consider the set of all polynomials that vanish at every point of V:

$$I(V) = \{\, f \in k[x_1, \ldots, x_n] \mid f(P) = 0 \text{ for all } P \in V \,\}. \tag{1.3}$$

Note that $I(V)$ is always a radical ideal of $k[x_1, \ldots, x_n]$. Moreover, if $W \subset V$, then $I(V) \subset I(W)$.

We thereby have two inclusion-reversing mappings between the set of radical ideals and the set of closed sets of $\mathbb{A}^n(k)$: $\mathfrak{a} \mapsto Z(\mathfrak{a})$ and $X \mapsto I(X)$. The precise relationship between them is given by Hilbert's Nullstellensatz.

Theorem 1.6 (Hilbert's Nullstellensatz). Let k be an algebraically closed field. Then the map

$$\Big\{\text{radical ideals } \mathfrak{a} \subset k[x_1, \ldots, x_n]\Big\} \longrightarrow \Big\{\text{closed sets } X \subset \mathbb{A}^n(k)\Big\} \tag{1.4}$$

defined by $\mathfrak{a} \mapsto Z(\mathfrak{a})$ is a bijection, with inverse defined by $X \mapsto I(X)$.

To prove the theorem, we have to show that $Z(I(X)) = X$ for an algebraic set X and $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for any ideal $\mathfrak{a}$. Three of the four required inclusions are straightforward and hold for any field k.

(i) $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$: If $f \in \sqrt{\mathfrak{a}}$, then $f^r \in \mathfrak{a}$ for some positive integer r. Then f^r vanishes on $Z(\mathfrak{a})$ and so f vanishes on $Z(\mathfrak{a})$ as well. Hence $f \in I(Z(\mathfrak{a}))$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(ii) $X \subset Z(I(X))$: If $P \in X$, then by definition of $I(X)$, $f(P) = 0$ for every $f \in I(X)$, so $P \in Z(I(X))$.
(iii) $Z(I(X)) \subset X$: Write $X = Z(\mathfrak{a})$ for an ideal $\mathfrak{a}$. By (i), we have $I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}} \supset \mathfrak{a}$. Taking zero sets, $Z(I(Z(\mathfrak{a}))) \subset Z(\mathfrak{a})$, that is, $Z(I(X)) \subset X$.

The real content of the Nullstellensatz is that if k is algebraically closed, then the inclusion $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$ is an equality. In other words, $\sqrt{\mathfrak{a}}$ is exactly the set of polynomial functions which vanish on the closed set $Z(\mathfrak{a})$.

We will deduce Theorem 1.6 from the following algebraic result, which will also be useful over fields that are not algebraically closed. The proof uses the fact that a polynomial ring $k[x]$ contains infinitely many irreducible monic polynomials. This fact can be seen using Euclid's proof of the infinitude of prime numbers: if $p_1, \dots, p_r \in k[x]$ were all the irreducible monic polynomials, then $p_1 \cdots p_r + 1$ is not divisible by any of the p_i, so there must exist an additional irreducible polynomial, a contradiction.

Lemma 1.7 (Zariski's Lemma). Let K/k be a field extension such that K is a finitely generated k-algebra. Then K is a finite algebraic extension of k.

Proof Write $K = k[x_1, \dots, x_n]$, where the x_i may be algebraically dependent. We proceed by induction on n. If $n = 1$, then $K = k[x_1]$. If this is a field, then $x_1^{-1} \in K$, so $x_1^{-1} = p(x_1)$ for some polynomial p. Hence x_1 satisfies the nontrivial algebraic equation $tp(t) - 1 = 0$, so it is algebraic over k.

If $n > 1$, we may write $K = k(x_1)[x_2, \dots, x_n]$. By induction, $x_2, \dots, x_n$ are algebraic over $k(x_1)$, so they satisfy equations of the form $x_i^{m_i} + a_{i,1}x_i^{m_i-1} + \cdots + a_{i,m_i} = 0$ with $a_{i,j} \in k(x_1)$. Letting g be the product of all the denominators appearing in the $a_{i,j}$, the coefficients of $x_i^{m_i} + a_{i,1}x_i^{m_i-1} + \cdots + a_{i,m_i} = 0$ lie in $k[x_1]_g$. Using these relations to reduce degrees, we see that K is generated as a $k[x_1]_g$-module by finitely many monomials $x_1^{e_1} \cdots x_n^{e_n}$. Since $k[x_1]_g$ is Noetherian, every submodule of the finitely generated module K is also finitely generated. In particular, $k(x_1)$ is finitely generated as a $k[x_1]_g$-module. This is only possible if x_1 is algebraic over k: if not, then $k[x_1]$ would be a polynomial ring in one variable, containing infinitely many irreducible monic polynomials $\{h_j\}_{j\in\mathbb{N}}$. But then the elements $1/h_j$ cannot all lie in a finitely generated $k[x_1]_g$-module, because only finitely many irreducible factors can appear in denominators of such elements. This is a contradiction, so x_1 is algebraic over k. Then since $x_2, \dots, x_n$ are algebraic over $k(x_1)$, they are algebraic over k as well. Hence K is a finitely generated algebraic k-algebra, hence a finite field extension. □

If k is algebraically closed, there are no nontrivial algebraic extensions, so $K = k$.

As a first consequence, we have the following result, which shows that $Z(\mathfrak{a})$ is nonempty whenever $\mathfrak{a}$ is a proper ideal:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Theorem 1.8 (Weak Nullstellensatz). Let k be an algebraically closed field.
(i) Every maximal ideal in $k[x_1, \dots, x_n]$ is of the form

$$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$$

for some point $(a_1, \dots, a_n)$ in $\mathbb{A}^n(k)$.
(ii) For an ideal $\mathfrak{a}$, the zero set $Z(\mathfrak{a})$ is empty if and only if $\mathfrak{a} = (1)$.

Proof (i): By Theorem 1.7, the field $k[x_1, \dots, x_n]/\mathfrak{m}$ is a finite extension of k, and therefore equals k since k is assumed to be algebraically closed. For each $i = 1, \dots, n$, let $a_i \in k$ be the image of x_i under the quotient map $\pi\colon k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]/\mathfrak{m} = k$. Then all the $x_i - a_i$ belong to the kernel of π. Since the ideal $(x_1 - a_1, \dots, x_n - a_n)$ is already maximal, it must be equal to $\mathfrak{m}$.

(ii): If $\mathfrak{a} = (1)$, then clearly $Z(\mathfrak{a}) = \varnothing$. Conversely, if $\mathfrak{a}$ is a proper ideal, there is a maximal ideal $\mathfrak{m}$ in $k[x_1, \dots, x_n]$ containing it. By (i), $\mathfrak{m}$ has the form $(x_1 - a_1, \dots, x_n - a_n)$, and consequently $(a_1, \dots, a_n) \in Z(\mathfrak{a})$, and so $Z(\mathfrak{a})$ is not empty. □

Proof of Hilbert's Nullstellensatz To prove the theorem, it suffices to show that for any ideal $\mathfrak{a} \subset k[x_1, \dots, x_n]$, we have $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$.

Let $f \in I(Z(\mathfrak{a}))$ and consider the k-algebra obtained by inverting f and modding out by $\mathfrak{a}$:

$$A = k[x_1, \dots, x_n, t]/(\mathfrak{a} + (tf - 1)).$$

If $A \neq 0$, then there exists a maximal ideal $\mathfrak{m} \subset A$. By the Weak Nullstellensatz, $\mathfrak{m}$ corresponds to a point $(a_1, \dots, a_n, b) \in \mathbb{A}^{n+1}(k)$ with $(a_1, \dots, a_n) \in Z(\mathfrak{a})$ and $b \cdot f(a_1, \dots, a_n) = 1$. In particular, $f(a_1, \dots, a_n) \neq 0$, which contradicts the assumption $f \in I(Z(\mathfrak{a}))$.

We deduce that $A = 0$. But note that A is isomorphic to the localization of $k[x_1, \dots, x_n]/\mathfrak{a}$ in the multiplicative set $S = \{1, f, f^2, \dots\}$. Therefore, $A = 0$ implies that there is an $N \in \mathbb{N}$ such that $f^N \in \mathfrak{a}$, and hence $f \in \sqrt{\mathfrak{a}}$. □

Example 1.9. It is important to note that Hilbert's Nullstellensatz only holds when the ground field is algebraically closed. For instance, for the ideal $\mathfrak{a} = (x^2 + y^2 + 1)$ in $\mathbb{R}[x, y]$ from Example 1.4 we have $I(Z(\mathfrak{a})) = \mathbb{R}[x, y]$ (vanishing at the empty set imposes no condition), so the inclusion $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$ is strict. △

Example 1.10. When $k = \mathbb{C}$, there is a quick way to prove Theorem 1.7 as follows. Suppose $K = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{m}$ is a finitely generated $\mathbb{C}$-algebra which is also a field. If there exists an element $t \in K$ which is not algebraic over $\mathbb{C}$, then the elements $\frac{1}{t-a} \in K$ for $a \in \mathbb{C}$ form an uncountable subset of K that is linearly independent over $\mathbb{C}$. However, this is impossible, because K is spanned over $\mathbb{C}$ by the monomials $x_1^{e_1} \cdots x_n^{e_n}$, so it has at most countable dimension as a $\mathbb{C}$-vector space.

See also Corollary C.29 for another proof of Theorem 1.7. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

1.3 The structure of algebraic sets

As a consequence of Hilbert's Nullstellensatz, many geometric properties of algebraic sets can be translated into algebraic properties of the ideals of the polynomial ring $k[x_1, \ldots, x_n]$. To describe these geometric properties, we recall a few notions from point-set topology.

A topological space X is called *irreducible* if it is nonempty and cannot be written as the union of two proper closed subsets; in other words, if $X = Z \cup Z'$ with Z and Z' closed, then $Z = X$ or $Z' = X$.

Proposition 1.11. An algebraic set $X \subset \mathbb{A}^n(k)$ is irreducible if and only if the ideal $I(X)$ is prime.

Proof Suppose $X = X_1 \cup X_2$ with X_1 and X_2 proper closed subsets. Then $I(X_1) \supset I(X)$ and $I(X_2) \supset I(X)$ are strict inclusions, so we may find $f_i \in I(X_i) - I(X)$ for $i = 1, 2$. Then $f_1 f_2 \in I(X)$, which means that $I(X)$ is not prime.

Conversely, suppose $I(X)$ is not prime, so there exist f_1, f_2 so that $f_i \notin I(X)$ for $i = 1, 2$, but $f_1 f_2 \in I(X)$. Then $X_1 = X \cap Z(f_1)$ and $X_2 = X \cap Z(f_2)$ are closed proper subsets and $X_1 \cup X_2 = X \cap Z(f_1 f_2) = X$, so X is not irreducible. □

The following is a preliminary definition of an affine variety. We will revisit the definition in Section 4.10, when we have introduced schemes.

Definition 1.12. An *affine variety* is an irreducible algebraic set in $\mathbb{A}^n(k)$.

Example 1.13. Affine space $\mathbb{A}^n(k)$ is irreducible as $k[x_1, \ldots, x_n]$ is an integral domain. △

Example 1.14. The closed set $Z(xyz) \subset \mathbb{A}^3(k)$ is not irreducible, as it can be written as a union of three planes $Z(x) \cup Z(y) \cup Z(z)$. Similarly, $Z(xz, yz) = Z(x, y) \cup Z(z) \subset \mathbb{A}^3(k)$ is not irreducible, being the union of a plane and a line. △

We say that a topological space X is *Noetherian* if all descending chains of closed subsets

$$X_0 \supset X_1 \supset X_2 \supset \cdots$$

in X eventually stabilize, that is, there exists an integer N such that $X_i = X_{i+1}$ for all $i \geqslant N$.

Proposition 1.15.
(i) Every algebraic set X is a Noetherian topological space.
(ii) Every Noetherian topological space has a unique decomposition

$$X = X_1 \cup \cdots \cup X_r \tag{1.5}$$

where $X_i \subset X$ is a closed irreducible subset and $X_i \not\subset X_j$ for all $i \neq j$.

Proof (i): If $X_0 \supset X_1 \supset X_2 \supset \cdots$ is a chain of closed subsets, then $I(X_0) \subset I(X_1) \subset I(X_2) \subset \cdots$ is a chain of ideals in $k[x_1, \ldots, x_n]$, which must stabilize because the polynomial ring is Noetherian. Since $X_i = Z(I(X_i))$, we see that the chain of closed subsets also stabilizes.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(ii): Suppose, for a contradiction, that X does not admit such a decomposition. Then X cannot be irreducible, so we may write $X = X_1 \cup X_1'$ where X_1, X_1' are proper closed subsets. As $X = X_1 \cup X_1'$, the statement in (ii) must fail for either X_1 or X_1'. Suppose that it fails for X_1. Then again X_1 cannot be irreducible, so we write $X_1 = X_2 \cup X_2'$. Continuing in this way, we obtain an infinite chain of closed subsets $X \supset X_1 \supset X_2 \supset \dots$, where each inclusion is strict, contradicting (i). Removing any redundant subsets (where $X_i \subset X_j$ for $i \neq j$) gives the desired decomposition (1.5)

For uniqueness, suppose $X = X_1 \cup \cdots \cup X_r$ and $X = Y_1 \cup \cdots \cup Y_s$ are two such decompositions. For each i, we have

$$X_i = \bigcup_{j=1}^{s} (X_i \cap Y_j).$$

Since X_i is irreducible, there exists a j such that $X_i = X_i \cap Y_j$, hence $X_i \subset Y_j$. Likewise, $Y_j \subset X_k$ for some k. Thus, $X_i \subset X_k$, and by the minimality condition ($X_i \not\subset X_\ell$ unless $i = \ell$), we must have $i = k$ and hence $X_i = Y_j$. Similarly, every Y_j must equal some X_i. Therefore, the two decompositions are the same up to reordering. □

The irreducible sets appearing in the decomposition (1.5) are called the *irreducible components* of X. They are precisely the closed irreducible subsets that are maximal with respect to inclusion.

Example 1.16. Consider the algebraic set $X = Z(I)$ in $\mathbb{A}^3(k)$, where

$$I = (xz - y^2, x^2 - y).$$

Let us find the irreducible components of X. Let $P = (a, b, c) \in X$ be a point. Then the second equation implies that $b = a^2$. Plugging this into the first equation, we get $ac - a^4 = 0$, which implies that either $a = 0$ or $c = a^3$. Hence P lies in one of the irreducible subsets $X_1 = Z(x, y)$ or $X_2 = Z(y - x^2, z - x^3)$. Conversely, any point in X_1 lies in X, and if $P = (a, b, c) \in X_2$, then $b = a^2$ and $c = a^3$ so that $ac - b^2 = a^4 - a^4 = 0$, and P lies in X. Hence we find that

$$X = Z(x, y) \cup Z(y - x^2, z - x^3).$$

This means that X is the union of the z-axis and the 'twisted cubic curve' (see Example 1.29). △

Example 1.17. Consider the algebraic set $Z(I) \subset \mathbb{A}^2(k)$, defined by the ideal

$$I = (y - x^2, x^2 + (y - 1)^2 - 1).$$

Over the real numbers, we recognize the points of $Z(I)$ as the intersection points of the parabola $y = x^2$ and the circle of radius 1 with center at $(0, 1)$. To find these intersection points, one can either argue as in the previous example, or directly write the ideal as an

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

intersection:

$$\begin{aligned} I &= (y - x^2, x^2 + (x^2 - 1)^2 - 1) \\ &= (y - x^2, x^2(x - 1)(x + 1)) \\ &= (y - x^2, x^2) \cap (y - x^2, x - 1) \cap (y - x^2, x + 1) \\ &= (y, x^2) \cap (y - 1, x - 1) \cap (y - 1, x + 1). \end{aligned} \tag{1.6}$$

This shows that $Z(I)$ consists of the three points $(0, 0)$, $(1, 1)$, and $(-1, 1)$. Here the decomposition of ideals contains more refined information than the decomposition of the zero set, as the factor (y, x^2) carries extra algebraic information at the origin. We will revisit this example when we discuss *non-reduced schemes* in Section 4.7. △

1.4 Regular and rational functions

Each algebraic set has a *coordinate ring*, which describes the polynomial functions on it.

Definition 1.18. For an algebraic set $X \subset \mathbb{A}^n(k)$, its *coordinate ring* is the quotient

$$A(X) = k[x_1, \ldots, x_n]/I(X),$$

where $I(X)$ is the ideal of polynomials that vanish on X.

Note that any polynomial in $k[x_1, \ldots, x_n]$ defines a function $X \to k$ by restriction, and two polynomials f and g define the same function on X exactly when their difference $f - g$ vanishes on X, that is, $f - g \in I(X)$. Hence $A(X)$ can be regarded as the ring of polynomial functions on X.

Now assume that X is an affine variety, so that $I(X)$ is prime. Then the coordinate ring $A(X)$ is an integral domain, so it has a fraction field, denoted $K(X)$. The field $K(X)$ is called the *function field* (or *field of rational functions*) on X, and its elements are called *rational functions*.

A rational function is by definition an equivalence class of fractions a/b with $a, b \in A(X)$ and $b \neq 0$, where two fractions a/b and c/d represent the same element if $ad = bc$ in $A(X)$. If $A(X)$ is a UFD, a rational function f has a minimal representation as a fraction a/b with $a, b \in A(X)$, unique up to multiplication of both numerator and denominator by a unit. In general, there may be different expressions.

While rational functions are formally defined as fractions of polynomials, we may interpret them as functions on open sets. A rational function $f \in K(X)$ is said to be *regular* at a point $P \in X$ if it can be written as $f = a/b$ with $a, b \in A(X)$ and $b(P) \neq 0$. If this holds, we define the *value* $f(P)$ of f at P to be $a(P)/b(P) \in k$. One checks that this is independent of the choice of representatives a/b. If f is regular at every point of some open set U, then it defines a function $f\colon U \to k$. For instance, an element $a/b \in K(X)$ will define a function $D(b) \to k$ over the open set $D(b) = \{\, P \in X \mid b(P) \neq 0 \,\}$. It may however be regular on a larger open set, as illustrated below:

Example 1.19. Consider the affine variety $X \subset \mathbb{A}^4(k)$ defined by the equation $xw - yz = 0$. In the function field $K(X)$, we have $x/y = z/w$, and the corresponding rational function is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

therefore regular over the open set $D(y) \cup D(w)$, which is strictly larger than both $D(y)$ and $D(w)$ (for instance, it contains both $(0,1,0,0)$ and $(0,0,0,1)$). △

Rational functions are uniquely determined by their values over open sets: if two rational functions $f, g \in K(X)$ are regular on some open set V and $f(P) = g(P)$ for all $P \in V$, then $f = g$ in $K(X)$. To see this, write $f = a/b$ and $g = c/d$ in $K(X)$. Shrinking V if necessary, we may assume that $a(P)/b(P) = c(P)/d(P)$ for every $P \in V$. Then the polynomial function $ad - bc$ vanishes on a dense subset of X, and therefore on all of X. But then $ad - bc = 0$ in $A(X)$ as well, and so $a/b = c/d$ in $K(X)$.

If $P \in X$ is a point, the set of rational functions which are regular at P forms a subring of $K(X)$ called the *local ring* of X at P:

$$\mathcal{O}_{X,P} = \{\, f \in K(X) \mid f \text{ is regular at } P \,\}. \tag{1.7}$$

The ring $\mathcal{O}_{X,P}$ is indeed a local ring with maximal ideal

$$\mathfrak{m}_P = \{\, f \in \mathcal{O}_{X,P} \mid f(P) = 0 \,\}.$$

In fact, if $P = (a_1, \ldots, a_n) \in X$ and we write $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)$ for the maximal ideal in $A(X)$, then

$$\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P}.$$

To see this, note that since $A(X)$ is an integral domain, the localization $A(X)_{\mathfrak{m}_P}$ is naturally a subring of $K(X)$. Then an element $f \in K(X)$ lies in $A(X)_{\mathfrak{m}_P}$ precisely when f can be written as a/b with $b \notin \mathfrak{m}_P$, which is equivalent to f being regular at P.

If $U \subset X$ is an open set, we set

$$\begin{aligned} \mathcal{O}_X(U) &= \{\, f \in K(X) \mid f \text{ is regular at every point of } U \,\} \\ &= \bigcap_{P \in U} \mathcal{O}_{X,P}. \end{aligned} \tag{1.8}$$

Note that each $\mathcal{O}_X(U)$ is a sub-k-algebra of $K(X)$. Moreover, if $V \subset U$, then any $f \in K(X)$ which is regular on U is also regular on V, so there is an inclusion $\mathcal{O}_X(U) \subset \mathcal{O}_X(V)$.

Example 1.20. Let $U = \mathbb{A}^1(k) - \{0\}$ denote the complement of the origin in $\mathbb{A}^1$. The function field of $\mathbb{A}^1(k)$ is $k(x)$, and a rational function f is regular in U if and only if it is of the form $f = a(x)/x^n$ for some polynomial $a(x)$. Hence $\mathcal{O}_{\mathbb{A}^1(k)}(U) = k[x, x^{-1}]$. △

This example generalizes as follows:

Proposition 1.21. Let X be an affine variety.

(i) If a rational function $g \in K(X)$ is regular at every point of X, then g is a polynomial function. In other words,

$$\mathcal{O}_X(X) = A(X).$$

(ii) More generally, for $f \in A(X)$ and $D(f) = \{\, P \in X \mid f(P) \neq 0 \,\}$, then

$$\mathcal{O}_X(D(f)) = A(X)_f.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof (i) follows from (ii) by taking $f = 1$.

(ii): Clearly $A(X)_f \subset \mathcal{O}_X(D(f))$. Conversely, given $g \in K(X)$, define the ideal $\mathfrak{a}_g = \{\, b \in A(X) \mid bg \in A(X) \,\}$. This ideal has the property that g is regular at P if and only if $P \notin Z(\mathfrak{a}_g)$. To see this, note that $P \notin Z(\mathfrak{a}_g)$ precisely when some $b \in \mathfrak{a}_g$ does not vanish at P, which is equivalent to g being expressible as a/b with $b(P) \neq 0$. Now, if g is regular on all of $D(f)$, then $Z(\mathfrak{a}_g) \subset Z(f)$, and the Nullstellensatz implies that $f^n \in \mathfrak{a}_g$ for some $n > 0$. But then $f^n g \in A(X)$, which shows that g is an element of $A(X)_f$. □

The above proposition is a sort of 'local-to-global principle': being regular is a local condition, which has to be satisfied near every point, but the conclusion is that a rational function which is regular at every point can be represented globally by a polynomial function.

Example 1.22. Consider the complement of the origin in $\mathbb{A}^2(k)$, $U = \mathbb{A}^2(k) - \{(0,0)\}$. We claim that every rational function $f \in \mathcal{O}_{\mathbb{A}^2(k)}(U)$ is the restriction of a polynomial function, that is, $\mathcal{O}_{\mathbb{A}^2(k)}(U) = k[x,y]$. To see this, identify the function field of $\mathbb{A}^2(k)$ with $k(x,y)$, so that f can be expressed as a quotient of two polynomials without common factors $f = a/b$. As $k[x,y]$ is a unique factorization domain, this expression is unique. This means that f cannot be regular at the points in $Z(b)$.

If b is a constant, then f is a polynomial, so the conclusion follows. If b is not constant, then $Z(b)$ contains infinitely many points. Indeed, for all but finitely many $a \in k$, the polynomial $b(a,y) \in k[y]$ is nonzero, and since k is algebraically closed it has a root. In particular, f fails to be regular at points of $Z(b)$ other than the origin, contradicting $f \in \mathcal{O}_{\mathbb{A}^2(k)}(U)$. △

1.5 Morphisms of affine varieties

Let X and Y be two affine varieties. For a continuous map $f\colon X \to Y$ and a regular function $g \in \mathcal{O}_Y(V)$, the *pullback of g by f* is defined as the composition

$$f^\sharp(g) = g \circ f,$$

which is a function on the open set $f^{-1}(V)$.

Definition 1.23. A *morphism of affine varieties* is a continuous map $f\colon X \to Y$ which pulls regular functions back to regular functions, that is, $f^\sharp(g) \in \mathcal{O}_X(f^{-1}(V))$ for every open set $V \subset Y$ and every $g \in \mathcal{O}_Y(V)$.

If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, the composition $g \circ f\colon X \to Z$ is a morphism. Indeed, this map is continuous, and it pulls back regular functions because

$$(g \circ f)^\sharp(h) = f^\sharp(g^\sharp(h)).$$

A morphism of affine varieties $f\colon X \to Y$ is said to be an *isomorphism* if it has an inverse map that is also a morphism.

Given n regular functions $f_1, \ldots, f_n \in A(X)$, we can define a morphism

$$\begin{aligned} f\colon X &\longrightarrow \mathbb{A}^n(k) \\ P &\longmapsto (f_1(P), \ldots, f_n(P)). \end{aligned} \tag{1.9}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Let us check that f is continuous. For any closed set $W = Z(g_1, \dots, g_r) \subset \mathbb{A}^n(k)$, we have

$$\begin{aligned} f^{-1}(W) &= \{\, P \in X \mid g_i(f_1(P), \dots, f_n(P)) = 0 \text{ for all } i = 1, \dots, r \,\} \\ &= Z(f^\sharp(g_1), \dots, f^\sharp(g_r)). \end{aligned}$$

Therefore $f^{-1}(W)$ is closed in X, so f is continuous.

Next, f pulls back regular functions to regular functions. Indeed, let $g \in k(y_1, \dots, y_n)$ be a rational function on $\mathbb{A}^n(k)$ and assume g is regular on an open set $V \subset \mathbb{A}^n(k)$. Let $P \in f^{-1}(V)$ and $Q = f(P)$. Locally around Q, we may write $g = a/b$ where a, b are polynomials in $y_1, \dots, y_n$ and $b(Q) \neq 0$. Then in a neighborhood of P, we have

$$f^\sharp(g)(x) = \frac{a(f_1(x), \dots, f_n(x))}{b(f_1(x), \dots, f_n(x))}.$$

After expanding, this can be written as a quotient of polynomials with denominator nonvanishing at P. Hence $f^\sharp(g) \in \mathcal{O}_X(f^{-1}V)$ and f is a morphism.

Proposition 1.24. Let X be an affine variety. Then every morphism $f\colon X \to \mathbb{A}^n(k)$ is of the form

$$f\colon X \longrightarrow \mathbb{A}^n(k), \quad f(P) = (f_1(P), \dots, f_n(P)) \tag{1.10}$$

for some $f_1, \dots, f_n \in A(X)$.

Proof Let $f\colon X \to \mathbb{A}^n(k)$ be a morphism. The coordinate functions $y_1, \dots, y_n$ are regular functions on $\mathbb{A}^n(k)$, so their pullbacks $f^\sharp(y_i)$ are regular on X. The map defined by the $f^\sharp(y_i)$ coincides with f, as both are defined by the same coordinate functions. □

Theorem 1.25 (The main theorem of affine varieties). Let X and Y be affine varieties. Then there is a one-to-one correspondence between morphisms $f\colon X \to Y$ and maps of k-algebras $\phi\colon A(Y) \to A(X)$.

Proof The correspondence is defined as follows: if $f\colon X \to Y$ is a morphism, then the pullback $f^\sharp\colon A(Y) \to A(X)$ is a map of k-algebras.

To define the inverse, let $\phi\colon A(Y) \to A(X)$ be a map of k-algebras. Suppose that $Y \subset \mathbb{A}^n(k)$ and let $y_1, \dots, y_n$ be the coordinate functions on $\mathbb{A}^n(k)$. Now set $f_i = \phi(y_i)$ for $i = 1, \dots, n$. The $f_1, \dots, f_n$ define a morphism

$$f\colon X \longrightarrow \mathbb{A}^n(k).$$

For any polynomial $h \in k[y_1, \dots, y_n]$, we have

$$f^\sharp(h)(x) = h(f(x)) = h(f_1(x), \dots, f_n(x)) = \phi(h(y_1, \dots, y_n))(x) \tag{1.11}$$

where the last equality holds because both sides are maps of k-algebras in h and take the same values $f_1, \dots, f_n$ on the generators $y_1, \dots, y_n$. This shows that $f^\sharp(h) = \phi(h) = 0$ for every $h \in I(Y)$, as h is zero in $A(Y)$. Therefore, the image of f is contained in $Y = Z(I(Y))$.

Finally, (1.11) shows that $f^\sharp = \phi$, so the correspondence is one-to-one. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

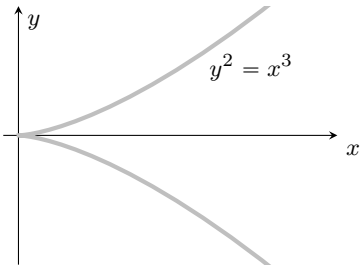


Figure 1.2 The cuspidal cubic curve

Corollary 1.26. Two affine varieties X and Y are isomorphic if and only if the k-algebras $A(X)$ and $A(Y)$ are isomorphic.

If $X \subset \mathbb{A}^m(k)$ and $Y \subset \mathbb{A}^n(k)$ are affine varieties, the theorem implies that any morphism $f\colon X \to Y$ is the restriction of a morphism of the form

$$\mathbb{A}^m(k) \longrightarrow \mathbb{A}^n(k), \quad P \mapsto (h_1(P), \dots, h_n(P)) \tag{1.12}$$

where $h_1, \dots, h_n \in k[x_1, \dots, x_m]$ are polynomials. We could have taken this as the definition of morphisms of affine varieties. However, as we will see, the above definition generalizes to more general contexts.

Examples

Example 1.27. Any linear map $f\colon \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ is a morphism. The components h_i of f are linear polynomials $h_i(x) = \sum_{j=1}^n a_{ij}x_j$. △

Example 1.28 (The cuspidal cubic curve)**.** Consider the algebraic set $X = Z(y^2 - x^3)$ in $\mathbb{A}^2(k)$. The affine coordinate ring of X is

$$A(X) = k[x, y]/(y^2 - x^3).$$

This is an integral domain because the polynomial $y^2 - x^3$ is irreducible. Consider the morphism $f\colon \mathbb{A}^1(k) \to \mathbb{A}^2(k)$ defined by $t \mapsto (t^2, t^3)$. The image of f is contained in $X \subset \mathbb{A}^2(k)$, and, in fact, f is a bijection between $\mathbb{A}^1(k)$ and X. Indeed, observe that $f(t) = (0, 0)$ only for $t = 0$. Moreover, for points in $X - \{(0, 0)\}$, the assignment $t = y/x$ defines the inverse. However, f is not an isomorphism. As $f^\sharp(x) = t^2$ and $f^\sharp(y) = t^3$, the induced ring map

$$f^\sharp\colon k[x, y]/(y^2 - x^3) \longrightarrow k[t]$$

has image $k[t^2, t^3]$ and so is not surjective.

This means that in order to check that a morphism is an isomorphism, it is not enough to check that it is injective and surjective; also the ring map $f^\sharp$ has to be an isomorphism. △

Example 1.29 (The Twisted Cubic)**.** Consider the morphism

$$f\colon \mathbb{A}^1(k) \longrightarrow \mathbb{A}^3(k), \qquad t \longmapsto (t, t^2, t^3).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

If x, y, z are coordinates on $\mathbb{A}^3(k)$, the image of f is the irreducible algebraic set $X = Z(y - x^2, z - x^3)$. X is called the *twisted cubic curve*. The word 'twisted' refers to the fact that X is not contained in a linear plane in $\mathbb{A}^3(k)$.

The corresponding map of k-algebras is defined by

$$\phi\colon k[x, y, z] \longrightarrow k[t]$$

mapping $x \mapsto t$, $y \mapsto t^2$, and $z \mapsto t^3$. Let us show that the kernel of ϕ is $\mathfrak{a} = (y - x^2, z - x^3)$. Clearly, $\mathfrak{a} \subset \operatorname{Ker} \phi$. Conversely, if $g \in k[x, y, z]$ is any polynomial, then using $y = x^2$ and $z = x^3$ to remove powers of y and z, we can write

$$g(x, y, z) = g(x, x^2, x^3) \mod \mathfrak{a}.$$

If $g \in \operatorname{Ker} \phi$, then $g(x, x^2, x^3)$ is the zero polynomial in x and so $g \in \mathfrak{a}$.

As ϕ is clearly surjective, we see that

$$A(X) = k[x, y, z]/\mathfrak{a} \simeq k[t],$$

and f induces an isomorphism $X \simeq \mathbb{A}^1(k)$. △

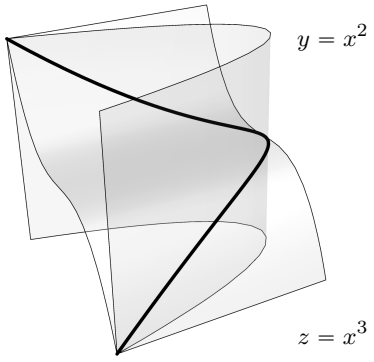


Figure 1.3 The twisted cubic curve

Example 1.30. Let $X = Z(xy - 1) \subset \mathbb{A}^2(k)$ and consider the first projection

$$f\colon X \longrightarrow \mathbb{A}^1(k), \qquad (x, y) \mapsto x.$$

The map f is a bijection onto $U = \mathbb{A}^1(k) - \{0\}$. The inverse is given by $g(x) = (x, x^{-1})$ (note that x^{-1} is indeed a regular function on $\mathbb{A}^1(k) - \{0\}$). On the algebraic side, this reflects the fact that $\mathcal{O}_{\mathbb{A}^1(k)}(U) = k[x, x^{-1}] \simeq k[x, y]/(xy - 1)$. △

Example 1.31. Consider the morphism $f\colon \mathbb{A}^2(k) \to \mathbb{A}^2(k)$ given by $(x, y) \mapsto (x, xy)$. Then the image of f is equal to $D(x) \cup \{(0, 0)\}$. This shows that the image of an algebraic set need not be either open or closed. △

Example 1.32 (Products)**.** If $X \subset \mathbb{A}^m(k)$ and $Y \subset \mathbb{A}^n(k)$ are algebraic sets, we can define the product $X \times Y \subset \mathbb{A}^{m+n}(k)$ as the subset

$$X \times Y = \{\, (a_1, \ldots, a_m, b_1, \ldots, b_n) \mid (a_1, \ldots, a_m) \in X, (b_1, \ldots, b_n) \in Y \,\}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This is indeed an algebraic set: if $X = Z(f_1, \dots, f_r)$ and $Y = Z(g_1, \dots, g_s)$, then $X \times Y = Z(f_1, \dots, f_r, g_1, \dots, g_s)$. Moreover, the projection maps $\pi_1 \colon \mathbb{A}^{m+n}(k) \to \mathbb{A}^m(k)$ and $\pi_2 \colon \mathbb{A}^{m+n}(k) \to \mathbb{A}^n(k)$ define morphisms $X \times Y \to X$ and $X \times Y \to Y$.

It is worth emphasizing that the topology on $X \times Y$ is generally not the product topology. For example, $\mathbb{A}^1(k) \times \mathbb{A}^1(k) \simeq \mathbb{A}^2(k)$ as algebraic sets, but $\mathbb{A}^2(k)$ has closed subsets that are not unions of products $A \times B$ with $A, B \subset \mathbb{A}^1(k)$ closed. The parabola $Z(y - x^2) \subset \mathbb{A}^2(k)$ is one such example. △

1.6 Exercises

Exercise 1.6.1. Show that the algebraic set $Z(y^2 - x^3 - 1) \subset \mathbb{A}^2(k)$ is irreducible.

Exercise 1.6.2. Let $f \colon \mathbb{A}^1(k) \to \mathbb{A}^1(k)$ be an isomorphism. Show that f is given by a linear polynomial $ax + b$, where $a, b \in k$ and $a \neq 0$.

Exercise 1.6.3. Let $X = Z(y^2 - xz, z - x^2 - 1) \subset \mathbb{A}^3(k)$. Show that X is isomorphic to the plane curve $Z(v^2 - u^3 - u) \subset \mathbb{A}^2(k)$.

Exercise 1.6.4. Find the irreducible compoents of the algebraic sets below.

a) $Z(x^2y, y^2x) \subset \mathbb{A}^2(k)$.
b) $Z(x^2 - y^2) \subset \mathbb{A}^2(k)$.
c) $Z(x - yz, xz - y^2) \subset \mathbb{A}^3(k)$.
d) $Z(x^2 + y^2, x^2 - z^2 - 1) \subset \mathbb{A}^3(k)$.
e) $Z(x^2 - y, xz - y^2, x^3 - xz) \subset \mathbb{A}^3(k)$.

Exercise 1.6.5. Consider the algebraic set $X = Z(x^2 - y^3, x^2 + y^2 - 1)$ in $\mathbb{A}^2(k)$.

a) Show that X consists of 6 points.
b) If $\pi \colon \mathbb{A}^2(k) \to \mathbb{A}^1(k)$ denotes the projection onto the x-axis, find the polynomial describing the closed set $Y = \pi(X)$ in $\mathbb{A}^1(k)$.

Exercise 1.6.6 (The diagonal)**.** Let X be an affine variety and consider the map

$$\Delta \colon X \longrightarrow X \times X, \qquad P \longmapsto (P, P).$$

a) Show that Δ is a morphism of affine varieties.
b) Let $X = \mathbb{A}^n(k)$, and let $x_1, \dots, x_n, y_1, \dots, y_n$ be coordinates on $\mathbb{A}^n(k) \times \mathbb{A}^n(k)$. Show that the ideal defining the image of Δ is given by
$$\mathfrak{a} = (x_1 - y_1, \dots, x_n - y_n)$$
c) In general, for an affine variety X, show that the image of Δ is closed in $X \times X$, and that Δ gives an isomorphism $X \to \Delta(X)$. HINT: Use the two projections $X \times X \to X$.

Exercise 1.6.7. Let $X \subset \mathbb{A}^4(k)$ be the algebraic set defined by the equation $xw - yz = 0$.

a) Show that $xw - yz$ is an irreducible polynomial.
b) Show that the rational function x/y is not regular in any open set containing the locus where $y = w = 0$.
c) Show that there is no single distinguished open set so that $D(b)$ contains all the points where x/y is regular.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

d) Compute $\mathcal{O}_X(U)$ for the open set $U = D(x) \cup D(y)$.

Exercise 1.6.8. Let $X \subset \mathbb{A}^3(k)$ be the closed subset defined by the equation $x^2+y^2+z^2 = 0$. Compute $I(X)$ when k has characteristic 2 and when k has characteristic not equal to 2.

Exercise 1.6.9. Let $x_0, \ldots, x_n$ be coordinates on $\mathbb{A}^{n+1}(k)$ and let $f \in k[x_1, \ldots, x_n]$.

a) Show that the algebraic set $X = Z(x_0 - f)$ is isomorphic to $\mathbb{A}^n(k)$.
b) For which f's is the algebraic set $X = Z(x_0^2 - f)$ irreducible?
c) Find a bijection between the open set $\mathbb{A}^n(k) - Z(f)$ in $\mathbb{A}^n(k)$ and the algebraic set $Z(x_0 f - 1)$ in $\mathbb{A}^{n+1}(k)$.

Exercise 1.6.10. Let A be a ring and let $\mathfrak{a}$ be an ideal

a) Show that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$, where the intersection is taken over all the prime ideals containing $\mathfrak{a}$. HINT: If $f \notin \sqrt{\mathfrak{a}}$ the ideal $\mathfrak{a}A_f$ is a proper ideal in the localization A_f, hence contained in a maximal ideal.
b) Using the Nullstellensatz, show that if $A = k[x_1, \ldots, x_n]$ over an algebraically closed field k, we have $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \mathfrak{m}$ where the intersection is over all the *maximal* ideals containing $\mathfrak{a}$.

Exercise 1.6.11. Let $X = Z(f)$ and $Y = Z(g)$ be two algebraic sets in $\mathbb{A}^2(k)$ with X irreducible. Show that either $X \cap Y$ is a finite set, or $X \subset Y$.

Exercise 1.6.12. Show that the image of the map

$$\phi\colon \mathbb{A}^1(k) \longrightarrow \mathbb{A}^3(k), \qquad t \mapsto (t^2, t^3, t^6)$$

is given by $V(x^3 - y^2, z - x^3)$. Show that ϕ is bijective. Is ϕ an isomorphism of affine varieties?

Exercise 1.6.13. Consider the ring map $\phi\colon k[x, y, z, w] \to k[s, t]$ induced by $(x, y, z, w) \mapsto (s^3, s^2t, st^2, t^3)$. Show that $\operatorname{Ker} \phi = (y^2 - xz, xw - yz, z^2 - yw)$. HINT: Introduce a grading on $k[x, y, z, w]$ by $\deg(x, y, z, w) = (1, 2, 3, 4)$. For $f \in \operatorname{Ker} \phi$, subtract terms from the ideal $(y^2 - xz, xw - yz, z^2 - yw)$ to reduce the total degree.

Exercise 1.6.14. Let $X = Z(y^2 - x^3 - x) \subset \mathbb{A}^2(k)$. Show that y/x and $(x^2 + 1)/y$ define the same rational function f on X. Show that f is regular on $U = X - \{(0, 0)\}$. Show that there are no polynomials $a, b \in k[x, y]$ so that f can be represented as a/b on all of U.

Exercise 1.6.15. Let X be a topological space.

a) Show a subset $Z \subset X$ is irreducible if and only if its closure $\overline{Z}$ is irreducible;
b) Show that X is irreducible if and only if every non-empty open subset is dense.
c) Show that X is irreducible if and only if every pair of nonempty open sets $U, V \subset X$ intersect.
d) Show that every nonempty open subset of an irreducible topological space is irreducible.
e) If $f\colon X \to Y$ is a continuous map, show that $f(X)$ is irreducible if X is.

Exercise 1.6.16. Show that any irreducible conic in $\mathbb{A}^2(k)$ is isomorphic to either $Z(y - x^2)$ or $Z(xy - 1)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 1.6.17 (Varieties defined by matrices)**.** Identify the space of complex $n \times n$ matrices with affine n^2-space $\mathbb{A}^{n^2}(\mathbb{C})$.

a) Show that the set of symmetric matrices, i.e. matrices such that $A^T = A$, is an algebraic set in $\mathbb{A}^{n^2}(\mathbb{C})$.
b) Show that the set $\mathrm{GL}_n(\mathbb{C})$ of invertible matrices is an open set in $\mathbb{A}^{n^2}(\mathbb{C})$.
c) Show that the set $\mathrm{SL}_n(\mathbb{C})$ of matrices with determinant 1 is an algebraic set in $\mathbb{A}^{n^2}(\mathbb{C})$. Is it irreducible?
d) Let $D \subset \mathbb{A}^{n^2}(\mathbb{C})$ denote the subset of diagonalizable matrices with n distinct eigenvalues. Show that D is an open set.
e) For a matrix A, let $P_A(t) = \det(tI - A)$ denote its characteristic polynomial. Let $W \subset \mathbb{A}^{n^2}(\mathbb{C})$ denote the set of matrices A such that $P_A(A) = 0$. Show that W is closed. Using *d)*, deduce the *Cayley–Hamilton theorem*, that $P_A(A) = 0$ for every matrix A.

Exercise 1.6.18. Identify $\mathbb{A}^{nm}(k)$ with the space of $m \times n$-matrices over the field k. For a given integer r, show that the set of matrices of rank less than r is an algebraic set. Is it irreducible?

Exercise 1.6.19 (Nilpotent matrices)**.** As in Exercise 1.6.17 identify the space of 2×2-matrices with $\mathbb{A}^4(k)$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (a, b, c, d)$. Show that the two ideals in $k[a, b, c, d]$

$$\mathfrak{a} = (a^2 + bc, ab + bd, ac + cd, bc + d^2), \qquad \mathfrak{b} = (ad - bc, a + d)$$

satisfy $\mathfrak{a} \neq \mathfrak{b}$ but $\sqrt{\mathfrak{a}} = \mathfrak{b}$. Furthermore, show that $Z(\mathfrak{a}) = Z(\mathfrak{b}) \subset \mathbb{A}^4(k)$ consists of the nilpotent matrices.

Exercise 1.6.20. Show that if X is a Hausdorff space, then the only irreducible subsets are the one-point sets $\{x\}$. Deduce that if X is an irreducible space with at least 2 points, then X is not Hausdorff.

Exercise 1.6.21. Show that a topological space is Noetherian if and only if every non-empty collection of closed subsets has a minimal element. Equivalently, every collection of open subsets has a maximal element.

Exercise 1.6.22 (The Frobenius morphism)**.** Let k be an algebraically closed field of positive characteristic p.

a) Show that the map $\phi\colon k[t] \to k[t]$ defined by $t \mapsto t^p$, is a map of k-algebras.
b) Describe the corresponding morphism $F\colon \mathbb{A}^1(k) \to \mathbb{A}^1(k)$. Show that F is bijective, but not an isomorphism.

Exercise 1.6.23. Let $X \subset \mathbb{A}^3(k)$ denote the twisted cubic curve of Example 1.29 and let $p\colon \mathbb{A}^3(k) \to \mathbb{A}^2(k)$ denote the projection map $(x, y, z) \mapsto (x, y)$. Show that $p(X)$ is closed in $\mathbb{A}^2(k)$ and compute the defining ideal in $k[x, y]$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

2

The spectrum of a ring

In the previous chapter we saw that the geometry of an algebraic set $X \subset \mathbb{A}^n(k)$ is reflected in the algebraic properties of its coordinate ring

$$A(X) = k[x_1, \ldots, x_n]/I(X).$$

By Hilbert's Nullstellensatz, the points $P = (a_1, \ldots, a_n)$ of X are in one-to-one correspondence with the maximal ideals $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)$ of $A(X)$. A point $P \in X$ lies in the closed set $Z(\mathfrak{a})$ if and only if $\mathfrak{a} \subset \mathfrak{m}_P$, so we may write $Z(\mathfrak{a}) = \{\, P \mid \mathfrak{m}_P \supset \mathfrak{a} \,\}$. In this way, the Zariski topology on X can be recovered entirely from the ideal structure of $A(X)$.

From an algebraic point of view, the ring $A(X)$ has rather nice properties: it is a finitely generated k-algebra and it is reduced, since $I(X)$ is a radical ideal. Conversely, every reduced finitely generated k-algebra arises as the coordinate ring of an algebraic set. Indeed, if $A = k[x_1, \ldots, x_n]/\mathfrak{a}$, then the algebraic set $X = Z(\mathfrak{a}) \subset \mathbb{A}^n(k)$ has coordinate ring A. Moreover, X is irreducible if and only if $A(X)$ is an integral domain.

This correspondence can be formulated categorically as follows. If AffVar/k denotes the category of affine varieties over k, then the assignment $X \mapsto A(X)$ defines a contravariant functor from AffVar/k to the category of k-algebras Alg/k. By the Main Theorem of Affine Varieties, for affine varieties X and Y there is a natural bijection

$$\mathrm{Hom}_{\mathsf{AffVar}/k}(X, Y) = \mathrm{Hom}_{\mathsf{Alg}/k}(A(Y), A(X)).$$

This means that the category of affine varieties over k is anti-equivalent to the category of finitely generated k-algebras that are integral domains.

Our goal is to generalize this picture to arbitrary rings. To do so, we must associate a geometric space to an arbitrary ring A. This space is called the *spectrum* of A, denoted by $\mathrm{Spec}\, A$. By definition $\mathrm{Spec}\, A$ is the set of all prime ideals, equipped with a Zariski topology similar to the one defined for affine varieties.

The idea of replacing maximal ideals by prime ideals lies at the heart of scheme theory. From a functorial perspective, this is very natural, because inverse images of prime ideals under ring maps are again prime, and hence every ring map $A \to B$ induces a map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$. If X and Y are affine varieties, the induced map $A(Y) \to A(X)$ does in fact pull maximal ideals back to maximal ideals, but this is generally not true for arbitrary ring maps (consider for instance the inclusion $\mathbb{Z} \to \mathbb{Q}$).

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

2.1 The spectrum of a ring

Let A be a ring. As always, A is commutative with 1.

Definition 2.1. The *spectrum* of A, denoted $\operatorname{Spec} A$, is the set of all prime ideals in A:

$$\operatorname{Spec} A = \{\, \mathfrak{p} \mid \mathfrak{p} \subset A \text{ is a prime ideal} \,\}.$$

The spectrum has a topology similar to the Zariski topology defined in Chapter 1. The closed sets are those of the form

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \mathfrak{a} \,\}, \tag{2.1}$$

where $\mathfrak{a} \subset A$ is an ideal. Note that $\mathfrak{a} \subset \mathfrak{b}$ implies $V(\mathfrak{a}) \supset V(\mathfrak{b})$.

To draw a closer analogy to the zero sets $Z(\mathfrak{a})$ from Chapter 1, we can think of the sets $V(\mathfrak{a})$ as the 'vanishing sets' of sets of functions on $\operatorname{Spec} A$. To explain this, we introduce the concept of a residue field.

Definition 2.2. For a point $\mathfrak{p} \in \operatorname{Spec} A$, we define the *residue field* $\kappa(\mathfrak{p})$ to be the field of fractions of the integral domain $A/\mathfrak{p}$.

The residue field $\kappa(\mathfrak{p})$ is canonically isomorphic to $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, where $A_{\mathfrak{p}}$ is the localization of A at $\mathfrak{p}$, which is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Given $f \in A$, we let $f(\mathfrak{p}) \in \kappa(\mathfrak{p})$ denote the image of f in the residue field $\kappa(\mathfrak{p}) = K(A/\mathfrak{p})$. In this way, we can view f as a function on $\operatorname{Spec} A$. Unlike the case of the coordinate ring of an affine variety, the values of f do not lie in a single fixed field k, but rather in all the residue fields $\kappa(\mathfrak{p})$, which vary from point to point.

With this notation, it makes sense to talk about the *vanishing set* of f, that is,

$$V(f) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \text{ in } \kappa(\mathfrak{p}) \,\}.$$

Note that $f(\mathfrak{p}) = 0$ if and only if $f \in \mathfrak{p}$, so this set is exactly the vanishing set associated with the ideal $\mathfrak{a} = (f)$, as defined in (2.1). More generally, if $\mathfrak{a} \subset A$ is an ideal, the set (2.1) can be written as

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \text{ for all } f \in \mathfrak{a} \,\}. \tag{2.2}$$

The advantage of expressing $V(\mathfrak{a})$ in this way is that all the identities for the $Z(\mathfrak{a})$ continue to hold for $V(\mathfrak{a})$, and the proofs are essentially the same. For instance, the analogue of Proposition 1.2 takes the following form. It shows that the vanishing sets form the closed sets of a topology on $\operatorname{Spec} A$, which we call the *Zariski topology*.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 2.3. The vanishing sets in $\operatorname{Spec} A$ satisfy the following properties.

(i) (Arbitrary intersections) For any collection of ideals $\{\mathfrak{a}_i\}_{i\in I}$ in A,

$$\bigcap_{i\in I} V(\mathfrak{a}_i) = V\Big(\sum_{i\in I} \mathfrak{a}_i\Big). \tag{2.3}$$

(ii) (Finite unions) For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A,

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b}). \tag{2.4}$$

Additionally, $V(A) = \varnothing$ and $V(0) = \operatorname{Spec} A$.

Proof The proof of Proposition 1.2 carries over word for word. □

Example 2.4 (Fields)**.** If K is a field, the prime spectrum $\operatorname{Spec} K$ has only one point, which corresponds to the only prime ideal in K, the zero ideal. △

Example 2.5 ($\operatorname{Spec}\mathbb{Z}$)**.** In the ring of integers $\mathbb{Z}$, there are two types of prime ideals: the zero ideal (0) and the maximal ideals (p), one for each prime number p.

As $\mathbb{Z}$ is a principal ideal domain, any ideal is of the form (n) for some integer n. It follows that for $n \neq 0$ the closed subsets are of the form $V(n) = \{(p_1), \dots, (p_r)\}$ where the p_i are the prime factors of n, whereas $V(0) = \operatorname{Spec}\mathbb{Z}$. This means that the closed sets are either finite sets of closed points or the whole space. Dually, the non-empty open sets are the complements of finite sets of closed points. This means that $\operatorname{Spec}\mathbb{Z}$ is not Hausdorff, as every nonempty open set contains (0).

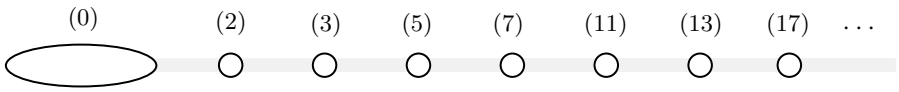


Figure 2.1 $\operatorname{Spec}\mathbb{Z}$

The residue field at a closed point (p) is equal to $\kappa((p)) = \mathbb{Z}/(p) = \mathbb{F}_p$, whereas the residue field at (0) is equal to $\mathbb{Q}$. Each element $f \in \mathbb{Z}$ gives rise to a function on $\operatorname{Spec}\mathbb{Z}$ with values in the various residue fields. For instance, $f = 17$ takes the values $f((0)) = 17$, $f((2)) = \bar{1}$, $f((3)) = \bar{2}$, $f((5)) = \bar{2}$, $f((7)) = \bar{3}, \dots$, in the fields $\mathbb{Q}, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \dots$, respectively, where the bar indicates the class modulo the relevant prime. △

Example 2.6 ($\operatorname{Spec} k[t]$)**.** Consider the polynomial ring $k[t]$ in one variable. Since $k[t]$ is a principal ideal domain, every ideal is of the form $(g(t))$ for some polynomial $g(t)$. Assuming that k is algebraically closed and that $g(t)$ is nonconstant, we can factor $g(t)$ as a product of terms of the form $t - a_i$, so that

$$V(g(t)) = \{(t - a_1), \dots, (t - a_n)\}.$$

Therefore, the closed sets of $\operatorname{Spec} k[t]$ are either the empty set, the whole space, or a finite set of closed points. As in the previous example, this space is not Hausdorff.

At the prime $\mathfrak{p} = (t - a)$, the residue field $\kappa(\mathfrak{p})$ is equal to $k[t]/(t - a) \simeq k$. Under this isomorphism, a polynomial $f \in k[t]$ maps to $f(a) \in k$. Hence the 'value' $f(\mathfrak{p}) \in \kappa(\mathfrak{p})$ agrees with the usual evaluation of the polynomial at $t = a$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The next lemma is about inclusions between the vanishing sets of $\operatorname{Spec} A$.

Lemma 2.7. For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A, the following hold:
(i) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$
(ii) $V(\mathfrak{a}) \subset V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$
(iii) $V(\mathfrak{a}) = \varnothing$ if and only if $\mathfrak{a} = A$
(iv) $V(\mathfrak{a}) = \operatorname{Spec} A$ if and only if $\mathfrak{a} \subset \sqrt{(0)}$.

Proof (i): For a prime ideal $\mathfrak{p}$, the condition $\mathfrak{p} \supset \mathfrak{a}$ is equivalent to $\mathfrak{p} \supset \sqrt{\mathfrak{a}}$, since prime ideals are radical. Hence $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(ii): We recall the following identity for the radical of an ideal (see page 473):

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}. \tag{2.5}$$

If $V(\mathfrak{a}) \subset V(\mathfrak{b})$, then the intersection defining $\sqrt{\mathfrak{b}}$ is taken over a larger set than that defining $\sqrt{\mathfrak{a}}$, giving $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$. Conversely, if $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$, then by (i),

$$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}) \subset V(\sqrt{\mathfrak{b}}) = V(\mathfrak{b}).$$

(iii): Clearly, $V(A) = \varnothing$. Conversely, if $\mathfrak{a} \neq A$, then it is contained in a maximal ideal so $V(\mathfrak{a})$ is nonempty.

(iv): By (ii), we have $V(\mathfrak{a}) = \operatorname{Spec} A = V(0)$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{(0)}$, which is equivalent to $\mathfrak{a} \subset \sqrt{(0)}$. □

As in the definition (1.3), we define the ideal of a subset $S \subset \operatorname{Spec} A$ by

$$I(S) = \{\, f \in A \mid f(\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in S \,\} = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}. \tag{2.6}$$

Note that $I(S)$ is a radical ideal.

Proposition 2.8. For any subset $S \subset \operatorname{Spec} A$, we have

$$V(I(S)) = \overline{S}. \tag{2.7}$$

In particular, the closure $\overline{\{\mathfrak{p}\}}$ of the one-point set $\{\mathfrak{p}\}$ is equal to $V(\mathfrak{p})$.

Proof First of all, $S \subset V(I(S))$, because if $\mathfrak{p} \in S$ then clearly $\mathfrak{p} \supset \bigcap_{\mathfrak{q} \in S} \mathfrak{q} = I(S)$ and so $\mathfrak{p} \in V(I(S))$. It remains to show that $V(I(S))$ is the smallest closed set containing S. If $V(\mathfrak{b})$ is any closed subset containing S, then for any $\mathfrak{p} \in S$, we have $\mathfrak{p} \in V(\mathfrak{b})$ and hence $\mathfrak{p} \supset \mathfrak{b}$. Taking the intersection over all $\mathfrak{p}$'s in S, we see that $I(S) \supset \mathfrak{b}$. Applying $V(-)$, we obtain $V(I(S)) \subset V(\mathfrak{b})$, as required. □

As a consequence, we get the following analogue of the Nullstellensatz:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Corollary 2.9. Let A be a ring. Then the map $\mathfrak{a} \mapsto V(\mathfrak{a})$ defines a bijection

$$\big\{\text{radical ideals } \mathfrak{a} \subset A\big\} \longrightarrow \{\text{closed sets } W \subset \operatorname{Spec} A\}, \tag{2.8}$$

with inverse defined by $W \mapsto I(W)$.

Proof If $\mathfrak{a}$ is any ideal, then $I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \sqrt{\mathfrak{a}}$ by the formula (2.5). Hence $I(V(-))$ is the identity map on radical ideals $\mathfrak{a}$. Conversely, if W is a closed set, then $V(I(W)) = \overline{W} = W$ by Proposition 2.8, so also $V(I(-))$ is the identity. □

Corollary 2.10. A point $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if $\mathfrak{p}$ is a maximal ideal.

Proof $V(\mathfrak{p}) = \{\mathfrak{p}\}$ says that $\mathfrak{p}$ is the only prime ideal which contains $\mathfrak{p}$, which precisely means that $\mathfrak{p}$ is maximal. □

Recall from Chapter 1 that a nonempty topological space X is *irreducible* if it cannot be written as the union of two proper closed subsets. The next result says that under the bijection (2.8), the irreducible closed subsets of $\operatorname{Spec} A$ correspond exactly to the prime ideals of A.

Proposition 2.11. Let A be a ring.
(i) A closed subset $V(\mathfrak{a})$ is irreducible if and only if $\sqrt{\mathfrak{a}}$ is prime.
(ii) The irreducible components of $V(\mathfrak{a})$ are precisely the $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal containing $\mathfrak{a}$.

Proof (i): Suppose that $V(\mathfrak{a})$ is irreducible, and let $fg \in \sqrt{\mathfrak{a}}$. Then $V(\mathfrak{a}) \subset V(fg) = V(f) \cup V(g)$. Since $V(\mathfrak{a})$ is irreducible, it must be contained in $V(f)$ or in $V(g)$. Hence $f \in \sqrt{\mathfrak{a}}$ or $g \in \sqrt{\mathfrak{a}}$, so $\sqrt{\mathfrak{a}}$ is prime. Conversely, suppose $\mathfrak{p} = \sqrt{\mathfrak{a}}$ is prime. If $V(\mathfrak{p}) \subset V(I) \cup V(J) = V(I \cap J)$ for ideals $I, J \subset A$, then $I \cap J \subset \mathfrak{p}$, so since $\mathfrak{p}$ is prime, either $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$. Therefore $V(\mathfrak{p}) \subset V(I)$ or $V(\mathfrak{p}) \subset V(J)$, and hence $V(\mathfrak{p})$ is irreducible.

(ii): By definition, the irreducible components of $V(\mathfrak{a})$ are the maximal irreducible closed subsets $Z \subset V(\mathfrak{a})$. By (i), we may write $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$, and we see that $\mathfrak{p}$ is minimal among those containing $\mathfrak{a}$. □

Corollary 2.12. $\operatorname{Spec} A$ is irreducible if and only if A has a unique minimal prime ideal, or equivalently, $\sqrt{(0)}$ is prime.

In particular, $\operatorname{Spec} A$ is irreducible when A is an integral domain. However, $\operatorname{Spec} A$ may be irreducible for other rings as well. For example, the ring $A = \mathbb{C}[t]/(t^2)$ is not an integral domain, and yet $\operatorname{Spec} A$ consists of a single point, corresponding to the prime ideal (t). The fact that such examples can appear when we allow nilpotent elements is expected. Keeping the analogy between ring elements and regular functions, there are nonzero functions which vanish at every point: the element t is nonzero in A, but it becomes zero in the residue field $\mathbb{C}[t]/(t) \simeq \mathbb{C}$.

Example 2.13. The spectrum $\operatorname{Spec} \mathbb{C}[x, y]/(y^2 + x^5 + 1)$ is irreducible, as the polynomial $y^2 + x^5 + 1$ is irreducible, and hence generates a prime ideal. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 2.14. The spectrum $\operatorname{Spec}\mathbb{C}[x,y,z,w]/(x^2,xy,y^2,xw-yz)$ is irreducible. Indeed, the nilradical is given by $\sqrt{(0)}=(x,y)$, and this is a prime ideal. △

Generic points and specialization

Corollary 2.10 shows that the Zariski topology on $\operatorname{Spec} A$ behaves differently from not only the Euclidean topology on manifolds, but also the usual Zariski topology on affine varieties. Indeed, a general ring A typically contains many prime ideals which are not maximal, so $\operatorname{Spec} A$ contains many non-closed points.

To understand these points, we introduce the following terminology. For two points $x, y \in X$ in a topological space X, we say that y is a *specialization* of x (and that x is a *generalization* of y) if $y \in \overline{\{x\}}$. In the case of $\operatorname{Spec} A$, Lemma 2.7 says that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, so $\mathfrak{q}$ is a specialization of $\mathfrak{p}$ precisely when $\mathfrak{p} \subset \mathfrak{q}$. So going to a specialization means passing to a larger prime ideal. In this way, closed points are the most specialized points, whereas non-closed points are more general.

Definition 2.15 (Generic points)**.** A point x in a closed subset Z of a topological space X is called a *generic point* for Z if $\overline{\{x\}} = Z$.

In $\operatorname{Spec} A$, every irreducible closed subset has the form $V(\mathfrak{p})$ and there is a unique generic point, namely $\mathfrak{p}$. Indeed, if $\mathfrak{p}, \mathfrak{q}$ are primes satisfying $V(\mathfrak{q}) = V(\mathfrak{p})$, then Lemma 2.7 implies both $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$, so $\mathfrak{p} = \mathfrak{q}$.

Example 2.16. If A is an integral domain, the zero ideal (0) is prime, and as $V(0) = \operatorname{Spec} A$, it is the generic point of all of $\operatorname{Spec} A$. This explains the 'fat' points in the pictures in Examples 2.5 and 2.29 – the closures of these points are the whole space. △

2.2 Affine spaces

The most important examples of prime spectra are the affine spaces.

Definition 2.17. For each non-negative integer n, we define the *affine n-space* as

$$\mathbb{A}^n = \operatorname{Spec}\mathbb{Z}[x_1,\dots,x_n].$$

More generally, for a ring R, we define the *affine n-space over R* by

$$\mathbb{A}^n_R = \operatorname{Spec} R[x_1,\dots,x_n].$$

Over a field k, each n-tuple $(a_1,\dots,a_n) \in k^n$ determines a maximal ideal $\mathfrak{m} = (x_1 - a_1,\dots,x_n - a_n)$ in the polynomial ring $A = k[x_1,\dots,x_n]$ and hence a point of $\mathbb{A}^n_k$. The residue field is equal to

$$A/\mathfrak{m} = k[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n) \simeq k.$$

Conversely, every maximal ideal $\mathfrak{m}$ in A with residue field $A/\mathfrak{m}$ isomorphic to k is of the form $(x_1 - a_1,\dots,x_n - a_n)$: the a_i are uniquely determined as the images of x_i via the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

quotient map $A \to A/\mathfrak{m} = k$. Therefore, there is a bijection between k^n and the points in $\operatorname{Spec} A$ with residue field isomorphic to k.

For these 'traditional points' of $\mathbb{A}^n_k$, we will frequently pass between the n-tuple $(a_1, \ldots, a_n) \in k^n$ and the corresponding maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$.

If k is algebraically closed, then Hilbert's Nullstellensatz tells us that every maximal ideal is of this form. This means that $\mathbb{A}^n(k)$, as introduced in Chapter 1, can be identified with a subset of the affine n-space $\mathbb{A}^n_k$, and the Zariski topology on $\mathbb{A}^n(k)$ is simply the subspace topology induced from $\mathbb{A}^n_k$.

In general, the affine space $\mathbb{A}^n_k$ also contains many points which are not of the 'traditional' type. There can be maximal ideals with residue fields not isomorphic to k, and prime ideals which are not maximal (the zero ideal for instance). So $\mathbb{A}^n_k$ is in general strictly larger than $\mathbb{A}^n(k)$. The differences between $\mathbb{A}^n_k$ and $\mathbb{A}^n(k)$ become even more apparent if k is not algebraically closed.

Example 2.18 (The affine line)**.** The prime spectrum $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ is called the *affine line over* k. The polynomial ring $k[t]$ is a principal ideal domain, so the prime ideals are either of the form $(f(t))$ where $f(t)$ is an irreducible polynomial, or the zero ideal (0). In the first case, the ideals are automatically maximal. Therefore, $\mathbb{A}^1_k$ has two types of points: the closed points and the generic point η.

When k is algebraically closed, the maximal ideals are all of the form $(t - a)$ for $a \in k$, and their residue fields are isomorphic to k.

When k is not algebraically closed, there will be other closed points in $\operatorname{Spec} k[t]$ besides the ones of the form $(t - a)$. An interesting example is when $k = \mathbb{R}$. By the Fundamental Theorem of Algebra, a nonzero prime ideal $\mathfrak{p}$ of $\mathbb{R}[t]$ is of the form $\mathfrak{p} = (f(t))$ where $f(t)$ is either linear, that is, $f(t) = t - a$ for some $a \in \mathbb{R}$; or f is quadratic with two conjugate complex non-real roots, that is, $f(t) = (t - a)(t - \bar{a})$ with $a \in \mathbb{C}$ but $a \notin \mathbb{R}$. The closed points in $\operatorname{Spec} \mathbb{R}[t]$ may therefore be identified with the set of pairs $\{a, \bar{a}\}$ with $a \in \mathbb{C}$. For instance, the maximal ideal $(t^2 + 1)$ corresponds to the pair $\{i, -i\}$. The residue field at this point is isomorphic to $\mathbb{R}[t]/(t^2 + 1) \simeq \mathbb{C}$.

In general, if a maximal ideal $\mathfrak{m}$ in $k[t]$ is generated by the irreducible polynomial $f(t)$, the residue field at the corresponding point in $\mathbb{A}^1_k$ is the extension of k obtained by adjoining a root of f.

Affine spaces over non-algebraically closed fields can be quite subtle. Even the affine line $\mathbb{A}^1_{\mathbb{Q}}$ is quite mysterious, as its closed points correspond to the monic irreducible polynomials in $\mathbb{Q}[t]$. △

Example 2.19 (The affine plane)**.** When k is algebraically closed, the prime ideals of $k[x, y]$ come in three types: the maximal ideals, which are all of the form $(x - a, y - b)$ for $a, b \in k$; the prime ideals of the form $\mathfrak{p} = (f)$, where $f \in k[x, y]$ is an irreducible polynomial; and the zero ideal (0).

The maximal ideals $(x - a, y - b)$ are the closed points of $\mathbb{A}^2_k$, corresponding to the usual points $(a, b) \in k^2$. The zero ideal (0) is the generic point of the whole space. For each irreducible polynomial f, the prime ideal (f) is the generic point of the irreducible closed subset $V(f) \subset \mathbb{A}^2_k$. In addition to (f), $V(f)$ contains the maximal ideals $(x - a, y - b)$ which contain (f), or equivalently, for which $f(a, b) = 0$. Each such maximal ideal is a specialization of the generic point (f). △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

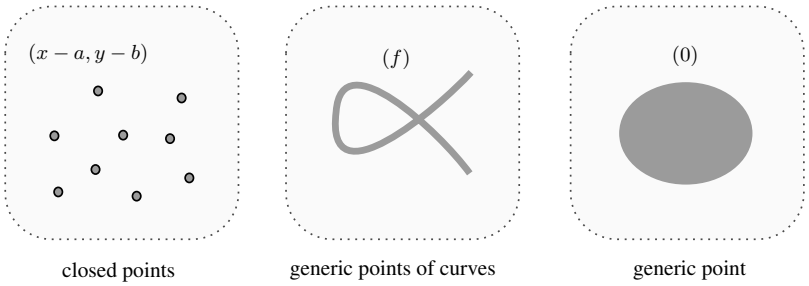


Figure 2.2 Points in $\mathbb{A}^2_k$

2.3 Distinguished open sets

The Zariski topology of $\operatorname{Spec} A$ has a convenient basis consisting of particularly simple open sets. For an element $f \in A$, we define the *distinguished open set*, $D(f)$ to be the complement of the closed set $V(f)$,

$$D(f) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \,\} = \operatorname{Spec} A - V(f).$$

In other words, $D(f)$ is exactly the set of points $\mathfrak{p} \in \operatorname{Spec} A$ at which $f(\mathfrak{p}) \neq 0$ in $\kappa(\mathfrak{p})$. These sets will play an important role in computations involving the spectrum of a ring.

Lemma 2.20. For any $f, g \in A$, we have
(i) $D(f) \cap D(g) = D(fg)$.
(ii) $D(g) \subset D(f)$ if and only if some power of g is a multiple of f, i.e., we may write $g^r = cf$ for some $c \in A$, $r \in \mathbb{N}$.

Proof (i): If $\mathfrak{p}$ is a prime ideal, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ hold if and only if $fg \notin \mathfrak{p}$.

(ii): The inclusion $D(g) \subset D(f)$ holds if and only if $V(f) \subset V(g)$, and by Lemma 2.7, this is equivalent to $(g) \subset \sqrt{(f)}$, which in turn means that $g^r \in (f)$ for some $r \in \mathbb{N}$. □

Lemma 2.21.
(i) The open sets $\{D(f)\}_{f \in A}$ form a basis for the topology of $\operatorname{Spec} A$.
(ii) A collection $\{D(f_i)\}_{i \in I}$ is an open cover of $\operatorname{Spec} A$ if and only if the f_i's generate the unit ideal. Equivalently, there are finitely many indices $i_1, \ldots, i_r \in I$ and $a_1, \ldots, a_r \in A$ such that

$$1 = a_1 f_{i_1} + \cdots + a_r f_{i_r}. \tag{2.9}$$

Proof (i): We need to show that every open subset U of $\operatorname{Spec} A$ can be written as the union of distinguished open sets. By definition of the Zariski topology, the complement of U is of the form $U^c = V(\mathfrak{a})$ with $\mathfrak{a} \subset A$ an ideal. If we choose a (possibly infinite) set of generators $\{f_i\}_{i \in I}$ for $\mathfrak{a}$, then we have

$$U = V(\mathfrak{a})^c = V\big(\sum_{i \in I}(f_i)\big)^c = \big(\bigcap_{i \in I} V(f_i)\big)^c = \bigcup_{i \in I} D(f_i). \tag{2.10}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(ii): By the identity (2.10) with $U = \operatorname{Spec} A$, the collection of open sets $\{D(f_i)\}_{i\in I}$ covers $\operatorname{Spec} A$ if and only if $V(\sum_{i\in I}(f_i)) = \varnothing$. By Lemma 2.7, this in turn happens if and only if $\sum_{i\in I}(f_i) = (1)$, or in other words, the f_i generate the unit ideal. But this happens if and only if 1 can be expressed as a combination of finitely many of the f_i's. □

In algebraic geometry, a topological space is said to be *quasi-compact* if every open covering has a finite subcover. This terminology is a bit unfortunate, as spaces with this property are usually called *compact*, but the term has become standard here.

Corollary 2.22. $\operatorname{Spec} A$ is quasi-compact.

Proof Let $\{U_i\}_{i\in I}$ be any open cover of $\operatorname{Spec} A$. Then each U_i can be covered by distinguished open sets $D(f_{ij})$ with $f_{ij} \in A$. Since the $D(f_{ij})$'s cover $\operatorname{Spec} A$, finitely many of them will suffice, by Lemma 2.21. As each U_i contains $D(f_{ij})$, we can cover $\operatorname{Spec} A$ with finitely many U_i as well. □

Example 2.23. In the affine line $\mathbb{A}^1_k$ over a field, every closed set is of the form $V(f)$ for some polynomial f, so every open set is a distinguished open set $D(f)$.

In $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, the set $U = \mathbb{A}^2_k - V(x,y)$ is open, but not of the form $D(f)$. Still, it can be written as $U = D(x) \cup D(y)$. △

Example 2.24 (The circle)**.** Consider $X = \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2-1)$. The maximal ideal $\mathfrak{m} = (x, y-1)$ corresponds to the point $(0,1)$ in X. Even though $\mathfrak{m}$ is not a principal ideal (Exercise 2.7.26), the complement $X - \{\mathfrak{m}\}$ is nevertheless a distinguished open set. Indeed, it is equal to $D(y-1)$, because in the quotient ring we have $x^2 = 1-y^2 = -(y-1)(y+1)$, so $x^2 \in (y-1)$ and hence $V(y-1) = \{\mathfrak{m}\}$. △

2.4 Maps between prime spectra

If $\phi\colon A \to B$ is a ring map, there is an induced map

$$f\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad \mathfrak{p} \longmapsto \phi^{-1}(\mathfrak{p}).$$

This is well-defined, because the inverse image $\phi^{-1}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset B$ is a prime ideal: that $ab \in \phi^{-1}(\mathfrak{p})$ means that $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, so at least one of $\phi(a)$ or $\phi(b)$ has to lie in $\mathfrak{p}$.

The next result implies that the above map is also continuous in the Zariski topology, because its inverse images of closed sets are closed.

Proposition 2.25. Let $\phi\colon A \to B$ be a ring map and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map of spectra. Then

(i) $f^{-1}V(\mathfrak{a}) = V(\phi(\mathfrak{a})B)$ for each ideal $\mathfrak{a} \subset A$.
(ii) $f^{-1}D(g) = D(\phi(g))$ for each $g \in A$.
(iii) $\overline{f(V(\mathfrak{b}))} = V(\phi^{-1}(\mathfrak{b}))$ for each ideal $\mathfrak{b}$ of B.

Proof (i): For any ideal $\mathfrak{a} \subset A$, we have the following equalities:

$$f^{-1}V(\mathfrak{a}) = \{\, \mathfrak{p} \subset B \mid \mathfrak{a} \subset \phi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \subset B \mid \phi(\mathfrak{a}) \subset \mathfrak{p} \,\} = V(\phi(\mathfrak{a})B).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(ii): For each element $g \in A$, we have

$$f^{-1}D(g) = \{\, \mathfrak{p} \subset B \mid g \notin \phi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \subset B \mid \phi(g) \notin \mathfrak{p} \,\} = D(\phi(g)).$$

(iii): By Proposition 2.8, the closure $\overline{f(V(\mathfrak{b}))}$ is equal to $V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in f(V(\mathfrak{b}))} \mathfrak{p} = \bigcap_{\mathfrak{b} \subset \mathfrak{q}} \phi^{-1}(\mathfrak{q}).$$

Here the second equality holds because $\mathfrak{p} \in f(V(\mathfrak{b}))$ implies that $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q}$ with $\mathfrak{b} \subset \mathfrak{q}$. Furthermore,

$$\mathfrak{a} = \bigcap_{\mathfrak{b} \subset \mathfrak{q}} \phi^{-1}(\mathfrak{q}) = \phi^{-1}\Big(\bigcap_{\mathfrak{b} \subset \mathfrak{q}} \mathfrak{q}\Big) = \phi^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\phi^{-1}(\mathfrak{b})}.$$

Hence $V(\mathfrak{a}) = V(\phi^{-1}(\mathfrak{b}))$, as required. □

We next study the behaviour of Spec under quotient maps and localization maps.

Proposition 2.26. For an ideal $\mathfrak{a} \subset A$, the quotient map $A \to A/\mathfrak{a}$ induces a homeomorphism

$$f\colon \operatorname{Spec}(A/\mathfrak{a}) \xrightarrow{\simeq} V(\mathfrak{a}) \subset \operatorname{Spec} A.$$

Proof If $\phi\colon A \to A/\mathfrak{a}$ denotes the quotient map, the map $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ gives an inclusion preserving one-to-one correspondence between the set of prime ideals in $A/\mathfrak{a}$ and the set of prime ideals in A containing $\mathfrak{a}$, with inverse given by $\mathfrak{q} \mapsto \mathfrak{q}/\mathfrak{a}$. This shows that f is a continuous bijection onto the closed subset $V(\mathfrak{a})$. To show that f is a homeomorphism, it suffices to show that it is closed. This follows from the equalities

$$f(V(\mathfrak{b}/\mathfrak{a})) = \big\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{b}/\mathfrak{a} \subset \mathfrak{p}/\mathfrak{a} \in \operatorname{Spec}(A/\mathfrak{a})\big\} = V(\mathfrak{b}).$$ □

In the special case where $\mathfrak{a} = \sqrt{(0)}$ is the nilradical of A, we have $V(\mathfrak{a}) = \operatorname{Spec} A$, and the proposition implies the following:

Corollary 2.27. Let A be a ring and let $A_{\mathrm{red}} = A/\sqrt{(0)}$ be the reduction of A. Then the map $A \to A_{\mathrm{red}}$ induces a homeomorphism

$$\operatorname{Spec}(A_{\mathrm{red}}) \simeq \operatorname{Spec}(A).$$

We next consider the maps of spectra induced by localization maps. The next result implies in particular that distinguished open sets $D(f)$ can themselves be viewed as prime spectra. This fact will be important later on.

Proposition 2.28. Let A be a ring and $S \subset A$ a multiplicative subset. The localization map $\ell\colon A \to S^{-1}A$ induces a canonical homeomorphism

$$\iota\colon \operatorname{Spec}(S^{-1}A) \xrightarrow{\simeq} \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\}. \tag{2.11}$$

In particular, for any $f \in A$, there is a canonical homeomorphism

$$\operatorname{Spec}(A_f) \xrightarrow{\simeq} D(f) \subset \operatorname{Spec} A.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof Write $D = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing\}$ for the right-hand side of (2.11).

Let $\mathfrak{q} \subset S^{-1}A$ be a prime ideal. Since $\mathfrak{q}$ does not contain 1, its inverse image $\ell^{-1}(\mathfrak{q})$ does not contain any elements of S. Hence $\iota(\mathfrak{q}) \in D$. Conversely, if $\mathfrak{p} \subset A$ is a prime ideal with $\mathfrak{p} \cap S = \varnothing$, then $\mathfrak{p}S^{-1}A$ is a prime ideal of $S^{-1}A$, and $\mathfrak{p} = \ell^{-1}(\mathfrak{p}S^{-1}A)$.

This shows that ι is a continuous bijection onto D. To show that ι is a homeomorphism, it is enough to show that it is closed. This follows from the following equalities:

$$\begin{aligned}\iota\big(V(\mathfrak{a})\big) &= \{\ell^{-1}(\mathfrak{q}) \mid \mathfrak{q} \supset \mathfrak{a}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \ell^{-1}(\mathfrak{a}) \text{ and } \mathfrak{p} \cap S = \varnothing\} \\ &= V\big(\ell^{-1}(\mathfrak{a})\big) \cap D,\end{aligned}$$

which is closed in D. Hence ι is a homeomorphism. □

Let $D(g) \subset D(f)$ be distinguished open sets. By Lemma 2.20 we may write $g^r = cf$ for some $c \in A$ and $r \in \mathbb{N}$. Then $f \cdot (c/g^r) = 1$ in A_g, so f is invertible there. This means that the localization map $A \to A_g$ factors uniquely as $A \to A_f \to A_g$.

Since $D(g) = D(g^r) = D(cf)$, we see that any distinguished open contained in $D(f)$ is of the form $D(cf)$ for some $c \in A$. Concretely, the localization map $A_f \to A_{cf}$ sending a/f^n to $ac^n/(cf)^n$, is the localization of A_f at $c/1$ and it identifies $D(cf)$ with $D(c/1)$ inside $\operatorname{Spec}(A_f)$.

Example 2.29 (Discrete valuation rings)**.** Consider a discrete valuation ring A, such as $k[t]_{(t)}$ or $\mathbb{Z}_{(p)}$. (See Appendix C for background on discrete valuation rings). The ring A has exactly two prime ideals, the maximal ideal $\mathfrak{m}$ and the zero ideal (0). Therefore, $\operatorname{Spec} A$ consists of just two points, $\operatorname{Spec} A = \{s, \xi\}$ where s corresponds to $\mathfrak{m}$ and ξ to (0). The closed sets are $\varnothing$, $\{s\}$ and $\{s, \xi\}$, while the open sets are $\varnothing$, $\operatorname{Spec} A$ and $\{\xi\}$. In particular, ξ is an open point. As before, $\operatorname{Spec} A$ is not Hausdorff, as there are no open sets that separate s and ξ. △



Figure 2.3 The spectrum of a DVR

Example 2.30 (Reduction modulo a prime p)**.** The reduction mod p map $\mathbb{Z} \to \mathbb{F}_p$ induces a map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. The one and only point in $\operatorname{Spec} \mathbb{F}_p$ is sent to the point in $\operatorname{Spec} \mathbb{Z}$ corresponding to the maximal ideal (p).

Likewise, the inclusion $\mathbb{Z} \subset \mathbb{Q}$ induces a map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$. It sends the unique point in $\operatorname{Spec} \mathbb{Q}$ to the generic point η of $\operatorname{Spec} \mathbb{Z}$. △

Example 2.31 (The twisted cubic curve)**.** Let k be a field. The ring map $\phi\colon k[x, y, z] \to k[t]$ given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ corresponds to a map of prime spectra

$$f\colon \mathbb{A}^1_k \longrightarrow \mathbb{A}^3_k.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

As ϕ is surjective, f is a homeomorphism onto the closed set defined by the ideal $\operatorname{Ker}\phi = (y - x^2, z - x^3)$. This is the scheme version of the twisted cubic curve from Example 1.29. △

Example 2.32. Let k be a field. The inclusion $k[x] \to k[x,y]/(xy-1)$ induces a morphism

$$\operatorname{Spec} k[x,y]/(xy-1) \longrightarrow \mathbb{A}^1_k = \operatorname{Spec} k[x].$$

Geometrically, this is the projection of the hyperbola $xy = 1$ onto the x-axis. The image of this morphism is the open subset $D(x) \subset \mathbb{A}^1_k$. In particular, the image is dense, but not closed in $\mathbb{A}^1_k$. △

2.5 Finitely generated k-algebras

Most of the rings in this book are finitely generated algebras over a field k. We will usually not assume that k is algebraically closed. The next proposition tells us how to work with closed points for such rings.

Proposition 2.33. Let k be a field and let A and B be finitely generated k-algebras.
(i) A point $P \in \operatorname{Spec} A$ is closed if and only if $\kappa(P)$ is a finite extension of k.
(ii) The set of closed points is dense in $\operatorname{Spec} A$.
(iii) If $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is induced by a map of k-algebras $\phi\colon A \to B$ and $P \in \operatorname{Spec} B$ is a closed point, then $f(P)$ is closed in $\operatorname{Spec} A$ and $\kappa(P)$ is a finite extension of $\kappa(f(P))$.

Proof We will use the following basic fact: If R is an integral domain containing a field k, and $\dim_k R < \infty$, then R is a field. Indeed, if $a \in R$ is nonzero, multiplication by a is a k-linear map $R \to R$, and it is injective because R is a domain. Since $\dim_k R < \infty$, it is also surjective, so $ab = 1$ for some $b \in R$.

(i): Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to P, so that P is closed if and only if $A/\mathfrak{p}$ is a field. If P is closed, then $\kappa(P) = A/\mathfrak{p}$ is a field which is a finitely generated k-algebra, hence a finite extension of k by Theorem 1.7. Conversely, if $\kappa(P)$ is finite over k, then $A/\mathfrak{p}$ is a domain with $\dim_k A/\mathfrak{p} \leqslant \dim_k \kappa(P) < \infty$, so it is a finite field extension of k.

(ii): As the sets $D(f)$ form a basis for the topology, it suffices to show that every nonempty $D(f) = \operatorname{Spec} A_f$ contains a point which is closed in $\operatorname{Spec} A$. Choose a maximal ideal $\mathfrak{m} \subset A_f$. Since A_f is a finitely generated k-algebra, $A_f/\mathfrak{m}$ is a finite extension of k by Theorem 1.7. The corresponding point P has $\kappa(P) = A_f/\mathfrak{m}$, so P is closed in X by (i).

(iii): Since $P \in \operatorname{Spec} B$ is closed, it corresponds to a maximal ideal $\mathfrak{m} \subset B$, and Q to the prime ideal $\mathfrak{q} = \phi^{-1}(\mathfrak{m}) \in \operatorname{Spec} A$. The map ϕ induces an injection of k-algebras $A/\mathfrak{q} \hookrightarrow B/\mathfrak{m} = \kappa(P)$. By (i), $\kappa(P)$ is finite over k, so $A/\mathfrak{q}$ is a domain of finite dimension over k, hence a field. Hence Q is closed, and $\kappa(Q) = B/\mathfrak{p}$ is a subfield of $\kappa(P)$, with $[\kappa(P) : \kappa(Q)] \leqslant [\kappa(P) : k] < \infty$. □

Example 2.34. The fact that the k-algebra defining X is finitely generated is essential. For instance, if $X = \operatorname{Spec} k[x]_{(x)}$, then there is a single closed point, corresponding to the maximal ideal (x), and this point is not dense in X. In contrast, the ring $k[x]_x = k[x, x^{-1}]$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is finitely generated and its spectrum is the open set $D(x) \subset \mathbb{A}^1_k$, where closed points are dense. △

Example 2.35 (The empty conic)**.** Let $A = \mathbb{R}[x,y]/(x^2+y^2+1)$ and let $X = \operatorname{Spec} A$. As x^2+y^2+1 is irreducible over $\mathbb{R}$, the ring A is an integral domain. Note that A has dimension 1, so the prime ideals are either (0) or a maximal ideal $\mathfrak{m}$. We claim that every maximal ideal is of the form

$$\mathfrak{m} = (ax+by+c), \qquad a,b,c \in \mathbb{R},\ (a,b) \neq (0,0).$$

First of all, any such ideal is maximal. Without loss of generality, we may assume that $a = 1$. Then, eliminating x,

$$\begin{aligned} A/\mathfrak{m} &= \mathbb{R}[x,y]/(x^2+y^2+1, x+by+c) \\ &= \mathbb{R}[y]/((by+c)^2+y^2+1). \end{aligned}$$

Now $(by+c)^2+y^2+1 = (b^2+1)y^2+2bc\,y+(c^2+1)$ has discriminant $-4(b^2+c^2+1) < 0$, so it has no real roots. Therefore $A/\mathfrak{m} \simeq \mathbb{C}$ is a field, and so $\mathfrak{m}$ is maximal.

Conversely, let $\mathfrak{m}$ be a maximal ideal of A. Then $A/\mathfrak{m}$ is a finite extension of $\mathbb{R}$, hence isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. It cannot be isomorphic to $\mathbb{R}$, because the images of x and y in $A/\mathfrak{m}$ would be real numbers satisfying $x^2+y^2+1=0$, which is impossible. Therefore we may assume that $A/\mathfrak{m} \simeq \mathbb{C}$. As $A/\mathfrak{m}$ is an $\mathbb{R}$-vector space of dimension 2, the elements $1, x, y$ must be $\mathbb{R}$-linearly dependent modulo $\mathfrak{m}$. So there is a linear form $ax+by+c \in \mathfrak{m}$ with $(a,b) \neq (0,0)$. As shown above, $(ax+by+c)$ is a maximal ideal, so we must have $\mathfrak{m} = (ax+by+c)$. △

2.6 Fibers

To understand a map of spectra $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$, it is often useful to understand the *fibers* of f, that is, the inverse images of points. If f is induced by a ring map $\phi\colon A \to B$ and $y \in \operatorname{Spec} A$ is a point corresponding to $\mathfrak{p}$ in A, the fiber $f^{-1}(y)$ consists of the primes $\mathfrak{q}$ in B such that $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.

If $y \in \operatorname{Spec} A$ is a closed point, so that $\mathfrak{p}$ is a maximal ideal, then $\{y\} = \overline{\{y\}} = V(\mathfrak{p})$, so Proposition 2.25 implies that $f^{-1}V(\mathfrak{p}) = V(\mathfrak{p}B)$. In particular, the fiber $f^{-1}(y)$ is a closed set, homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$.

If y is not closed, the fiber $f^{-1}(y)$ may or may not be closed in $\operatorname{Spec} B$. While we still have $f^{-1}V(\mathfrak{p}) = V(\mathfrak{p}B)$, this closed set may contain other primes than the ones mapping to y. For instance, in the situation when $\mathfrak{p} = (0)$ is prime in A, then $V(\mathfrak{p}B) = \operatorname{Spec} B$.

To describe the fiber, we consider the localization $B_{\mathfrak{p}} = S^{-1}B$ in the multiplicative set

$$S = \phi(A - \mathfrak{p}).$$

The idea is that the 'extra primes' in $V(\mathfrak{p}B)$ which do not map to $\mathfrak{p}$ will localize to the unit ideal in $B_{\mathfrak{p}}$.

Consider the composition

$$\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \longrightarrow \operatorname{Spec}(B_{\mathfrak{p}}) \longrightarrow \operatorname{Spec}(B), \tag{2.12}$$

which is induced by the localization map $B \to B_{\mathfrak{p}}$ and the quotient map $B_{\mathfrak{p}} \to B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 2.36. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a map between prime spectra and let $\mathfrak{p} \in \operatorname{Spec} A$. Then the composition (2.12) induces a homeomorphism

$$f^{-1}(\mathfrak{p}) \simeq \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}).$$

If $\mathfrak{p}$ is a closed point, then $f^{-1}(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$.

Proof The first map is a homeomorphism onto the subset $V(\mathfrak{p}B_{\mathfrak{p}})$ in $\operatorname{Spec}(B_{\mathfrak{p}})$ (by Proposition 2.25), and the second is a homeomorphism onto the set of primes $\mathfrak{q} \in \operatorname{Spec} B$ disjoint from $S = \phi(A - \mathfrak{p})$ (Proposition 2.28). Therefore the composition is a homeomorphism onto the set of prime ideals $\mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{q} \cap S = \varnothing$ and $\mathfrak{q}B_{\mathfrak{p}} \supset \phi(\mathfrak{p})B_{\mathfrak{p}}$. The second inclusion implies that $\mathfrak{q} \supset \phi(\mathfrak{p})$ and hence $\phi^{-1}(\mathfrak{q}) \supset \mathfrak{p}$. The first equality then implies that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ (if $\phi^{-1}(\mathfrak{q}) \supset \mathfrak{p}$ were strict, then elements in $\phi^{-1}(\mathfrak{q}) - \mathfrak{p}$ would map to elements of $\mathfrak{q} \cap S$). Therefore, the image of (2.12) consists of precisely those prime ideals $\mathfrak{q}$ in B so that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, that is, $f(\mathfrak{q}) = \mathfrak{p}$. □

In computations involving fibers, it is useful to keep in mind that localization commutes with taking quotients. In particular, $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \simeq S^{-1}(B/\mathfrak{p}B)$.

Example 2.37. Consider the map

$$f\colon \operatorname{Spec} \mathbb{C}[x,y] \longrightarrow \operatorname{Spec} \mathbb{C}[z],$$

induced by the ring map $\mathbb{C}[z] \to \mathbb{C}[x,y]$ defined by $z \mapsto xy$. Let us compute the fibers $f^{-1}(\mathfrak{p})$ over the maximal ideals $\mathfrak{p} = (z-a)$. Note that

$$B/\mathfrak{p}B = \mathbb{C}[x,y]/(xy-a).$$

There are two cases. If $a \neq 0$, then $xy - a$ is an irreducible polynomial, and then $\operatorname{Spec}(B/\mathfrak{p}B)$ is irreducible. This is intuitive, as it corresponds to the hyperbola $V(xy-a)$ in $\mathbb{A}^2_{\mathbb{C}}$. If $a = 0$, we are left with $\operatorname{Spec} \mathbb{C}[x,y]/(xy)$, which is not irreducible. It has two components, the two coordinate axes $V(x)$ and $V(y)$.

Let us also consider the fiber over the generic point η of $\operatorname{Spec} \mathbb{C}[z]$, which corresponds to $\mathfrak{p} = (0)$. In this case, the ring $(B/\mathfrak{p}B)_{\mathfrak{p}}$ is the localization of B with respect to the multiplicative set $S = \mathbb{C}[z] - \{0\}$; that is, the ring

$$\mathbb{C}(z)[x,y]/(xy-z).$$

This is again an integral domain, so the fiber $f^{-1}(\eta)$ is irreducible. This fiber may be regarded as a hyperbola in the affine plane $\mathbb{A}^2_{\mathbb{C}(z)}$ over the field $\mathbb{C}(z)$. △

Example 2.38 (Double covers)**.** Let k be a field of characteristic different from 2 and let $g \in k[x_1, \dots, x_n]$ be a nonzero polynomial. Consider the spectrum

$$X = \operatorname{Spec} k[x_1, \dots, x_n, y]/(y^2 - g).$$

The inclusion $k[x_1, \dots, x_n] \to k[x_1, \dots, x_n, y]/(y^2-g)$ induces a map

$$f\colon X \longrightarrow \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n].$$

We call X the *double cover* of $\mathbb{A}^n_k$ associated to g.

The name 'double cover' comes from the fibers of f. For a point $a = (a_1, \dots, a_n) \in k^n$,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

let $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ be the corresponding maximal ideal. Then the fiber $f^{-1}(\mathfrak{m})$ is the spectrum of the ring

$$\frac{k[x_1, \ldots, x_n, y]}{(y^2 - g, x_1 - a_1, \ldots, x_n - a_n)} \simeq \frac{k[y]}{(y^2 - g(a))}.$$

There are three cases, depending on the value $b = g(a)$:

(i) If $b \neq 0$ and b is a square in k, say $b = c^2$, then the polynomial $y^2 - b$ factors as $(y - c)(y + c)$. By the Chinese Remainder Theorem, the fiber is

$$\operatorname{Spec}\Big(\frac{k[y]}{(y-c)} \times \frac{k[y]}{(y+c)}\Big),$$

which is the disjoint union of two copies of $\operatorname{Spec} k$.

(ii) If $b \neq 0$, but b is not a square in k, then the fiber equals $\operatorname{Spec} k(\sqrt{b})$, where $k(\sqrt{b})$ is a quadratic field extension of k. This fiber is a single point, but with 'multiplicity 2' (in the sense that the degree of the field extension $k \subset k(\sqrt{b})$ is 2).

(iii) If $b = 0$, the fiber is $\operatorname{Spec} k[y]/(y^2)$, which is a single point. Again there is a 'multiplicity 2', accounted for by the presence of nilpotent elements in the ring $(k[y]/(y^2)$ has dimension 2 as a k-vector space).

△

Example 2.39. The inclusion $\mathbb{R}[t] \subset \mathbb{C}[t]$ induces a map

$$\pi\colon \operatorname{Spec} \mathbb{C}[t] \longrightarrow \operatorname{Spec} \mathbb{R}[t].$$

By Example 2.18 there are three cases to consider for the fiber $\pi^{-1}(y)$ of a point $y \in \mathbb{A}^1_{\mathbb{R}}$.

(i) The point y corresponds to the maximal ideal $(t - a)$ with $a \in \mathbb{R}$. The fiber is equal to

$$\pi^{-1}(y) = \operatorname{Spec}\left(\mathbb{C}[t]/(t-a)\right) \simeq \operatorname{Spec} \mathbb{C}.$$

So the fiber is a single closed point with residue field $\mathbb{C}$.

(ii) The point y is a closed point corresponding to $\mathfrak{p} = (f(t))$ where $f \in \mathbb{R}[t]$ has two conjugate complex roots $a, \bar{a}$. Then

$$\pi^{-1}(y) = \operatorname{Spec}\left(\mathbb{C}[t]/(f(t))\right) \simeq \operatorname{Spec}\left(\frac{\mathbb{C}[t]}{(t-a)} \times \frac{\mathbb{C}[t]}{(t-\bar{a})}\right).$$

Here the fiber consists of two closed points, both with residue field $\mathbb{C}$.

(iii) The point y is the generic point η. Then $\pi^{-1}(\eta)$ is the spectrum of the localization $S^{-1}\mathbb{C}[t] = \mathbb{C}(t)$ where $S = \mathbb{R}[t] - (0)$. Therefore, $\pi^{-1}(\eta)$ is the generic point with residue field $\mathbb{C}(t)$.

Complex conjugation acts on the fibers in this example. More precisely, consider the conjugation map $\mathbb{C}[t] \to \mathbb{C}[t]$ on the polynomial ring $\mathbb{C}[t]$ given by conjugating the coefficients of the polynomials; that is, sending a polynomial $\sum_i a_i t^i$ to $\sum_i \bar{a}_i t^i$. This defines an automorphism

$$\iota\colon \operatorname{Spec} \mathbb{C}[t] \longrightarrow \operatorname{Spec} \mathbb{C}[t].$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The subring $\mathbb{R}[t] \subset \mathbb{C}[t]$ is fixed by the conjugation map, so the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[t] & \xrightarrow{\quad\iota\quad} & \operatorname{Spec}\mathbb{C}[t] \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \operatorname{Spec}\mathbb{R}[t] & \end{array}$$

Therefore, the group $G = \langle \iota \rangle \simeq \mathbb{Z}/2$ acts by automorphisms on the fibers of π, and $\operatorname{Spec}\mathbb{R}[t]$ can be viewed as the quotient space of $\operatorname{Spec}\mathbb{C}[t]$ by G; i.e. the space of G-orbits. Indeed, by Example 2.18, the closed points of $\operatorname{Spec}\mathbb{R}[t]$ correspond exactly to the orbits of G and the generic point of $\operatorname{Spec}\mathbb{C}[t]$ is invariant and corresponds to the generic point of $\operatorname{Spec}\mathbb{R}[t]$. △

Example 2.40 (The Gaussian integers)**.** The inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$ induces a morphism

$$f\colon \operatorname{Spec}\mathbb{Z}[i] \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

We will study $\operatorname{Spec}\mathbb{Z}[i]$ by studying the fibers of this map. If $p \in \mathbb{Z}$ is a prime, the fiber over $(p) \in \operatorname{Spec}\mathbb{Z}$ is given by $V((p)\mathbb{Z}[i])$. This is homeomorphic to the spectrum

$$\operatorname{Spec}\left(\mathbb{Z}[i]/p\right) \simeq \operatorname{Spec}\left(\frac{\mathbb{F}_p[x]}{(x^2+1)}\right).$$

The ring $\mathbb{F}_p[x]/(x^2+1)$ is not an integral domain precisely when the equation $x^2+1=0$ has a solution modulo p. This happens if and only if $p = 2$ or $p \equiv 1 \mod 4$. By Fermat's Two Squares Theorem, this is equivalent to $p = a^2 + b^2$ being a sum of two squares, or in other words, that $p = (a+ib)(a-ib)$ factors in $\mathbb{Z}[i]$. This means that there are three cases:

(i) $p = 2$. Then $(2)\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i]$. Therefore, $f^{-1}((2))$ consists of the single point corresponding to the prime ideal $(1+i)$.

(ii) $p \equiv 1 \pmod 4$. In this case, we may factor $p = (a+ib)(a-ib)$ in $\mathbb{Z}[i]$ and so the fiber consists of the two points with residue fields $\mathbb{F}_p$,

$$f^{-1}((p)) = V((p)\mathbb{Z}[i]) = \{(a+ib), (a-ib)\}.$$

(iii) $p \equiv 3 \pmod 4$. Then the ideal (p) stays prime in $\mathbb{Z}[i]$ and the fiber $f^{-1}((p))$ consists of the single point x corresponding to $(p)\mathbb{Z}[i]$. The residue field is isomorphic to $\mathbb{F}_p[x]/(x^2+1) \simeq \mathbb{F}_{p^2}$, the field with p^2 elements.

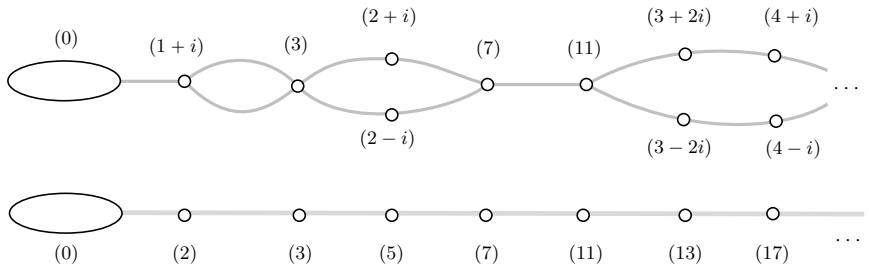


Figure 2.4 The spectrum $\operatorname{Spec}\mathbb{Z}[i]$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Note that the complex conjugation map $\sigma\colon \mathbb{Z}[i] \to \mathbb{Z}[i]$ acts on $\operatorname{Spec}\mathbb{Z}[i]$ and the fibers of the map f. For instance, the primes sitting over (5) are $(2+i)$ and $(2-i)$, and complex conjugation sends one to the other. We therefore picture $\operatorname{Spec}\mathbb{Z}[i]$ as a curve lying above $\operatorname{Spec}\mathbb{Z}$, with σ permuting the points in each fiber.

△

Example 2.41 (The affine line $\mathbb{A}^1_{\mathbb{Z}}$)**.** Consider the affine line $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$ and the morphism $f\colon \operatorname{Spec}\mathbb{Z}[t] \to \operatorname{Spec}\mathbb{Z}$ induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[t]$.

There are two cases for a fiber $f^{-1}(y)$ of a point $y \in \operatorname{Spec}\mathbb{Z}$. If y corresponds to the closed point $(p) \in \operatorname{Spec}\mathbb{Z}$, the fiber $f^{-1}(y)$ consists of all primes $\mathfrak{p} \subset \mathbb{Z}[t]$ such that $\mathfrak{p} \cap \mathbb{Z} = (p)$. According to Proposition 2.36, this is given by

$$V((p)\mathbb{Z}[t]) = \operatorname{Spec}(\mathbb{Z}[t]/(p)\mathbb{Z}[t]) = \mathbb{A}^1_{\mathbb{F}_p}.$$

Likewise, the fiber over the generic point of $\operatorname{Spec}\mathbb{Z}$ is, by Proposition 2.36, the spectrum of the localization $S^{-1}\mathbb{Z}[t] = \mathbb{Q}[t]$, where $S = \mathbb{Z} - (0)$, that is,

$$f^{-1}(\eta) = \operatorname{Spec}\mathbb{Q}[t] = \mathbb{A}^1_{\mathbb{Q}}.$$

Therefore, all fibers of $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ are affine lines.

The affine line $\mathbb{A}^1_{\mathbb{Z}}$ is shown in Figure 2.5. In the figure, we have depicted the two closed sets $V(6x+1)$ and $V(x^2+1)$ (in dashed and gray lines respectively). Note that $V(6x+1)$ is disjoint from the fibers above the primes 2 and 3 (why?). The closed subset $V(x^2+1)$ should be compared to Example 2.40. △

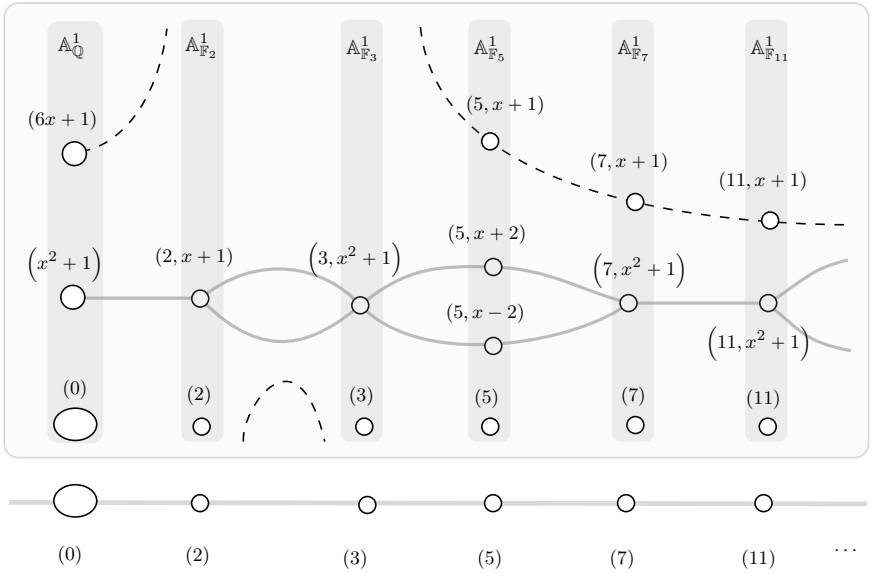


Figure 2.5 The affine line $\operatorname{Spec}\mathbb{Z}[x]$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

2.7 Exercises

Exercise 2.7.1. a) Show that $\operatorname{Spec} A = \varnothing$ if and only if A is the zero ring.
b) Show that $D(f) = \varnothing$ if and only if f is nilpotent.
c) Show that $\operatorname{Spec} A$ consists of a single point if and only if $\sqrt{(0)}$ is a maximal ideal in A.

Exercise 2.7.2. Let A be a Noetherian ring. Show that $\operatorname{Spec} A$ is a finite set with the discrete topology if and only if A is an Artinian ring. HINT: Write A as a product of its localizations.

Exercise 2.7.3. Describe $\operatorname{Spec} \mathbb{Z}[\frac{1}{18}]$.

Exercise 2.7.4. Describe the following prime spectra
a) $\operatorname{Spec} \mathbb{C}[x]/(x^3 + x^2)$
b) $\operatorname{Spec} \mathbb{R}[x]/(x^3 + x^2)$.
c) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 + 2)$
d) $\operatorname{Spec} \mathbb{Z}[x]/(x^3 - 2)$
e) $\operatorname{Spec} \mathbb{Z}[x, y]/(x^2 - y, y^2 + 1)$

Exercise 2.7.5. Show that $\operatorname{Spec} \overline{\mathbb{Q}}[x]$ is homeomorphic to $\operatorname{Spec} \mathbb{Z}$.

Exercise 2.7.6. Describe the spectrum of the ring $A = \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod 2 \}$.

Exercise 2.7.7. Describe $\operatorname{Spec} \mathbb{Z}[x, y, z]/(xz, yz)$.

Exercise 2.7.8. Show that $\operatorname{Spec} A$ is Hausdorff if and only if every prime ideal $\mathfrak{p}$ is maximal.

Exercise 2.7.9. Show that the closed points in $\operatorname{Spec} A$ are dense if and only if $\sqrt{0} = \bigcap_{\mathfrak{m} \subset A} \mathfrak{m}$, the intersection of all maximal ideals in A. HINT: Corollary 2.8 on page 22.

Exercise 2.7.10. Let $X = \operatorname{Spec} A$. Show that the intersection of all nonempty open sets $U \subset X$ is equal to the set of generic points of X.

Exercise 2.7.11. Show that a topological space X is Noetherian if and only if every open subset of X is quasi-compact.

Exercise 2.7.12. Show that the infinite disjoint union $\bigsqcup_{i=1}^{\infty} \operatorname{Spec} \mathbb{Z}$ is not homeomorphic to the spectrum of a ring.

Exercise 2.7.13. Let $\mathfrak{p}$ be a prime ideal in a ring A. Show that there is a canonical inclusion $A/\mathfrak{p} \hookrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ which allows us to identify $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ with the fraction field of $A/\mathfrak{p}$.

Exercise 2.7.14. Let A be a ring and $\mathfrak{p} \subset A$ be a prime ideal. Show that the image of the canonical map $\operatorname{Spec}(A_{\mathfrak{p}}) \to \operatorname{Spec}(A)$ consists of the points $\mathfrak{q}$ such that $\mathfrak{q} \subset \mathfrak{p}$, that is, $\mathfrak{p} \in V(\mathfrak{q})$.

Exercise 2.7.15. a) Describe $\operatorname{Spec} \mathbb{C}[x, y]_{(x)}$ and $\operatorname{Spec} \mathbb{C}[x, y]_{(x,y)}$.
b) Describe the fibers of the map $\operatorname{Spec} \mathbb{C}[x]_{(x)}[y] \to \operatorname{Spec} \mathbb{C}[x]_{(x)}$. Which fibers are closed?
c) Describe $\operatorname{Spec} \mathbb{C}[[x, y]]$ and the map $\operatorname{Spec} \mathbb{C}[[x, y]] \to \operatorname{Spec} \mathbb{C}[x, y]$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 2.7.16. Let A be a ring, let $\mathfrak{a}$ be an ideal in A and let $\{f_i\}_{i\in I}$ be elements from $\mathfrak{a}$. Show that the open distinguished sets $D(f_i)$ cover $\operatorname{Spec} A - V(\mathfrak{a})$ if and only if some power of each element $f \in \mathfrak{a}$ lies in the ideal generated by the f_i's.

Exercise 2.7.17. Find an open set of $\mathbb{A}^1_{\mathbb{Z}}$ which is not a distinguished open set.

Exercise 2.7.18. Let $A = \mathbb{Q}[t_0, t_1, \dots]$ and let $\mathfrak{m}$ be the maximal ideal $\mathfrak{m} = (t_0, t_1, \dots)$. Show that $U = \operatorname{Spec} A - \mathfrak{m}$ is not quasi-compact and that U is not homeomorphic to the spectrum of a ring.

Exercise 2.7.19. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a map induced by a ring map $\phi\colon A \to B$.

a) Show that if ϕ is injective, then the image of f is dense in $\operatorname{Spec} A$.
b) More generally, show that $f(\operatorname{Spec} B)$ is dense if and only if $\operatorname{Ker}\phi \subset \sqrt{(0)}$.
c) Define $\phi\colon \mathbb{Q}[x] \to \mathbb{C}$ by sending x to π. Show that ϕ is injective. Describe the morphism $\operatorname{Spec}\mathbb{C} \to \mathbb{A}^1_{\mathbb{Q}}$.

Exercise 2.7.20. Consider the map $f\colon \operatorname{Spec}\mathbb{C}[x,y,z]/(xz-y) \to \operatorname{Spec}\mathbb{C}[x,y]$. Show that on the level of closed points, f sends (a, ab, b) to (a, ab), and the generic point to the generic point. Show that the image equals $D(x) \cup V(x,y)$. In particular it is neither open nor closed.

Exercise 2.7.21.

a) Study the fibers of the morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ induced by the ring map $\mathbb{C}[y] \to \mathbb{C}[t]$ sending $y \mapsto t^n$.
b) Study the fibers of the morphism $f\colon \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ induced from the map $\mathbb{C}[x,y] \to \mathbb{C}[t]$ sending $x \mapsto t^3$ and $y \mapsto t^5$.

Exercise 2.7.22. Describe all fibers of the following morphisms.

a) $\operatorname{Spec}\mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec}\mathbb{C}[x]$
b) $\operatorname{Spec}\mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec}\mathbb{C}[x]$
c) $\operatorname{Spec}\mathbb{Z}[x,y]/(xy-n) \to \operatorname{Spec}\mathbb{Z}$, where n is a nonzero integer.
d) $\operatorname{Spec}\mathbb{C}[x,y]/(x^2+xy+1) \to \operatorname{Spec}\mathbb{C}[y]$.

Exercise 2.7.23. Let $A = \mathbb{C}[x,y,z]/(xy, xz, yz)$ and $X = \operatorname{Spec} A$. Consider the map $f\colon X \to \mathbb{A}^1_{\mathbb{C}}$ corresponding to the ring map $\mathbb{C}[t] \to A$ sending $t \mapsto x+y+z$. Describe all fibers of f. HINT: The set of closed points is the union of the three coordinate axes in $\mathbb{C}^3$.

Exercise 2.7.24. Let $A = \mathbb{Z}[x,y]/(x^2-y^2-5)$ and consider the morphism $f\colon \operatorname{Spec} A \to \operatorname{Spec}\mathbb{Z}$. Compute the fibers over (0), (2), (3) and (5). What happens if you replace A with the ring $\mathbb{Z}[x,y]/(3x^2-3y^2-15)$?

Exercise 2.7.25. Let $k = \mathbb{F}_p$. Show that the polynomial $x^p - x \in k[x]$ takes the value 0 at every closed point of $\mathbb{A}^1_k$, but it is nonzero.

Exercise 2.7.26. Let $A = \mathbb{R}[x,y]/(x^2+y^2-1)$ and $\mathfrak{m} = (x, y-1)$.

a) Show that $\mathfrak{m}^2 = (y-1)$.
b) Show that $A \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[t, t^{-1}]$ via $t = x + iy$, and that $A^\times = \mathbb{R}^\times$.
c) Suppose that $\mathfrak{m} = (f)$. Show that $f^2 = c(y-1)$ for some $c \in \mathbb{R}^\times$.
d) Write $f = a(y) + xb(y)$ with $a, b \in \mathbb{R}[y]$. Comparing the coefficients of 1 and x, show that $a(y)b(y) = 0$. Conclude that $\mathfrak{m}$ is not principal.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

3

The structure sheaf

In this chapter, we define the *structure sheaf* on the spectrum of a ring $X = \operatorname{Spec} A$. This construction generalizes the notion of regular functions on an affine variety. As in Chapter 1, to each open set $U \subset X$ we associate a ring $\mathcal{O}_X(U)$ whose elements may be viewed as regular functions on U.

More generally, a *sheaf* is a structure that assigns to each open set an algebraic object such as a ring, a module, or a set of morphisms, in a way which is compatible with restriction to smaller open sets. At first, it is helpful to think of sheaves as defining 'distinguished functions' on a space (regular, continuous, holomorphic, and so on), but the notion is far more general. The main point is that the assignment satisfies the two sheaf axioms *locality* and *gluing*, which says that sections are determined by their local restrictions, and compatible local sections can be uniquely glued together.

In this chapter, we introduce the general theory of sheaves and build up to the definition of the structure sheaf in Section 3.5. The theory of sheaves will be expanded in later chapters, where we develop more advanced techniques and study their applications to schemes.

3.1 Presheaves and sheaves

The concept of a sheaf may be defined for any topological space, and the theory is best studied at this level of generality. We begin with the definition of a *presheaf*.

Definition 3.1 (Presheaf)**.** Let X be a topological space. A *presheaf of abelian groups* $\mathcal{F}$ on X consists of the following two sets of data:

(i) For each open $U \subset X$, an abelian group $\mathcal{F}(U)$.

(ii) For each pair of nested opens $V \subset U$, a map of groups

$$\rho_{UV} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

called the *restriction map*. These data are required to satisfy the following two conditions:

(a) For any open $U \subset X$, we have $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$.

(b) For any three open subsets $W \subset V \subset U$, one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We will usually write $s|_V$ for $\rho_{UV}(s)$ when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called *sections* (or *sections* of $\mathcal{F}$ over U). It is also common to write $\Gamma(U, \mathcal{F})$ for the group $\mathcal{F}(U)$; Γ is the 'global sections'-functor (it is functorial in both U and $\mathcal{F}$).

The notion of a presheaf is not confined to presheaves of abelian groups. One can also consider presheaves of sets, rings, vector spaces etc. In fact, for any category C one may

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

define presheaves with values in C. The definition is essentially the same as for presheaves of abelian groups, the only difference being that one requires that the $\mathcal{F}(U)$ are objects from C, and of course, that restriction maps are all morphisms in C. We are certainly going to meet sheaves with more structure beyond that of abelian groups, e.g. sheaves of rings, but they will usually have an underlying structure of abelian group, so we start with these. We will also encounter sheaves of sets. Most of the results we establish for sheaves of abelian groups can be proved for sheaves of sets as well, as long as they can be formulated in terms of sets, and the proofs are essentially the same.

Definition 3.2 (Sheaf)**.** A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the following two conditions for every open set $U \subset X$ and every open cover $\{U_i\}_{i\in I}$ of U:

(i) (Locality axiom) If $s, t \in \mathcal{F}(U)$ are sections such that

$$s|_{U_i} = t|_{U_i}$$

for all i, then $s = t$.

(ii) (Gluing axiom) Given a collection of sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \qquad \text{for all } i, j \in I,$$

then there exists a (unique) section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

These two axioms mirror the properties of continuous functions mentioned in the introduction. The Locality axiom says that sections are uniquely determined from their restrictions to smaller open sets. The Gluing axiom says that you are allowed to patch together local sections to a global one, provided they agree on overlaps.

There is a convenient way of formulating the two sheaf axioms simultaneously. For each open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of an open set $U \subset X$, there is a *sheaf exact sequence*

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j), \tag{3.1}$$

where the maps α and β are defined by $\alpha(s) = (s|_{U_i})_i$, and $\beta(s_i) = (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$. Then $\mathcal{F}$ is a sheaf if and only if these sequences are exact. Indeed, exactness at $\mathcal{F}(U)$ means that α is injective, i.e. that $s|_{U_i} = 0$ for all i implies that $s = 0$ (this is equivalent to the Locality axiom). Exactness in the middle means that $\operatorname{Ker}\beta = \operatorname{Im}\alpha$; that is, elements s_i satisfying $s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j} = 0$ come from an element $s \in \mathcal{F}(U)$ (the Gluing axiom).

When $\mathcal{F}$ is a sheaf, since $\mathcal{F}(U) = \operatorname{Ker}\beta$, we can often use the sequence to compute $\mathcal{F}(U)$ if the $\mathcal{F}(U_i)$'s and the $\mathcal{F}(U_i \cap U_j)$'s are known.

Remark 3.3 (The empty set). For a sheaf $\mathcal{F}$ of abelian groups, it is common to include the condition $\mathcal{F}(\varnothing) = 0$. This is essentially forced upon us by the sheaf axioms: the empty set is covered by the empty covering, and the sheaf axioms requires $\mathcal{F}(\varnothing)$ to be the empty product $\prod_{\varnothing}$ in the category of abelian groups, which is the trivial group (the terminal object).

A *morphism* (or simply a *map*) *of (pre)sheaves* $\phi\colon \mathcal{F} \to \mathcal{G}$ is a collection of maps of abelian groups $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open set in X, which are required to be compatible with the restriction maps, in the sense that the following diagram commutes for

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

each inclusion $V \subset U$ of open sets:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow[\phi_U]{} & \mathcal{G}(U) \\ \big\downarrow{\scriptstyle \rho_{UV}} & & \big\downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V). \end{array} \tag{3.2}$$

In this way, presheaves of abelian groups on X, together with their morphisms form a category, which is denoted by $\mathsf{PAb}(X)$. Likewise, we have the category of sheaves of abelian groups, $\mathsf{Ab}(X)$, which forms a full subcategory.

As usual, a map ϕ between two (pre)sheaves $\mathcal{F}$ and $\mathcal{G}$ is an *isomorphism* if it has a two-sided inverse, i.e. a map $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\phi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \phi = \mathrm{id}_{\mathcal{F}}$. This is the same as saying that ϕ_U is an isomorphism for every open set U.

A presheaf $\mathcal{G}$ is a *subpresheaf* of a presheaf $\mathcal{F}$ if $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open $U \subset X$, and such that the restriction maps of $\mathcal{G}$ are the restrictons of those of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $\mathcal{G}$ is naturally called a *subsheaf*.

Examples

Example 3.4 (Continuous functions)**.** Take $X = \mathbb{R}^n$ and let $C(U, \mathbb{R})$ denote the presheaf whose sections over an open set U is the ring of continuous real valued functions on U, and whose restriction maps ρ_{UV} are the usual restrictions of functions. Then $C(-, \mathbb{R})$ is a sheaf of rings (functions can be added and multiplied), and both sheaf axioms are satisfied. Indeed, any function $f\colon X \to \mathbb{R}$ which restricts to zero on an open covering of X takes the value zero at every point, so it is the zero function. Also, given continuous functions $f_i\colon U_i \to \mathbb{R}$ that agree on the overlaps $U_i \cap U_j$, we can form the continuous function $f\colon U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ for any i such that $x \in U_i$. △

Example 3.5. More generally, for any two topological spaces X and Y, the presheaf $C(-, Y)$ of continuous maps $f\colon U \to Y$ forms a sheaf (it is a sheaf of sets, because we cannot in general add or multiply maps).

Furthermore, any presheaf of the form

$$\mathcal{F}(U) = \{\, f\colon U \to Y \mid f \text{ satisfies } (*) \,\} \tag{3.3}$$

is a subsheaf of $C(-, Y)$, provided $(*)$ is a condition that can be checked locally. For instance, we can take $(*)$ to mean 'differentiable', 'smooth', 'analytic', and so on. Locality holds because it holds in $C(-, Y)$. Given local sections $f_i \in \mathcal{F}(U_i)$, agreeing on the overlaps, we can glue them to an element $f \in C(U, Y)$, and the result lies in $\mathcal{F}(U)$, because the restriction of f to U_i is equal to f_i, which satisfies $(*)$. △

Example 3.6 (A presheaf which is not a sheaf)**.** To give an example of a presheaf which is not a sheaf, we take a presheaf as in (3.3) where $(*)$ is not a local condition.

Let $X = \mathbb{C} - \{0\}$, and let $\mathscr{A}_X$ denote the sheaf of complex analytic functions (which is a sheaf by Example 3.5). Inside $\mathscr{A}_X$ we find a subpresheaf defined by

$$\mathcal{F}(U) = \{\, f \in \mathscr{A}_X(U) \mid f = g^2 \text{ for some } g \in \mathscr{A}_X(U) \,\}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

While $\mathcal{F}$ satisfies the Locality axiom (being a subpresheaf of a sheaf), the Gluing axiom fails. Consider for instance, the function $f(z) = z$ in $\mathcal{F}(X)$. For each point $a \in \mathbb{C} - \{0\}$, we can find an open neighbourhood U_a and a local square root g_a such that $g_a^2 = z$. The g_a can moreover be chosen to agree on the overlaps $U_a \cap U_{a'}$ by consistently choosing signs. However, there is no continuous function $g(z)$ on $\mathbb{C} - \{0\}$ such that $g(z)^2 = z$, as any such function would change sign when z transveses a loop around the origin. △

Example 3.7 (Constant sheaves)**.** For any space X and any abelian group A, one can define the *constant presheaf* by setting $A(U) = A$ for any nonempty open set U and $A(\varnothing) = 0$, with restriction maps given by the identity $A \to A$ between nonempty opens and by the zero map $A \to 0$ otherwise.

This is not a sheaf in general. For instance, if $X = U_1 \sqcup U_2$ is a disjoint union, and $A = \mathbb{Z}$, then any choice of integers $a_1, a_2 \in \mathbb{Z}$ define sections of $A(U_1)$ and $A(U_2)$. These sections trivially agree over $U_1 \cap U_2 = \varnothing$, because they both map to 0 in $A(\varnothing) = 0$. However, if $a_1 \neq a_2$, they do not glue to a section over X as $A(X) = \mathbb{Z}$. In fact, the constant presheaf is a sheaf if and only if any two non-empty open subsets of X have non-empty intersection. Algebraic varieties with the Zariski topology are examples of such spaces.

There is a quick fix for this, namely to define

$$A_X(U) = \{\, f \colon U \to A \mid f \text{ is locally constant} \,\}$$

for every nonempty open set $U \subset X$, and again set $A_X(\varnothing) = 0$. Hence by replacing 'constant' by 'locally constant', we obtain a sheaf (e.g. by Example 3.5).

The new presheaf A_X is called the *constant sheaf* on X with value A. Despite its name, the sheaf A_X is not quite constant: it is only constant on each connected component. If U is connected, we have $A_X(U) = A$, but if U has r connected components $U_1, \ldots, U_r$, then $A_X(U) \simeq A \times \cdots \times A$ (with r factors) since a locally constant function is determined by its values on each U_i. △

3.2 Stalks

Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X. For any point $x \in X$, we can associate an abelian group $\mathcal{F}_x$ called the *stalk* of $\mathcal{F}$ at x. Intuitively, this group is designed to capture the behavior of sections of $\mathcal{F}$ in small neighbourhoods of x.

Formally, the *stalk* $\mathcal{F}_x$ is defined as the direct limit of the groups $\mathcal{F}(U)$ as U runs through the directed set of open neighbourhoods U of x (ordered by inclusion)[1]:

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

In more concrete terms, $\mathcal{F}_x$ is constructed as follows. Consider the disjoint union $\coprod_{U \ni x} \mathcal{F}(U)$, where the elements are pairs (s, U) with U an open neighbourhood of x and s is a section in $\mathcal{F}(U)$. We identify sections that coincide near x by declaring (s, U) and (s', U') to be equivalent, written $(s, U) \sim (s', U')$, if there is an open $V \subset U \cap U'$ with $x \in V$ such that

$$s|_V = s'|_V.$$

[1] For background on direct limits, see Appendix C

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The relation $\sim$ is reflexive and symmetric by definition. For transitivity, suppose $(s, U) \sim (s', U')$ and $(s', U') \sim (s'', U'')$. Then there exist open neighbourhoods $V \subset U \cap U'$ and $V' \subset U' \cap U''$ of x such that $s|_V = s'|_V$ and $s'|_{V'} = s''|_{V'}$. The restrictions of s and s'' then coincide over the open set $V' \cap V$, and so $(s, U) \sim (s'', U'')$. Therefore, $\sim$ is an equivalence relation. The stalk $\mathcal{F}_x$ at $x \in X$ is then the set of equivalence classes

$$\mathcal{F}_x = \coprod_{U \ni x} \mathcal{F}(U) / \sim .$$

The equivalence class of a section (s, U) is called the *germ* of s and it is denoted by s_x.

When $\mathcal{F}$ is a presheaf of abelian groups, the stalks $\mathcal{F}_x$ are all abelian groups. This is not completely obvious, because sections over different open sets can not be added directly. However, if (s, U) and (s', U') are given, the restrictions $s|_V$ and $s'|_V$ to any open $V \subset U \cap U'$ can be added within $\mathcal{F}(V)$, and this suffices to define an abelian group structure on $\mathcal{F}_x$.

For each open set U containing x, there is a canonical map sending a section to its germ:

$$\mathcal{F}(U) \longrightarrow \mathcal{F}_x, \qquad s \longmapsto s_x.$$

This is compatible with restrictions: if $V \subset U$ is another open set containing x, we have $s_x = (s|_V)_x$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x. \\ {\scriptstyle \rho_{UV}}\downarrow & \nearrow & \\ \mathcal{F}(V) & & \end{array} \tag{3.4}$$

When working with sheaves and stalks, it is useful to keep in mind the following facts:

- If $s \in \mathcal{F}(U)$ satsifies $s_x = 0 \in \mathcal{F}_x$, then there exists an open set $V \subset U$ containing x such that $s|_V = 0$.
- Every element of $\mathcal{F}_x$ is the germ s_x of some section s defined on a neighbourhood of x.
- A sheaf $\mathcal{F}$ is the zero sheaf if and only if all stalks are zero, that is, $\mathcal{F}_x = 0$ for all $x \in X$.

A map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a map between the stalks

$$\phi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x.$$

This map sends the equivalence class of (s, U) to the equivalence class of $(\phi_U(s), U)$. This is well-defined because ϕ commutes with restrictions: if $(s, U) \sim (s', U')$ and s and s' coincide on an open set $V \subset U \cap U'$, then the diagram (3.2) gives

$$\phi_U(s)|_V = \phi_V(s|_V) = \phi_V(s'|_V) = \phi_{U'}(s')|_V.$$

One also verifies that $(\phi \circ \psi)_x = \phi_x \circ \psi_x$ and $(\mathrm{id}_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}$, so the assignments $\mathcal{F} \mapsto \mathcal{F}_x$ and $\phi \mapsto \phi_x$ define a functor from the category of sheaves to the category of abelian groups.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

3.3 Sheaves defined on a basis

Recall that a *basis* for the topology on a space X is a collection of open subsets $\mathscr{B}$ such that every open set of X can be written as a union of members of $\mathscr{B}$. In many situations, it is convenient to define a sheaf by specifying it only the open sets of a basis. In this section, we show that this always determines a unique sheaf on X.

A *presheaf on* $\mathscr{B}$ (or *$\mathscr{B}$-presheaf*) F consists of (i) for each $U \in \mathscr{B}$, an abelian group $F(U)$, and (ii) for each pair $V \subset U$ in $\mathscr{B}$, a restriction map $\rho_{UV} \colon F(U) \to F(V)$ satisfying $\rho_{UU} = \mathrm{id}_{F(U)}$ and $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for each triple $W \subset V \subset U$ of open sets in $\mathscr{B}$. A presheaf on $\mathscr{B}$ is furthermore a *sheaf on* $\mathscr{B}$ if it satisfies the Locality and Gluing axioms with respect to covers by basis elements. Precisely, this means that for every open set $U \in \mathscr{B}$ and any open cover $\{U_i\}_{i\in I}$ with $U_i \in \mathscr{B}$, the following sequence is exact:

$$0 \longrightarrow F(U) \longrightarrow \prod_{i\in I} F(U_i) \longrightarrow \prod_{V\subset U_i\cap U_j,\ V\in\mathscr{B}} F(V) \tag{3.5}$$

where the maps are defined by restrictions and differences of restrictions, as before.

Proposition 3.8. Let $\mathscr{B}$ be a basis for a topological space X. Then:
(i) Every $\mathscr{B}$-sheaf F has a unique extension to a sheaf $\mathcal{F}$ on X.
(ii) Every morphism of $\mathscr{B}$-sheaves $\phi_0 \colon F \to G$ extends uniquely to a morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$.

The basic idea is that a a section of $\mathcal{F}$ over an open set U is obtained by covering U by basis elements $V_i \in \mathscr{B}$ and taking sections $s_i \in \mathcal{F}(V_i)$ such that

$$s_i|_W = s_j|_W \quad \text{whenever } W \in \mathscr{B} \text{ and } W \subset V_i \cap V_j.$$

In other words, $\mathcal{F}(U)$ is the set of all compatible collections of sections of F over basis elements covering U. If we define $\mathcal{F}(U)$ and $\mathcal{G}(U)$ in this way, we can define the extended sheaf map $\phi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ by sending (s_i) to $(\phi_0)_{V_i}(s_i)$.

Proof To define the group $\mathcal{F}(U)$ in a way that is independent of any particular cover, we consider the 'biggest cover' of U, where we include *every* $V \in \mathscr{B}$ contained in U. More precisely, let $\mathscr{B}_U \subset \mathscr{B}$ consist of the open sets in $\mathscr{B}$ which are contained in U and define

$$\mathcal{F}(U) = \left\{ (s_V) \in \prod_{V\in\mathscr{B}_U} F(V) \;\middle|\; s_V|_W = s_W,\ \forall\, W \subset V \text{ in } \mathscr{B} \right\}. \tag{3.6}$$

An element of $\mathcal{F}(U)$ is given by a set of sections $s_V \in F(V)$, one for each V, such that whenever $W \subset V$ in $\mathscr{B}_U$, we have $s_V|_W = s_W$.

Note that if $U' \subset U$, then $\mathscr{B}_{U'} \subset \mathscr{B}_U$, and so the projection maps give us restriction maps $\mathcal{F}(U) \to \mathcal{F}(U')$. This makes $\mathcal{F}$ into a presheaf. We next check the sheaf axioms.

Locality: Suppose $s = (s_V) \in \mathcal{F}(U)$ is a collection of compatible elements and $\{U_i\}_{i\in I}$ is a covering of U such that $s|_{U_i} = 0$ for every i. Let $V \in \mathscr{B}_U$ be any open set. Since $\mathscr{B}$ is a basis, we can find a covering $\mathcal{V}$ of V consisting of open sets $B \in \mathscr{B}$ such that for each $B \in \mathcal{V}$, there is an index $i(B)$ such that $B \subset U_{i(B)}$. Now $s|_{U_i} = 0$ means that $s_B = 0$ for every $B \subset U_i$ and so $s_V|_B = s_B = 0$ for every $B \in \mathcal{V}$. In particular, by the Locality axiom for F, we must have $s_V = 0$. Since this happens for every V, we see that $s = 0$ as well.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Gluing: Let $\{U_i\}_{i\in I}$ be an open cover of U and let $s^i = (s^i)_V \in \mathcal{F}(U_i)$ be a collection of elements so that $s^i|_{U_i\cap U_j} = s^j|_{U_i\cap U_j}$ for all i, j. This means that for each i and j, and $B \subset U_i \cap U_j$ in $\mathscr{B}$, we have

$$(s^i)_B = (s^j)_B. \tag{3.7}$$

Fix $V \in \mathcal{B}_U$. We would like to define an element $s_V \in F(V)$ from the s^i and then set $s = (s_V)$ in $\mathcal{F}(V)$. As above, we choose an open cover $\mathcal{V}$ of V by open sets $B \in \mathscr{B}$ such that each B is contained in some $U_{i(B)}$. For each $B \in \mathcal{V}$, we define $s_B = (s^{i(B)})_B \in F(B)$ (by (3.7), this is independent of the choice of index $i(B)$). Then the elements s_B glue to a unique element $s_V \in F(V)$, because if $W \in \mathscr{B}$ is contained in $B \cap B'$ with $B, B' \in \mathcal{V}$, then by (3.7),

$$s_B|_W = (s^{i(B)})_B|_W = (s^{i(B)})_W = (s^{i(B')})_W = (s^{i(B')})_{B'}|_W = s_{B'}|_W.$$

By uniqueness, the elements s_V constructed in this way are moreover compatible, so it makes sense to define $s = (s_V) \in \mathcal{F}(U)$. It is clear that $s|_{U_i} = s^i$ for every i. The presheaf $\mathcal{F}$ therefore satisfies the Gluing axiom, and hence $\mathcal{F}$ is a sheaf.
Note that if U is an open set in $\mathscr{B}$, there is an obvious map $F(U) \longrightarrow \mathcal{F}(U)$. sending $t \in F(U)$ to the collection $(t|_V)_V \in \mathcal{F}(U)$. This is in fact an isomorphism, with inverse defined by the projection $\mathcal{F}(U) \to F(U)$ to the 'U-th component'. Therefore, $\mathcal{F}$ agrees with F over opens in $\mathscr{B}$.

(ii): Saying that $\phi_0 \colon F \to G$ is a map of $\mathscr{B}$-sheaves amounts to saying that the following diagram commutes for each pair $V' \subset V$ of opens in $\mathscr{B}$:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{(\phi_0)_V} & G(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(V') & \xrightarrow[(\phi_0)_{V'}]{} & G(V'). \end{array}$$

Taking the products over all open subsets $V \in \mathscr{B}_U$, we obtain a natural map $\mathcal{F}(U) \to \mathcal{G}(U)$ which extends ϕ_0. These maps are moreover compatible with the restriction maps, so we get a map of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$.

The extension $\mathcal{F}$ is unique: If $\mathcal{G}$ is another extension, satisfying $\mathcal{F}(U) = \mathcal{G}(U)$ for every $U \in \mathscr{B}$, then since $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we may use the expression (3.6) to compute $\mathcal{F}(U)$ and $\mathcal{G}(U)$ for any open set. Using this, we obtain a canonical isomorphism of sheaves $\mathcal{F} \simeq \mathcal{G}$. A similar argument shows that the extended morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ in (ii) is unique. □

3.4 The pushforward of a sheaf

Given a continuous map $f \colon X \to Y$ and a sheaf $\mathcal{F}$ on X, we define the *pushforward* $f_*\mathcal{F}$ to be the sheaf on U defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U),$$

and the restriction maps $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ to be those coming from $\mathcal{F}$.

It is straightforward to verify that the sheaf axioms are satisfied for $f_*\mathcal{F}$. Indeed, if $\{U_i\}_{i\in I}$ is an open covering of U, then $\{f^{-1}U_i\}$ is an open covering of $f^{-1}U$. A set of gluing

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

data for $f_*\mathcal{F}$ and the given covering consists of sections $s_i \in \Gamma(U_i, f_*\mathcal{F}) = \Gamma(f^{-1}U_i, \mathcal{F})$ that agree on the intersections. This means that they coincide in $\Gamma(U_i \cap U_j, f_*\mathcal{F})$, which equals $\Gamma(f^{-1}U_i \cap f^{-1}U_j, \mathcal{F})$, and they may therefore be glued together to a section in $\Gamma(f^{-1}U, \mathcal{F}) = \Gamma(U, f_*\mathcal{F})$, as $\mathcal{F}$ is a sheaf. The Locality axiom follows for $f_*\mathcal{F}$, because it holds for $\mathcal{F}$.

If $g\colon X \to Y$ and $f\colon Y \to Z$ are continuous maps, and $\mathcal{F}$ is a sheaf on X, there is an equality of sheaves on Z

$$(f \circ g)_*\mathcal{F} = f_*(g_*\mathcal{F}). \tag{3.8}$$

Indeed, both sheaves have value $\Gamma(g^{-1}f^{-1}(V), \mathcal{F})$ for an open set $V \subset Z$.

Example 3.9. If $Y = \{*\}$ consists of a single point and $f\colon X \to Y$ is the one and only map, then $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$, seen as a constant sheaf on Y. △

Example 3.10 (Skyscraper sheaves)**.** Let $\iota\colon \{x\} \to X$ be the inclusion of a closed point in X and let $\mathcal{A}$ be the constant sheaf of an abelian group A on $\{x\}$. The *skyscraper sheaf* of A at x, denoted $A(x)$, is the pushforward sheaf $\iota_*\mathcal{A}$. Explicitly, the sections are given by

$$A(x)(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

△

3.5 The structure sheaf on the spectrum of a ring

We now define the structure sheaf on $\operatorname{Spec} A$. To motivate the definition, consider an affine variety $X \subset \mathbb{A}^n(k)$ be an affine variety with coordinate ring $A = A(X)$. For each open set $U \subset X$, the ring $\mathcal{O}_X(U)$ consists of the rational functions that are regular at every point of U. The elements of $\mathcal{O}_X(U)$ can be viewed either as elements of the function field $K(X)$, or as functions $f\colon U \to k$. Moreover, each restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ can be seen as an inclusion of subrings of $K(X)$, or as an actual restriction map on functions.

For a distinguished open set $D(f)$, the ring of regular functions $\mathcal{O}_X(D(f))$ is simply the localization A_f (Proposition 1.21). This is intuitive, because the elements of the form a/f^n define regular functions on $D(f)$. Moreover, if $D(g) \subset D(f)$, the restriction map corresponds to the canonical localization map $A_f \to A_g$.

In fact, the sheaf $\mathcal{O}_X$ is completely determined by $\mathcal{O}_X(D(f)) = A_f$ for each $f \in A$. Indeed, inside $K(X)$, we have

$$\mathcal{O}_X(U) = \bigcap_{D(f)\subset U} \mathcal{O}_X(D(f)). \tag{3.9}$$

If one tries to carry out the above construction for a general ring A, there are a few issues that arise. First, there may not exist a natural field k for which we can view elements of $\mathcal{O}_X(U)$ as functions $U \to k$. More importantly, the localization maps $A_f \to A_g$ may fail to be injective (so an intersection like (3.9) does not make sense). This happens already in the simple example when $A = k[x, y]/(xy)$, which corresponds to the union of x-axis and the y-axis in the affine plane. Here the nonzero element $x \in A$ maps to 0 via the localization

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

map $A \to A_y$. So this behaviour is not a big surprise; it naturally appears once we allow reducible spaces into our discussion.

Motivated by this discussion, we consider the basis of distinguished open sets $D(f)$. We would like $\mathcal{O}_X$ to satisfy $\mathcal{O}_X(D(f)) \simeq A_f$ for every $f \in A$, with restriction maps $\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$ corresponding to the localization maps $A_f \to A_g$ whenever $D(f) \supset D(g)$. There is a small subtlety here, namely that two different f's may define the same distinguished open set U (for instance, $D(f) = D(f^2)$). A nice way to resolve this is to work with a more canonical localization:

Definition 3.11. For a distinguished open set U, we define

$$\mathcal{O}(U) = S(U)^{-1}A. \tag{3.10}$$

where $S(U) = \{\, s \in A \mid D(s) \supset U \,\}$ is the multiplicative subset of elements $s \in A$ that vanish nowhere on U.

The elements of $\mathcal{O}(U)$ are fractions a/s so that $s(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in U$. If $U \supset V$ are two distinguished open sets, then $S(U) \subset S(V)$, so by the universal property of localization, there is a canonical map

$$S(U)^{-1}A \longrightarrow S(V)^{-1}A,$$

which we use to define the restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$.

Note that for $U = D(f)$, we have $f \in S(U)$, and so there is a canonical localization map $A_f \to S(U)^{-1}A$. We claim that this is an isomorphism:

Lemma 3.12. When $U = D(f)$, there is a canonical isomorphism $S(U)^{-1}A \simeq A_f$.

Proof This map is injective: if $a/f^n \in A_f$ maps to 0 in $S(U)^{-1}A$, there is an $s \in S(U)$ such that $sa = 0$. By definition, we have $D(f) \subset D(s)$, which is equivalent to being able to write $f^N = cs$ for some $c \in A$ and $N \in \mathbb{N}$. Multiplying $sa = 0$ by c now shows that $f^N a = 0$ in A, and hence $a/f^n = 0$ in A_f. The map is also surjective: if $a/s \in S(U)^{-1}A$, then writing $f^N = cs$ as before, we have $a/s = (ca)/f^N$ in $S(U)^{-1}A$, which is in the image of $A_f \to S(U)^{-1}A$. □

In light of this, we will from now on always identify $\mathcal{O}(D(f)) = A_f$, and $\mathcal{O}(D(f)) \to \mathcal{O}(D(g))$ with $A_f \to A_g$ whenever $D(g) \subset D(f)$.

We now prove that $\mathcal{O}$ is a $\mathscr{B}$-sheaf. This will follow from the following algebraic lemma:

Lemma 3.13. Let R be a ring and let $f_1, \ldots, f_r \in R$ be elements generating the unit ideal. For any R-module M, the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\alpha} \prod_{i=1}^r M_{f_i} \xrightarrow{\beta} \prod_{i,j=1}^r M_{f_i f_j} \tag{3.11}$$

where $\alpha(s) = (s/1, \ldots, s/1)$ and $\beta(s_1, \ldots, s_r)_{i,j} = s_i/1 - s_j/1$.

Proof Note that (3.11) is a sequence of R-modules. It is therefore exact if and only if it

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is exact after being localized at every prime ideal $\mathfrak{p}$. After reordering the indices, we may assume that $\mathfrak{p} \in D(f_1)$. Since $f_1 \notin \mathfrak{p}$, the localization of (3.11) at $\mathfrak{p}$ factors via localizing at f_1. As localization is exact and commutes with finite direct products, it suffices to show that the following sequence is exact:

$$0 \longrightarrow M_{f_1} \xrightarrow{a} \prod_{i=1}^r M_{f_1 f_i} \xrightarrow{b} \prod_{i,j=1}^r M_{f_1 f_i f_j}. \tag{3.12}$$

In this sequence, f_1 is invertible.

Injectivity of a: For $s \in M_{f_1}$, if $a(s) = (s/1, \ldots, s/1) = (0, \ldots, 0)$, then in particular $s/1 = 0$ in $M_{f_1 f_1} = M_{f_1}$. But this means that $s = 0$.

Exactness in the middle: The containment $\operatorname{Ker} b \supset \operatorname{Im} a$ is clear. Conversely, take $(s_1, \ldots, s_r) \in \prod_{i=1}^r M_{f_1 f_i}$ in $\operatorname{Ker} b$. Then for all i, j, we have $s_i/1 = s_j/1$ in $M_{f_1 f_i f_j}$. In particular, for $j = 1$, we have $s_i/1 = s_1/1$ in $M_{f_1 f_i f_1} = M_{f_1 f_i}$. This shows that $(s_1, \ldots, s_r) = a(s_1)$ and hence $\operatorname{Ker} b \subset \operatorname{Im} a$. □

Proposition 3.14. *$\mathcal{O}$ is a sheaf on the basis of distinguished open sets.*

Proof *Case 1. Finite covers.* Suppose that $D(f)$ is a distinguished open set and $\{D(f_i)\}_{i=1}^r$ is a finite cover of $D(f)$. Then applying Lemma 3.13 to $R = A_f$ and $M = A_f$ gives an exact sequence

$$0 \longrightarrow A_f \xrightarrow{\alpha} \prod_{i=1}^r A_{f_i} \xrightarrow{\beta} \prod_{i,j=1}^r A_{f_i f_j}. \tag{3.13}$$

The Locality and Gluing axioms follow immediately from this.

Case 2. Arbitrary covers. Suppose now that $\{D(f_i)\}_{i \in I}$ is a general cover. Since $D(f)$ is quasi-compact, we may pick a finite subset $J \subset I$, so that $\{D(f_j)\}_{j \in J}$ forms a subcover.

Locality axiom: If $s \in A_f$ maps to zero in A_{f_i} for every $i \in I$, then in particular, it maps to zero in each A_{f_i} for $i \in J$, so by Case 1, we have $s = 0$ in A_f.

Gluing axiom: Given compatible elements $s_i \in A_{f_i}$ for $i \in I$ (meaning $s_i/1 = s_j/1$ in $A_{f_i f_j}$ for all $i, j \in I$), then Case 1 provides a unique $s \in A_f$ such that $s_i = s/1 \in A_{f_i}$ for all $i \in J$. We claim that s in fact induces s_i for all $i \in I$. For this, fix an index $\alpha \in I$. Applying Case 1 to the finite covering consisting of $D(f_i)$, $i \in J \cup \{\alpha\}$, we get an element $s' \in A_f$ such that $s'/1 = s_i$ in A_{f_i} for all $i \in J$ and also $s'/1 = s_\alpha$. As s and s' coincide in each A_{f_i} for $i \in J$, we must also have $s = s'$ in A_f by uniqueness in Case 1, and hence $s/1 = s_\alpha$ holds in A_{f_α} as well. □

Using Proposition 3.8 on page 43, we may now make the following definition:

Definition 3.15. The *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ is the unique sheaf extending the $\mathscr{B}$-sheaf $\mathcal{O}$.

One should note that the formula $\mathcal{O}_{\operatorname{Spec} A}(U) = S(U)^{-1}A$ does not hold for an arbitrary open set $U \subset \operatorname{Spec} A$ (see Exercise 3.7.15). Instead, to compute the ring $\mathcal{O}_{\operatorname{Spec} A}(U)$, one covers U by distinguished open sets $D(f_i)$, $i \in I$, and then

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \left\{ (s_i) \in \prod_{i \in I} A_{f_i} \;\middle|\; s_i/1 = s_j/1 \text{ in } A_{f_i f_j} \text{ for all } i, j \in I \right\}. \tag{3.14}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

In practice, most computations with $\mathcal{O}_{\operatorname{Spec} A}$ will only involve distinguished open sets.

Proposition 3.16.
(i) $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$.
(ii) If $P \in \operatorname{Spec} A$ is a point corresponding to a prime ideal $\mathfrak{p}$, then

$$\mathcal{O}_{\operatorname{Spec} A, P} = A_{\mathfrak{p}}.$$

Proof We defined $\mathcal{O}_{\operatorname{Spec} A}$ so that $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ for every $f \in A$. Taking $f = 1$, we see that (i) holds.

For (ii): Since the distinguished open sets form a basis for the topology on $\operatorname{Spec} A$, any element of $\mathcal{O}_{\operatorname{Spec} A, P}$ may be represented by a pair $(D(f), \sigma)$ where $P \in D(f)$ and $\sigma = a/f^n \in \mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$. We define

$$\rho \colon \mathcal{O}_{\operatorname{Spec} A, P} \longrightarrow A_{\mathfrak{p}}. \tag{3.15}$$

by sending $(D(f), a/f^n) \mapsto a/f^n$. This is well-defined because if $(D(f), a/f^n)$ and $(D(g), b/g^m)$ represent the same germ, then they agree over the open set $D(fg)$, which means that $a/f^n = b/g^m$ in A_{fg}, and hence they map to the same element in $A_{\mathfrak{p}}$.

The map ρ is surjective because any $a/s \in A_{\mathfrak{p}}$ with $s \in A - \mathfrak{p}$ is in the image of a section $a/s \in A_s = \mathcal{O}_X(D(s))$. For injectivity: if $(D(f), a/f^n)$ maps to zero in $A_{\mathfrak{p}}$, then there is some $g \in A - \mathfrak{p}$ such that $ga = 0$. But then a/f^n maps to zero in A_{fg}. This means that a/f^n restricts to zero in $\mathcal{O}_X(D(fg))$ and hence it is zero in $\mathcal{O}_{\operatorname{Spec} A, P}$. □

Remark 3.17. The structure sheaf behaves particularly well when the base ring A is an integral domain. In this case, all the localizations A_f are subrings of the field of fractions K of A, and for $D(g) \subset D(f)$, the localization maps $A_f \to A_g$ are simply inclusions. This implies that $\mathcal{O}_X(U)$ is a subring of K for any open set U. Indeed, covering U by distinguished open sets $\{D(f_i)\}_{i \in I}$, the formula (3.14) shows that $\mathcal{O}_X(U)$ is identified with the intersection

$$\mathcal{O}_X(U) = \bigcap_{i \in I} A_{f_i} = \bigcap_{i \in I} \bigcap_{\mathfrak{p} \in D(f_i)} A_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

Consequently, when A is an integral domain, the intuition from affine varieties is valid: the rings $\mathcal{O}_X(U)$ are subrings of the fraction field of A, and the elements $f \in \mathcal{O}_X(U)$ are precisely those for which, for each point $P \in U$, we may write $f = a/b$ and $b(P) \neq 0$.

In algebraic geometry, the localizations A_f and $A_{\mathfrak{p}}$ serve different purposes. The ring A_f is used to study the open set obtained by removing the hypersurface $V(f)$, whereas $A_{\mathfrak{p}}$ is used to study the space in a neighbourhood of the point $\mathfrak{p}$. It is this local nature of $A_{\mathfrak{p}}$ that motivates the terms *localization* and *local ring*.

Example 3.18. For a field K, the structure sheaf $\mathcal{O}_{\operatorname{Spec} K}$ is a constant sheaf with the value K at the single point of $\operatorname{Spec} K$. △

Example 3.19. The structure sheaf of $\operatorname{Spec} \mathbb{Z}$ satisfies $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(n)) = \mathbb{Z}[\frac{1}{n}]$ for each natural number n. The stalks of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ at the closed point (p) is equal to $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}, (p)} = \mathbb{Z}_{(p)}$ and at the generic point the stalk is equal to $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}, (0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 3.20. Let $X = \operatorname{Spec} \mathbb{C}[t]$. Then the stalk of $\mathcal{O}_X$ at the generic point $\xi = (0)$ is equal to $\mathcal{O}_{X,\xi} = \mathbb{C}(t)$. Each closed point $P \in X$ corresponds to a maximal ideal $(t-a)$, and the stalk of $\mathcal{O}_X$ at P is equal to $\mathcal{O}_{X,P} = \mathbb{C}[t]_{(t-a)}$. △

Example 3.21. Consider a discrete valation ring A, as in Example 2.29. The space $X = \operatorname{Spec} A$ has two points, s and ξ, and there are three open sets: $\varnothing$, $\{\xi\}$, and X. The maximal ideal of A is principal, so we may write $\mathfrak{m} = (t)$ for $t \in A$. Then $\{\xi\} = D(t)$ is a distinguished open in X. The structure sheaf takes the following values at the open sets:

$$\mathcal{O}_X(\varnothing) = 0, \qquad \mathcal{O}_X(X) = A, \qquad \mathcal{O}_X(\{\xi\}) = A_t = K(A).$$

The stalks are given by $\mathcal{O}_{X,s} = A_{(t)} = A$ and $\mathcal{O}_{X,\xi} = A_{(0)} = K(A)$. △

Example 3.22. Let $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ and consider the open subset

$$U = \mathbb{A}^2_k - V(x,y) = D(x) \cup D(y)$$

In analogy with Example 1.22, we will show that the restriction map $\mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k) \to \mathcal{O}_{\mathbb{A}^2_k}(U)$ is an isomorphism. In particular, $\mathcal{O}_{\mathbb{A}^2_k}(U) = k[x,y]$.

To do this, we use the sheaf sequence (3.1) for the covering $\{D(x), D(y)\}$ of U. In view of the equalities $\mathcal{O}_{\mathbb{A}^2_k}(D(x)) = k[x,y]_x$, $\mathcal{O}_{\mathbb{A}^2_k}(D(y)) = k[x,y]_y$, and $\mathcal{O}_{\mathbb{A}^2_k}(D(xy)) = k[x,y]_{xy}$, the sheaf sequence takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(U) \xrightarrow{\alpha} k[x,y]_x \oplus k[x,y]_y, \xrightarrow{\beta} k[x,y]_{xy}.$$

Here $\alpha(s) = (s|_{D(x)}, s|_{D(y)})$ and β sends $(a/x^n, b/y^m)$ to $a/x^n - b/y^m$. The group $\mathcal{O}_{\mathbb{A}^2_k}(U)$ is identified with the kernel of β, which consists of pairs $(a/x^n, b/y^m)$ such that $a/x^n = b/y^m$ in $k[x,y]_{xy}$. This equality implies that $ay^m = bx^n$ in $k[x,y]$. As the polynomial ring $k[x,y]$ is a UFD, we conclude that $a = cx^n$ and $b = cy^m$ for some $c \in k[x,y]$. But this means that any section of $\mathcal{O}_X(U)$ is of the form (c,c) with $c \in k[x,y]$. Those sections are exactly the restrictions from $\mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k) = k[x,y]$, as we wanted to show. △

The construction of the structure sheaf can be extended to any A-module. For an A-module M, define a presheaf $\widetilde{M}$ on the base of distinguished opens $D(f) \subset \operatorname{Spec} A$ by

$$\widetilde{M}(D(f)) = M_f,$$

with restriction maps $M_f \to M_g$ the canonical localization maps whenever $D(g) \subset D(f)$. The same proof as in Proposition 3.14 shows that $\widetilde{M}$ satisfies the two $\mathscr{B}$-sheaf axioms, so it extends uniquely to a sheaf on $\operatorname{Spec} A$, which we again denote $\widetilde{M}$.

Proposition 3.23. (i) The global sections of $\widetilde{M}$ recovers the module,

$$\Gamma(\operatorname{Spec} A, \widetilde{M}) = M.$$

(ii) For $P \in \operatorname{Spec} A$ corresponding to a prime ideal $\mathfrak{p}$, the stalk is given by

$$\widetilde{M}_P = M_{\mathfrak{p}}.$$

This construction is functorial in M. For any A-linear map $\phi\colon M \to N$, there is an

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

induced map of sheaves $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$. To define this sheaf map, it suffices by Proposition 3.8 to specify it over the distinguished open sets. Over $D(f)$, we define $\widetilde{\phi}_{D(f)}\colon \widetilde{M}(D(f)) \longrightarrow \widetilde{N}(D(f))$ to be the A_f-linear map

$$\phi_f\colon M_f \to N_f, \qquad m/f^n \mapsto \phi(m)/f^n.$$

To see that this defines a map of $\mathscr{B}$-sheaves, we note that the following diagram commutes, where $f, g \in A$ are two elements satisfying $D(g) \subset D(f)$.

$$\begin{array}{ccc} M_f & \xrightarrow{\phi_f} & N_f \\ \downarrow & & \downarrow \\ M_g & \xrightarrow{\phi_g} & N_g \end{array}$$

Indeed, writing $g^r = af$, the composition $M_f \to M_g \to N_g$ sends m/f^n to $\phi(ma^n)/g^{rn} = a^n\phi(m)/g^{rn}$, which is is the same as the image via $M_f \to N_f \to N_g$. From this, we

We also have $\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi}$, whenever ϕ and ψ are composable A-linear maps. Therefore, the assignment $M \mapsto \widetilde{M}$ defines a functor from the category Mod_A of A-modules to the category of sheaves on $\operatorname{Spec} A$.

3.6 Application: Connectedness of $\operatorname{Spec} A$

A topological space is *connected* if it cannot be written as a disjoint union of two proper open subsets. A natural question is which rings A have connected prime spectrum. A simple disconnected example is $\operatorname{Spec}(\mathbb{C} \times \mathbb{C})$ which consists of two points, namely $\mathfrak{p} = \mathbb{C} \times 0$ and $\mathfrak{q} = 0 \times \mathbb{C}$ with neither contained in the other. As an application of the structure sheaf, we prove that this example is typical: $\operatorname{Spec} A$ is disconnected if and only if A is a product of two nonzero rings.

Proposition 3.24. A spectrum $\operatorname{Spec} A$ is disconnected if and only if A is isomorphic to a direct product $A = A_1 \times A_2$ of nonzero rings A_1 and A_2.

Proof Suppose $A = A_1 \times A_2$ where A_1 and A_2 are nonzero. The elements $e_1 = (1, 0)$ and $e_2 = (0, 1)$ they satisfy the relations $e_1^2 = e_1$, $e_1e_2 = 0$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$ (that is, they are *orthogonal idempotents*).

The spectrum $\operatorname{Spec} A$ decomposes as the disjoint union $\operatorname{Spec} A = V(e_1) \cup V(e_2)$ of the two disjoint closed sets $V(e_i)$. Indeed, since $e_1e_2 = 0$, every prime $\mathfrak{p}$ must contain either e_1 or e_2, so the $V(e_i)$'s cover $\operatorname{Spec} A$. Moreover, since $e_1 + e_2 = 1$, we have $V(e_1) \cap V(e_2) = V(e_1, e_2) = \varnothing$, so the subsets are disjoint.

Conversely, suppose $X = U_1 \cup U_2$, where U_1, U_2 are disjoint nonempty open subsets. The sheaf exact sequence takes the form

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1 \cap U_2) = 0.$$

By exactness, we see that the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$, which is a ring map, is an isomorphism. Therefore $A = \mathcal{O}_X(X)$ is a nontrivial product of rings. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 3.25. Any irreducible space is also connected, but the converse does not hold. For instance, $\operatorname{Spec} k[x,y]/(xy)$ corresponds to the union of two intersecting lines $V(xy) = V(x) \cup V(y)$ in $\mathbb{A}^2_k$, so it is connected but not irreducible. △

3.7 Exercises

Exercise 3.7.1. Let X be the set with two elements with the discrete topology. Find a presheaf on X for which the Locality axiom fails. HINT: Define $\mathcal{F}(X)$ to be much bigger than $\mathcal{F}(\{p\})$ and $\mathcal{F}(\{q\})$.

Exercise 3.7.2. Let $X \subset \mathbb{C}$ be an open set and let $\mathscr{A}_X$ be the presheaf of holomorphic functions, that is, the sections $\mathscr{A}_X(U)$ is the ring of complex differentiable functions on U.

a) Show that $\mathscr{A}_X$ is a *subsheaf* of the sheaf of complex-valued continuous functions $U \to \mathbb{C}$.
b) Show that the stalk $\mathscr{A}_{X,P}$ can be identified with the ring of power series that converge in a neighbourhood of P.
c) Show that the differential operator $D = d/dz$ defines a map of sheaves $D\colon \mathscr{A}_X \to \mathscr{A}_X$.
d) Show that $\mathcal{F}(U) = \{\, f \in \mathscr{A}_X(U) \mid Df = 0 \,\}$ defines a subsheaf $\mathcal{F}$ of $\mathscr{A}_X$.
e) Show that if U is a connected open subset of X, one has $\mathcal{F}(U) = \mathbb{C}$.
f) For a not necessarily connected set U, show that $\mathcal{F}(U) = \prod_{\pi_0(U)} \mathbb{C}$ where the product is taken over the set $\pi_0(U)$ of connected components of U. Deduce that $\mathcal{F}$ is the constant sheaf with value $\mathbb{C}$.

Exercise 3.7.3. Let $X \subset \mathbb{C}$ be an open set.

a) For $U \subset X$ open, define $\mathcal{K}(U)$ to be the group of meromorphic functions on U (these are functions holomorphic on all of U except for a set of isolated points, where they have poles). Show that $\mathcal{K}(U)$ is a sheaf of groups which contains the sheaf of holomorphic functions $\mathcal{A}_X$ as a subsheaf.
b) For points $a_1, \ldots, a_r \in X$ and $n_1, \ldots, n_r$ natural numbers, define $\mathcal{F}(U)$ to be the set of meromorphic functions in U which are holomorphic away from the a_i's and having a pole order bounded by n_i at a_i. Show that $\mathcal{F}$ is a sheaf of abelian groups.
c) Are $\mathcal{K}$ and $\mathcal{F}$ sheaves of rings?

Exercise 3.7.4. Let $X = S^1$ be the unit circle in $\mathbb{C}$. Let $\mathcal{F}$ be the sheaf of continuous $\mathbb{C}$-valued functions on S^1. Let $\mathcal{G}$ be the sheaf of $\mathbb{C}^\times$-valued functions on S^1 (this is a sheaf of abelian groups under multiplication). Show that the exponential function defines a map of sheaves $\exp\colon \mathcal{F} \to \mathcal{G}$. Show that the presheaf $U \mapsto \operatorname{Im} \exp(U)$ js not a sheaf.

Exercise 3.7.5. Let X be a topological space and let $Z \subset X$ be a closed subset with inclusion $\iota\colon Z \to X$. For a sheaf $\mathcal{F}$ on Z, describe the stalks of $\iota_*\mathcal{F}$.

Exercise 3.7.6. Let X be a topological space and define $\mathcal{F}$ by setting $\mathcal{F}(X) = 0, \mathcal{F}(\varnothing) = 0$ and $\mathcal{F}(U) = \mathbb{Z}$ for every other open set, and with restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ equal to the identity if U and V are different from $\varnothing$ and X, and the zero map otherwise. Show that $\mathcal{F}$ is a presheaf. Is $\mathcal{F}$ a sheaf?

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 3.7.7 (The sheaf of homomorphisms)**.** Given two presheaves $\mathcal{F}$ and $\mathcal{G}$, we may form a presheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$ by letting the sections over an open U be given by

$$\mathcal{H}om(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{\mathsf{Ab}(U)}(\mathcal{F}|_U, \mathcal{G}|_U). \tag{3.16}$$

If $V \subset U$, the restriction map sends $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ to the restriction $\phi|_V\colon \mathcal{F}|_V \to \mathcal{G}|_V$. Show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is a sheaf whenever $\mathcal{G}$ is a sheaf.

Exercise 3.7.8. Let $\mathcal{F}$ be a sheaf and let $s, t \in \mathcal{F}(U)$ be two sections. Show that $s = t$ if and only if $s_x = t_x$ for every $x \in U$.

Exercise 3.7.9. Let $\phi, \psi\colon \mathcal{F} \to \mathcal{G}$ be maps of presheaves and assume that $\mathcal{G}$ is a sheaf. Prove that $\phi = \psi$ if and only if ϕ and ψ induce the same maps on all stalks, i.e. $\phi_x = \psi_x$ for every $x \in X$. HINT: Use Exercise 3.7.8.

Exercise 3.7.10. Let X be a topological space and $x \in X$ a point that is not necessarily closed. Let $\iota\colon \{x\} \to X$ be the inclusion. Let $\mathcal{A}$ be the constant sheaf on $\{x\}$ on the group A. Show that the stalk of $\iota_*\mathcal{A}$ at a point $y \in X$ is equal to A if $y \in \overline{x}$ and 0 otherwise.

Exercise 3.7.11. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X, let $\mathcal{U}$ be an open covering, and let $\theta_U\colon \mathcal{F}|_U \to \mathcal{G}|_U$ be morphisms with $\theta_U|_{U\cap U'} = \theta_{U'}|_{U\cap U'}$ for all $U, U' \in \mathcal{U}$.

a) Show that there is a unique morphism $\theta\colon \mathcal{F} \to \mathcal{G}$ with $\theta|_U = \theta_U$ for all U. HINT: The open sets contained in some $U \in \mathcal{U}$ form a basis.
b) Show that θ is an isomorphism if every θ_U is. HINT: Apply (a) to the inverses.

Exercise 3.7.12 (The restriction of a sheaf)**.** Let $U \subset X$ be an open subset of a topological space X and let $\mathcal{F}$ be a sheaf on X. Define $\mathcal{F}|_U$ by the assignment $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subset U$.

a) Show that $\mathcal{F}|_U$ defines a sheaf on U, and that the stalks at points in U are equal to the corresponding stalks of $\mathcal{F}$.
b) Let $\mathcal{G}$ be a sheaf on U. Let $\iota\colon U \to X$ be the inclusion map. Show that $\iota_*(\mathcal{G})|_U \simeq \mathcal{G}$.

Exercise 3.7.13. Let A be a ring and let M be an A-module. Show that for each $f \in A$, there is a natural isomorphism $\widetilde{M}_f \simeq \widetilde{M}|_{D(f)}$, as sheaves on $D(f) = \mathrm{Spec}(A_f)$.

Exercise 3.7.14. Let $X = \mathbb{R}$ with the standard topology.

a) Let $\mathcal{F}$ be the subpresheaf of $C(-,\mathbb{R})$ defined by

$$\mathcal{F}(U) = \{\, f\colon U \to \mathbb{R} \mid f \text{ is bounded} \,\}$$

Show that $\mathcal{F}$ is not a sheaf.
b) Let $\mathcal{G}$ be the subpresheaf of $C(-,\mathbb{R})$ defined by

$$\mathcal{G}(U) = \{\, f\colon U \to \mathbb{R} \mid f(0) = f(1) \,\}$$

Show that $\mathcal{G}$ is not a sheaf.

Exercise 3.7.15. Let $X = \mathrm{Spec}\, A$ where $A = \mathbb{Q}[x,y,z]/(x^2, xy, xz)$. Show that if $U = D(x) \cup D(y) = X - \{(0,0,0)\}$, then $\mathcal{O}_X(U) \neq S(U)^{-1}A$. HINT: Imitate Example 3.22.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 3.7.16. Describe the topological space $X = \operatorname{Spec} \mathbb{C}[x,y]_{(x,y)}$. Compute $\mathcal{O}_X(U)$ where $U = X - \{(x,y)\}$.

Exercise 3.7.17. Let $A = \mathbb{Q}[x,y,z,w]/(xz, xw, yz, yw)$ and $X = \operatorname{Spec} A$.

a) Show that, as a set, X is the union of two affine planes intersecting at the origin.

b) Let $U \subset X$ be the complement of the origin and consider the subset

$$S(U) = \{f \in A \mid f(P) \neq 0 \text{ for all } P \in U\}$$

Prove that the natural map $S(U)^{-1}A \to \mathcal{O}_X(U)$ is *not* an isomorphism.

c) What happens if X is instead the union of two affine lines intersecting at a single point?

Exercise 3.7.18. Let X be the spectrum of a DVR, as described in Example 2.29. Show that any presheaf on X is automatically a sheaf.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

4

Schemes

4.1 Locally ringed spaces

We would like to define a scheme to be a space which is 'locally affine', that is, one where every point has a neighborhood that is isomorphic to the spectrum of a ring. In order to make such a definition precise, we need a suitable category of spaces to work with. To this end, we use the two pieces of data we have: the topological space $\operatorname{Spec} A$ and its sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$.

Definition 4.1 (Locally ringed spaces)**.** A *locally ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that all the stalks $\mathcal{O}_{X,x}$ are local rings.

For a point $x \in X$, the maximal ideal of $\mathcal{O}_{X,x}$ will be denoted by $\mathfrak{m}_x$ and the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is written $\kappa(x)$. If U is a neighbourhood of x and $s \in \mathcal{O}_X(U)$, we write $s_x \in \mathcal{O}_{X,x}$ for its germ in $\mathcal{O}_{X,x}$. The image of s in $\kappa(x)$ will be denoted by $s(x)$. Note that $s(x) = 0$ if and only if $s_x \in \mathfrak{m}_x$.

To make this into a category, we need to define what a morphism between two locally ringed spaces is. Reflecting the above definition, a morphism between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ has two components, one map between the underlying topological spaces X and Y and one on the level of sheaves. Note that it does not make sense to talk about morphisms $\mathcal{O}_Y \to \mathcal{O}_X$, as these sheaves live on different spaces. Instead, once a continuous map $f\colon X \to Y$ is specified, we specify a map of sheaves of rings $\mathcal{O}_Y \to f_*\mathcal{O}_X$ on Y. Concretely, this means that for all open subsets $U \subset Y$, there are ring maps

$$f_U^\sharp\colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}U),$$

compatible with the restriction maps; that is, making the following diagrams commute for all $V \subset U \subset Y$:

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f_U^\sharp} & \mathcal{O}_X(f^{-1}U) \\ {\scriptstyle \rho_{UV}}\downarrow & & \downarrow{\scriptstyle \rho_{f^{-1}U\, f^{-1}V}} \\ \mathcal{O}_Y(V) & \xrightarrow[f_V^\sharp]{} & \mathcal{O}_X(f^{-1}V). \end{array} \tag{4.1}$$

The intuition comes from the setting of affine varieties. As we saw in Section 1.5, a morphism of affine varieties $f\colon X \to Y$ is precisely a continuous map such that the pullback $g \circ f$ is a

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

regular function on $f^{-1}(U)$ whenever g is regular on U. In other words, the pullback is a ring map $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$. Note that if g vanishes at a point $y \in Y$, then its pullback $g \circ f$ vanishes at every point $x \in f^{-1}(y)$. In other words, if $g \in \mathfrak{m}_y$, then $f^\sharp(g) \in \mathfrak{m}_x$ for each such x and y.

For a general locally ringed space, we do not have the luxury of speaking about functions into some fixed field k, so the sheaf map $f^\sharp$ has to be specified as part of the definition of a morphism. Given this, the condition of 'pulling back vanishing functions' translates into the following requirement. For a point $x \in X$ and $y = f(x)$, the morphism $f^\sharp$ induces a map of local rings

$$f_x^\sharp \colon \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}, \tag{4.2}$$

This map is defined as follows[1]: pick an element $s_y \in \mathcal{O}_{Y,y}$ and represent it as the germ of a section $s \in \mathcal{O}_Y(W)$ over some open set $W \subset Y$. Then the section $t = f_W^\sharp(s)$ is a section of $\mathcal{O}_X(f^{-1}W)$. We define $f_x^\sharp(s_y)$ to be the germ of this section at x, i.e. $f_x^\sharp(s_y) = t_x \in \mathcal{O}_{X,x}$. This makes sense because $f^{-1}W$ contains x. Using the definition of the stalk, one checks that this does not depend on the choice of open set W containing y.

In the case of affine varieties, the map (4.2) sends a rational function $g \in \mathcal{O}_{Y,y}$ which is regular at $y = f(x)$ to $g \circ f$ which is regular at x. By what we said above, for every $x \in X$ with $y = f(x)$, the map $f_x^\sharp$ sends the maximal ideal $\mathfrak{m}_y$ into $\mathfrak{m}_x$. In other words, $f_x^\sharp$ is a *map of local rings*.

Definition 4.2 (Morphisms of locally ringed spaces)**.** A *morphism*, or simply *map*, of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$, where
(i) $f \colon X \to Y$ is a continuous map,
(ii) $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on Y,
such that for every $x \in X$, the induced map on stalks

$$f_x^\sharp \colon \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a map of local rings; that is, $f_x^\sharp(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$.

For simplicity, we usually omit the sheaf components from our notation; that is, we write X instead of $(X, \mathcal{O}_X)$ and $f \colon X \to Y$ instead of $(f, f^\sharp)$.

Morphisms of locally ringed spaces can be composed. Given $f \colon X \to Y$ and $g \colon Y \to Z$, we define the morphism $X \to Z$ by the composition $g \circ f$ on the level of topological spaces and the sheaf map $(g \circ f)^\sharp$ is defined over an open set $U \subset Z$ by the composition

$$\mathcal{O}_Z(U) \xrightarrow{g^\sharp} \mathcal{O}_Y(g^{-1}U) \xrightarrow{f^\sharp} \mathcal{O}_X((g \circ f)^{-1}U) = \Gamma(U, (g \circ f)_*\mathcal{O}_X).$$

An *isomorphism of locally ringed spaces* is a morphism $f \colon X \to Y$ which admits an inverse morphism. In other words, there is a morphism $g \colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In more concrete terms, this means that (i) f is a homeomorphism, and (ii) for every open $U \subset Y$, the map $f_U^\sharp \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ is an isomorphism of rings.

[1] The map (4.2) is not the map obtained by taking stalks of $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Generally $(f_*\mathcal{O}_X)_y \neq \mathcal{O}_{X,x}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\operatorname{Spec} A$ *as a locally ringed space*

For a ring A, the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is by design a locally ringed space. The set $\operatorname{Spec} A$ is endowed with the Zariski topology and the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is a sheaf of rings with stalks which are local rings of the form $A_{\mathfrak{p}}$. In this section, we show that ring maps induce maps of locally ringed spaces.

Let $\phi\colon A \to B$ be a ring map and write $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We define a morphism

$$(f, f^\sharp)\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y) \tag{4.3}$$

as follows. On the level of topological spaces, we define $f\colon X \to Y$ by

$$f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}). \tag{4.4}$$

The sheaf map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is defined on the basis of distinguished open sets. Consider a distinguished open set $D(g) \subset Y = \operatorname{Spec} A$. Then,

$$f^{-1}(D(g)) = D(\phi(g)). \tag{4.5}$$

The sections involved are

$$\begin{aligned} \mathcal{O}_Y(D(g)) &= A_g, \\ f_*\mathcal{O}_X(D(g)) &= \mathcal{O}_X(D(\phi(g))) = B_{\phi(g)}. \end{aligned}$$

We define $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ over $D(g)$ by the localization map

$$A_g \longrightarrow B_{\phi(g)}, \qquad \frac{a}{g^n} \longmapsto \frac{\phi(a)}{\phi(g)^n}. \tag{4.6}$$

Note that if $\phi(g) = 0$, then $B_{\phi(g)} = 0$, and this is the zero map.

These maps define a morphism of $\mathscr{B}$-sheaves: if $D(h) \subset D(g)$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(D(g)) & \longrightarrow & \mathcal{O}_X(D(\phi(g))) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(D(h)) & \longrightarrow & \mathcal{O}_X(D(\phi(h))) \end{array} \qquad = \qquad \begin{array}{ccc} A_g & \longrightarrow & B_{\phi(g)} \\ \downarrow & & \downarrow \\ A_h & \longrightarrow & B_{\phi(h)} \end{array}$$

In the diagram on the left, the vertical maps are restriction maps, while on the right, the vertical maps are localization maps of rings.

Finally, we check that for each point $x \in X$, the induced map $f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a map of local rings. Let $\mathfrak{p}$ be the prime ideal corresponding to x, so that $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is the prime ideal corresponding to $f(x) \in \operatorname{Spec} A$. From the paragraph following (4.2), we can understand the map $f_x^\sharp$ as follows: For an element $t \in \mathcal{O}_{Y,f(x)} = A_{\mathfrak{q}}$, writing it as $t = a/g^n$ allows us to view it as the germ of a section $a/g^n \in \mathcal{O}_Y(D(g))$ for some open set $D(g)$ containing $\mathfrak{q}$. Note that $\mathfrak{q} \in D(g)$ means that $g \notin \mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Then $f_x^\sharp(t)$ is the image of t via the composition

$$A_g \longrightarrow B_{\phi(g)} \longrightarrow B_{\mathfrak{p}} = \mathcal{O}_{X,x}. \tag{4.7}$$

Now, if $t \in \mathfrak{q}A_{\mathfrak{q}}$ lies in the maximal ideal, then we can write $t = a/g^n$ with $a \in \mathfrak{q} = \phi^{-1}(\mathfrak{p})$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Then $\phi(a)/\phi(g)^n \in \mathfrak{p}B_{\phi(g)}$, and hence $f_x^\sharp(t) \in \mathfrak{p}B_\mathfrak{p}$ lies in the maximal ideal of $\mathcal{O}_{X,x}$. Hence $f_x^\sharp$ is a map of local rings, and consequently $(f, f^\sharp)$ is a morphism of locally ringed spaces.

The next example shows a pair $(f, f^\sharp)$ that is not a morphism of locally ringed spaces.

Example 4.3. Let $X = \operatorname{Spec} \mathbb{C}(t)$ and $Y = \mathbb{A}^1_\mathbb{C} = \operatorname{Spec} \mathbb{C}[t]$. There is a natural map $f\colon X \to Y$ induced by the inclusion $\mathbb{C}[t] \subset \mathbb{C}(t)$. On the level of topological spaces, X consists of a single point ξ, and f maps ξ to the generic point γ of Y. The corresponding stalk map $f_\xi^\sharp\colon \mathcal{O}_{Y,\gamma} \to \mathcal{O}_{X,\xi}$ is the identity map

$$\mathcal{O}_{Y,\gamma} = \mathbb{C}[t]_{(0)} = \mathbb{C}(t) \longrightarrow \mathbb{C}(t) = \mathcal{O}_{X,\xi},$$

which is certainly a map of local rings.

On the other hand, we could try to define a map $g\colon X \to Y$ by sending ξ to some other point $y \in \mathbb{A}^1_\mathbb{C}$, say the origin, corresponding to the maximal ideal $(t) \subset \mathbb{C}[t]$. This map is certainly continuous. For the sheaf map, note that $(g_*\mathcal{O}_X)(U) = \mathbb{C}(t)$ if U contains y and 0 otherwise. We can therefore define $g^\sharp\colon \mathcal{O}_Y \to g_*\mathcal{O}_X$ as follows: for an open set $U \subset Y$, $g_U^\sharp$ is the inclusion map $\mathcal{O}_Y(U) \subset \mathbb{C}(t)$ if $y \in U$, and the zero map otherwise. This is indeed a map of sheaves of rings. However, the map on stalks is given by

$$g_\xi^\sharp\colon \mathcal{O}_{Y,y} = \mathbb{C}[t]_{(t)} \longrightarrow \mathbb{C}(t) = \mathcal{O}_{X,\xi}$$

which sends t to t. Hence the maximal ideal of $\mathcal{O}_{Y,y}$ maps to the unit ideal in $\mathbb{C}(t)$. Therefore, $(g, g^\sharp)$ is not a morphism of locally ringed spaces. Geometrically, the element t vanishes at $y \in Y$, but its pullback, the image of t in $\mathbb{C}(t)$, does not vanish at ξ. △

4.2 Schemes

We are now in a position to define schemes.

Definition 4.4. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

If X is a locally ringed space and $U \subset X$ is an open subset, the restricted sheaf $\mathcal{O}_X|_U$ makes $(U, \mathcal{O}_X|_U)$ into a locally ringed space. Indeed, the stalks of $\mathcal{O}_X|_U$ are the same as those of $\mathcal{O}_X$ at every point of U. In particular, it makes sense to ask when a locally ringed space is locally isomorphic to an affine scheme:

Definition 4.5 (Schemes)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme: there is an open cover $\{U_i\}_{i\in I}$ of open subsets of X such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to some affine scheme $(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$.

Thus a scheme consists of a topological space X together with a sheaf of rings $\mathcal{O}_X$, such that every point of X has an open neighbourhood isomorphic, as a locally ringed space, to the spectrum of a ring.

A *morphism*, or *map*, between two schemes X and Y is simply a morphism of locally

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

ringed spaces. In this way, schemes form a category, denoted Sch. It contains the category of affine schemes AffSch as a full subcategory.

4.3 Morphisms into an affine scheme

The following is a fundamental result in the theory of schemes.

Theorem 4.6. Let X be a scheme and let A be a ring. Then there is a canonical bijection

$$\mathrm{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) = \mathrm{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_X(X)), \tag{4.8}$$

defined by sending $(f, f^\sharp)\colon X \to \operatorname{Spec} A$ to the ring map $f^\sharp_{\operatorname{Spec} A}\colon A \to \mathcal{O}_X(X)$.

Proof *Injectivity of* (4.8): Write $Y = \operatorname{Spec} A$. Let $(f, f^\sharp)$ and $(g, g^\sharp)$ be two morphisms of schemes $X \to Y$ and assume that $f^\sharp$ and $g^\sharp$ induce the same ring map $\phi\colon A \to \mathcal{O}_X(X)$.

Let $x \in X$ and write $\mathfrak{p} \subset A$ for the prime ideal corresponding to $y = f(x)$. If $\ell\colon A \to A_\mathfrak{p}$ denotes the localization map and $\rho_x\colon \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ denotes the germ map, then there is a commutative diagram

$$\begin{array}{ccccc} A & = & \mathcal{O}_Y(Y) & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ \downarrow{\scriptstyle \ell} & & \downarrow & & \downarrow{\scriptstyle \rho_x} \\ A_\mathfrak{p} & = & \mathcal{O}_{Y,f(x)} & \xrightarrow{f^\sharp_x} & \mathcal{O}_{X,x} \end{array} \tag{4.9}$$

From the diagram, we see that

$$\mathfrak{p} = \ell^{-1}(\mathfrak{p}A_\mathfrak{p}) \overset{(*)}{=} \ell^{-1}((f^\sharp_x)^{-1}(\mathfrak{m}_x)) = \phi^{-1}(\rho_x^{-1}(\mathfrak{m}_x)). \tag{4.10}$$

For the equality $(*)$, we use the fact that $f^\sharp_x$ is a map of local rings to obtain $\mathfrak{p}A_\mathfrak{p} = (f^\sharp_x)^{-1}(\mathfrak{m}_x)$. As the right-hand side of (4.10) only depends on ϕ and x, we see that $\mathfrak{p}$ is determined by ϕ and hence $f(x)$ and $g(x)$ must be equal, as they correspond to the same prime ideal in A. Therefore, f and g are equal on the level of topological spaces. We will from now on write f for this map.

Next we show that $f^\sharp = g^\sharp$ as maps of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$. As the sheaves involved are on $Y = \operatorname{Spec} A$, it suffices to show that they agree over the distinguished open sets. For $h \in A$, consider the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(Y) & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(D(h)) & \xrightarrow{f^\sharp_{D(h)}} & \mathcal{O}_X(f^{-1}D(h)) \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_h & \xrightarrow{f^\sharp_{D(h)}} & \mathcal{O}_X(f^{-1}D(h)) \end{array}$$

By the universal property of localization, a morphism $A_h \to \mathcal{O}_X(f^{-1}D(h))$ exists if and only if h becomes invertible in $\mathcal{O}_X(f^{-1}D(h))$, in which case it is unique. This means that $f^\sharp_{D(h)} = g^\sharp_{D(h)}$ as ring maps. As this happens for any $h \in A$, we find that $f^\sharp = g^\sharp$.

Surjectivity of (4.8): Let $\phi\colon A \to \mathcal{O}_X(X)$ be a ring map. We will define a morphism of schemes $f\colon X \to Y$ such that $f^\sharp_Y = \phi$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

In light of (4.4), there is a natural choice for the map on topological spaces. For $x \in X$, we set $f(x)$ to be the prime ideal in A that is the preimage of $\mathfrak{m}_x$ under the composition

$$A \xrightarrow{\phi} \mathcal{O}_X(X) \xrightarrow{\rho_x} \mathcal{O}_{X,x}.$$

First we check that this assignment is continuous. For a section $s \in \mathcal{O}_X(X)$, define the set

$$D(s) = \{\, x \in X \mid s_x \notin \mathfrak{m}_x \,\}.$$

This set is open in X, because if $s_x \notin \mathfrak{m}_x$ for some point $x \in X$, then s_x is invertible in $\mathcal{O}_{X,x}$, and there is a neighbourhood $V \subset D(s)$ of x such that $s|_V \in \mathcal{O}_X(V)$ is invertible (see Lemma 4.7).

Consider a distinguished open set $D(g) \subset \operatorname{Spec} A$. Then

$$\begin{aligned} f^{-1}(D(g)) &= \{\, x \in X \mid f(x) \in D(g) \,\} \\ &= \{\, x \in X \mid g \notin (\rho_x \circ \phi)^{-1}(\mathfrak{m}_x) \,\} \\ &= \{\, x \in X \mid \rho_x(\phi(g)) \notin \mathfrak{m}_x \,\} \\ &= \{\, x \in X \mid (\phi(g))_x \notin \mathfrak{m}_x \,\} \\ &= D(\phi(g)). \end{aligned}$$

Since $D(\phi(g))$ is open in X, and the sets $D(g)$ form a basis for the topology on $\operatorname{Spec} A$, we conclude that f is continuous.

By Lemma 4.7, the restriction $\phi(g)|_{D(\phi(g))}$ is invertible in $\mathcal{O}_X(D(\phi(g)))$. Therefore, by the universal property of localization, the ring map ϕ induces a ring map

$$A_g \longrightarrow \mathcal{O}_X(D(\phi(g))). \tag{4.11}$$

Explicitly, (4.11) sends a/g^n to $\phi(a)|_{D(\phi(g))} \cdot (\phi(g)|_{D(\phi(g))})^{-n}$. Moreover, if $D(h) \subset D(g)$, then we have the following diagram

$$\begin{array}{ccc} A_g & \longrightarrow & \mathcal{O}_X(D(\phi(g))) \\ \downarrow & & \downarrow \\ A_h & \longrightarrow & \mathcal{O}_X(D(\phi(h))) \end{array} \tag{4.12}$$

which commutes by the uniqueness part in the universal property of localization.

We define $f^\sharp_{D(g)}$ to be the ring map (4.11). By (4.12), these maps are compatible with the restriction maps, and so we get a map of sheaves $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

To see that $(f, f^\sharp)$ is a morphism of locally ringed spaces, we need to check that for each $x \in X$, the induced map $f^\sharp_x \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a map of local rings. That is, we need to check that $f^\sharp_x(\mathfrak{p}A_\mathfrak{p}) \subset \mathfrak{m}_x$. But this follows by looking at the commutative diagram (4.9), because $\mathfrak{p}$ is exactly the inverse image of $\mathfrak{m}_x$ under the composition $\rho_x \circ \phi$.

Finally, if we set $g = 1$ in the definition of $f^\sharp$, then $\phi(g) = 1$ and (4.11) is simply given by the original ring map $\phi \colon A \to \mathcal{O}_X(X)$. This means that ϕ is induced by $(f, f^\sharp)$. □

For the surjectivity part of the above proof, we used the following little lemma:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Lemma 4.7. Let X be a scheme and let $s \in \mathcal{O}_X(U)$ be a section.
(i) If s_x is invertible in $\mathcal{O}_{X,x}$ for some point $x \in U$, then there is an open set V containing x such that $s|_V$ is invertible in $\mathcal{O}_X(V)$.
(ii) If s_x is invertible in $\mathcal{O}_{X,x}$ for every $x \in U$, then s is invertible in $\mathcal{O}_X(U)$.

Proof (i): Let $t_x \in \mathcal{O}_{X,x}$ be the inverse of s_x, so that $s_x \cdot t_x = 1$ in $\mathcal{O}_{X,x}$. As all elements of $\mathcal{O}_{X,x}$ are germs, we may find an open set V containing x and a section $t_V \in \mathcal{O}_X(V)$ such that $(t_V)_x = t_x$. Then $(t_V)_x \cdot s_x = 1$ in $\mathcal{O}_{X,x}$, so shrinking V if necessary, we may assume that $s|_V \cdot t_V = 1$ in $\mathcal{O}_X(V)$.

(ii): By the first part, we may find an open covering $\{V_i\}_{i\in I}$ of U and sections $t_i \in \mathcal{O}_X(V_i)$ such that $t_i \cdot s|_{V_i} = 1$ in $\mathcal{O}_X(V_i)$. As inverses are unique, we must have $t_i|_{V_i\cap V_j} = t_j|_{V_i\cap V_j}$ for every i and j, and so the t_i glue to a section $t \in \mathcal{O}_X(U)$. By the Locality axiom, the section t has the property that $st = 1$ in $\mathcal{O}_X(U)$, because st and '1' both restrict to the same section over each V_i. Hence s is invertible in $\mathcal{O}_X(U)$. □

In the above proof, we never used the fact that X had an open covering consisting of affine schemes. This means that the theorem and the proof are valid even in the case where X is a general locally ringed space.

Corollary 4.8. For any scheme X, there is a canonical morphism $\theta\colon X \to \operatorname{Spec}\mathcal{O}_X(X)$. If $f\colon X \to \operatorname{Spec} A$ is any morphism to an affine scheme, then there is a unique morphism $h\colon \operatorname{Spec}\mathcal{O}_X(X) \to \operatorname{Spec} A$ such that $f = h \circ \theta$.

Proof The morphism θ is obtained by taking the identity map in the right-hand side of 4.6. Morphisms $f\colon X \to \operatorname{Spec} A$ are in one-to-one correspondence with ring maps $A \to \mathcal{O}_X(X)$; the factorization $f = h \circ \theta$ comes from the composition $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(X)$. □

Example 4.9. For a scheme X, there is one and only one ring map $\mathbb{Z} \to \mathcal{O}_X(X)$ (ring maps are required to send 1 to 1 and this determines the map completely). By the theorem, this means that there is a *unique* morphism $X \to \operatorname{Spec}\mathbb{Z}$. In categorical terms, this means that $\operatorname{Spec}\mathbb{Z}$ is the *terminal object* in the category of schemes. △

Example 4.10 (Morphisms to $\mathbb{A}^1$ and $\mathcal{O}_X(X)$)**.** For a ring A, there is a bijection between ring maps $\phi\colon \mathbb{Z}[t] \to A$ and elements of A (that is, ϕ is determined uniquely by the image of t). Therefore, by Theorem 4.6, the map $f \mapsto f^\sharp_{\mathbb{A}^1}(t)$ defines a bijection

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \mathbb{A}^1) = \mathcal{O}_X(X).$$

That is, an element of $\mathcal{O}_X(X)$ is the same thing as a morphism

$$f\colon X \longrightarrow \mathbb{A}^1.$$

Thus the global sections of the structure sheaf $\mathcal{O}_X$ do indeed correspond to some sort of 'regular functions' on X, not into a field k, but into the affine line over $\mathbb{Z}$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The main theorem of affine schemes

The main consequence of Theorem 4.6 is that the category of affine schemes, AffSch, is dual to the category of rings, Rings. To formulate this precisely, we note that the assignment $A \mapsto \operatorname{Spec} A$ defines a functor

$$\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \longrightarrow \mathsf{AffSch}. \tag{4.13}$$

For each ring map $\phi\colon A \to B$, this sends ϕ to the morphism $(f, f^\sharp)\colon \operatorname{Spec} B \to \operatorname{Spec} A$. It follows from the definitions that $\operatorname{Spec}\phi \circ \operatorname{Spec}\psi = \operatorname{Spec}(\psi \circ \phi)$, whenever ϕ and ψ are composable ring maps.

In the opposite direction, there is the global sections functor Γ which sends each affine scheme X to the ring $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. For a morphism of affine schemes $f\colon X \to Y$ the associated sheaf map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a ring map on global sections:

$$f_Y^\sharp\colon \mathcal{O}_Y(Y) \longrightarrow \Gamma(Y, f_*\mathcal{O}_X) = \mathcal{O}_X(X).$$

By Theorem 4.6, these two functors give an equivalence of categories, analogous to Theorem 1.25 for affine varieties.

Theorem 4.11 (Main Theorem of Affine Schemes). The two functors

$$\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \longrightarrow \mathsf{AffSch}, \qquad \Gamma\colon \mathsf{AffSch} \longrightarrow \mathsf{Rings}^{\mathrm{op}}$$

define an equivalence of categories between $\mathsf{Rings}^{\mathrm{op}}$ and AffSch.

Proof As $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$ for any ring A, we have an equality of functors $\Gamma \circ \operatorname{Spec} = \mathrm{id}_{\mathsf{Rings}^{\mathrm{op}}}$. Conversely, starting with an affine scheme X, Corollary 4.8 shows that X is canonically isomorphic to $\operatorname{Spec}(\mathcal{O}_X(X))$. Therefore $\operatorname{Spec} \circ \Gamma$ is equivalent to $\mathrm{id}_{\mathsf{AffSch}}$. □

To summarize, affine schemes X are completely characterized by their rings of global sections $\mathcal{O}_X(X)$, both on the level of topological spaces and the structure sheaf. Morphisms between affine schemes $X \to Y$ are in bijective correspondence with ring homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. In particular, a morphism between two affine schemes is an isomorphism if and only if the corresponding ring map is an isomorphism.

4.4 Schemes over a ring

Recall that an algebra over a ring A is a ring B equipped with a ring map $A \to B$. A morphism of A-algebras is a ring map $B \to B'$ which is compatible with the ring maps, in the sense that $A \to B'$ factors as $A \to B \to B'$.

The scheme analogue of this concept is the following. A *scheme over* A or simply A-*scheme*, is a scheme X together with a morphism $X \to \operatorname{Spec} A$. A *morphism of* A*-schemes* is a morphism $f\colon X \to Y$ which is compatible with the two structure morphisms to $\operatorname{Spec} A$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array} \tag{4.14}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

We denote the category of schemes over A by Sch/A and the full subcategory of affine schemes over A by AffSch/A.

An affine scheme $\operatorname{Spec} B$ has the structure of an A-scheme precisely when B is an A-algebra. Similarly, a morphism $\operatorname{Spec} B' \to \operatorname{Spec} B$ is a morphism over A precisely when the corresponding ring map $B \to B'$ is a map of A-algebras. More generally, Theorem 4.8 takes the following relative form.

Theorem 4.12. Let A be a ring. Then the category AffSch/A of affine schemes over A is dual to the category of A-algebras Alg/A via the functor $X \mapsto \mathcal{O}_X(X)$.

Example 4.13. By Example 4.9, every scheme has a unique structure as a $\mathbb{Z}$-scheme. On the level of categories, there is an equivalence $\mathsf{Sch}/\mathbb{Z} = \mathsf{Sch}$. △

Example 4.14. $\mathbb{A}^1_{\mathbb{Q}}$ is in a unique way a scheme over $\mathbb{Q}$. It does not admit the structure of an $\mathbb{R}$-scheme because $\mathbb{Q}[x]$ is not an $\mathbb{R}$-algebra. △

One motivation for introducing this concept is that it allows us to precisely say what it means for a scheme, or a morphism of schemes, to be defined over a field or a ring. If X is a scheme over a field k, then all of the rings $\mathcal{O}_X(U)$ are k-algebras, so informally, X is locally defined by polynomial equations with coefficients in k.

Likewise, if $f\colon X \to Y$ is a morphism of k-schemes, the ring maps $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ are maps of k-algebras. Geometrically, this means that morphisms of k-schemes preserve the constant functions coming from k. For example, over $\mathbb{C}$, a morphism of $\mathbb{C}$-schemes cannot send i to $-i$ or to any other complex number; the base field remains intact.

Example 4.15. The scheme $X = \operatorname{Spec} \mathbb{C}$ can be regarded as a scheme over $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$. The set of self-morphisms $X \to X$ depends on this choice:

(i) As a $\mathbb{C}$-scheme, the only morphism $X \to X$ is the identity, so $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{C}}(X, X)$ has a single element.

(ii) As an $\mathbb{R}$-scheme, there are two morphisms: the identity and the one induced by complex conjugation $\mathbb{C} \to \mathbb{C}$, so $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{R}}(X, X)$ has two elements.

(iii) As a $\mathbb{Z}$-scheme, the set of self-morphisms can be identified with the set of field automorphisms $\mathbb{C} \to \mathbb{C}$, so $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{Z}}(X, X)$ is an uncountable group. △

If k is a field, then any n-tuple of polynomials $f_1, \dots, f_n$ in $k[x_1, \dots, x_m]$ determines a morphism $\mathbb{A}^m_k \to \mathbb{A}^n_k$ via the ring map $\phi\colon k[y_1, \dots, y_n] \to k[x_1, \dots, x_m]$ which sends y_1 to f_1, y_2 to f_2 etc. For simplicity, we will continue to denote such morphisms in 'coordinate form'

$$f\colon \mathbb{A}^m_k \longrightarrow \mathbb{A}^n_k, \qquad (x_1, \dots, x_m) \mapsto (f_1(x), \dots, f_n(x)). \tag{4.15}$$

By Theorem 4.12 every k-morphism $\mathbb{A}^m_k \to \mathbb{A}^n_k$ is of this form.

4.5 Open embeddings and open subschemes

If X is a scheme and $U \subset X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on U making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is moreover a scheme, because if X is covered by affines $V_i = \operatorname{Spec} A_i$, then each $U \cap V_i$ is open in V_i, and hence can be covered

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

by distinguished open subsets, which are all affine schemes. We call $(U, \mathcal{O}_X|_U)$ an *open subscheme* of X and refer to the scheme structure on U as the *induced scheme structure*.

When referring to an 'open affine subset' in X or an 'open affine covering' of X, we will always implicitly assume that the open sets are endowed with their canonical scheme structures, so that they are open *subschemes*.

A morphism of schemes $\iota\colon V \to X$ is called an *open embedding* if it is an isomorphism onto an open subscheme of X.

Example 4.16. The open set $U = \mathbb{A}^1_k - V(x)$ is an open subscheme of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. Note the isomorphism $U \simeq \operatorname{Spec} k[x, x^{-1}] = \operatorname{Spec} k[x, y]/(xy - 1)$ of schemes. △

Example 4.17 (Distinguished open subsets)**.** More generally, each distinguished open set $D(f)$ in an affine scheme $\operatorname{Spec} A$ is an open subscheme. It is affine, isomorphic to $\operatorname{Spec} A_f$. Indeed, by Proposition 2.28 the map $\iota\colon \operatorname{Spec} A_f \to \operatorname{Spec} A$ corresponding to the localization map $A \to A_f$ is a homeomorphism onto $D(f)$, and it follows readily from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_{D(f)}$ coincides with the structure sheaf on $\operatorname{Spec} A_f$. △

The next example shows that an open subscheme of an affine scheme may not be affine.

Example 4.18 (A non-affine scheme)**.** The open subset $U = \mathbb{A}^2_k - V(x, y)$ of $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ is not an affine scheme. This is a consequence of Example 3.22, which showed that the restriction map $\iota^\sharp\colon \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k) \to \mathcal{O}_{\mathbb{A}^2_k}(U)$ is an isomorphism. Indeed, if U were affine, the inclusion $\iota\colon U \to \mathbb{A}^2_k$, being induced by the ring map $\iota^\sharp$, would be an isomorphism by the Main Theorem of Affine Schemes (Theorem 4.11), but obviously this is not the case. △

Example 4.19. $\operatorname{Spec} k \to \operatorname{Spec} k[x]/(x^2)$ is not an open embedding, even though it is a homeomorphism onto an open set. △

4.6 Closed embeddings and closed subschemes

In this section, we define *closed subschemes* of a scheme. Intuitively, a closed subscheme of a scheme X is a scheme Z which is embedded as a closed subset $Z \subset X$. Unlike open subschemes, however, the underlying subset alone does not determine the scheme structure; we also need to specify the structure sheaf.

The main example to keep in mind is $\operatorname{Spec}(A/\mathfrak{a})$, which embeds naturally as the closed subset $V(\mathfrak{a}) \subset \operatorname{Spec} A$ (Proposition 2.25). In general, a closed subscheme is a scheme $(Z, \mathcal{O}_Z)$ together with a morphism $\iota\colon Z \to X$, which locally looks like the map $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$. We formalize this in the following definition.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Definition 4.20. A morphism $\iota\colon Z \to X$ is called a *closed embedding* if there is an affine open cover $\{U_i\}_{i\in I}$ of X such that for each $i \in I$:

(i) $\iota^{-1}(U_i)$ is affine, and

(ii) the ring map

$$\iota^\sharp\colon \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_Z(\iota^{-1}U_i)$$

is surjective.

In this case, we say that Z is a *closed subscheme* of X. Two closed subschemes Z, Z' are considered *equal* if there is an isomorphism $\phi : Z \to Z'$ such that $\iota = \iota' \circ \phi$.

Even if a closed subscheme Z is defined in terms of a morphism into X, we usually view it as a closed subset of X equipped with a scheme structure. Indeed, let $U_i = \operatorname{Spec} A_i$ be as in the definition and set $\mathfrak{a}_i = \operatorname{Ker}(\mathcal{O}_X(U_i) \to \mathcal{O}_Z(\iota^{-1}U_i))$. Then $\mathcal{O}_Z(\iota^{-1}U_i) \simeq A_i/\mathfrak{a}_i$ and the restriction $\iota^{-1}(U_i) \to U_i$ is identified with the map $\operatorname{Spec}(A_i/\mathfrak{a}_i) \to \operatorname{Spec}(A_i)$. It follows that $\iota(Z) \cap U_i = V(\mathfrak{a}_i) \subset U_i$, so $V = \iota(Z)$ is closed in X. We equip this closed subset with a structure sheaf by defining $\mathcal{O}_V$ to be $\iota_*\mathcal{O}_Z$.

For brevity, we will write $V(\mathfrak{a})$ for both the closed subset $V(\mathfrak{a}) \subset \operatorname{Spec} A$ and the corresponding closed subscheme. The intended meaning will usually be clear from the context.

Example 4.21. Let $\mathbb{A}^4_k = \operatorname{Spec} k[x, y, z, w]$ for a field k. The three ideals

$$I_1 = (x, y), \qquad I_2 = (x^2, y), \qquad I_3 = (x^2, xy, y^2, xw - yz),$$

have the same radical (x, y), and hence give rise to the same closed subset $V(x, y) \subset \mathbb{A}^4_k$. However, they define different closed subschemes of $\mathbb{A}^4_k$, as the structure sheaves are not isomorphic. △

The above definition tells us what a closed subscheme looks like locally, but not how to determine all closed subschemes of a given scheme. Even when the scheme is affine this is not clear: every ideal $\mathfrak{a} \subset A$ gives a closed subscheme $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$, but the definition could also allow for closed subschemes defined from some other open covering of $\operatorname{Spec} A$. Nevertheless, it is a theorem that every closed subscheme of $\operatorname{Spec} A$ arises from an ideal of A.

Proposition 4.22. Let $X = \operatorname{Spec} A$ be an affine scheme. The map $\mathfrak{a} \mapsto \operatorname{Spec}(A/\mathfrak{a})$ defines a one-to-one correspondence between the set of ideals of A and the set of closed subschemes of X. In particular, every closed subscheme of an affine scheme is also affine.

We defer the proof of this fact to Section 10.12, as it is more naturally proved using sheaves.

Example 4.23 (Graphs)**.** Let $g_1, \dots, g_r \in A[x_1, \dots, x_n]$ and consider the morphism

$$\iota\colon \mathbb{A}^n_A \longrightarrow \mathbb{A}^{n+r}_A, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, g_1(x), \dots, g_r(x)) \tag{4.16}$$

Then ι is a closed embedding. Indeed, it is induced by the surjective ring map

$$\phi\colon A[x_1, \dots, x_n, y_1, \dots, y_r] \longrightarrow A[x_1, \dots, x_n], \qquad x_i \mapsto x_i,\ y_j \mapsto g_j.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The corresponding ideal is $I = \operatorname{Ker} \phi = (y_1 - g_1, \ldots, y_r - g_r)$. △

For future reference, we say that a morphism $f\colon Y \to X$ is a *locally closed embedding* if there exists an open subscheme $U \subset X$ such that f factors via a closed embedding $\iota\colon Y \to U$. For example, the morphism from $\operatorname{Spec} k[x]_x$ onto the subset $V(y) \cap D(x) \subset \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ is a locally closed embedding.

4.7 Reduced schemes

We define a scheme X to be *reduced* if for each open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is a reduced ring (that is, it contains no nonzero nilpotent elements). The property of being reduced is a local condition: it can be checked on stalks.

Lemma 4.24. A scheme X is reduced if and only if for every point $P \in X$, the local ring $\mathcal{O}_{X,P}$ is a reduced ring.

Proof Suppose X is reduced. Let $P \in X$ be a point and let $\sigma \in \mathcal{O}_{X,P}$ be an element such that $\sigma^n = 0$. We may represent σ as the germ of a section $s \in \mathcal{O}_X(U)$ for some $U \subset X$. As $s^n_P = 0$ in $\mathcal{O}_{X,P}$, we may after possibly shrinking U assume that also $s^n = 0$ in $\mathcal{O}_X(U)$. By assumption $\mathcal{O}_X(U)$ is reduced, so $s = 0$, and hence $\sigma = 0$ as well.

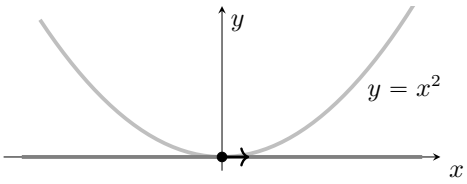
Conversely, let $U \subset X$ be an open set and let $s \in \mathcal{O}_X(U)$. If s is nonzero, then it has a nonzero germ in at least one stalk $\mathcal{O}_{X,P}$. Therefore, if s is nilpotent then so is its germ, and hence $\mathcal{O}_{X,P}$ is not reduced. □

Example 4.25. An affine scheme $X = \operatorname{Spec} A$ is reduced precisely when A is a reduced ring. So for instance $\operatorname{Spec} \mathbb{Z}$ and $\mathbb{A}^n_k$ are reduced, but $\operatorname{Spec}(\mathbb{Z}/18)$ and $\operatorname{Spec} k[x]/(x^3)$ are not. △

Non-reduced schemes appear naturally when we consider intersections of closed subschemes. Suppose for instance we have an affine scheme $X = \operatorname{Spec} A$ and two closed subschemes Y and Z, defined by ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A, respectively. Then the *scheme-theoretic intersection* of Y and Z is defined as the closed subscheme of $\operatorname{Spec} A$ defined by the ideal $\mathfrak{a} + \mathfrak{b}$. Note that there is no reason to expect that the ideal $\mathfrak{a} + \mathfrak{b}$ is a prime ideal in A, even if $\mathfrak{a}$ and $\mathfrak{b}$ are. This means that the scheme-theoretic intersection may fail to be reduced or irreducible even if both Y and Z are integral. This is natural and very important. It means that the scheme-theoretic intersection more accurately captures the multiplicities of an intersection.

Example 4.26. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, and consider the two subschemes $Y = V(y - x^2)$ and $Z = V(y)$. The scheme-theoretic intersection of these is given by the ideal $(y - x^2, y) = (x^2, y)$, which is not a radical ideal in $k[x,y]$. The nilpotent elements of $k[x,y]/(x^2,y) = k[x]/(x^2)$ in some sense account for the 'tangency' of the intersection $Y \cap Z$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026



Example 4.27. Consider the parabola $V(y - x^2)$ and the circle $V(x^2 + (y-1)^2 - 1)$ in $\mathbb{A}^2_k$. The calculation of Example 1.17 shows that the scheme-theoretic intersection is defined by the ideal

$$(y, x^2) \cap (y - 1, x - 1) \cap (y - 1, x + 1).$$

This scheme is also non-reduced, and the ideal (y, x^2) reflects the fact that the two curves intersect at the origin $(0, 0)$ with 'multiplicity 2', unlike the two other intersection points, which have multiplicity 1. Geometrically, this corresponds to the fact that the parabola and the circle have a common tangent at $(0, 0)$. △

4.8 Integral schemes, dominant morphisms, and function fields

A scheme X is defined to be *integral* if $\mathcal{O}_X(U)$ is an integral domain for every affine open set $U \subset X$. As the name suggests, an affine scheme $X = \operatorname{Spec} A$ is integral if and only if A is an integral domain. This follows from the following proposition:

Proposition 4.28. A scheme X is integral if and only if it is irreducible and reduced.

Proof Assume that X is integral. To show X is reduced, let $P \in X$ be a point and choose an affine open $U = \operatorname{Spec} A$ containing it so that P corresponds to a prime ideal $\mathfrak{p}$. Then by assumption, A is an integral domain, so the local ring $\mathcal{O}_{X,P} = A_{\mathfrak{p}}$ has no zerodivisors. For irreducibility: if X is not irreducible, then there exist two nonempty open sets U, V such that $U \cap V = \varnothing$. But then the sheaf sequence implies that $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$, which is not an integral domain.

Conversely, suppose that X is irreducible and reduced. Let $U = \operatorname{Spec} A \subset X$ be an affine open set. Suppose that $f, g \in \mathcal{O}_X(U) = A$ are elements so that $fg = 0$; we have to show that $f = 0$ or $g = 0$. Note that $U = \operatorname{Spec} A = V(0) = V(fg) = V(f) \cup V(g)$. Since X is irreducible, the open subset U is irreducible, so we must have, say $V(f) = U$. This means that f is nilpotent in A. But as X is reduced, there are no nilpotent elements in A, and so $f = 0$. □

Example 4.29. $\mathbb{A}^n_k$ and $\operatorname{Spec} \mathbb{Z}$ are integral schemes. The schemes $\operatorname{Spec}(\mathbb{C}[x]/x^2)$ and $\operatorname{Spec}(\mathbb{Z}/60)$ are not. △

An important property of integral schemes is that they have *function fields*, analogous to the field of rational functions on an affine variety (as discussed in Chapter 1).

To define $K(X)$, we first note that any integral scheme X has a unique generic point ξ. Indeed, take any nonempty affine open $U = \operatorname{Spec} A$. As X is integral, A is an integral domain, and the zero ideal (0) in A is the unique point which is dense in U, and hence in

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

X. Conversely, any generic point of X is contained in every nonempty open subset, and in particular in U.

Definition 4.30. For an integral scheme X, we define its *function field* by

$$K(X) = \mathcal{O}_{X,\xi}$$

where $\xi \in X$ is the generic point.

Concretely, for any affine subscheme $U = \operatorname{Spec} A \subset X$, the local ring $\mathcal{O}_{X,\xi} = A_{(0)}$ is canonically identified with the field of fractions $K(A)$ of A.

Example 4.31. The function field of $\operatorname{Spec} \mathbb{Z}$ is equal to $\mathcal{O}_{\operatorname{Spec}\mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$. △

Example 4.32. The function field of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ is equal to the field $k(x_1, \dots, x_n)$ of rational functions in $x_1, \dots, x_n$. △

Example 4.33 (The quadratic cone)**.** The quadratic cone $X = \operatorname{Spec} k[x, y, z]/(y^2 - xz)$ is an integral scheme, as $y^2 - xz$ is irreducible. To compute the function field $K(X)$, we express z as $z = x^{-1}y^2$, and we find

$$K(X) = K\big(k[x, y, z]/(y^2 - xz)\big) \simeq k(x, y). \qquad \triangle$$

In Remark 3.17, we saw that each $\mathcal{O}_X(U)$ is naturally a subring of $K(X)$ when X is an integral affine scheme. This extends to integral schemes in general.

Proposition 4.34. Let X be an integral scheme.

(i) For an open set $U \subset X$, there is a natural inclusion $\mathcal{O}_X(U) \subset K(X)$.

(ii) For a point $P \in X$, there is a natural inclusion $\mathcal{O}_{X,P} \subset K(X)$.

With these identifications, for any nonempty open set $U \subset X$ we have

$$\mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_{X,P} \subset K(X). \tag{4.17}$$

Proof (i): For $V = \operatorname{Spec} A$ affine, the germ map $\mathcal{O}_X(V) \to \mathcal{O}_{X,\xi}$ corresponds to the inclusion $A \subset K(A)$. This implies that for an arbitrary open subset $U \subset X$, the germ map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is injective: if $s \in \mathcal{O}_X(U)$ satisfies $s_\xi = 0$, then for any affine $V \subset U$, the factorization $\mathcal{O}_X(U) \to \mathcal{O}_X(V) \hookrightarrow \mathcal{O}_{X,\xi}$ forces $s|_V = 0$ and hence $s = 0$ by the Locality axiom. Hence each $\mathcal{O}_X(U)$ embeds as a subring of $K(X)$ via the germ map. The statement (ii) then follows because $\mathcal{O}_{X,P}$ is the direct limit of the $\mathcal{O}_X(U) \subset K(X)$ for U containing P.

We now prove the equality (4.17). The '$\subset$'-inclusion is clear. For the reverse inclusion, suppose $f \in K(X)$ lies in $\mathcal{O}_{X,P}$ for all $P \in U$. For each $P \in U$, we may pick an affine open neighbourhood V_P of P and sections $h_P \in \mathcal{O}_X(V_P)$ so that the germ of h_P at ξ is equal to f. By the injectivity of $\mathcal{O}_X(V_P \cap V_Q) \to \mathcal{O}_{X,\xi}$, the h_P must agree on the overlaps $V_P \cap V_Q$, and hence they glue to a section of $\mathcal{O}_X(U)$ which maps to f. □

If $P \in X$ is a point and $f \in K(X)$ is a rational function, we say that f is *defined* at P if $f \in \mathcal{O}_{X,P}$. The subset $U \subset X$ where f is defined is open (Exercise 4.11.12). The set $X - U$ is called the *indeterminacy locus* of f.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 4.35. If A is an integral domain, then a rational function $a/b \in K(A)$ is defined on the open set $D(b) \subset \operatorname{Spec} A$. It may also be defined on a larger open set, as in Example 1.19. △

A morphism $f\colon X \to Y$ is called *dominant* if its image $f(X) \subset Y$ is dense in Y, that is, if $f(X)$ intersects every nonempty open subset of Y. Note that this condition depends only on the underlying topological spaces of X and Y, not on their scheme structures.

Example 4.36. The inclusion $\mathbb{A}^1_k - \{0\} \to \mathbb{A}^1_k$ is dominant, since $\mathbb{A}^1_k - \{0\}$ is a dense open subset of $\mathbb{A}^1_k$. More generally, any open embedding of a dense open subset is dominant. △

Example 4.37. The morphism $\mathbb{A}^2_k \to \mathbb{A}^2_k$, $(x, y) \mapsto (x, xy)$, is dominant: its image is $D(x) \cup \{(0,0)\}$ (see Example 1.31), and $D(x)$ is dense in $\mathbb{A}^2_k$. △

Example 4.38. Let $\phi\colon A \to B$ be a ring map and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map. Then by Proposition 2.25(iii),

$$\overline{f(\operatorname{Spec} B)} = \overline{f(V(0))} = V(\operatorname{Ker} \phi).$$

Hence f is dominant if and only if $\operatorname{Ker} \phi \subset \sqrt{(0)}$. If A is reduced, this is equivalent to ϕ being injective. △

Proposition 4.39. Let $f\colon X \to Y$ be a morphism of integral schemes. Then the following are equivalent:

(i) f is dominant
(ii) For all nonempty affine open subsets $U \subset X$ and $V \subset Y$ with $f(U) \subset V$, the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective.
(iii) f maps the generic point of X to the generic point of Y.

Proof Let ξ and γ denote the generic points of X and Y, respectively.

(i) ⇒ (ii): Assume f is dominant. Let $V = \operatorname{Spec} A \subset Y$ and $U = \operatorname{Spec} B \subset X$ be nonempty affine opens with $f(U) \subset V$, and let $\phi : A \to B$ be the induced map. Since X is irreducible, U is dense, so $f(X) = f(\overline{U}) \subset \overline{f(U)}$. Hence $Y = \overline{f(X)} \subset \overline{f(U)} \subset Y$, hence $\overline{f(U)} = Y$. Intersecting with V gives $\overline{f(U)} \cap V = V$, so $U \to V$ is dominant. As A and B are integral domains, Example 4.38 shows that $A \to B$ is injective.

(ii) ⇒ (iii): Choose a nonempty affine open $V = \operatorname{Spec} A \subset Y$ with $f(\xi) \in V$, and a nonempty affine open $U = \operatorname{Spec} B \subset f^{-1}(V)$ with $\xi \in U$. Let $\phi : A \to B$ be the induced map; by (ii) it is injective. The point $\xi \in U$ is the zero ideal $(0) \subset B$, and its image in V is $\phi^{-1}((0)) = \operatorname{Ker}(\phi) = (0) \subset A$, the generic point of V, hence γ. So $f(\xi) = \gamma$.

(iii) ⇒ (i): Assume $f(\xi) = \gamma$. Then $\gamma \in f(X)$ implies that $Y = \overline{\{\gamma\}} \subset \overline{f(X)} \subset Y$, so $\overline{f(X)} = Y$ and f is dominant. □

An important property of dominant morphisms is that they induce pullback maps on function fields. More precisely, let X and Y be integral schemes with generic points ξ and γ respectively, and let $f\colon X \to Y$ be dominant. By Proposition 4.39, $f(\xi) = \gamma$, so the map $f^\sharp$ on stalks gives a map of local rings $\mathcal{O}_{Y,\gamma} \to \mathcal{O}_{X,\xi}$, hence a map

$$f^\sharp\colon K(Y) \longrightarrow K(X). \tag{4.18}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This is a formal version of the pull back as discussed in Section 1.5. If $g \in K(Y)$, we may find an open set $V \subset Y$ such that $g \in \mathcal{O}_Y(V)$. If we think of this as a morphism $g\colon V \to \mathbb{A}^1$, then $f^\sharp(g) \in K(X)$ lies in $\mathcal{O}_X(f^{-1}V)$ and corresponds to the composition $g \circ f\colon f^{-1}V \to \mathbb{A}^1$.

In this setting it really is important that $f(X)$ is dense in Y. For instance, if $f\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ is the closed embedding of the 'x-axis', i.e., induced by the ring map $k[x,y] \to k[x,y]/y$, then there is no reasonable way to pull back the element $y^{-1} \in k(x,y)$ to $K(\mathbb{A}^1_k) = k(x)$.

4.9 Residue fields and ring-valued points

As in Chapter 2, it is useful to organize the points of a scheme according to their residue fields. A point P of a scheme X has a local ring $\mathcal{O}_{X,P}$ with maximal ideal $\mathfrak{m}_P$ and residue field $\kappa(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$. If P lies in an affine open set $U = \operatorname{Spec} A$ and corresponds to a prime ideal $\mathfrak{p} \subset A$, then $\mathcal{O}_{X,P} = A_\mathfrak{p}$, $\mathfrak{m}_P = \mathfrak{p}A_\mathfrak{p}$ and $\kappa(P) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Each point is also represented by a canonical morphism

$$\operatorname{Spec}\kappa(P) \longrightarrow X, \tag{4.19}$$

which sends the unique point of $\operatorname{Spec}\kappa(P)$ to P. In the notation above, it is the composition

$$\operatorname{Spec}\kappa(P) \to \operatorname{Spec} A_\mathfrak{p} \to U \to X.$$

Here the left-most map is a closed embedding, induced by $A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. The composition $\operatorname{Spec}\kappa(P) \to X$ is a closed embedding if and only if P is a closed point.

More generally, we can try to understand a scheme by looking at morphisms from an affine scheme into it. For a ring R, a morphism $\operatorname{Spec} R \to X$ is called an *R-valued point* or simply *R-point*. The set of R-valued points is denoted by

$$X(R) = \operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} R, X). \tag{4.20}$$

If X is affine, say $X = \operatorname{Spec} A$, then by Theorem 4.6 we may identify

$$X(R) = \operatorname{Hom}_{\mathsf{Rings}}(A, R).$$

That is, giving an R-valued point of $\operatorname{Spec} A$ is the same thing as giving a ring map $A \to R$.

If $R \to S$ is a ring map, then there is a map of sets $X(R) \to X(S)$ sending $\operatorname{Spec} R \to X$ to the composition $\operatorname{Spec} S \to \operatorname{Spec} R \to X$. Likewise, if $f\colon X \to Y$ is a morphism, then we get an induced map $f(R)\colon X(R) \to Y(R)$ sending $\operatorname{Spec} R \to X$ to $\operatorname{Spec} R \to X \to Y$.

Example 4.40. For the affine n-space $\mathbb{A}^n = \operatorname{Spec}\mathbb{Z}[x_1, \dots, x_n]$, we have

$$\mathbb{A}^n(R) = R^n.$$

Indeed, by the above, the R-valued points are in one-to-one correspondence with ring maps $\phi\colon \mathbb{Z}[x_1, \dots, x_n] \to R$, and each such is uniquely determined by a choice of where to send the n generators $x_1, \dots, x_n$.

In particular, if k is a field, then $\mathbb{A}^n(k) = k^n$, which recovers the notation $\mathbb{A}^n(k)$ used in Chapter 1. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 4.41. More generally, let $X = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r)$, where $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$. Then $X(R)$ corresponds to the ring maps $\phi\colon \mathbb{Z}[x_1, \dots, x_n] \to R$ that send each f_i to zero. Therefore, we may identify

$$X(R) = \big\{a \in R^n \mid f_1(a) = \cdots = f_r(a) = 0\big\}.$$

In this way, the scheme X packages the solution sets of the system $f_1(a) = \cdots = f_r(a) = 0$ in *every* ring R into a single geometric space. △

Example 4.42. For a concrete example, consider the scheme

$$X = \operatorname{Spec} \mathbb{Z}[u, v]/(u^2 + v^2 - 1).$$

Then $X(\mathbb{R})$ is identified with the points of the unit circle in $\mathbb{R}^2$, while $X(\mathbb{Z})$ consists of the four points $(\pm 1, 0), (0, \pm 1)$. △

The most important special case is when R is a field. Every point $P \in X$ is a $\kappa(P)$-valued point via the canonical map $\operatorname{Spec} \kappa(P) \to X$. The next result says that any morphism from the spectrum of a field into X factors uniquely through one of these.

Proposition 4.43. Let X be a scheme and let K be a field.

(i) Giving a K-valued point of X is equivalent to giving a point $P \in X$ and a map of fields $\kappa(P) \to K$.

(ii) If $K \subset L$ is a field extension, then $X(K) \to X(L)$ is injective.

Proof (i): Fix a point $P \in X$ and let $U = \operatorname{Spec} A$ be an affine open neighbourhood containing P. Since $\operatorname{Spec} K$ consists of a single point, any morphism $\operatorname{Spec} K \to X$ with image P factors uniquely as $\operatorname{Spec} K \to U$ followed by the inclusion. We therefore reduce to proving the proposition for $X = \operatorname{Spec} A$ affine. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to P. By Theorem 4.6, morphisms $f\colon \operatorname{Spec} K \to \operatorname{Spec} A$ correspond to ring maps $\phi\colon A \to K$. The condition that f has image P is equivalent to $\phi^{-1}(0) = \mathfrak{p}$. In that case, ϕ factors uniquely as $A \to A/\mathfrak{p} \to K$, where $A/\mathfrak{p} \to K$ is injective. Since K is a field, any such map $A/\mathfrak{p} \to K$ factors uniquely as $A/\mathfrak{p} \to \kappa(\mathfrak{p}) \to K$. Hence morphisms $\operatorname{Spec} K \to X$ with image P are in bijection with the maps of fields $\kappa(\mathfrak{p}) \to K$, as required.

(ii): With the notation above, the map $X(K) \to X(L)$ keeps the point P and sends $\kappa(P) \to K$ to the composition $\kappa(P) \to K \hookrightarrow L$, and this assignment is injective. □

There is also a relative version of R-valued points. For a scheme X over a ring A and an A-algebra R, an *R-valued point* of X is a morphism of A-schemes $\operatorname{Spec} R \to X$. The set of R-valued points is denoted by

$$X(R) = \operatorname{Hom}_{\mathsf{Sch}/A}(\operatorname{Spec} R, X).$$

Even though the base ring is not present in the notation $X(R)$, it will usually be clear from the context which category we are working in.

To see why working over a fixed base is natural, consider $X = \operatorname{Spec} \mathbb{C}$ as a $\mathbb{C}$-scheme. Geometrically, X is a single point, so we want $X(\mathbb{C})$ to be a single point as well. There are infinitely many morphisms $\operatorname{Spec} \mathbb{C} \to X$ (Example 4.15), but only one of them is a morphism of $\mathbb{C}$-schemes.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

When X is a scheme over a field k, a k-point is a morphism of k-schemes $\operatorname{Spec} k \to X$. By Proposition 4.43, this is the same as a point $P \in X$ for which the structure map $k \to \kappa(P)$ is an isomorphism. If $X = \operatorname{Spec} A$ is affine, with A a k-algebra, then k-points correspond to maps of k-algebras $A \to k$, so we may identify $X(k) = \operatorname{Hom}_{\mathsf{Alg}/k}(A, k)$.

4.10 Affine varieties as schemes

In Chapter 1, we defined an affine variety to be an irreducible closed algebraic set in affine space $\mathbb{A}^n(k)$, where k is an algebraically closed field. We now give the scheme-theoretic version.

An *affine variety over a field* k is a k-scheme of the form

$$X = \operatorname{Spec}\big(k[x_1, \ldots, x_n]/\mathfrak{a}\big), \tag{4.21}$$

where $\mathfrak{a} \subset k[x_1, \ldots, x_n]$ is a prime ideal. A *morphism of affine varieties over* k is a morphism of the corresponding k-schemes.

When the field k is algebraically closed, this definition recovers the previous one. Indeed, if $V \subset \mathbb{A}^n(k)$ is an affine variety in the sense of Chapter 1, then its coordinate ring $A(V) = k[x_1, \ldots, x_n]/I(V)$ is precisely of the form (4.21), and $X = \operatorname{Spec} A(V)$ is an affine variety in the new sense. By Proposition 2.33, the points of V are identified with the closed points of X. Moreover, for any open set $U \subset X$, the ring $\mathcal{O}_X(U)$ consists of the rational functions $f \in K(X)$ which are regular at every point of U, Example 3.17, so the structure sheaf recovers exactly the regular functions from before.

This correspondence also works well with respect to morphisms. Morphisms of classical affine varieties $V \to W$ are in one-to-one correspondence with k-algebra maps $A(W) \to A(V)$, and these are in turn in bijection with morphisms of k-schemes $X \to Y$. This means that the category of classical affine varieties is equivalent to the category of affine varieties in the new sense.

The advantage of the new definition is that it also makes sense over fields that are not algebraically closed. In that case, a closed point need not come from a tuple $(a_1, \ldots, a_n)$ in k^n.

Example 4.44. The following schemes are affine varieties, over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}_p$ respectively:

$$\mathbb{A}^n_{\mathbb{C}}, \quad \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2), \quad \operatorname{Spec} \mathbb{F}_p[x, y, z]/(y^2 - xz).$$

The following schemes are not affine varieties (over any field):

$$\operatorname{Spec} \mathbb{Z}, \quad \operatorname{Spec} \mathbb{C}[x, y]/(x^2 + y^2), \quad \operatorname{Spec} \mathbb{F}_p[x, y]/(y^2). \qquad \triangle$$

4.11 Exercises

Exercise 4.11.1. Find an example of a morphism of schemes $f\colon X \to Y$ which is both injective and surjective, but not an isomorphism.

Exercise 4.11.2. Let k be a field. For which n and s is the scheme $U = \mathbb{A}^n_k - V(x_1, \ldots, x_s)$ affine?

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 4.11.3. Let $A = k[x,y,z]/(xyz)$ and $X = \operatorname{Spec} A$. Compute $\mathcal{O}_{X,P}$ where P corresponds to the prime ideal $\mathfrak{p} = (x-1,y,z)$. Show that $yz \neq 0$ in $\mathcal{O}_{X,P}$, but takes the value 0 for all points in a neighbourhood of P.

Exercise 4.11.4 (The cuspidal cubic)**.** Let k be a field and consider the morphism

$$\operatorname{Spec} k[t] \longrightarrow \operatorname{Spec} k[x,y]/(y^2 - x^3), \qquad t \mapsto (t^2, t^3).$$

a) Show that f is a homeomorphism.
b) Let $P \in \mathbb{A}^1_k$ be the origin (corresponding to the ideal (t) in $k[t]$). Compute the map $f^\sharp_P \colon \mathcal{O}_{X,f(P)} \to \mathcal{O}_{\mathbb{A}^1_k,P}$ and show that it is not an isomorphism. Conclude that the cuspidal cubic is not even 'locally isomorphic' to $\mathbb{A}^1_k$ near the origin.

Exercise 4.11.5. Show that $\operatorname{Spec} \overline{\mathbb{Q}}[x]$ and $\operatorname{Spec} \mathbb{Z}$ are homeomorphic, but not isomorphic as schemes.

Exercise 4.11.6 (Finite schemes)**.**

a) Show that if a scheme X consists of one or two points, then X is affine. Show that if X has an underlying topological space which is a finite and discrete set, then it is affine. HINT: Consider the canonical map $X \to \operatorname{Spec} \mathcal{O}_X(X)$.
b) Consider a set X consisting of three elements, x, y, z. Define a topology on X where the open sets are defined as $\varnothing, \{x\}, \{x,y\}, \{x,z\}$, and X itself. On X, define a sheaf of rings $\mathcal{O}_X$ where $\mathcal{O}_X(\{x\})$ corresponds to the field $\mathbb{Q}(t)$, while for the open sets $\mathcal{O}_X(\{x,y\}), \mathcal{O}_X(\{x,z\})$, and $\mathcal{O}_X(X)$ we assign the localized ring $\mathbb{Q}[t]_{(t)}$. Show that $(X, \mathcal{O}_X)$ is a scheme which is not isomorphic to an affine scheme.

Exercise 4.11.7. Let X be a scheme. Show that the map $p \mapsto \overline{\{p\}}$ defines a bijection between the points of X and the set of irreducible closed subsets of X.

Exercise 4.11.8. Let $f \colon X \to Y$ be a morphism. Show that if $g \in \mathcal{O}_Y(Y)$ corresponds to the morphism $g \colon Y \to \mathbb{A}^1$, then $f^\sharp(g) \in \mathcal{O}_X(X)$ corresponds to the composition $X \to Y \to \mathbb{A}^1$.

Exercise 4.11.9. Show that $\mathbb{A}^1_{\mathbb{C}}$ embeds as a closed subscheme of $\mathbb{A}^2_{\mathbb{R}}$. Can $\mathbb{A}^1_{\mathbb{R}}$ be a closed subscheme of some $\mathbb{A}^n_{\mathbb{C}}$?

Exercise 4.11.10. Let X be a scheme and let $P \in X$ be a point.

a) Show that there is a canonical morphism $f \colon \operatorname{Spec} \mathcal{O}_{X,P} \to X$.
b) Show that the map $\iota_P \colon \operatorname{Spec} \kappa(P) \to X$ defined in the text factors via f.
c) Show that f induces a homeomorphism between $\operatorname{Spec}(\mathcal{O}_{X,P})$ and the subset $W \subset X$ of points $w \in X$ such that $w \in \overline{\{P\}}$. Show that W is the intersection of all open neighbourhoods containing P.
d) Compute the image of f when:
   (i) P is the generic point of an irreducible scheme.
   (ii) P is a closed point of $\mathbb{A}^2_{\mathbb{C}}$.

Exercise 4.11.11. Let $X = \{a,b,c\}$ be a set with three elements. Let X have the coarsest topology so that the two subsets $U = \{a,b\}$, and $V = \{a,c\}$ are open. Define a presheaf

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\mathcal{O}_X$ by $\mathcal{O}_X(U) = \mathcal{O}_X(V) = \mathbb{C}[t]_{(t)}$ and $\mathcal{O}_X(U \cap V) = \mathbb{C}(t)$ with the restriction map given by the inclusion $\mathbb{C}[t]_{(t)} \subset \mathbb{C}(t)$.

a) Show that $\mathcal{O}_X$ is a sheaf.
b) Show that $(X, \mathcal{O}_X)$ is a scheme.
c) Show that $(X, \mathcal{O}_X)$ is not affine.

Exercise 4.11.12. Let X be an integral scheme and let $f \in K(X)$. Show that the set of points where f is defined, $\{ P \in X \mid f \in \mathcal{O}_{X,P} \}$, is open in X.

Exercise 4.11.13. Let X be a scheme.

a) Show that any irreducible closed subset $Z \subset X$ has a unique generic point. HINT: Reduce to the affine case.
b) What are the possible underlying topologies of affine schemes that are Hausdorff?
c) Show that X satisfies the zeroth separation axiom (they are T_0); that is, given two points x and y in X, there is an open subset of X containing one of them but not the other.

Exercise 4.11.14 (The sheaf of units)**.** Let X be a scheme with structure sheaf $\mathcal{O}_X$. We say that $s \in \mathcal{O}_X(U)$ is a *unit* if there exists a multiplicative inverse $s^{-1} \in \mathcal{O}_X(U)$.

a) Show that $s \in \mathcal{O}_X(U)$ is a unit if and only if for all $x \in U$, the germ s_x is a unit in the ring $\mathcal{O}_{X,x}$; that is, if and only if s_x does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.
b) We let $\mathcal{O}_X^\times(U)$ denote the subgroup of units in $\mathcal{O}_X(U)$. Show that $\mathcal{O}_X^\times(U)$ is a sub*sheaf* of $\mathcal{O}_X$.

Exercise 4.11.15. Let $X = \operatorname{Spec} k[x, y, z, w]/(xw - yz)$ and consider the open set $U = X - V(x, y)$. Use the strategy as in Example 4.18 to compute $\mathcal{O}_X(U)$. Prove that U is not affine.

Exercise 4.11.16. Let $f\colon Y \to X$ be a morphism.

a) Suppse that there is a covering $\{U_i\}$ of X such that each $f^{-1}(U_i) \to U_i$ is a closed embedding. Show that f is a closed embedding.
b) Suppse that there are open sets U_i in X such that $f^{-1}(U_i)$ covers Y; each $f^{-1}(U_i) \to U_i$ is a closed embedding; and $f(X)$ is closed in X. Show that f is a closed embedding.

Exercise 4.11.17. Describe the following schemes and the structure sheaf on them.

a) $\operatorname{Spec} \mathbb{C}[t]/(t^2 + 1)$
b) $\operatorname{Spec} \mathbb{R}[t]/(t^3 - t^2)$
c) $\operatorname{Spec} \mathbb{F}_3[t]/(t^3 - 1)$.

Exercise 4.11.18. Let K be a finite field extension of $\mathbb{Q}$ and let $X = \operatorname{Spec} K$. Show that $\operatorname{Hom}_{\mathsf{Sch}}(X, X)$ can be identified with the Galois group $\operatorname{Gal}(K/\mathbb{Q})$.

Exercise 4.11.19. Describe the schemes $\operatorname{Spec} A$ for

a) $A = \mathbb{Z}[x]/(3x - 1)$
b) $A = \mathbb{Z}[x]/(9x - 3)$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

c) $A = \mathbb{Z}[x]/(3x^2 - 1)$

Which of these are isomorphic/homeomorphic?

Exercise 4.11.20. Let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the morphism associated to a ring map $\phi\colon A \to B$. Show that the sheaf map $f^\sharp\colon \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_{\operatorname{Spec} B}$ is given by $\widetilde{\phi} : \widetilde{A} \to \widetilde{B}$, where we regard B as an A-module via ϕ.

Exercise 4.11.21. Let X be a scheme and let X_{red} be its reduction. Show that if k is a field, then there is a natural bijection $X(k) = X_{\text{red}}(k)$. Generalize the statement to reduced rings.

Exercise 4.11.22. Let $\mathbb{A}^3_k = \operatorname{Spec} k[x, y, z]$. Compute the scheme-theoretic intersection of the two subschemes $V(z - xy^2)$ and $V(z)$. Is it irreducible? Reduced?

Exercise 4.11.23 (The reduction of a scheme)**.** Let X be a scheme and let $\mathscr{B}$ be the basis of affine open subsets of X. For a ring R write $R_{\text{red}} = R/\sqrt{0}$ where $\sqrt{0}$ is the nilradical.

a) Define a presheaf on the basis $\mathscr{B}$ by $\mathcal{F}(V) = \mathcal{O}_X(V)_{\text{red}}$. Show that this assignment is a $\mathscr{B}$-sheaf of rings. If $X = \operatorname{Spec} A$, show that the resulting sheaf on $\operatorname{Spec} A$ is the sheaf associated to A_{red}.
b) Let $\mathcal{O}_{X_{\text{red}}}$ denote the sheaf extending the $\mathscr{B}$-sheaf of *a*). Show that $(X, \mathcal{O}_{X_{\text{red}}})$ is a scheme. Moreover, if $X = \operatorname{Spec} A$ is affine then $(X, \mathcal{O}_{X_{\text{red}}}) \simeq \operatorname{Spec}(A_{\text{red}})$.
c) Show that there is a canonical closed embedding $r_X\colon X_{\text{red}} \longrightarrow X$.
d) Prove the universal property: for any reduced scheme Y and any morphism $f\colon Y \to X$ there exists a unique morphism $g\colon Y \to X_{\text{red}}$ with $f = r_X \circ g$.
e) Show that for any morphism $f\colon X \to Y$ there is a unique morphism $f_{\text{red}}\colon X_{\text{red}} \longrightarrow Y_{\text{red}}$ such that $f \circ r_X = r_Y \circ f_{\text{red}}$.

Exercise 4.11.24 (Morphisms to an open subscheme)**.** Let $U \subset X$ be an open subscheme. Show that the natural map $\operatorname{Hom}_{\mathsf{Sch}}(T, U) \to \operatorname{Hom}_{\mathsf{Sch}}(T, X)$ is injective with image consisting of those morphisms whose image is contained in U.

Exercise 4.11.25 ($\mathbb{A}^1$ minus a point)**.** Consider the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$.

a) For a prime number p, show that $\mathfrak{m} = (t, p)$ defines a closed point of $\mathbb{A}^1$.
b) Let $X = \mathbb{A}^1 - \{\mathfrak{m}\}$. Show that $U = D(p) \cup D(t)$ and that $\mathcal{O}_X(U) = \mathbb{Z}[t]$.
c) Is U an affine scheme?
d) What happens if we replace $\mathbb{Z}$ by a field k in this example?

Exercise 4.11.26. Let X be a scheme and let $P, Q \in X$ be two points. Show that there exists an open subset $U \subset X$ which contains exactly one of P, Q.

Exercise 4.11.27. Let X be a quasi-compact scheme and let $f\colon X \to \operatorname{Spec} A$ be the morphism corresponding to a ring map $\varphi\colon A \to \mathcal{O}_X(X)$. Show that

$$\overline{f(X)} \;=\; V(\operatorname{Ker} \varphi) \subset \operatorname{Spec} A. \tag{4.22}$$

In particular, f is dominant if and only if $\operatorname{Ker} \varphi \subset \sqrt{(0)}$. If A is reduced, this is equivalent to φ being injective.

Exercise 4.11.28. Let X be a scheme and let $g_0, \ldots, g_n \in \mathcal{O}_X(X)$. Show that if the g_i generate the unit ideal in $\mathcal{O}_X(X)$, then the $D(g_i)$ cover X. Show that the converse holds if X is affine, but not when $X = \mathbb{A}^2_k - \{(0, 0)\}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

5

Gluing

By definition, a scheme is a space which is locally built from affine open subschemes. Over the next few sections, we will explain how to conversely construct schemes by gluing schemes along open subschemes. These gluing techniques are fundamental in scheme theory. As we will see shortly, they give a vast collection of new examples of schemes. They are also used heavily in general existence proofs, such as the construction of fiber products.

5.1 Gluing two schemes together

We begin with the most basic case, where we glue *two* schemes X_1 and X_2 along a common open subscheme. Concretely, this means that we are given open subschemes $X_{12} \subset X_1$ and $X_{21} \subset X_2$ together with an isomorphism

$$\tau\colon X_{12} \longrightarrow X_{21}.$$

As a topological space, the glued scheme X is the quotient space

$$X = (X_1 \sqcup X_2) \,/\sim$$

where $\sim$ is the equivalence relation generated by $x \sim \tau(x)$ for all $x \in X_{12} \subset X_1$. Let

$$q\colon X_1 \sqcup X_2 \to X$$

be the quotient map and write $g_i = q|_{X_i}\colon X_i \to X$ for $i = 1, 2$. By the definition of the quotient topology, a subset $V \subset X$ is open if and only if $g_i^{-1}V$ is open in X_i for $i = 1, 2$. In particular, each map g_i is open and its image $U_i = g_i(X_i)$ is an open set in X. Moreover, U_1 and U_2 form an open cover of X.

We next define the structure sheaf $\mathcal{O}_X$ on X. Over an open set $V \subset X$, the preimages $g_1^{-1}V \subset X_1$ and $g_2^{-1}V \subset X_2$ are open and τ identifies $g_1^{-1}V \cap X_{12}$ with $g_2^{-1}V \cap X_{21}$. A section of $\mathcal{O}_X(V)$ is a pair of sections

$$(s_1, s_2) \in \mathcal{O}_{X_1}(g_1^{-1}(V)) \times \mathcal{O}_{X_2}(g_2^{-1}(V))$$

which agree on the overlap via $\tau^\sharp$, that is,

$$s_1\big|_{g_1^{-1}(V)\cap X_{12}} = \tau^\sharp\Big(s_2\big|_{g_2^{-1}(V)\cap X_{21}}\Big).$$

For $V' \subset V$, the restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(V')$ sends (s_1, s_2) to $(s_1|_{g_1^{-1}V'}, s_2|_{g_2^{-1}V'})$.

It is straightforward to check that $\mathcal{O}_X$ is a sheaf and that $(X, \mathcal{O}_X)$ is a locally ringed space. Moreover, the maps $g_i\colon X_i \to X$ are open embeddings onto $U_i \subset X$ for $i = 1, 2$. This

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

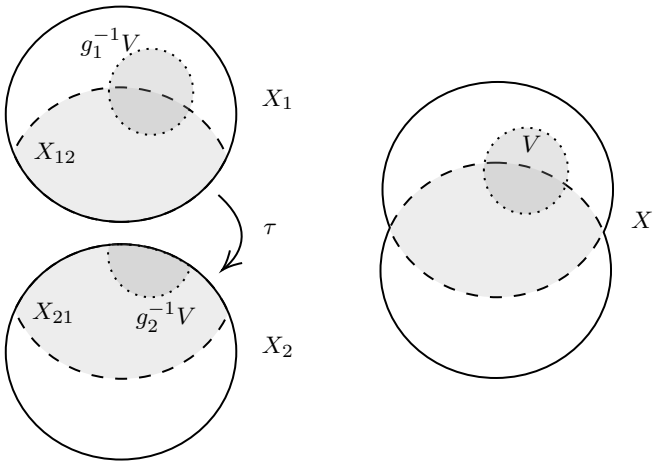


Figure 5.1 Gluing two schemes together

means that every point of X has an open neighbourhood which is affine, and hence X is a scheme. All of these statements will be proved in greater generality below.

A basic example to keep in mind is the case where both X_1 and X_2 are affine, say $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, and are glued together along distinguished open subsets $D(a)$ and $D(b)$ for some $a \in A$ and $b \in B$. The gluing isomorphism τ is specified by a ring isomorphism between the localizations

$$\varphi\colon B_b \longrightarrow A_a.$$

The resulting scheme X can be studied entirely in terms of A, B and φ. For example, the ring of global sections has the following description:

$$\mathcal{O}_X(X) = \{\, (f,g) \in A \times B \mid f/1 = \varphi(g/1) \text{ in } A_a \,\}. \tag{5.1}$$

This basic construction gives rise to a surprisingly rich set of examples (see Section 5.4).

Our next goal is to understand how to glue schemes from general collections of affine pieces. To prepare for this, we first establish some general results on gluing sheaves and morphisms of sheaves. The proofs in the next sections are somewhat technical and consist largely of routine verifications, so on a first reading the reader may wish to focus on the statements and return to the details later.

5.2 Gluing sheaves

Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. For convenience, we will denote the pairwise and triple intersections by

$$U_{ij} = U_i \cap U_j, \qquad U_{ijk} = U_i \cap U_j \cap U_k, \quad i,j,k \in I.$$

The simplest case of gluing involves morphisms of sheaves. The following proposition gives the precise conditions under which this can be done:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 5.1 (Gluing morphisms for sheaves). Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Suppose we are given, for each $i \in I$, a morphism of sheaves

$$\phi_i \colon \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i},$$

such that for all $i, j \in I$, the restrictions agree on overlaps:

$$\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}} \quad \text{for all } i, j \in I \tag{5.2}$$

Then there exists a unique morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$ for all i.

Proof For clarity, we will drop the subscripts from the sheaf maps ϕ_V and simply write $\phi(s)$ for the image of a section $s \in \mathcal{F}(V)$ under the map on sections over V.

Let $V \subset X$ be open. The sets $V_i = V \cap U_i$ form an open cover of V. Given $s \in \mathcal{F}(V)$, consider for each i the section $\phi_i(s|_{V_i}) \in \mathcal{G}(V_i)$. On overlaps we have

$$\phi_i(s|_{V_{ij}}) = \phi_i(s)|_{V_{ij}} = \phi_j(s)|_{V_{ij}} = \phi_j(s|_{V_{ij}}),$$

where the second equality follows from (5.2). Since $\mathcal{G}$ is a sheaf, the local sections $\phi_i(s|_{V_i})$ glue to a unique section in $\mathcal{G}(V)$. We define $\phi(s)$ to be this glued section.

By construction, this assignment is compatible with restrictions and additive, hence defines a morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$. Moreover, for any i and any $V \subset U_i$, the construction reduces to $\phi(s) = \phi_i(s)$, so $\phi|_{U_i} = \phi_i$.

For uniqueness, suppose $\psi \colon \mathcal{F} \to \mathcal{G}$ is another morphism with $\psi|_{U_i} = \phi_i$ for all i. Then for each $V \subset X$, $s \in \mathcal{F}(V)$ and $i \in I$,

$$\psi(s)|_{V_i} = \psi(s|_{V_i}) = \phi_i(s|_{V_i}) = \phi(s)|_{V_i}.$$

Since the open sets $V_i = V \cap U_i$ cover V, we must have $\psi(s) = \phi(s)$ by the Locality axiom for $\mathcal{G}$, and hence $\psi = \phi$. □

To glue sheaves, suppose we are given a sheaf $\mathcal{F}_i$ on each open set U_i. We want to construct a global sheaf $\mathcal{F}$ on X that restricts to $\mathcal{F}_i$ for every U_i. Precisely, this means that there is an isomorphism of sheaves on U_i

$$\nu_i \colon \mathcal{F}|_{U_i} \longrightarrow \mathcal{F}_i.$$

These isomorphisms must be compatible on the overlaps. On $U_{ij} = U_i \cap U_j$, ν_i and ν_j restrict to isomorphisms $\nu_i|_{U_{ij}} \colon \mathcal{F}|_{U_{ij}} \simeq \mathcal{F}_i|_{U_{ij}}$ and $\nu_j|_{U_{ij}} \colon \mathcal{F}|_{U_{ij}} \simeq \mathcal{F}_j|_{U_{ij}}$, so the restrictions $\mathcal{F}_i|_{U_{ij}}$ and $\mathcal{F}_j|_{U_{ij}}$ are isomorphic via $\tau_{ij} = \nu_j \circ \nu_i^{-1}$. Since the τ_{ij} are of this form, they automatically satisfy the following relations:

(i) $\tau_{ii} = \mathrm{id}$
(ii) $\tau_{ji} = \tau_{ij}^{-1}$ on U_{ij}
(iii) $\tau_{ik} = \tau_{jk} \circ \tau_{ij}$ on $U_i \cap U_j \cap U_k$.

The last identity follows because $\tau_{jk} \circ \tau_{ij} = \nu_k \circ \nu_j^{-1} \circ \nu_j \circ \nu_i^{-1} = \nu_k \circ \nu_i^{-1} = \tau_{ik}$.

The condition (iii) is called the *cocycle condition*. In fact, it formally implies the two others, as can be seen by setting $j = k = i$ and $k = i$, respectively.

The following proposition shows that conversely, given local sheaves $\mathcal{F}_i$ together with compatible isomorphisms τ_{ij}, one can glue them together to a global sheaf $\mathcal{F}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 5.2 (Gluing Sheaves). Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover. Suppose we are given a sheaf $\mathcal{F}_i$ on each U_i together with sheaf isomorphisms

$$\tau_{ij}\colon \mathcal{F}_i|_{U_i\cap U_j} \longrightarrow \mathcal{F}_j|_{U_i\cap U_j}$$

such that for every $i, j, k \in I$

$$\tau_{ik} = \tau_{jk}\circ\tau_{ij}, \qquad \text{on } U_i\cap U_j\cap U_k. \tag{5.3}$$

Then there is a sheaf $\mathcal{F}$ on X together with isomorphisms $\nu_i\colon \mathcal{F}|_{U_i}\simeq\mathcal{F}_i$ such that the following diagram commutes for all $i, j \in I$:

$$\begin{array}{ccc} & \mathcal{F}|_{U_{ij}} & \\ {\scriptstyle\nu_i}\swarrow & & \searrow{\scriptstyle\nu_j} \\ \mathcal{F}_i|_{U_{ij}} & \xrightarrow{\quad\tau_{ij}\quad} & \mathcal{F}_j|_{U_{ij}} \end{array} \tag{5.4}$$

The sheaf $\mathcal{F}$ is unique up to isomorphism.

Proof We define the sheaf $\mathcal{F}$ by setting, for $V\subset X$,

$$\mathcal{F}(V) = \Big\{(s_i)_{i\in I}\in\prod_{i\in I}\mathcal{F}_i(V\cap U_i) \;\Big|\; \tau_{ij}\big(s_i|_{V\cap U_{ij}}\big) = s_j|_{V\cap U_{ij}} \text{ for all } i,j\Big\}. \tag{5.5}$$

For an inclusion $W\subset V$, the restriction map $\mathcal{F}(V)\to\mathcal{F}(W)$ is defined componentwise, sending $(s_i)_i$ to $(s_i|_{W\cap U_i})_i$. These maps make $\mathcal{F}$ into a presheaf; we next check the sheaf axioms.

Locality axiom: If $s=(s_i)_i$ and $t=(t_i)_i$ are elements of $\mathcal{F}(V)$ with $s|_{W_\alpha}=t|_{W_\alpha}$ for an open cover $V=\bigcup_\alpha W_\alpha$, then for each i the components satisfy $s_i|_{W_\alpha\cap U_i}=t_i|_{W_\alpha\cap U_i}$. Since each $\mathcal{F}_i$ is a sheaf and $\{W_\alpha\cap U_i\}$ covers $V\cap U_i$, we conclude $s_i=t_i$ for every i, hence $s=t$.

Gluing axiom: Let $V=\bigcup_\alpha W_\alpha$ and suppose given sections $s^{(\alpha)}\in\mathcal{F}(W_\alpha)$ with $s^{(\alpha)}|_{W_\alpha\cap W_\beta}=s^{(\beta)}|_{W_\alpha\cap W_\beta}$ for all α,β. Write $s^{(\alpha)}=(s_i^{(\alpha)})_i$. Fix an index $i\in I$. The family $\{s_i^{(\alpha)}\}_\alpha$ is compatible on the cover $\{W_\alpha\cap U_i\}$ of $V\cap U_i$, so since $\mathcal{F}_i$ is a sheaf, there is a unique section $s_i\in\mathcal{F}_i(V\cap U_i)$ with $s_i|_{W_\alpha\cap U_i}=s_i^{(\alpha)}$ for every α. We must check that the tuple $(s_i)_i$ satisfies the compatibility relations defining $\mathcal{F}(V)$. On each $W_\alpha\cap U_{ij}$ we have

$$\tau_{ij}\big(s_i|_{W_\alpha\cap U_{ij}}\big)=\tau_{ij}\big(s_i^{(\alpha)}|_{W_\alpha\cap U_{ij}}\big)=s_j^{(\alpha)}|_{W_\alpha\cap U_{ij}}=s_j|_{W_\alpha\cap U_{ij}}.$$

Since the sets $W_\alpha\cap U_{ij}$ cover $V\cap U_{ij}$, it follows that $\tau_{ij}(s_i|_{V\cap U_{ij}})=s_j|_{V\cap U_{ij}}$. Hence $(s_i)_i\in\mathcal{F}(V)$, and by construction its restriction to each W_α equals $s^{(\alpha)}$.

For each i define $\nu_i\colon\mathcal{F}|_{U_i}\to\mathcal{F}_i$ by projection to the i-th coordinate: if $V\subset U_i$ and $s=(s_j)_j\in\mathcal{F}(V)$, set $\nu_i(s)=s_i\in\mathcal{F}_i(V)$. This is a morphism of sheaves. It has an inverse on U_i constructed as follows. Given $t\in\mathcal{F}_i(V)$ with $V\subset U_i$, define for each j

$$t_j=\tau_{ij}\big(t|_{V\cap U_{ij}}\big)\in\mathcal{F}_j(V\cap U_j).$$

The relations (i)–(iii) for the τ's ensure that the $(t_j)_j$ satisfies the compatibility conditions, hence define an element of $\mathcal{F}(V)$. Projection to the i-th coordinate returns t, so ν_i is an

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

isomorphism and the inverse is given by $t \mapsto (t_j)_j$. By construction, the diagram (5.4) commutes for all i, j.

Finally, $\mathcal{F}$ is unique up to unique isomorphism. If $\mathcal{F}'$ is another sheaf on X with isomorphisms $\nu_i' : \mathcal{F}'|_{U_i} \simeq \mathcal{F}_i$ satisfying the same compatibility, define

$$\beta_i = \nu_i'^{-1} \circ \nu_i \colon \mathcal{F}|_{U_i} \longrightarrow \mathcal{F}'|_{U_i}.$$

On U_{ij} we have $\nu_j \circ \nu_i^{-1} = \tau_{ij} = \nu_j' \circ \nu_i'^{-1}$, hence $\nu_j'^{-1} \circ \nu_j = \nu_i'^{-1} \circ \nu_i$ on U_{ij}, and so $\beta_j = \beta_i$ on U_{ij}. Thus the family $\{\beta_i\}$ glue to a unique morphism $\beta \colon \mathcal{F} \to \mathcal{F}'$ with $\beta|_{U_i} = \beta_i$. Applying the same construction to the inverses ν_i^{-1} and $\nu_i'^{-1}$ yields an inverse $\mathcal{F}' \to \mathcal{F}$. Hence β is an isomorphism. Moreover, any isomorphism $\gamma \colon \mathcal{F} \to \mathcal{F}'$ that is compatible with the identifications must restrict to $\nu_i'^{-1} \circ \nu_i = \beta_i$ on each U_i, and by uniqueness of gluing morphisms these local restrictions determine γ uniquely; hence $\gamma = \beta$. □

The collection $(\mathcal{F}_i, \tau_{ij})$ from Proposition 5.2 is often called *gluing data* for sheaves (see Exercise 5.5.7). Describing a sheaf in terms of its gluing data is often a convenient way to present it. A common situation is when the open sets of the cover are affine, say $U_i = \operatorname{Spec} A_i$, and each $\mathcal{F}_i$ is of the form $\widetilde{M}_i$ for some A_i-module M_i. This leads to the class of quasi-coherent sheaves, which we will study in Chapter 10.

5.3 Gluing schemes

The following result tells us when we can glue morphisms of schemes.

Proposition 5.3 (Gluing morphisms of schemes). Let X and Y be schemes and let $\{U_i\}_{i \in I}$ be an open cover of X. Suppose we are given morphisms $f_i \colon U_i \to Y$ such that

$$f_i|_{U_{ij}} = f_j|_{U_{ij}} \quad \text{for all } i, j,$$

Then there exists a unique morphism $f \colon X \to Y$ such that $f|_{U_i} = f_i$ for every $i \in I$.

Proof On the underlying topological space we define $f(x) = f_i(x)$ if $x \in U_i$. This is well-defined because $f_i(x) = f_j(x)$ whenever $x \in U_{ij}$, and it is continuous since each f_i is continuous.

To construct the sheaf map $f^\sharp$, let $V \subset Y$ be open and take $s \in \mathcal{O}_Y(V)$. For each i the morphism f_i gives a section $t_i = f_i^\sharp(s) \in \mathcal{O}_X(f^{-1}V \cap U_i)$. Over the overlaps U_{ij} the assumption $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ implies that

$$t_i|_{f^{-1}V \cap U_{ij}} = t_j|_{f^{-1}V \cap U_{ij}}.$$

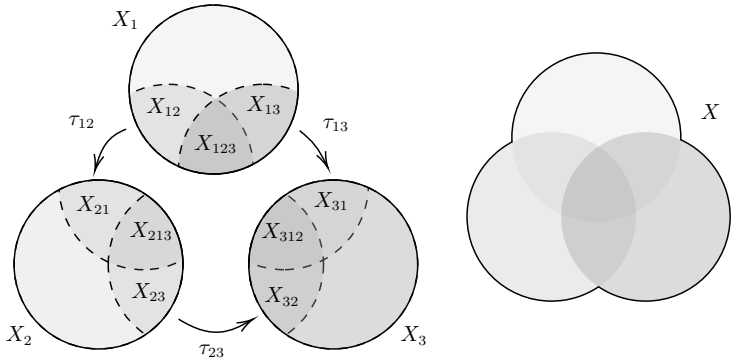
The sections $\{t_i\}_i$ therefore glue to a unique section $t \in \mathcal{O}_X(f^{-1}V)$, and we set $f^\sharp(s) = t$.

This construction is compatible with restriction maps, so $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings. As pair $(f, f^\sharp)$ restricts to $(f_i, f_i^\sharp)$ over each U_i, it defines a morphism of locally ringed spaces $f \colon X \to Y$.

For uniqueness, if $g \colon X \to Y$ is another morphism with $g|_{U_i} = f_i$, then the underlying maps of spaces f and g agree on the cover $\{U_i\}$ and hence coincide globally. The same holds for $f^\sharp$ and $g^\sharp$, since two sheaf maps which agree on an open cover must agree everywhere. Thus $f = g$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

When we talk about gluing schemes, the set-up is as follows. We start with a collection of schemes $\{X_i\}_{i\in I}$, and for each i, j, an open subscheme $X_{ij} \subset X_i$. The goal is to produce a new scheme X by gluing together all the X_i along these open subschemes. Concretely, we identify the open sets $X_{ij} \subset X_i$ and $X_{ji} \subset X_j$ via isomorphisms $\tau_{ij}\colon X_{ij} \simeq X_{ji}$. There are three gluing conditions, directly analogous to the ones we saw for sheaves, which must be satisfied in order for the gluing construction to work.



Proposition 5.4 (Gluing conditions for schemes). Let $\{X_i\}_{i\in I}$ be a collection of schemes. For each $i, j \in I$, let $X_{ij} \subset X_i$ be an open subscheme, with $X_{ii} = X_i$, and let

$$\tau_{ij}\colon X_{ij} \xrightarrow{\sim} X_{ji}$$

be an isomorphism such that for all i, j, k we have $\tau_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$ and

$$\tau_{ik} = \tau_{jk} \circ \tau_{ij} \qquad \text{on } X_{ij} \cap X_{ik}. \tag{5.6}$$

Then there exists a scheme X together with open embeddings $g_i\colon X_i \hookrightarrow X$ such that for all i, j the following diagram commutes:

$$\begin{array}{ccccc} X_{ij} & \hookrightarrow & X_i & & \\ \downarrow{\scriptstyle \tau_{ij}} & & & \searrow^{g_i} & \\ & & & & X \\ X_{ji} & \hookrightarrow & X_j & \nearrow_{g_j} & \end{array}$$

The scheme X has the following universal property: if $f_i\colon X_i \to Y$ are morphisms such that $f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ \tau_{ij}$ for all i, j, then there exists a unique morphism $f\colon X \to Y$ such that $f \circ g_i = f_i$ for all i.

Proof Let $\bigsqcup_{i\in I} X_i$ be the disjoint union of the topological spaces X_i. Define an equivalence relation $\sim$ on this set by setting for $x \in X_i$ and $y \in X_j$,

$$x \sim y \iff x \in X_{ij},\ y \in X_{ji} \text{ and } y = \tau_{ij}(x).$$

(In particular, when $i = j$, $x \in X_i$ is only equivalent to itself.) The cocycle condition (5.6) ensures that $\sim$ is an equivalence relation. Let

$$X = \bigsqcup_{i\in I} X_i/\sim$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

be the quotient space with the quotient topology, and let $\pi\colon \bigsqcup_i X_i \to X$ be the quotient map. For each i denote by $g_i\colon X_i \to X$ the restriction of π. Note that g_i is injective, since no two distinct points of the same X_i are identified by $\sim$.

We next show that g_i is an open map, hence a homeomorphism onto its image $U_i = g_i(X_i)$. If $V \subset X_i$ is open then by the definition of the quotient topology, $g_i(V) \subset X$ is open if and only if $\pi^{-1}(g_i(V))$ is open in the disjoint union. This preimage equals

$$\pi^{-1}(g_i(V)) = \bigsqcup_j \tau_{ij}(V \cap X_{ij}).$$

As each τ_{ij} is a homeomorphism onto an open subset, the right-hand side is open, so $\pi^{-1}(g_i(V))$ is open and hence g_i is an open map.

Next we glue the structure sheaves. On each open U_i push forward the structure sheaf of X_i along the homeomorphism g_i and let

$$\mathcal{O}_{U_i} = (g_i)_*\mathcal{O}_{X_i}.$$

Using the identification $U_{ij} = g_i(X_{ij}) = g_j(X_{ji})$, the given scheme isomorphisms $\tau_{ij}\colon X_{ij} \to X_{ji}$ induce isomorphisms of sheaves

$$\tau_{ij}^\sharp\colon \mathcal{O}_{U_j}|_{U_{ij}} \xrightarrow{\sim} \mathcal{O}_{U_i}|_{U_{ij}}.$$

The assumption 5.6 ensures that the conditions for gluing sheaves in Proposition 5.2 are satisfied, so the sheaves $\mathcal{O}_{U_i}$ glue to a sheaf of rings $\mathcal{O}_X$ on X whose restriction to each U_i is $\mathcal{O}_{U_i}$. The stalk of $\mathcal{O}_X$ at a point $x \in X_i$ is canonically isomorphic to the stalk $\mathcal{O}_{X_i,x}$, hence is a local ring. Therefore $(X, \mathcal{O}_X)$ is a locally ringed space. Moreover, any point $x \in X$ we can choose an index i with $x \in U_i$ and then choose an affine open neighbourhood $V \subset X_i$ of the corresponding point. The image $g_i(V) \subset U_i \subset X$ is an open neighbourhood of x and is isomorphic (via g_i) to the affine scheme V, hence $g_i(V)$ is affine. Thus X has an open affine cover and so is a scheme.

Finally, we prove the universal property. If $f_i\colon X_i \to Y$ are morphisms to a scheme Y satisfying $f_i|_{X_{ij}} = f_j \circ \tau_{ij}$ for all i, j, then by the universal property of the quotient space, the maps f_i assemble to a continuous map $f\colon X \to Y$. The compatible maps on structure sheaves $f_i^\sharp$ glue (by Proposition 5.1) to a morphism $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, and the pair $(f, f^\sharp)$ is a morphism of schemes with $f \circ g_i = f_i$. Uniqueness follows from the corresponding uniqueness statement for gluing morphisms of schemes. □

5.4 First examples

In the examples below, k will be a field.

The affine line with two origins

Consider two copies of $\mathbb{A}^1_k$, say $X_1 = \operatorname{Spec} k[x]$ and $X_2 = \operatorname{Spec} k[u]$. We identify the open subsets $X_{12} = D(x) = \operatorname{Spec} k[x, x^{-1}] \subset X_1$ and $X_{21} = D(u) = \operatorname{Spec} k[u, u^{-1}] \subset X_2$ using the ring isomorphism $\tau\colon k[u, u^{-1}] \to k[x, x^{-1}]$, $u \mapsto x$. The resulting scheme X has an affine covering consisting of two $\mathbb{A}^1_k$'s which overlap outside the origin. However, as

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

the gluing process does nothing over the origins of each $\mathbb{A}^1_k$, there are now *two* points in X that replace the origin. X is called the *affine line with two origins*.



The scheme X is not affine. To see this, we compute $\mathcal{O}_X(X)$ using (5.1):

$$\begin{aligned}\mathcal{O}_X(X) &= \{\, (p(x), q(x)) \in k[x] \times k[x] \mid p(x) = q(x) \text{ in } k[x, x^{-1}] \,\} \\ &\simeq k[x].\end{aligned}$$

Under this identification, the inclusion $\iota\colon X_1 = \mathbb{A}^1_k \to X$ induces the identity map $\iota^\sharp\colon k[x] \to k[x]$ on global sections. Now if X were affine, this would imply that ι is an isomorphism, which is clearly not the case (it is not surjective, since the image misses one of the two origins).

The affine line with two origins X is both irreducible and reduced. The function field is isomorphic to $K = k(x)$. The two local rings $\mathcal{O}_{X,0_1}$ and $\mathcal{O}_{X,0_2}$ are both equal to the subring $k[x]_{(x)}$ inside K. This is somewhat unsettling: any rational function which is regular at 0_1 is automatically regular at 0_2 and it takes the same value there. This is related to the property of *separatedness*, which we will discuss in Chapter 13.

The projective line

Consider two copies of the affine line $X_1 = \operatorname{Spec} k[x]$ and $X_2 = \operatorname{Spec} k[u]$. We again identify the open subsets $X_{12} = D(x) = \operatorname{Spec} k[x, x^{-1}]$ and $X_{21} = D(u) = \operatorname{Spec} k[u, u^{-1}]$, but this time using the ring isomorphism $\varphi\colon k[u, u^{-1}] \to k[x, x^{-1}]$, sending $u \mapsto x^{-1}$.The resulting scheme is called the *projective line over* k, and it is denoted by $\mathbb{P}^1_k$.

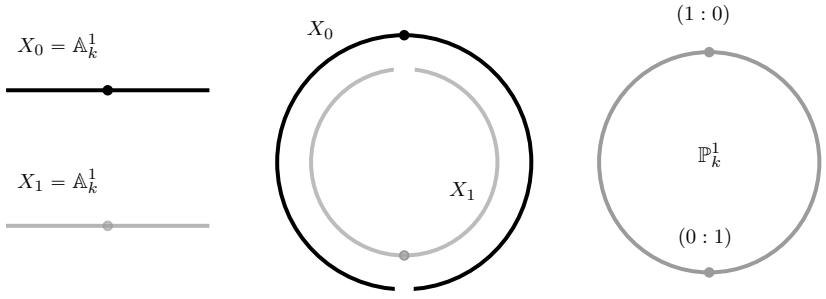


Figure 5.2 $\mathbb{P}^1(k)$ obtained by gluing two affine lines.

As a set, $\mathbb{P}^1_k$ consists of the points in X_1, which is a copy of $\mathbb{A}^1_k$, along with the points of $X_2 - D(u) = V(u)$, that is, the origin in $X_2 = \mathbb{A}^1_k$. In other words, $\mathbb{P}^1_k$ can be thought of as $\mathbb{A}^1_k$ together with a *point at infinity*.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The global sections of $\mathcal{O}_{\mathbb{P}^1_k}$ is computed using (5.1):

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) &= \{\, (p(x), q(u)) \in k[x] \times k[u] \mid p(x) = q(x^{-1}) \text{ in } k[x, x^{-1}] \,\} \\ &\simeq k,\end{aligned}$$

as the condition $p(x) = q(x^{-1})$ forces p and q to be constants. In particular, $\mathbb{P}^1_k$ is not affine: if it were, it would be $\operatorname{Spec} k$, which is clearly not the case.

We will study this scheme, along with higher-dimensional projective spaces $\mathbb{P}^n_k$ in more detail in Chapter 6.

Line bundles on $\mathbb{P}^1_k$

In this section, we will study a family of examples, constructed by gluing two affine planes $\mathbb{A}^2_k$ along distinguished opens. Let $n \in \mathbb{Z}$, and let x, t be variables and set $u = x^{-1}$ and $v = x^n t$. Consider the affine schemes

$$X_1 = \operatorname{Spec} k[x, t], \qquad X_2 = \operatorname{Spec} k[u, v] = \operatorname{Spec} k[x^{-1}, x^n t], \tag{5.7}$$

These are glued along $X_{12} = \operatorname{Spec} k[x, x^{-1}, t]$, which is a distinguished open set in both X_1 and X_2 (it arises by inverting x in $k[x, t]$ and x^{-1} in $k[x^{-1}, x^n t]$). We obtain a scheme, which we denote by X.

The scheme X admits a morphism $\pi\colon X \to \mathbb{P}^1_k$. This is defined by gluing together the two projections $\operatorname{Spec} k[x, t] \to \mathbb{A}^1_k = \operatorname{Spec} k[x]$ and $\operatorname{Spec} k[u, v] \to \operatorname{Spec} k[u]$. These glue, by the commutative diagram

$$\begin{array}{ccccc} k[x,t] & \hookrightarrow & k[x,x^{-1},t] & \hookleftarrow & k[x^{-1},x^n t] \\ \uparrow & & \uparrow & & \uparrow \\ k[x] & \hookrightarrow & k[x,x^{-1}] & \hookleftarrow & k[x^{-1}] \end{array} \tag{5.8}$$

So the morphism π is locally the projection map $\mathbb{A}^2_k \to \mathbb{A}^1_k$ onto the first factor, but there is a 'twist': over the intersection, the vertical coordinates are related by $v = x^n t$.

The diagram (5.8) also shows that if $P \in \operatorname{Spec} k[x]$ is a point, then $\pi^{-1}(P)$ is contained in X_1. If P corresponds to a maximal ideal $\mathfrak{m} \subset k[x]$, then the fiber is given by

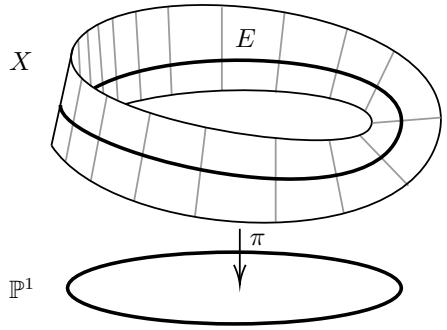
$$\pi^{-1}(P) = \operatorname{Spec}(k[x, t]/\mathfrak{m}k[x, t]) = \operatorname{Spec}(\kappa(P)[t]) \simeq \mathbb{A}^1_{\kappa(P)}.$$

This is the reason for the term 'line bundle'. We picture X as a family of affine lines parameterized by the base space $\mathbb{P}^1_k$.

In X_1, the closed subscheme $V(t)$, corresponding to 'the x-axis', is isomorphic to $\mathbb{A}^1_k$ and similarly for $V(v) \subset X_2$. Since x is invertible in $k[x, x^{-1}, t]$, the two ideals (t), $(v) = (x^n t)$ are equal and hence glue to a closed subscheme $E \subset X$. Since E is constructed by gluing $\operatorname{Spec} k[x, t]/(t) \simeq \operatorname{Spec} k[x]$ and $\operatorname{Spec} k[u, v]/(v) \simeq \operatorname{Spec} k[x^{-1}]$, it is isomorphic to $\mathbb{P}^1_k$.

We also see that the composition $E \to X \to \mathbb{P}^1_k$ is an isomorphism, as it restricts to the identity maps $\operatorname{Spec} k[x] \to \operatorname{Spec} k[x]$ and $\operatorname{Spec} k[u] \to \operatorname{Spec} k[u]$ over the two open sets. In other words, E is a *section* of the morphism π.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026



Since X is glued together from two copies of $\mathbb{A}^2_k$, we see that it is reduced. It is also irreducible, hence integral, because they intersect along a dense open set. The function field is $K(X) = k(x,t)$.

By the sheaf exact sequence, the ring of global sections $\mathcal{O}_X(X)$ is given by

$$\mathcal{O}_X(X) = k[x,t] \cap k[x^{-1}, x^n t]. \tag{5.9}$$

inside $k(x,t)$. If $n \geqslant 0$, then

$$k[x,t] \cap k[x^{-1}, x^n t] = k[x^n t, x^{n-1} t, \ldots, t] \tag{5.10}$$

whereas if $n < 0$,

$$k[x,t] \cap k[x^{-1}, x^n t] = k.$$

Example 5.5 ($n = 1$)**.** The case $n = 1$ is a famous example in algebraic geometry. It is called the *blow-up of* $\mathbb{A}^2_k$ *at the origin* (see Figure A.1). We have

$$\mathcal{O}_X(X) = k[t, xt] = k[t, v].$$

The inclusions of $k[t,v]$ into the two polynomial rings induce a morphism

$$\sigma\colon X \longrightarrow \mathbb{A}^2_k = \operatorname{Spec} k[t,v].$$

This morphism is called the 'blow-up'. The reason is that it collapses the subscheme E to the origin $P = (0,0) \in \mathbb{A}^2_k$. Indeed, over X_1, the morphism is $\operatorname{Spec} k[t,x] \to \operatorname{Spec} k[t,v]$ sending $(t,x) \mapsto (t, xt)$ which collapses $V(t)$ to the origin $(t,v) \in \mathbb{A}^2_k$. The example is symmetric: over X_2, the morphism is given by $(u,v) \mapsto (uv, v)$, which collapses $V(v)$ to the origin.

On the other hand, σ induces an isomorphism $X - E \to \mathbb{A}^2_k - P$. Indeed, over X_1, the inclusion $k[t,xt] \subset k[x,t]$, used to define σ, becomes an equality after inverting t. Likewise, $k[t,v]_v = k[u,v]_v$, so σ restricts to an isomorphism over $D(t)$ and $D(v)$. Under σ, the fibers of π are mapped to lines in $\mathbb{A}^2_k$ passing through the origin.

The scheme X is yet another example of a non-affine scheme. In fact, the canonical morphism $X \to \operatorname{Spec} \mathcal{O}_X(X)$ is exactly the morphism σ which collapses the non-affine subscheme $E \simeq \mathbb{P}^1_k$ to a point. △

Example 5.6 ($n = 2$)**.** Here the ring $R = \mathcal{O}_X(X)$ has the following description:

$$R = k[x^2 t, xt, t] \simeq k[a,b,c]/(b^2 - ac)$$

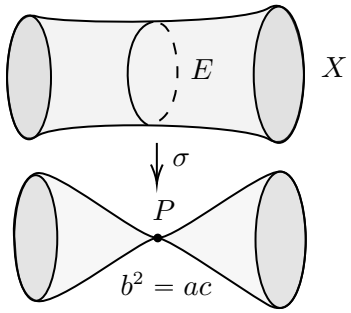
This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

via $a = x^2t$, $b = xt$ and $c = t$.

The inclusions $R \subset k[x,t]$ and $R \subset k[x^{-1}, x^2t]$ induce morphisms $\sigma_1 \colon X_1 \to \operatorname{Spec} R$ and $\sigma_2 \colon X_2 \to \operatorname{Spec} R$ which glue to a morphism $\sigma \colon X \to \operatorname{Spec} R$. As in the $n = 1$ case, this morphism is surjective and collapses the subscheme E to the origin $P = (0,0,0) \in \operatorname{Spec} R$, corresponding to the maximal ideal $(a,b,c) \subset R$.

To see the isomorphism $X - E \simeq \operatorname{Spec} R - P$, note that $\operatorname{Spec} R - P$ is covered by the two distinguished open sets $D(a)$ and $D(c)$ (the open set $D(b) = D(b^2) = D(ac) = D(a) \cap D(c)$ is not needed). Similarly, the complement $X - E$ is covered by the distinguished open subsets $D(t) \subset X_1$ and $D(v) \subset X_2$. Over $D(t)$, the inclusion $R \subset k[x,t]$ localizes to the equality $R_c = k[x,t]_t$, and similarly over $D(v)$. Therefore, σ restricts to an isomorphism $X - E \simeq \operatorname{Spec} R - P$ and the fiber over P is exactly E.

The morphism σ is helpful for understanding the quadratic cone. In the terminology of Chapter 16, $\operatorname{Spec} R$ has a *singular point* at the origin, and the morphism $X \to \operatorname{Spec} R$ resolves this singularity. △



Hyperelliptic curves

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree n in $k[x]$. Write $n = 2g+1$ if n is odd and $n = 2g+2$ if n is even. Consider the two affine schemes $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, where

$$A = k[x,y]/(y^2 - p(x)), \qquad B = k[u,v]/(v^2 - u^{2g+2}p(u^{-1})).$$

Here, the element $u^{2g+2}p(u^{-1})$ is in fact a polynomial in u:

$$u^{2g+2}p(u^{-1}) = \begin{cases} a_n + a_{n-1}u + \cdots + a_0 u^{2g+2}, & n = 2g+2, \\ a_n u + a_{n-1}u^2 + \cdots + a_0 u^{2g+2}, & n = 2g+1. \end{cases}$$

The gluing of X_1 and X_2 is achieved by identifying the localizations $A_x \simeq B_u$ using $u = x^{-1}$ and $v = x^{-g-1}y$. This is well-defined because

$$v^2 - u^{2g+2}p(u^{-1}) = x^{-2(g+1)}y^2 - x^{-2g-2}p(x) = x^{-(2g+2)}(y^2 - p(x)).$$

The resulting scheme X is called a *hyperelliptic curve*. Note that the gluing formulas are similar to those of the previous section. In fact, our subscheme X is a closed subscheme of the previous example.

As a set, X consists of the points of $X_1 = \operatorname{Spec} A$, along with the points of $\operatorname{Spec} B$ outside

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$D(u) = D(x)$, that is, the points of $V(u) \subset \operatorname{Spec} B$. The number of points in $V(u)$ depends on whether n is even or odd. If n is odd, then the equation $v^2 = a_n u + \cdots + a_0 u^{2g+2}$ shows that $V(u) = V(u, v)$ consists of a single point. If n is even, then $V(u) = V(u, v^2 - a_n)$ which consists of two points if a_n has a square root in k and $\operatorname{char} k \neq 2$ and a single point otherwise.

As in the previous example, the morphisms $\operatorname{Spec} A \to \operatorname{Spec} k[x]$ and $\operatorname{Spec} B \to \operatorname{Spec} k[x^{-1}]$ glue to a morphism $\pi\colon X \to \mathbb{P}^1_k$. It is a *double cover*; over each affine, π restricts to a double cover of $\mathbb{A}^1_k$, as discussed in Example 2.38.

5.5 Exercises

Exercise 5.5.1. Show that for $n = 0$, the scheme X in 5.4 is isomorphic to $\mathbb{P}^1_{k[t]}$.

Exercise 5.5.2. Let X be the scheme associated to the case $n = -1$ in Section 5.4. Show that X is isomorphic to $\mathbb{P}^2_k - \{(0:0:1)\}$ and that the morphism $\pi\colon X \to \mathbb{P}^1_k$ is identified with the projection $\mathbb{P}^2_k - \{P\} \to \mathbb{P}^1_k$ from $(0:0:1)$. Use this to explain that $\mathcal{O}_X(X) = k$.

Exercise 5.5.3. Let X be the scheme associated to the case $n = 3$ in Section 5.4. Show that the ring $\mathcal{O}_X(X)$ is isomorphic to the homogeneous coordinate ring of the projective twisted cubic curve. Describe the canonical morphism $X \to \operatorname{Spec} \mathcal{O}_X(X)$.

Exercise 5.5.4. Let X, $\mathcal{F}_i$, τ_{ij} be as in Proposition 5.2.

a) Let $\mathcal{B}$ be the collection of all open sets $U \subset X$ such that $U \subset U_i$ for at least one index $i \in I$. Show that $\mathscr{B}$ is a basis for the topology on X.

b) For each $U \in \mathcal{B}$, choose an index $i = i(U) \in I$ such that $U \subset U_i$, and set $\mathcal{F}(U) = \mathcal{F}_i(U)$. For an inclusion $V \subset U$ in $\mathcal{B}$, let $i = i(U)$ and $j = i(V)$, so that also $V \subset U_i \cap U_j = U_{ij}$. Define the restriction maps by

$$\rho_{U,V}\colon \mathcal{F}(U) = \mathcal{F}_i(U) \longrightarrow \mathcal{F}_i(V) \xrightarrow{\tau_{ji}|_V} \mathcal{F}_j(V) = \mathcal{F}(V) \tag{5.11}$$

Show that this makes $\mathcal{F}$ into a $\mathscr{B}$-presheaf.

c) Show that τ_{ij} defines an isomorphism $\nu_{i,V}\colon \mathcal{F}(V) \to \mathcal{F}_i(V)$. Use this to define an isomorphism of presheaves $\mathcal{F}|_{U_i} \to \mathcal{F}_i$. Deduce that $\mathcal{F}_i$ is a $\mathscr{B}$-*sheaf*.

d) Show that the extended sheaf $\mathcal{F}$ and ν_i satisfy the conclusion in Proposition 5.2.

Exercise 5.5.5 (Disjoint unions)**.** a) Let $\{X_i\}_{i\in I}$ be an arbitrary collection of schemes. Show that the disjoint union $\coprod_{i\in I} X_i$ exists as a scheme.

b) Let $X_1, \ldots, X_r$ be finitely many affine schemes. Show that $\coprod_{i=1}^r \operatorname{Spec}(A_i) = \operatorname{Spec}\left(\prod_{i=1}^r A_i\right)$.

c) Give an example showing that $\coprod_{i\in I} X_i$ may not be affine. HINT: Affine schemes are quasi-compact.

Exercise 5.5.6. Let X be the affine line with two origins, as defined in Section 5.4.

a) Imitate the construction of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$ to form a collection of sheaves $\mathcal{O}_X(m)$ on X, one for each integer m.

b) Show that $\mathcal{O}_X(m)$ and $\mathcal{O}_X(n)$ are not isomorphic unless $m = n$. HINT: Consider the behaviour of sections at the two origins.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 5.5.7 (Gluing data)**.** Let $\{U_i\}_{i\in I}$ be an open cover of a scheme X. A *gluing datum for sheaves* is a collection $(\mathcal{F}_i, \tau_{ij})$ where $\mathcal{F}_i$ is a sheaf on U_i and $\tau_{ij}\colon \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$ is a collection of isomorphisms satisfying $\tau_{ik} = \tau_{jk}\circ\tau_{ij}$ on $U_i \cap U_j \cap U_k$.

A *morphism of gluing data* $(\mathcal{F}_i, \tau_{ij}) \to (\mathcal{G}_i, \delta_{ij})$ is a collection of sheaf morphisms $\theta_i\colon \mathcal{F}_i \to \mathcal{G}_i$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_i|_{U_i\cap U_j} & \xrightarrow{\tau_{ij}} & \mathcal{F}_j|_{U_i\cap U_j} \\ {\scriptstyle \theta_i|_{U_i\cap U_j}}\downarrow & & \downarrow{\scriptstyle \theta_j|_{U_i\cap U_j}} \\ \mathcal{G}_i|_{U_i\cap U_j} & \xrightarrow[\delta_{ij}]{} & \mathcal{G}_j|_{U_i\cap U_j} \end{array}$$

Show that the gluing data form a category $\mathsf{GluingData}$ and that the functor $\mathsf{Ab}(X) \to \mathsf{GluingData}$, $\mathcal{F} \to (\mathcal{F}|_{U_i}, \mathrm{id}_{\mathcal{F}|_{U_i\cap U_j}})$ is an equivalence of categories.

Exercise 5.5.8 (Morphisms to a glued scheme)**.** Let X be the scheme obtained by the gluing data X_i, τ_{ij} as in Proposition 5.4. Show that to give a morphism $h\colon Y \to X$ is equivalent to giving a covering $\{Y_i\}_{i\in I}$ of Y and morphisms $h_i\colon Y_i \to X_i$ such that $h_i^{-1}(X_{ij}) = Y_i \cap Y_j$ and the following diagrams commute of all $i, j \in I$:

$$\begin{array}{ccc} Y_i \cap Y_j & \xrightarrow{g_i} & X_{ij} \\ & \searrow^{g_j} & \downarrow{\scriptstyle \tau_{ij}} \\ & & X_{ji}. \end{array}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

6

Projective schemes

Projective space and projective schemes play a central role in algebraic geometry. Most schemes encountered in practice are projective or at least open subsets of projective schemes (such schemes are called *quasi-projective*). In fact, it is surprisingly difficult to find naturally occurring examples of schemes that can be shown not to be quasi-projective.

Projective schemes often have better global properties than affine schemes. Over the complex numbers, for example, complex projective space $\mathbb{CP}^n$, and hence every closed subvariety, is compact in the classical topology, whereas any affine variety of positive dimension is not. Also, the image of a morphism between projective varieties is always closed, unlike in the affine case, where images can be more complicated.

These good properties come from the way projective space is constructed, by adding extra 'points at infinity' to affine space. This compactification makes many geometric statements behave more uniformly. For instance, in the affine plane two parallel lines never meet, but in the projective plane they intersect at a point at infinity. Because of this, many geometric theorems are most naturally stated and proved in the projective setting, with the affine case obtained as a consequence.

In this chapter, we approach projective schemes in two stages. In the first three sections we work concretely and define projective space as a set, then as a scheme glued from $n+1$ affine charts, and then we describe its closed subschemes, the projective schemes. In the final section, do we treat the general *Proj construction*, which produces a scheme out of an arbitrary graded ring. At the end we consider several classical examples, such as Veronese embeddings and weighted projective spaces.

6.1 Projective space

Traditionally, the projective n-space over a field k is defined as the quotient space

$$\mathbb{P}^n(k) = \Big(k^{n+1} - \{0\}\Big)/\sim \tag{6.1}$$

where we say that two $(n+1)$-tuples $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ are equivalent if there exists a $\lambda \in k^\times$ so that $b_i = \lambda a_i$ for all i. Geometrically, $\mathbb{P}^n(k)$ parameterizes the 1-dimensional subspaces of k^{n+1}, or in other words, the lines through the origin.

The equivalence class of an $(n+1)$-tuple $(x_0, \dots, x_n)$ is usually written $(x_0 : \cdots : x_n)$. So for instance, $(1:3) = (-1:-3)$ and $(1:2:3) = (2:4:6)$. The variables $x_0, \dots, x_n$ are called *homogeneous coordinates*. Despite the name, they are not coordinates in the usual sense. Indeed, a polynomial $f \in k[x_0, \dots, x_n]$ does not in general define a function on

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\mathbb{P}^n(k)$, since a point is only determined up to scaling and $f(\lambda x_0, \ldots, \lambda x_n)$ need not equal $f(x_0, \ldots, x_n)$. However, f is homogeneous of degree d, then

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n).$$

So although the value $f(P)$ is not well-defined, the condition $f(P) = 0$ is independent of the choice of representative. Homogeneous polynomials are therefore exactly those which give well-defined zero sets in $\mathbb{P}^n(k)$. For the same reason, a ratio

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)}$$

of two homogeneous polynomials of the same degree defines a function wherever $g \neq 0$.

The simplest such functions are the ratios x_j/x_i of two coordinates, and these will play a central role throughout the chapter. These are defined precisely on the subset

$$U_i = \{\, (x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid x_i \neq 0 \,\}.$$

Since a nonzero tuple has at least one nonzero entry, these sets cover $\mathbb{P}^n(k)$:

$$\mathbb{P}^n(k) = \bigcup_{i=0}^{n} U_i.$$

For each i there is a bijection

$$\begin{aligned} \phi_i \colon \mathbb{A}^n(k) &\longrightarrow U_i \\ (y_1, \ldots, y_n) &\longmapsto (y_1 : \cdots : y_{i-1} : 1 : y_i : \cdots : y_n), \end{aligned} \tag{6.2}$$

with inverse sending a point $(x_0 : \cdots : x_n) \in U_i$ to

$$\left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right) \in \mathbb{A}^n(k). \tag{6.3}$$

So we see that $\mathbb{P}^n(k)$ is a union of $n+1$ copies of affine space $\mathbb{A}^n(k)$ with affine coordinates given by the ratios in (6.3). We can define a topology on $\mathbb{P}^n(k)$ by declaring a set $W \subset \mathbb{P}^n(k)$ to be open if and only if $W \cap U_i$ is open in U_i for all $i = 0, \ldots, n$.

There is a second, complementary way to picture this decomposition. Note that the complement $\mathbb{P}^n(k) - U_0$ consists of all the points $(x_0 : \cdots : x_n)$ where $x_0 = 0$. This can be naturally identified with $\mathbb{P}^{n-1}(k)$ via the map $(y_0 : \cdots : y_{n-1}) \mapsto (0 : y_1 : \cdots : y_{n-1})$. We may write

$$\mathbb{P}^n(k) = U_0 \cup \mathbb{P}^{n-1}(k) = \mathbb{A}^n(k) \cup \mathbb{P}^{n-1}(k)$$

Thinking of $\mathbb{P}^n(k)$ as $\mathbb{A}^n(k)$ together with a copy of $\mathbb{P}^{n-1}(k)$ 'at infinity' is often a more useful way of thinking about the points of projective space.

Example 6.1. The projective line is the union $\mathbb{P}^1(k) = U_0 \cup U_1$. We identify $U_0 = \{\, (x_0 : x_1) \mid x_0 \neq 0 \,\}$ with $\mathbb{A}^1(k)$ via $z \mapsto (1 : z)$, so that $z = x_1/x_0$ is the affine coordinate on U_0; similarly U_1 is identified with $\mathbb{A}^1(k)$ via $w \mapsto (w : 1)$, with affine coordinate $w = x_0/x_1$. On the overlap $U_0 \cap U_1$, where both coordinates are nonzero, the two are related by $w = z^{-1}$. Thus $\mathbb{P}^1(k)$ is obtained by gluing two copies of $\mathbb{A}^1(k)$ along $\mathbb{A}^1(k) - \{0\}$ using the homeomorphism $z \mapsto z^{-1}$. Since $\mathbb{P}^1(k) = U_0 \cup \{(0 : 1)\}$, we may also view $\mathbb{P}^1(k)$ as the affine line with a single 'point at infinity' adjoined. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 6.2. The projective plane is covered by three open sets,

$$\mathbb{P}^2(k) = U_0 \cup U_1 \cup U_2$$

, each identified with an affine plane $\mathbb{A}^2(k)$. On U_0 the affine coordinates are $x = x_1/x_0$ and $y = x_2/x_0$, while on U_1 they are $u = x_0/x_1$ and $v = x_2/x_1$; over $U_0 \cap U_1$ these are related by $u = x^{-1}$ and $v = x^{-1}y$. Writing $\mathbb{P}^2(k) = U_0 \cup \mathbb{P}^1(k)$, with $\mathbb{P}^1(k)$ the line $\{x_0 = 0\}$ at infinity, we may think of $\mathbb{P}^2(k)$ as $\mathbb{A}^2(k)$ with a 'line at infinity' added.

This leads to a space where the theory of intersections is much nicer than in affine space. For instance, consider two parallel lines L_1, L_2 in $\mathbb{A}^2_k$ defined by the linear equations

$$L_1 \colon ay_1 + by_2 + c = 0$$
$$L_2 \colon ay_1 + by_2 + d = 0$$

where $c \neq d$. If we identify $\mathbb{A}^2(k)$ with $U_0 \subset \mathbb{P}^2(k)$, the closures $\overline{L}_1$ and $\overline{L}_2$ are given by the equations

$$\overline{L}_1 \colon ax_1 + bx_2 + cx_0 = 0$$
$$\overline{L}_2 \colon ax_1 + bx_2 + dx_0 = 0$$

These lines *do* meet, namely in the point $(0 : -b : a)$, on the line at infinity. In fact, any two distinct lines in $\mathbb{P}^2(k)$, will always intersect a point in $\mathbb{P}^2(k)$. This is the most classical example illustrating that projective varieties have better geometric properties than affine varieties. △

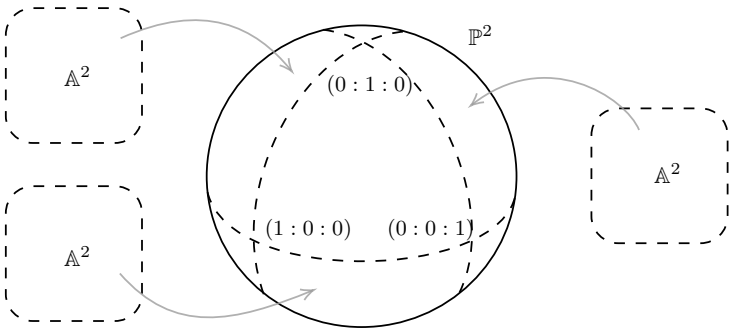


Figure 6.1 The projective plane

Projective space as a scheme

We now construct projective space as a scheme. The construction works over any base ring A. The basic idea is to construct $\mathbb{P}^n_A$ from $n+1$ copies of affine n-space $\mathbb{A}^n_A$ with the ratios x_j/x_i as affine coordinates.

Fix a ring A and consider the graded ring $R = A[x_0, \ldots, x_n]$. with its standard grading in which each x_i has degree 1. For each $i = 0, \ldots, n$, consider the localization $R_{x_i} =$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$A[x_0, \dots, x_n, x_i^{-1}]$ and its subring of degree 0 elements

$$R_i = (R_{x_i})_0 = A\Big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\Big].$$

Since R_i is isomorphic to a polynomial ring in n variables, each $U_i = \operatorname{Spec} R_i$ is isomorphic to the affine space $\mathbb{A}^n_A$.

To glue U_i and U_j, note that x_j/x_i on U_i should become invertible exactly where $x_j \neq 0$, that is, on the overlap. Inverting it gives the ring of degree-zero elements after inverting both x_i and x_j,

$$R_{ij} = (R_{x_i x_j})_0 = A\Big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\Big] = A\Big[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\Big].$$

viewed inside $A[x_0, \dots, x_n, x_i^{-1}, x_j^{-1}]$. By construction, $R_{ij} = R_{ji}$ and this ring can be obtained either from inverting x_j/x_i in $(R_{x_i})_0$ or by inverting x_i/x_j in $(R_{x_j})_0$.

We may therefore U_i and U_j by identifying $U_{ij} = D(x_j/x_i) \subset U_i$ with $U_{ji} = D(x_i/x_j) \subset U_j$ using the isomorphism $\tau_{ij} \colon U_{ij} \to U_{ji}$ induced by $R_{ij} = R_{ji}$. The cocycle condition holds automatically: for each triple i, j, k the intersection $U_{ijk} = D(x_j/x_i) \cap D(x_k/x_i)$ inside U_i is identified with the spectrum of the ring $R_{ijk} = (R_{x_i x_j x_k})_0$. Since

$$A\Big[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}, \frac{x_i}{x_k}\Big] = A\Big[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}, \frac{x_j}{x_k}\Big] = A\Big[\frac{x_0}{x_k}, \dots, \frac{x_n}{x_k}, \frac{x_k}{x_i}, \frac{x_k}{x_j}\Big].$$

we have $R_{ijk} = R_{jik} = R_{kij}$, and the cocycle condition $\tau_{ik} = \tau_{jk} \circ \tau_{ij}$ holds on U_{ijk} because the restrictions of τ_{ij}, τ_{ik} and τ_{jk} are all induced by the identity map on R_{ijk}.

Definition 6.3. The scheme $\mathbb{P}^n_A$ obtained by gluing the affines $U_i = \operatorname{Spec}(R_i)$ is called the *projective n-space over A*. The projective space over $\mathbb{Z}$ is called the *projective n-space* and it is denoted by $\mathbb{P}^n$.

Note that the rings R_i are A-algebras, so each U_i is a scheme over A and comes with a structure map $U_i \to \operatorname{Spec} A$. The same holds for the overlap rings R_{ij}, and the structure maps glue to a morphism $\pi \colon \mathbb{P}^n_A \to \operatorname{Spec} A$, making $\mathbb{P}^n_A$ an A-scheme.

If $\operatorname{Spec} A$ is irreducible, then so is $\mathbb{P}^n_A$, since each $U_i \simeq \mathbb{A}^n_A$ is irreducible and each $U_i \cap U_j$ is dense in U_i. Likewise, $\mathbb{P}^n_A$ is reduced if A is, because $\mathbb{A}^n_A$ is reduced. This means that $\mathbb{P}^n_A$ is integral if A is an integral domain.

Example 6.4. The projective line $\mathbb{P}^1_A$ over A is constructed by gluing two copies of the affine line $\mathbb{A}^1_A$, $U_0 = \operatorname{Spec} A[u]$ and $U_1 = \operatorname{Spec} A[v]$ where $u = x_1/x_0$ and $v = x_0/x_1$. These are glued along $U_{01} = \operatorname{Spec} A[u, u^{-1}] = \operatorname{Spec} A[v, v^{-1}]$ which is a distinguished open set in both U_0 and U_1. △

Example 6.5. The *projective plane* $\mathbb{P}^2_A$ is constructed by gluing together the three affine planes $U_i = \operatorname{Spec} R_i$ for $i = 0, 1, 2$. It is often convenient to rewrite these charts using the 'U_0-coordinates' $x = x_1/x_0$ and $y = x_2/x_0$. We can then express the other ratios in terms of x and y. For instance $x_2/x_1 = x^{-1}y$. The three affine opens become

$$U_0 = \operatorname{Spec} A[x, y], \quad U_1 = \operatorname{Spec} A[x^{-1}, yx^{-1}], \quad U_2 = \operatorname{Spec} A[y^{-1}, xy^{-1}]. \quad \triangle$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

There is a morphism of schemes

$$\pi\colon \mathbb{A}^{n+1}_A - V(x_0,\dots,x_n) \longrightarrow \mathbb{P}^n_A \tag{6.4}$$

which can be seen as a scheme-analogue of the quotient map $k^{n+1}-\{0\} \to \mathbb{P}^n(k)$.

To define it, we work locally over U_i. The inclusions $R_i \subset A[x_0,\dots,x_n,x_i^{-1}]$ determine morphisms of affine schemes

$$\pi_i\colon \operatorname{Spec} A[x_0,\dots,x_n,x_i^{-1}] \longrightarrow U_i.$$

Note that the scheme of the left can be identified with the distinguished open set $D(x_i) \subset \mathbb{A}^{n+1}_A$, and they form an affine cover of $D(x_0)\cup\cdots\cup D(x_n) = \mathbb{A}^{n+1}_A - V(x_0,\dots,x_n)$. Gluing follows from the fact that all maps are induced by inclusions of the localizations into $A[x_0,\dots,x_n,x_i^{-1},x_j^{-1}]$.

This is one way we can view $\mathbb{P}^n_A$ as a quotient space of $\mathbb{A}^{n+1}_A - V(x_0,\dots,x_n)$. Inside $A[x_0,\dots,x_n,x_i^{-1}]$, we can consider the elements which are invariant under scaling, i.e.,

$$p(tx_0,\dots,tx_n) = p(x_0,\dots,x_n) \tag{6.5}$$

and these are exactly the degree 0 elements, namely $A[x_0/x_i,\dots,x_n/x_i]$. Note that (6.5) should be considered as an equality in the ring $A[x_0,\dots,x_n,x_i^{-1},t]$.

Proposition 6.6. For any field k, the k-rational points of $\mathbb{P}^n$ is given by

$$\mathbb{P}^n(k) = \Big(k^{n+1}-\{0\}\Big)/\sim . \tag{6.6}$$

Proof The image of a k-point $\iota\colon \operatorname{Spec} k \to \mathbb{P}^n$ is a single point in $\mathbb{P}^n$, so it must be contained in one of the standard opens U_i. Therefore, the k-points of $\mathbb{P}^n(k)$ is the union of the k-points of the U_i. Identifying $U_i(k) = \mathbb{A}^n(k) = k^n$, we define $\tau\colon k^{n+1}-\{0\} \to \mathbb{P}^n(k)$ by the map

$$\tau(a_0,\dots,a_n) = \Big(\frac{a_0}{a_i},\dots,\frac{a_{i-1}}{a_i},\frac{a_{i+1}}{a_i},\dots,\frac{a_n}{a_i}\Big) \in U_i(k) \quad \text{if } a_i \neq 0$$

Then τ is well-defined, surjective, and $\tau(a_0,\dots,a_n) = \tau(b_0,\dots,b_n)$ precisely when $(b_0,\dots,b_n) = \lambda\cdot(a_0,\dots,a_n)$ for some $\lambda \in k^\times$. □

The structure sheaf of $\mathbb{P}^n_A$

The following computation is very important.

Proposition 6.7. For each ring A, we have $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

Proof By the sheaf sequence associated to the covering $U_0,\dots,U_n$, a global section of $\mathcal{O}_{\mathbb{P}^n_A}$ consists of elements $s_i \in \Gamma(U_i,\mathcal{O}_{\mathbb{P}^n_A})$ that agree on the overlaps $U_i\cap U_j$. Over U_i we have $\Gamma(U_i,\mathcal{O}_{\mathbb{P}^n_A}) = R_i$ and a section s_i is a polynomial of the form

$$g_i\Big(\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\Big) \tag{6.7}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The compatibility condition on the overlaps means that

$$g_j\Big(\frac{x_0}{x_j},\dots,\frac{x_n}{x_j}\Big) = g_i\Big(\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\Big). \tag{6.8}$$

The only way (6.8) can hold is that g_i and g_j are of degree 0 as polynomials in the ratios x_k/x_i and x_k/x_j. That is, g_i and g_j are equal to the same element in A. Conversely, if we choose any element $a \in A$ and set $g_i = a$ for every $i = 0,\dots,n$, then the sections clearly glue to a global section of $\mathcal{O}_{\mathbb{P}^n_A}$. □

In particular, for a field k, the space of global sections of $\mathcal{O}_{\mathbb{P}^n_k}$ is just the 'constants', k. When $k = \mathbb{C}$, this can be seen as an algebraic analogue of Liouville's theorem, that the only global holomorphic functions on $\mathbb{CP}^1$ are the constant functions.

Projective space also gives a good example of a scheme which is not affine: if $\mathbb{P}^1_k$ were affine, it would have to be isomorphic to $\operatorname{Spec} k$ by Theorem 4.11, but this is clearly not the case. In fact, $\mathbb{P}^1_{\mathbb{C}}$ is related to the first example of a non-affine scheme of Example 4.18, namely the affine plane with the origin removed, $\mathbb{A}^2_{\mathbb{C}} - V(x,y)$.

6.2 Projective schemes

Having constructed $\mathbb{P}^n_A$, we now describe its closed subschemes. By the discussion in Section 6.1, these should be described by ideals generated by homogeneous polynomials.

Let $\mathfrak{a} \subset A[x_0,\dots,x_n]$ be a homogeneous ideal. For each i, define

$$V_i = \operatorname{Spec}\big(A[x_0,\dots,x_n,x_i^{-1}]/\mathfrak{a}\big)_0. \tag{6.9}$$

The surjection $A[x_0,\dots,x_n,x_i^{-1}]_0 \longrightarrow \big(A[x_0,\dots,x_n,x_i^{-1}]/\mathfrak{a}\big)_0$ shows that V_i is a closed subscheme of U_i. These subschemes glue together to form a closed subscheme of $\mathbb{P}^n_A$, which we denote by $V_+(\mathfrak{a})$. Concretely, on U_i it is defined by the ideal

$$(\mathfrak{a}_{x_i})_0 = \Big\{ x_i^{-\deg F} F \in A\big[\tfrac{x_0}{x_i},\dots,\tfrac{x_n}{x_i}\big] \;\Big|\; F \in \mathfrak{a} \text{ homogeneous} \Big\}.$$

The operation $F \mapsto x_i^{-\deg F} F$ appearing here deserves a comment, since it the way to pass between the homogeneous equations on $\mathbb{P}^n_A$ and the affine equations on each chart. With $R = A[x_0,\dots,x_n]$ given the standard grading, there is for each i a natural isomorphism

$$(R_{x_i})_0 \xrightarrow{\sim} R/(x_i - 1),$$

under which a homogeneous polynomial F maps to its *dehomogenization* with respect to x_i, the polynomial obtained by setting $x_i = 1$. Equivalently, dehomogenizing F means dividing by the appropriate power of x_i to reach degree 0, namely passing to $x_i^{-\deg F} F \in R_i$. The reverse operation, *homogenization*, takes a polynomial g in the affine coordinates and multiplies each monomial by the power of x_i needed to make the total degree equal to $\deg g$. So the equations of a projective scheme on the chart U_i are obtained by dehomogenizing its defining homogeneous equations, and conversely.

Example 6.8. Let $F = x_1^5 - 7x_0^2x_2^3$ and $X = V_+(F) \subset \mathbb{P}^2_k$. In the open set $U_0 = \operatorname{Spec} k[u,v]$, where $u = \frac{x_1}{x_0}$ and $v = \frac{x_2}{x_0}$, dehomogenizing by setting $x_0 = 1$ gives

$$X \cap U_0 = V(u^5 - 7v^3).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Likewise, in the charts $U_1 = \operatorname{Spec} k[s,t]$ and $U_2 = \operatorname{Spec} k[a,b]$, where $s = x_0/x_1$, $t = x_2/x_1$, $a = x_0/x_2$, and $b = x_1/x_2$, the same procedure gives

$$V(1 - 7s^2t^3), \qquad V(b^5 - 7a^2).$$

Conversely, homogenizing $u^5 - 7v^3$ recovers $x_1^5 - 7x_0^2x_2^3$. △

Definition 6.9. An A-scheme X called *projective over* A if it is isomorphic to a closed subscheme of the form $V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$ for some homogeneous ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$. An A-scheme X is *quasi-projective over* A if it admits an open embedding (over A) into a projective scheme.

We will show later (Theorem 12.14) that every closed subscheme of $\mathbb{P}^n_A$ is of the form $V_+(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$.

For a field k, an (irreducible) *hypersurface* in $\mathbb{P}^n_k$ is the subscheme defined by a single irreducible homogeneous polynomial of positive degree. A hypersurface is a *quadric*, *cubic*, or *quartic* according as its degree is 2, 3, or 4. A *conic* is a quadric in $\mathbb{P}^2_k$, that is, the subscheme defined by an irreducible homogeneous polynomial of degree 2 in $k[x_0, x_1, x_2]$.

A *linear subspace* of $\mathbb{P}^n_k$ is a nonempty subscheme defined by the vanishing of homogeneous polynomials of degree 1. Cutting out c independent linear forms gives a subspace isomorphic to $\mathbb{P}^{n-c}_k$, so every linear subspace is isomorphic to $\mathbb{P}^r_k$ with r its dimension. A *hyperplane* is a linear subspace of codimension 1, defined by a single degree-1 form; a *line* is a linear subspace of dimension 1, and a *plane* one of dimension 2. Viewing $\mathbb{A}^n_k$ as the chart $U_0 \subset \mathbb{P}^n_k$, intersecting a projective linear subspace with $\mathbb{A}^n_k$ yields, when nonempty, an affine linear subspace, that is, a translate of a linear subspace of k^n.

Unlike the case for affine schemes, different ideals can correspond to the same closed subscheme. Here are two examples.

Example 6.10. If $\mathfrak{a} = (x_0, \ldots, x_n)$, then the corresponding closed subscheme is empty. Indeed, over U_i, the dehomogenized ideal contains $x_i/x_i = 1$, so $\mathfrak{a}$ defines the empty scheme in each U_i. The same argument applies to any ideal $\mathfrak{a}$ such that $\sqrt{\mathfrak{a}} = (x_0, \ldots, x_n)$. For this reason, $(x_0, \ldots, x_n)$ is called the *irrelevant ideal*. △

Example 6.11. In $\mathbb{P}^1_k$, the two ideals (x_0) and $(x_0^2, x_0x_1) = (x_0) \cap (x_0^2, x_1)$ both define the point $(0 : 1)$. △

6.3 An important family of sheaves on $\mathbb{P}^n_A$

Projective space $\mathbb{P}^n_A$ carries a special collection of sheaves $\mathcal{O}_{\mathbb{P}^n_A}(m)$, where m is an integer. These sheaves are *locally free of rank 1*, in the sense that they are locally isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^n_A}$. We will study such sheaves more systematically in Chapter 11.

The sheaves $\mathcal{O}_{\mathbb{P}^n_A}(m)$ are closely related to homogeneous polynomials. While homogeneous polynomials of degree $m \geqslant 1$ in $k[x_0, \ldots, x_n]$ do not define regular functions on $\mathbb{P}^n_k$, they do define global sections of the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(m)$. Intuitively, over an open set $U \subset \mathbb{P}^n_k$, the sections of $\mathcal{O}_{\mathbb{P}^n_k}(m)$ are the ratios

$$\frac{f(x_0, \ldots, x_n)}{g(x_0, \ldots, x_n)} \quad \text{with } \deg f - \deg g = m$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

where f, g are homogeneous and $g(P) \neq 0$ for all $P \in U$.

To define these sheaves by gluing, we use the standard open cover $U_i = \operatorname{Spec} R_i$ where $R_i = A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Over U_i, define the free rank 1 R_i-module

$$M_i = A\Big[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\Big]x_i^m$$

with generator x_i^m. Note that since $x_i^m = (x_i/x_j)^m x_j^m$, there is an equality

$$A\Big[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\Big]x_i^m = A\Big[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\Big]x_j^m \tag{6.10}$$

In other words, we may identify the localizations

$$(M_i)_{x_j/x_i} = (M_j)_{x_i/x_j} \tag{6.11}$$

as modules over the ring $R_i[\frac{x_i}{x_j}] = R_j[\frac{x_j}{x_i}]$. The two generators x_i^m and x_j^m are related by the formula

$$g\Big(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\Big)x_i^m = \Big(\frac{x_i}{x_j}\Big)^m g\Big(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\Big)x_j^m. \tag{6.12}$$

By the equality (6.11), the two sheaves $\widetilde{M}_i$ and $\widetilde{M}_j$ on U_i restrict to the same sheaf on $U_{ij} = \operatorname{Spec}(R_i[x_i/x_j])$, namely the tilde of the module (6.10) (see Exercise 3.7.13). As the isomorphisms are identity maps, the gluing conditions are satisfied, and the sheaves $\widetilde{M}_i$ glue to a sheaf on $\mathbb{P}^n_A$. We denote it by $\mathcal{O}_{\mathbb{P}^n_A}(m)$.

In the special case $m = 0$, we have $M_i = R_i$ and the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(0) = \mathcal{O}_{\mathbb{P}^n_A}$ is simply the structure sheaf. The next computation generalizes Proposition 6.7:

Proposition 6.12. $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ can be identified with the free A-module of homogeneous polynomials of degree m in the variables $x_0, \ldots, x_n$.

Proof By the sheaf sequence associated to the covering $U_0, \ldots, U_n$, a global section of $\mathcal{O}_{\mathbb{P}^n_A}(m)$ consists of sections $s_i \in \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_A}(m))$ that agree on the overlaps $U_i \cap U_j$. Over U_i we have $\Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_A}(m)) = M_i$ and a section s_i is an element of the form

$$g_i\Big(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\Big)x_i^m \in M_i. \tag{6.13}$$

The compatibility condition on overlaps implies that

$$g_j\Big(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\Big)x_j^m = g_i\Big(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\Big)x_i^m. \tag{6.14}$$

If $m < 0$, then the left-hand side contains only terms with negative powers of x_j, whereas the right-hand side contains only negative powers of x_i. There is therefore only one tuple satisfying (6.14), namely where $g_0 = \cdots = g_n = 0$.

If $m \geqslant 0$, we see that the g_j are determined by g_0. By comparing degrees in (6.14), we see that g_0 must have degree $\leqslant m$ as a polynomial in the ratios $x_1/x_0, \ldots, x_n/x_0$. This means that $F = g_0 x_0^m$ is a homogeneous polynomial of degree m in $x_0, \ldots, x_n$ with coefficients in A, and $s_i = g_i x_i^m = F$ for all $i = 0, \ldots, n$. Conversely, any homogeneous polynomial F of degree m determines a valid $(n+1)$-tuple $(s_0, \ldots, s_n)$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

6.4 The Proj construction

The construction of $\mathbb{P}^n_A$, and of its closed subschemes $V_+(\mathfrak{a})$, both proceeded by gluing degree-zero localizations of a graded ring. We now carry out this idea in full generality. Starting from any graded ring

$$R = \bigoplus_{d \geqslant 0} R_d \tag{6.15}$$

we construct a scheme $\operatorname{Proj} R$, called the *projective spectrum* of R. See Appendix C for background and notation on graded rings and modules.

The Proj construction is parallel to that of the spectrum of a ring in Chapter 2. In both constructions, the underlying topological space is defined in terms of prime ideals in the ring, and the structure sheaf from localizations. There are however two notable differences. First, $\operatorname{Proj} R$ is not functorial in R, in the sense that graded ring maps do not always induce maps between the corresponding projective spectra. Second, different graded rings R may give rise to isomorphic projective spectra.

Definition 6.13. Let R be a graded ring and let $R_+ = \bigoplus_{d>0} R_d$ be its irrelevant ideal. The *projective spectrum* of R, $\operatorname{Proj} R$ is defined as the set of homogeneous prime ideals of R that do not contain R_+, that is,

$$\operatorname{Proj}(R) = \big\{\mathfrak{p} \in \operatorname{Spec} R - V(R_+) \mid \mathfrak{p} \text{ homogeneous}\big\}.$$

The reason for excluding primes containing the irrelevant ideal R_+ can already be seen in Example 6.10. In that example, when $R = k[x_0, \ldots, x_n]$ with the standard grading, the irrelevant ideal $(x_0, \ldots, x_n)$ corresponds to the origin $(0, \ldots, 0)$ in $\mathbb{A}^{n+1}(k)$, and this point does not determine a well-defined point of $\mathbb{P}^n(k)$.

The set $\operatorname{Proj} R$ is equipped with the *Zariski topology* , in which the closed sets are of the form

$$V_+(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{p} \supset \mathfrak{a} \,\}$$

with $\mathfrak{a}$ a homogeneous ideal. The three topology axioms follow from the identities below, together with $V_+(0) = \operatorname{Proj} R$ and $V_+(R) = \varnothing$.

Lemma 6.14. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\{\mathfrak{a}_i\}_{i \in I}$ be homogeneous ideals. Then:
(i) If $\mathfrak{a} \subset \mathfrak{b}$, then $V_+(\mathfrak{b}) \subset V_+(\mathfrak{a})$.
(ii) $V_+(\sum \mathfrak{a}_i) = \bigcap V_+(\mathfrak{a}_i)$.
(iii) $V_+(\mathfrak{a}\mathfrak{b}) = V_+(\mathfrak{a} \cap \mathfrak{b}) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b})$.
(iv) $V_+(\sqrt{\mathfrak{a}}) = V_+(\mathfrak{a})$.

Proof The proofs of the four statements are identical to those of Proposition 2.3 for $\operatorname{Spec} R$ (the arguments there remain valid under the additional constraints that the prime ideals are homogeneous and do not contain the irrelevant ideal). □

As for the spectrum of a ring, $\operatorname{Proj} R$ has a collection of *distinguished open sets*, which form a basis for the topology. For a homogeneous $f \in R$ of positive degree, define

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} R \,|\, \mathfrak{p} \not\ni f\}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

That is, $D_+(f)$ is the set of homogeneous prime ideals in R that do not contain the irrelevant ideal R_+, and do not contain f. It is clear that $D_+(f)$ is an open set, being the complement of the closed set $V_+(f)$.

Lemma 6.15. Let R be a graded ring.

(i) For two homogeneous elements f and g of positive degree, we have $D_+(f) \cap D_+(g) = D_+(fg)$.

(ii) The sets $D_+(f)$, where f ranges over homogeneous elements of R of positive degree, form a basis for the topology on $\operatorname{Proj} R$.

Proof (i): For a homogeneous prime $\mathfrak{p}$, we have $fg \notin \mathfrak{p}$ if and only if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.

(ii): A closed set $V_+(\mathfrak{a})$ is equal to the intersection of the $V_+(f)$'s for the homogeneous $f \in \mathfrak{a} \cap R_+$. Taking complements, we see that any open set $\operatorname{Proj} R - V_+(\mathfrak{a})$ is the union of the corresponding $D_+(f)$'s. □

In analogy with the standard covering of $\mathbb{P}^n_A$, we will use the degree 0 localizations of R to define the structure sheaf on $\operatorname{Proj} R$. More precisely, for a nonzero homogeneous element $f \in R$, the ring R_f is naturally a $\mathbb{Z}$-graded ring, by defining $\deg(a/f^n) = \deg a - n \deg f$. It therefore makes sense to consider the subring of degree 0 elements $(R_f)_0$. In the case $R = A[x_0, \ldots, x_n]$ with the standard grading and $f = x_i$, this is exactly the ring R_i used to construct $\mathbb{P}^n_A$.

If $f, g \in R$ are homogeneous elements of positive degree, the localization map $R_f \to R_{fg}$ induces a ring map

$$(R_f)_0 \longrightarrow (R_{fg})_0. \tag{6.16}$$

The ring on the right can in turn be viewed as a localization of the ring on the left at the degree 0 element $u = g^{\deg f} f^{-\deg g}$. Indeed, in R_f the element f is already invertible, and after inverting u the element $g^{\deg f} = u f^{\deg g}$ also becomes invertible. It follows that g is invertible as well, so we may identify $(R_f)_u$ with R_{fg}. Since u has degree 0, this identification preserves the $\mathbb{Z}$-grading. Taking the degree 0 part, we find

$$((R_f)_0)_u = (R_{fg})_0. \tag{6.17}$$

The following proposition is the projective analogue of the homeomorphism between $D(f) \subset \operatorname{Spec} A$ and $\operatorname{Spec} A_f$, which sends a prime $\mathfrak{p} \in D(f)$ to the prime ideal $\mathfrak{p}A_f$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 6.16. Let R be a graded ring and let $f \in R_+$ be a nonzero element of positive degree. Then the map $\mathfrak{p} \mapsto (\mathfrak{p}R_f)_0$ defines a homeomorphism

$$\phi\colon D_+(f) \longrightarrow \operatorname{Spec}(R_f)_0. \tag{6.18}$$

These maps are compatible in the sense that the following diagram

$$\begin{array}{ccc} D_+(f) & \longrightarrow & \operatorname{Spec}(R_f)_0 \\ \uparrow & & \uparrow \\ D_+(fg) & \longrightarrow & \operatorname{Spec}(R_{fg})_0 \end{array} \tag{6.19}$$

commutes for each $f, g \in R_+$ homogeneous.

The reader may safely skip the details of the following proof on a first reading. It parallels the gluing construction for $\mathbb{P}^n_A$ using degree-0 localizations, but it is more technical because f need not have degree 1.

Proof First of all, ϕ is continuous because it is equal to the composition

$$D_+(f) \to D(f) \simeq \operatorname{Spec}(R_f) \to \operatorname{Spec}(R_f)_0. \tag{6.20}$$

where the first map comes from the inclusion of $D_+(f)$ into $\operatorname{Spec}(R)$ and the last map is induced by $(R_f)_0 \subset R_f$. We next show that it is bijective.

Let $d = \deg f$. The following observation will be useful: if $x \in R_{nd}$, and $\mathfrak{p}$ is a homogeneous prime ideal such that $\mathfrak{p} \not\ni f$, then

$$xf^{-n} \in \phi(\mathfrak{p}) \iff x \in \mathfrak{p}. \tag{6.21}$$

This is because $xf^{-n} \in \phi(\mathfrak{p})$ is equivalent to $x/1 \in \mathfrak{p}R_f$, and this is equivalent to $x \in \mathfrak{p}$, because $f \notin \mathfrak{p}$.

Injectivity of ϕ: If $\phi(\mathfrak{p}) = \phi(\mathfrak{p}')$, and $a \in R_n$, then applying (6.21) to $x = a^d$ shows that $a^d \in \mathfrak{p}$ if and only if $a^d \in \mathfrak{p}'$. But as $\mathfrak{p}$ is prime, it contains a^d if and only if it contains a, and the same applies to $\mathfrak{p}'$. Hence $\mathfrak{p} = \mathfrak{p}'$,

Surjectivity of ϕ: We construct the inverse of ϕ as follows. Given a prime ideal $\mathfrak{q} \subset (R_f)_0$, we define $\mathfrak{p} = \bigoplus_{n \geqslant 0} \mathfrak{p}_n$ where

$$\mathfrak{p}_n = \big\{ a \in R_n \;\mid\; a^d \cdot f^{-n} \in \mathfrak{q} \big\}.$$

In other words, $\mathfrak{p}_n$ consists of the homogeneous elements of degree n that land in $\mathfrak{q}$ when 'dehomogenized'. The first thing to check is that $\mathfrak{p}$ is a homogeneous prime ideal. It is clear that it is closed under multiplication by elements of R: if $t \in R_m$, and $a^d f^{-n} \in \mathfrak{q}$, then $(ta)^d f^{-m-n} \in \mathfrak{q}$ in $(R_f)_0$, so $ta \in \mathfrak{p}_{n+m}$. It is also closed under addition: if $a, b \in \mathfrak{p}_n$, then by definition, we have $a^d \cdot f^{-n} \in \mathfrak{q}$ and $b^d \cdot f^{-n} \in \mathfrak{q}$. Expanding $(a+b)^{2d} \cdot f^{-2n}$, we obtain a sum where every term is a multiple of either $a^d \cdot f^{-n}$ or $b^d \cdot f^{-n}$. Therefore $(a+b)^{2d} \cdot f^{-2n} \in \mathfrak{q}$, and hence $(a+b)^d \cdot f^{-n} \in \mathfrak{q}$, because $\mathfrak{q}$ is prime.

To show that $\mathfrak{p}$ is prime, it suffices to check that $ab \in \mathfrak{p}$ implies either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ for a and b homogeneous. That $a \in R_m$ and $b \in R_n$ satisfy $ab \in \mathfrak{p}$ means that $(ab)^d f^{-(m+n)} \in \mathfrak{q}$. Since f is a unit in R_f, we must have $(ab)^d \in \mathfrak{q}$, and hence either $a^d f^{-m} \in \mathfrak{q}$ or $b^d f^{-n} \in \mathfrak{q}$. Therefore we either have $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$, and so $\mathfrak{p}$ is prime.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Finally, we check that $\phi(\mathfrak{p}) = \mathfrak{q}$. This follows from the equivalences

$$af^{-n} \in \mathfrak{q} \iff a^d f^{-nd} \in \mathfrak{q} \iff a \in \mathfrak{p}_{nd} \iff af^{-n} \in \phi(\mathfrak{p}) \tag{6.22}$$

which follow from (6.21) and the primality of $\mathfrak{p}$ and $\mathfrak{q}$.

Hence ϕ is bijective. To prove that ϕ is a homeomorphism, it suffices to show that it is open. Since the open sets of the form $D_+(fg)$ form a basis for the open sets contained in $D_+(f)$, it suffices to show that for any homogeneous element $g \in R$, we have

$$\phi(D_+(fg)) = D(g^d f^{-\deg g}).$$

For $\mathfrak{p} \in D_+(f)$, the following series of equivalences hold because $\mathfrak{p}R_f$ is a prime ideal:

$$\begin{aligned} \mathfrak{p} \in D_+(fg) &\iff g \notin \mathfrak{p} \\ &\iff g^d f^{-\deg g} \notin (\mathfrak{p}R_f)_0 = \phi(\mathfrak{p}) \\ &\iff \phi(\mathfrak{p}) \in D(g^d f^{-\deg g}) \end{aligned}$$

Hence $\phi(D_+(fg)) = D(g^d f^{-\deg g})$.

Finally, we show that the diagram (6.19) commutes. The right vertical map is induced by the localization map $\ell\colon R_f \to R_{fg}$. If $\mathfrak{p} \in D_+(fg)$ is a homogeneous prime ideal such that $\mathfrak{p} \not\ni f$ and $\mathfrak{p} \not\ni g$, then we have $\ell^{-1}(\mathfrak{p}R_{fg}) = \mathfrak{p}R_f$. Since ℓ preserves the gradings, we have

$$\ell^{-1}(\mathfrak{p}R_{fg} \cap (R_{fg})_0) = \mathfrak{p}R_f \cap (R_f)_0.$$

This shows that $\mathfrak{p}$ maps to $\mathfrak{p}R_f \cap (R_f)_0$ via both compositions in (6.19). □

We now define the structure sheaf $\mathcal{O}_{\operatorname{Proj} R}$. Let $\mathscr{B}$ be the basis for the topology on $\operatorname{Proj} R$ consisting of the distinguished open subsets. We define a presheaf on $\mathscr{B}$ by setting[1]

$$\mathcal{O}(D_+(f)) = (R_f)_0. \tag{6.23}$$

The restriction map $\mathcal{O}(D_+(f)) \to \mathcal{O}(D_+(fg))$ is defined to be the ring map $(R_f)_0 \to (R_{fg})_0$. In this way, we obtain a $\mathscr{B}$-presheaf $\mathcal{O}$. To see that this is a $\mathscr{B}$-sheaf, fix a homogeneous element $f \in R_d$ and use the homeomorphism

$$\phi_f\colon D_+(f) \simeq \operatorname{Spec}(R_f)_0.$$

If $g \in R_e$ is another homogeneous element, then the open subset $D_+(fg) \subset D_+(f)$ is mapped homeomorphically onto the distinguished open subset

$$D\left(g^d/f^e\right) \subset \operatorname{Spec}(R_f)_0.$$

Moreover, there is a canonical identification

$$\mathcal{O}(D_+(fg)) = (R_{fg})_0 = ((R_f)_0)_{g^d/f^e} = \Gamma\left(D\left(g^d/f^e\right), \mathcal{O}_{\operatorname{Spec}(R_f)_0}\right).$$

These are compatible with the restriction maps, by the commutativity of the diagram

$$\begin{array}{ccc} \Gamma(D_+(f), \mathcal{O}) & \longrightarrow & \Gamma(\operatorname{Spec}(R_f)_0, \mathcal{O}_{\operatorname{Spec}(R_f)_0}) \\ \downarrow & & \downarrow \\ \Gamma(D_+(fg), \mathcal{O}) & \longrightarrow & \Gamma(D(g^d/f^e), \mathcal{O}_{\operatorname{Spec}(R_f)_0}) \end{array} \qquad \begin{array}{ccc} (R_f)_0 & = & (R_f)_0 \\ \downarrow & & \downarrow \\ (R_{fg})_0 & \longrightarrow & ((R_f)_0)_{g^d/f^e} \end{array}$$

[1] As in the affine case, it is possible to use a canonical localization which only depends on the set $D_+(f)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

where the vertical maps are localization maps, and the horizontal maps are the canonical isomorphisms. Hence, the presheaf $\mathcal{O}$ on the basis of open subsets of $D_+(f)$ is transported by ϕ_f to the usual structure sheaf on the affine scheme $\operatorname{Spec}(R_f)_0$, restricted to its basis of distinguished open sets. Since the latter is a sheaf, it follows that $\mathcal{O}$ is a sheaf on the basis $\mathscr{B}$.

Therefore $\mathcal{O}$ extends uniquely to a sheaf on $\operatorname{Proj} R$, which we denote by $\mathcal{O}_{\operatorname{Proj} R}$. By what we just showed, it is isomorphic to $\operatorname{Spec}(R_f)_0$ over each $D_+(f)$. In particular, the stalks of $\mathcal{O}_{\operatorname{Proj} R}$ are local rings and the locally ringed space $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ is a scheme.

For each homogeneous f of positive degree, we have $\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) = (R_f)_0$ is an R_0-algebra. This implies that $\mathcal{O}_{\operatorname{Proj} R}(U)$ is an R_0-algebra for every open set $U \subset \operatorname{Proj} R$ (covering U by $D_+(f)$'s). In particular, applying Theorem 4.6 to $R_0 \to \mathcal{O}_{\operatorname{Proj} R}(\operatorname{Proj} R)$, we get a canonical morphism $\pi\colon \operatorname{Proj} R \to \operatorname{Spec} R_0$ which over $D_+(f)$ is identified with $\operatorname{Spec}((R_f)_0) \to \operatorname{Spec}(R_0)$. Hence $\operatorname{Proj} R$ is a scheme over R_0.

The construction above has real content. For instance, it shows that the open subscheme $\mathbb{P}^2_k - V_+(x_0^2 + x_1^2 + x_2^2)$ is affine, namely $\operatorname{Spec}\big((k[x_0, x_1, x_2]_{x_0^2+x_1^2+x_2^2})_0\big)$, which would have been awkward to prove by explicit gluing. This is the main advantage of the Proj construction over the explicit gluing construction in Section 6.1. Once the necessary gluing has been done, later arguments reduce to statements about graded rings and modules.

Functoriality and the base locus

In contrast to the Spec-construction, the Proj construction is not entirely functorial. While a map of graded rings $\phi\colon R \to S$ induces a map on homogeneous prime ideals via $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$, this does not always induce a morphism. The issue is that the preimage $\phi^{-1}(\mathfrak{p})$ may contain the irrelevant ideal R_+ and it does not define a point of $\operatorname{Proj} R$.

To remedy this, we define the *base locus* of ϕ to be the closed set

$$V_+\left(\phi(R_+)\right) \subset \operatorname{Proj} S.$$

This consists of precisely the prime ideals $\mathfrak{p}$ in S for which $\phi^{-1}(\mathfrak{p}) \supset R_+$. This means that for homogeneous primes $\mathfrak{p}$ outside $V_+(\phi(R_+))$, the preimage $\phi^{-1}(\mathfrak{p})$ gives a well-defined point of $\operatorname{Proj} R$.

Proposition 6.17. A map of graded rings $\phi\colon R \to S$ induces a morphism

$$F\colon \operatorname{Proj} S - V_+(\phi(R_+)) \longrightarrow \operatorname{Proj} R$$

given on points by $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$.

Proof The complement $U = \operatorname{Proj} S - V_+(\phi(R_+))$ is open in $\operatorname{Proj} S$ and has a canonical scheme structure. By the above remarks, the map $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ is well-defined, and it is also continuous, because it is the restriction of $\operatorname{Spec} S \to \operatorname{Spec} R$.

We define the sheaf map $F^\sharp\colon \mathcal{O}_{\operatorname{Proj} R} \to F_*\mathcal{O}_U$ on the basis of distinguished opens $D_+(f)$. Note that as f runs through the homogeneous elements of R_+, the $D_+(\phi(f))$ cover U, and

$$F^{-1}(D_+(f)) = D_+(\phi(f)) \tag{6.24}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Indeed, for $\mathfrak{p} \in U$,

$$F(\mathfrak{p}) \in D_+(f) \iff f \notin \phi^{-1}(\mathfrak{p}) \iff \phi(f) \notin \mathfrak{p} \iff \mathfrak{p} \in D_+(\phi(f))$$

Over $D_+(f)$, we have $\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) = (R_f)_0$ and $(F_*\mathcal{O}_U)(D_+(f)) = (S_{\phi(f)})_0$. We define the map $F^\sharp$ over $D_+(f)$ to be the degree 0 part of the localization

$$(R_f)_0 \longrightarrow (S_{\phi(f)})_0. \tag{6.25}$$

For two homogeneous $f, g \in R_+$, the commutative diagram

$$\begin{array}{ccc} (R_f)_0 & \longrightarrow & (S_{\phi(f)})_0 \\ \downarrow & & \downarrow \\ (R_{fg})_0 & \longrightarrow & (S_{\phi(fg)})_0 \end{array}$$

shows that the maps (6.25) are compatible with restrictions and define a map of sheaves $F^\sharp\colon \mathcal{O}_{\operatorname{Proj} R} \to F_*\mathcal{O}_U$. Restricting $(F, F^\sharp)$ to a distinguished open set $D_+(f) \subset \operatorname{Proj} S$, we obtain the morphism $\operatorname{Spec}(S_{\phi(f)})_0 \to \operatorname{Spec}(R_f)_0$. This means that $F^\sharp$ induces a map between the local rings, so $(F, F^\sharp)$ is a morphism of schemes. □

Example 6.18 (Projections)**.** The inclusion $k[x_0, \dots, x_r] \hookrightarrow k[x_0, \dots, x_n]$ has base locus $V_+(x_0, \dots, x_r)$, so we obtain a morphism

$$\mathbb{P}^n_k - V_+(x_0, \dots, x_r) \longrightarrow \mathbb{P}^r_k.$$

On the level of k-points, this is given by the projection $(a_0 : \cdots : a_n) \mapsto (a_0 : \cdots : a_r)$. △

Closed embeddings

Let $\mathfrak{a}$ be a homogeneous ideal in the graded ring R. The quotient map $\phi\colon R \to R/\mathfrak{a}$ is a map of graded rings, and it satisfies $\phi(R_+) = (R/\mathfrak{a})_+$. Indeed, a prime $\mathfrak{q} \in \operatorname{Spec}(R/\mathfrak{a})$ contains $(R/\mathfrak{a})_+$ if and only if $\phi^{-1}(\mathfrak{q})$ contains $\phi^{-1}((R/\mathfrak{a})_+) = R_+$. The base locus is therefore empty, and we obtain a morphism

$$\iota\colon \operatorname{Proj} R/\mathfrak{a} \longrightarrow \operatorname{Proj} R$$

with image $V_+(\mathfrak{a})$.

We claim that ι is a closed embedding. It suffices to check this over each $D_+(f)$ for $f \in R$ homogeneous. There $\iota^{-1}D_+(f) = D_+(\phi(f))$, and the restriction of ι to this set is identified with the morphism

$$\operatorname{Spec}\left((R/\mathfrak{a})_f\right)_0 \longrightarrow \operatorname{Spec}\left(R_f\right)_0$$

induced by the degree 0 part of the localization $R_f \to (R/\mathfrak{a})_f$. Since this map is surjective, $\iota|_{\iota^{-1}D_+(f)}$ is a closed embedding.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Veronese subrings

Let R be a graded ring. For a positive integer n, the *Veronese subring* is defined by

$$R^{(n)} = \bigoplus_{d \geqslant 0} R_{nd}$$

The inclusion $R^{(n)} \subset R$ preserves the gradings and induces a morphism

$$\nu\colon \operatorname{Proj}(R) \longrightarrow \operatorname{Proj}(R^{(n)}).$$

To see that the base locus is empty, note that the irrelevant ideal of $R^{(n)}$ is generated by all elements in R whose degree is positive and divisible by n. Then $\phi(R^{(n)}_+)$ defines the empty set, since any prime $\mathfrak{p} \subset R$ such that $R_+ \cap R^{(n)} \subset \mathfrak{p}$ must contain all of R_+: for any $a \in R_+$, we have $a^n \in R_+ \cap R^{(n)}$, and so $a \in \mathfrak{p}$ as well.

Proposition 6.19. The morphism ν is an isomorphism.

Proof Both $\operatorname{Proj}(R)$ and $\operatorname{Proj}(R^{(n)})$ are covered by distinguished open sets of the form $D_+(f^n)$ where $f \in R_+$ is homogeneous. Over such an open set, the morphism ν is induced by the inclusion $(R^{(n)}_{f^n})_0 \subset (R_{f^n})_0$. This inclusion is actually an equality: if $g/f^{ns} \in (R_{f^n})_0$, then $\deg g = ns \cdot \deg f$, so $g \in R^{(n)}$ and g/f^{ns} belongs to the left-hand side. This means that ν restricts to an isomorphism of schemes over an open covering of $\operatorname{Proj}(R^{(n)})$, and hence it is an isomorphism. □

Example 6.20 (Rational normal curves)**.** For $R = k[x_0, x_1]$, the Veronese subring $R^{(2)}$ is equal to $k[x_0^2, x_0x_1, x_1^2]$. This ring is isomorphic to $k[y_0, y_1, y_2]/(y_1^2 - y_0y_2)$. So $\mathbb{P}^1_k = \operatorname{Proj} R$ is isomorphic to a plane conic $V_+(y_1^2 - y_0y_2)$ in $\mathbb{P}^2_k = \operatorname{Proj} k[u_0, u_1, u_2]$.

More generally, the Veronese ring $R^{(d)}$ is generated by the $d + 1$ monomials of degree d, $x_0^d, x_0^{d-1}x_1, \ldots, x_1^d$. We obtain a closed embedding

$$\nu\colon \mathbb{P}^1_k \longrightarrow \mathbb{P}^d_k.$$

onto the closed subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^n_k$ defined by the ideal $\mathfrak{a}$ generated by the 2×2-minors of the matrix

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}. \tag{6.26}$$

This closed subscheme is called the *rational normal curve of degree* d.

The case $d = 3$ is perhaps especially interesting: it is the *twisted cubic curve*, which is the projective version of the curve appearing in Example 1.29. △

Morphisms between projective spaces

Another important case is when $R = A[x_0, \ldots, x_n]$ and $S = A[y_0, \ldots, y_m]$ with $\phi\colon R \to S$ is defined by

$$\phi\colon A[y_0, \ldots, y_m] \longrightarrow A[x_0, \ldots, x_n], \quad y_i \mapsto f_i$$

where $f_0, \ldots, f_m$ are homogeneous polynomials of the same degree d. In that case, the base locus of ϕ is the closed subscheme $V_+(f_0, \ldots, f_m)$. Strictly speaking, to make ϕ into a map

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

of graded rings, we need to adjust the grading by defining $\deg(y_i) = d$ for each i (doing this does not change the Proj by Proposition 6.19). We will for brevity write

$$\begin{aligned} \mathbb{P}^m_A - V_+(f_0, \dots, f_n) &\longrightarrow \mathbb{P}^n_A \\ (x_0 : \cdots : x_m) &\longmapsto (f_0(x) : \cdots : f_n(x)) \end{aligned}$$

for the morphism defined by $f_0, \dots, f_n$.

The base locus $V_+(f_0, \dots, f_n)$ is empty if and only if the radical of $(f_0, \dots, f_n)$ contains irrelevant ideal $(x_0, \dots, x_n)$. In this situation, we obtain an actual morphism $\mathbb{P}^m_A \to \mathbb{P}^n_A$.

Example 6.21. For an integer $n \in \mathbb{N}$, we can define an 'n-th power map' by

$$\begin{aligned} f \colon \mathbb{P}^1_A &\longrightarrow \mathbb{P}^1_A \\ (x_0 : x_1) &\longmapsto (x_0^n : x_1^n) \end{aligned}$$

The base locus is empty, as the radical of (x_0^n, x_1^n) is equal to the irrelevant ideal (x_0, x_1). △

Weighted projective spaces

In the polynomial ring $R = A[x_0, \dots, x_n]$, we do not need to assume that the variables x_i have degree 1. We obtain a graded ring by letting $x_0, \dots, x_n$ have degrees $d_0, \dots, d_n$ for any set of positive integers d_i. The degree 0 localizations $R_i = (R_{x_i})_0$ still makes sense and there are equalities of subrings of $A[x_0^{\pm 1}, \dots, x_n^{\pm n}]$

$$(R_i)_{x_j/x_i} = (R_{x_i x_j})_0 = (R_j)_{x_j/x_i}$$

and the $U_i = \operatorname{Spec}(R_i)$ can be glued along $U_{ij} = D(x_j/x_i)$ using the identity maps as gluing isomorphisms. Once again, the conditions for gluing schemes are satisfied because they reduce to the identity map on $\operatorname{Spec}(R_{x_i x_j})_0$.

The resulting scheme is called the *weighted projective space* and it is denoted by $\mathbb{P}_A(d_0, \dots, d_n)$ or simply $\mathbb{P}(d_0, \dots, d_n)$ if there is no confusion about the base field.

Weighted projective planes share many similarities with the classical projective space. For instance, if k is a field, the k-points of $\mathbb{P}_k(d_0, \dots, d_n)$ is given by the set of $(n+1)$-tuples $(a_0, \dots, a_n)$ modulo the equivalence relation $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if $b_i = \lambda^{d_i} a_i$ for some $\lambda \in k^\times$.

Example 6.22 (The weighted projective space $\mathbb{P}(1,1,d)$)**.** Let $R = k[x_0, x_1, y]$ where the variables have degrees 1, 1, and d respectively. The weighted projective space $\mathbb{P}(1,1,d) = \operatorname{Proj} R$ is covered by three open affines $D_+(x_0), D_+(x_1), D_+(y)$. The first two charts are straightforward. We have

$$(R_{x_0})_0 = k\Big[\frac{x_1}{x_0}, \frac{y}{x_0^d}\Big], (R_{x_1})_0 = k\Big[\frac{x_0}{x_1}, \frac{y}{x_1^d}\Big].$$

Both are polynomial rings in two variables over k, so $D_+(x_0)$ and $D_+(x_1)$ are both isomorphic to the affine plane $\mathbb{A}^2_k$.

The third chart is more intricate. The degree-zero part of R_y is the k-subalgebra generated by the monomials $x_0^d y^{-1}, x_0^{d-1} x_1 y^{-1}, \dots, x_1^d y^{-1}$, so that

$$(R_y)_0 = k\Big[\frac{x_0^d}{y}, \frac{x_0^{d-1} x_1}{y}, \dots, \frac{x_1^d}{y}\Big].$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

When $d \geqslant 2$, this is not a polynomial ring. For instance, for $d = 2$, the maps $U \mapsto x_0^2 y^{-1}$, $V \mapsto x_0 x_1 y^{-1}$, $W \mapsto x_1^2 y^{-1}$ define an isomorphism

$$k\Big[\frac{x_0^2}{y}, \frac{x_0 x_1}{y}, \frac{x_1^2}{y}\Big] \simeq k[U, V, W]/(V^2 - UW).$$

The inclusion $S = k[x_0, x_1] \hookrightarrow R$ defines a morphism $\mathbb{P}(1, 1, d) - \{(0 : 0 : 1)\} \to \mathbb{P}^1_k$. Over the open set $D_+(x_0)$, it is given by $k[x_1/x_0] \to k[x_1/x_0, y/x_0^d]$, so the map looks like the first projection $\mathbb{A}^2_k \to \mathbb{A}^1_k$. △

Example 6.23. The weighted projective plane $\mathbb{P}(1, 2, 3)$ is the Proj of the ring $R = k[x_0, x_1, x_2]$ where the variables x_0, x_1, x_2 have degrees $1, 2, 3$ respectively. The affine opens are defined by the rings

$$(R_{x_0})_0 = k\left[\frac{x_1}{x_0^2}, \frac{x_2}{x_0^3}\right], \quad (R_{x_1})_0 = k\left[\frac{x_0^2}{x_1}, \frac{x_0 x_2}{x_1^2}, \frac{x_1^2}{x_1^3}\right], \quad (R_{x_2})_0 = k\left[\frac{x_0^3}{x_2}, \frac{x_0 x_1}{x_2}, \frac{x_1^3}{x_2^2}\right].$$

The first chart $\operatorname{Spec}(R_{x_0})_0$ is isomorphic to an affine plane $\mathbb{A}^2_k$, but the two others are not. For instance,

$$(R_{x_2})_0 \simeq k[U, V, W]/(UW - V^3).$$

△

Example 6.24. Weighted projective spaces are often used as ambient spaces for other projective schemes. Given a graded ring R generated by homogeneous elements $x_0, \ldots, x_n$ of degrees $d_0, \ldots, d_n$ respectively, there is a graded surjection $R_0[x_0, \ldots, x_n] \to R$. This means that there is a closed embedding $\operatorname{Proj} R \to \mathbb{P}(d_0, \ldots, d_n)$ over R_0.

For a concrete example, consider the hyperelliptic curves from Section 5.4. While the equation $w^2 = p(x_0, x_1)$ where p is homogeneous of degree $2d$, does not define a closed subscheme of $\mathbb{P}^2_k$, it does define a closed subscheme in the weighted projective space $\mathbb{P}(1, 1, d) = \operatorname{Proj} k[x_0, x_1, w]$ with $\deg(w) = d$. In fact, every hyperelliptic curve constructed earlier embed as closed subschemes of $\mathbb{P}(1, 1, d)$ defined by this equation.

In the notation of Section 5.4, the affine opens X_1 is given by $D_+(x_0) = \operatorname{Spec} k[x, y]/(y^2 - p(x))$ with $x = x_1/x_0$, $y = w/x_0^d$, and similarly for $X_2 = D_+(x_1)$. While $\mathbb{P}(1, 1, d)$ is covered by the three open sets $D_+(x_0)$, $D_+(x_1)$, and $D_+(w)$, the hyperelliptic curve requires only two, since it is entirely contained in $D_+(x_0) \cup D_+(x_1) = \mathbb{P}(1, 1, d) - V_+(x_0, x_1)$. △

6.5 Exercises

Exercise 6.5.1. Let $X \subset \mathbb{P}^n_k$ be a hypersurface of degree d. Show that $\mathbb{P}^n_k - X$ is affine.

Exercise 6.5.2. Let $\mathfrak{a} \subset k[x, y, z]$ be the ideal (xy, xz, yz). Show that $A = R/\mathfrak{a}$ is graded ring and describe each homogeneous component A_n.

Exercise 6.5.3. Consider the $\mathbb{Q}$-scheme $X = \operatorname{Proj}(\mathbb{Q}[x, y, z]/(2x^2 + y^2 - 5z^2))$. Show that $X(\mathbb{Q}) = \varnothing$.

Exercise 6.5.4. Show that $\operatorname{Proj} k[x_0, x_1]/(x_0^2, x_1^2) = \varnothing$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 6.5.5. Let x and y be two points in $\mathbb{P}^n_k$. Prove there is an open affine $U \subset \mathbb{P}^n_k$ containing both x and y.

Exercise 6.5.6 (Projective Nullstellensatz)**.** Let k be an algebraically closed field. Show that the map $\mathfrak{a} \mapsto Z(\mathfrak{a})$ defines a bijection between the radical homogeneous ideals that do not contain $(x_0, \dots, x_n)$ and the closed sets $X \subset \mathbb{P}^n_k$.

Exercise 6.5.7. Let $R = k[x_0, x_1, x_2]$ where x_0, x_1, x_2 have degrees $3, 5, 7$. Consider the standard grading on $k[U, V, W]$ and define $R \to k[U, V, W] \mapsto R$ by $x_0 \mapsto U^3$, $x_1 \mapsto V^5$, $x_2 \mapsto W^7$. Show that this induces a morphism $\pi\colon \mathbb{P}^2_k \to \operatorname{Proj} R$. When k is algebraically closed, describe its fibers over the closed points of $\operatorname{Proj} R$.

Exercise 6.5.8. Let $R = \mathbb{C}[x_0, x_1, x_2]$ and let the symmetric group S_3 act on R by permuting the variables.

a) Show that the invariant polynomials of R are generated by $s_1 = x_0 + x_1 + x_2$, $s_2 = x_0x_1 + x_0x_2 + x_1x_2$ and $s_3 = x_0x_1x_2$.
b) Show that s_1, s_2, s_3 define a morphism $\pi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}(1, 2, 3)$.
c) Describe the fibers of π.
d) Show that $\mathbb{P}(1, 2, 3)$ is homeomorphic to the quotient space $\mathbb{P}^2/S_3$.

Exercise 6.5.9. Let $U = \mathbb{P}^2_k - \{(0 : 0 : 1)\}$. Show that $\mathcal{O}_{\mathbb{P}^2_k}(U) \simeq k$. Is U a projective scheme over k?

Exercise 6.5.10 (Projective closures of affine varieties)**.** Let k be a field.

a) View $\mathbb{A}^n_k$ as the standard open subset $D_+(x_0) \subset \mathbb{P}^n_k$. Let $X \subset \mathbb{A}^n_k$ be an affine variety. Let $\overline{X} \subset \mathbb{P}^n_k$ be the *projective closure* of X, that is, the closed subscheme defined by the ideal I generated by homogeneous polynomials $F \in k[x_0, \dots, x_n]$ such that the dehomogenization with respect to x_0 lies in the prime ideal defining X. Show that I is prime and that $\overline{X}$ is an integral projective scheme.
b) Compute the projective closure of the affine curve $V(x^3 + xy + 1 = 0) \subset \mathbb{A}^2_{\mathbb{C}}$.

Exercise 6.5.11. Let k be an infinite field. Show that a Laurent polynomial $f \in k[x_0, \dots, x_n, x_i^{-1}]$ is invariant under the scaling operation $x_i \mapsto tx_i$ if and only if it belongs to the subring $k[x_0/x_i, \dots, x_n/x_i]$.

Exercise 6.5.12. Let k be a field. Show that the k-points of the weighted projective plane $\mathbb{P}(d_0, \dots, d_n)$ are given by equivalence classes of $(n + 1)$-tuples $(a_0, \dots, a_n)$ where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if $b_i = \lambda^{d_i} a_i$ for all i.

Exercise 6.5.13 (Affine cones)**.** Let $X \subset \mathbb{P}^n_k$ be a closed subscheme defined by a homogeneous ideal $I \subset k[x_0, \dots, x_n]$. Let $C(X) \subset \mathbb{A}^{n+1}_k$ denote the closed subscheme defined by I inside $\mathbb{A}^{n+1}_k = \operatorname{Spec} k[x_0, \dots, x_n]$. Show that there is a morphism

$$\pi\colon C(X) - V(x_0, \dots, x_n) \longrightarrow X.$$

Show that there is a bijection between the k-points of X and the set of lines in $C(X)$ defined over k.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 6.5.14. Let $X = \operatorname{Spec} A$ be an affine scheme over a field k. Show that every morphism $\mathbb{P}^1_k \to X$ is constant, i.e. it factors through some k-valued point of X.

Exercise 6.5.15. Show that $\mathbb{P}^1_A$ is not affine for any ring A.

Exercise 6.5.16. Describe the closed subsets of $\mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{R}}$ and $\mathbb{P}^1_{\mathbb{Q}}$.

Exercise 6.5.17. Let $a = (a_0, \dots, a_n) \in A^{n+1}$ be an $(n+1)$-tuple generating the unit ideal. Show that the homogeneous ideal generated by the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ a_0 & a_1 & \cdots & a_n \end{pmatrix} \tag{6.27}$$

defines a closed subscheme of $\mathbb{P}^n_A$ corresponding to the A-valued point $\iota_a \colon \operatorname{Spec} A \to \mathbb{P}^n_A$ defined in (**??**).

Exercise 6.5.18. Let $X = \mathbb{P}^1_k$. Show that any element $\mathcal{O}_X(X)$ corresponding to a map $X \to \mathbb{A}^1$ factors via a 'constant map' $\operatorname{Spec} k \to \mathbb{A}^1$.

Exercise 6.5.19. Let k be a field and $\mathrm{GL}_2 = \operatorname{Spec} k[x_{11}, x_{12}, x_{21}, x_{22}, \frac{1}{\det}]$ where $\det = x_{11}x_{22} - x_{12}x_{21}$.

a) Write t and s for the coordinates on the two standard charts of $\mathbb{P}^1_k$. Show that $\mathrm{GL}_2 = D(x_{11}) \cup D(x_{21})$, and that $t \mapsto x_{21}/x_{11}$ and $s \mapsto x_{11}/x_{21}$ glue to a morphism $f \colon \mathrm{GL}_2 \to \mathbb{P}^1_k$, sending a matrix to its first column.
b) Show that f is surjective. Deduce that the image of a morphism from an affine scheme need not be contained in an affine open subset of the target.
c) Describe the fibers of f.

Exercise 6.5.20 (Open subschemes as Proj)**.** Let A be a ring, let $X = \operatorname{Spec} A$, and let $U \subset X$ be an open subscheme. Define $I = \{f \in A : D(f) \subset U\}$. That is, I is the ideal of functions whose nonvanishing locus is contained in U. Let T be a variable of degree 1, and

$$R = A \oplus IT \oplus IT^2 \oplus IT^3 \oplus \cdots \subset A[T].$$

a) Show that $\operatorname{Proj} R$ is covered by the opens $D_+(fT)$, for $f \in I$.
b) Show that $D_+(fT) \simeq D(f)$, compatibly with the structure morphism $\operatorname{Proj} R \to \operatorname{Spec} A$. Show that these identifications agree on overlaps, and conclude that $\operatorname{Proj} R \simeq U$ over $\operatorname{Spec} A$.
c) Apply this to $A = k[x, y]$ and $U = \mathbb{A}^2_k - \{(0,0)\}$. Identify I, write down R, and conclude that $\operatorname{Proj} R \simeq \mathbb{A}^2_k - \{(0,0)\}$. Compare $\operatorname{Proj} R$ with $\operatorname{Proj} S$ where $S = \bigoplus_{n \geqslant 0} I^n T^n$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

7

Fiber products and the functor of points

Introduction

In this chapter, we discuss fiber products of schemes, a fundamental and remarkably versatile construction in scheme theory. Given two morphisms of schemes $X \to S$ and $Y \to S$, the *fiber product* $X \times_S Y$ is a scheme equipped with projection morphisms $p_X \colon X \times_S Y \to X$ and $p_Y \colon X \times_S Y \to Y$. The fiber product generalizes the product of two affine varieties as discussed in Example 1.32, but its role in algebraic geometry is much broader.

The main goal of this chapter is to construct the fiber product $X \times_S Y$ as a scheme and explore some of its properties. When X, Y, S are affine, $X \times_S Y$ is constructed via the tensor product of the corresponding rings. For general schemes, the fiber product is constructed by taking affine covers and gluing together the affine fiber products in a compatible way.

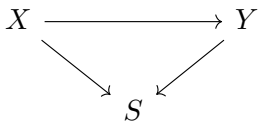
The existence of fiber products is one of the most important features of the category of schemes. It also gives another compelling reason for transitioning from varieties to schemes: even if X, Y, S are affine varieties, the fiber product may fail to be a variety, yet it always exists as a scheme. Many fundamental constructions are defined in terms of fiber products, including base change, intersections and scheme-theoretic fibers.

Towards the end of the chapter, we will explain R-valued points and Yoneda's Lemma, which allow us to understand fiber products in terms of the functor of points.

7.1 Relative schemes

Recall from Section 4.4 that a scheme over a ring A is a scheme equipped with a structure morphism $X \to \operatorname{Spec} A$. There is an important generalization where we replace the base $\operatorname{Spec} A$ by an arbitrary scheme.

For a scheme S, a *scheme over* S (or simply an S*-scheme*) is a scheme X together with a morphism $X \to S$. As before, the schemes over S form a category Sch/S, where a morphism between two S-schemes $X \to S$ and $Y \to S$ is a morphism $X \to Y$ that is compatible with the structure maps, that is, makes the following diagram commute:



The power of this definition lies in viewing X as a family of schemes parametrized by the points of S. That is, we think of the S-scheme X as a space *over* S, which we may picture

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

as a vertical arrow:

$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

This perspective turns out to be very useful in scheme theory, for instance when discussing properties of morphisms and how geometric properties behave in fibers.

Example 7.1. The scheme $X = \operatorname{Spec} k[x,y,t]/(y^2 - x^3 - t)$ can be viewed as a scheme over the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[t]$. Geometrically, one pictures X as a family of plane curves $y^2 = x^3 + t$ parametrized by the coordinate t on $\mathbb{A}^1_k$. △

Example 7.2. The Möbius strip scheme

$$X = \operatorname{Spec} \mathbb{R}[x,y,u,v]/(vx - uy, x^2 + y^2 - 1)$$

can be viewed as a scheme over $\mathbb{R}$, but one can also view it as a scheme over $S = \operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2 - 1)$. The latter perspective offers a bit more geometric insight, as all the fibers of $X \to S$ are affine lines. △

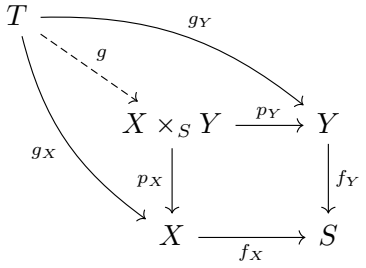
7.2 Fiber products

The notion of a fiber product is meaningful in any category C. Although our main concern will be the category of schemes, we give the definition in a general setting.

A *fiber product* of two arrows $f_X \colon X \to S$ and $f_Y \colon Y \to S$ in a category C is an object $X \times_S Y$ together with two arrows $p_X \colon X \times_S Y \to X$ and $p_Y \colon X \times_S Y \to Y$ such that

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow[f_X]{} & S \end{array} \tag{7.1}$$

commutes, and such that the following universal property holds: for any two arrows $g_X \colon T \to X$ and $g_Y \colon T \to Y$ in C such that $f_X \circ g_X = f_Y \circ g_Y$, there is a *unique* arrow $g \colon T \to X \times_S Y$ such that $p_X \circ g = g_X$ and $p_Y \circ g = g_Y$. The universal property can be visualized in the following diagram:



(7.2)

We call a commutative diagram (7.1) a *Cartesian diagram* or a *Cartesian square*.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Given two morphisms over S, $\phi\colon Z \to X$ and $\psi\colon W \to Y$, we can also form their product

$$\phi \times_S \psi\colon Z \times_S W \longrightarrow X \times_S Y, \tag{7.3}$$

which is the unique morphism induced by the compositions $\phi \circ p_Z$ and $\psi \circ p_W$ where p_Z, p_W are the projections of $Z \times_S W$. By construction, we have $p_X \circ (\phi \times_S \psi) = \phi \circ p_Z$ and $p_Y \circ (\phi \times_S \psi) = \psi \circ p_W$.

When a fiber product exists, it is unique up to a *unique isomorphism*. Indeed, if (W, p_X, p_Y) and (W', p'_X, p'_Y) are two fiber products of X and Y over S, then by the universal property of W' there is a unique morphism $\theta\colon W \to W'$ with $p_X = p'_X \circ \theta$ and $p_Y = p'_Y \circ \theta$. Likewise, the universal property of W yields a unique morphism $\theta'\colon W' \to W$ with $p'_X = p_X \circ \theta'$ and $p'_Y = p_Y \circ \theta'$. Uniqueness implies $\theta' \circ \theta = \mathrm{id}_W$ and $\theta \circ \theta' = \mathrm{id}_{W'}$, so θ is an isomorphism. For this reason, we allow ourselves to speak of *the* fiber product $X \times_S Y$ when it exists.

It is not so hard to come up with examples of categories which do not have all fiber products. For instance, the fiber product does not exist in the familiar category of differentiable manifolds, and also not in the category of affine varieties, as we will see later.

In the category of sets, the fiber product has a concrete description. Given two maps of sets $f_X\colon X \to S$ and $f_Y\colon Y \to S$, their fiber product $X \times_S Y$ is the subset of $X \times Y$ consisting of pairs (x, y) which map to the same element of S:

$$X \times_S Y = \{\, (x, y) \in X \times Y \mid f_X(x) = f_Y(y) \,\}.$$

The projections p_X and p_Y are the usual projection maps $p_X(x, y) = x$ and $p_Y(x, y) = y$.

It is clear that $X \times_S Y$ satisfies the universal property: given any two maps $g_X\colon T \to X$ and $g_Y\colon T \to Y$ such that $f_X \circ g_X = f_Y \circ g_Y$, the required unique map $g\colon T \to X \times_S Y$ is given by $g(z) = (g_X(z), g_Y(z))$.

Example 7.3. If $S = \{*\}$ has one point, $X \times_S Y$ is the usual Cartesian product $X \times Y$. △

Example 7.4. For subsets $X, Y \subset S$ with their inclusion maps into S, the fiber product is equal to the intersection: $X \times_S Y = X \cap Y$. △

Example 7.5. For a map $f\colon X \to S$ and $s \in S$, the fiber product with the inclusion $\{s\} \hookrightarrow S$ is the fiber (preimage) over s, that is, $X \times_S \{s\} = f^{-1}(s)$.

More generally, the fiber product decomposes as a disjoint union of products of fibers:

$$X \times_S Y \;=\; \bigsqcup_{s \in S} f_X^{-1}(s) \times f_Y^{-1}(s).$$

This explains the name *fiber product*: the fibers of the map $X \times_S Y \to S$ are precisely the Cartesian products of the fibers of the two maps f_X and f_Y. △

7.3 Fiber products of schemes

The following existence theorem is the main result of this chapter.

Theorem 7.6 (Existence of fiber products). For any schemes $X \to S$ and $Y \to S$ over a base scheme S, the fiber product $X \times_S Y$ exists as a scheme.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

When $S = \operatorname{Spec} A$ is affine, the fiber product $X \times_S Y$ is often denoted by $X \times_A Y$.

Before we proceed, it is worth emphasising that one cannot construct the fiber product $X \times_S Y$ by taking the fiber product of the underlying sets and then defining a structure sheaf on that set. In fact, the underlying set of a product of schemes can be very different from the product of the underlying sets of X and Y. This is not all that surprising, since for any reasonable definition of the fiber product, we should have $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$. On the other hand, there are more points in $\mathbb{A}^2_k$ than pairs of points of the two factors.

Products of affine schemes

We begin by constructing fiber products of affine schemes. The main observation is that the category of affine schemes is equivalent to the opposite category of rings $\mathsf{Rings}^{\mathrm{op}}$, and that the tensor product of algebras enjoys a universal property *dual* to the one of the fiber product.

Consider A-algebras B_1 and B_2 with structure maps $\alpha_i \colon A \to B_i$ for $i = 1, 2$. Then the tensor product $B_1 \otimes_A B_2$ is also an A-algebra, via $(b_1 \otimes b_2)(c_1 \otimes c_2) = (b_1 c_1) \otimes (b_2 c_2)$. Moreover, there are maps of A-algebras

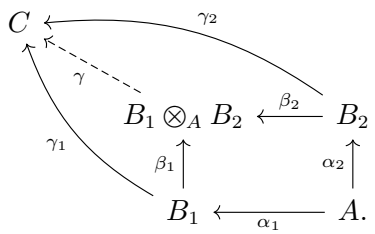
$$\begin{aligned} \beta_1 &\colon B_1 \longrightarrow B_1 \otimes_A B_2, \quad b_1 \mapsto b_1 \otimes 1 \\ \beta_2 &\colon B_2 \longrightarrow B_1 \otimes_A B_2, \quad b_2 \mapsto 1 \otimes b_2 \end{aligned}$$

These fit into a commutative diagram

$$\begin{array}{ccc} B_1 \otimes_A B_2 & \xleftarrow{\beta_2} & B_2 \\ {\scriptstyle\beta_1}\uparrow & & \uparrow{\scriptstyle\alpha_2} \\ B_1 & \xleftarrow[\alpha_1]{} & A \end{array} \tag{7.4}$$

where commutativity holds because $\alpha_1(a) \otimes 1 = 1 \otimes \alpha_2(a)$ by definition of the tensor product $B_1 \otimes_A B_2$.

Moreover, the tensor product is *universal* among diagrams such as (7.4). More precisely, suppose that $\gamma_i \colon B_i \to C$ are maps of A-algebras, i.e. $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$ (i.e., they fit into a commutative diagram analogous to (7.4), but with the β_i's replaced by the γ_i's). The assignment $(b_1, b_2) \mapsto \gamma_1(b_1) \cdot \gamma_2(b_2)$ is A-bilinear and hence induces a unique A-algebra homomorphism $\gamma \colon B_1 \otimes_A B_2 \to C$ with $\gamma(b_1 \otimes b_2) = \gamma_1(b_1) \cdot \gamma_2(b_2)$, which has the property $\gamma \circ \beta_i = \gamma_i$, as expressed in the following commutative diagram:



(7.5)

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Applying Spec to (7.4), we arrive at the diagram

$$\begin{array}{ccc} \operatorname{Spec}(B_1 \otimes_A B_2) & \xrightarrow{p_2} & \operatorname{Spec} B_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ \operatorname{Spec} B_1 & \longrightarrow & \operatorname{Spec} A, \end{array} \tag{7.6}$$

and $\operatorname{Spec}(B_1 \otimes_A B_2)$ is universal among affine schemes sitting in a diagram like (7.6). Hence $\operatorname{Spec}(B_1 \otimes_A B_2)$ serves as the fiber product in the category AffSch of affine schemes. The next result says that it satisfies the universal property of fiber products in the larger category of schemes.

Proposition 7.7. For morphisms $f_i \colon \operatorname{Spec} B_i \to \operatorname{Spec} A$ for $i = 1, 2$, the scheme $\operatorname{Spec}(B_1 \otimes_A B_2)$ with the two projections p_1 and p_2 is the fiber product of the $\operatorname{Spec} B_i$'s in the category of schemes Sch.

Proof Unravelled, this means: for any scheme T and morphisms $g_i \colon T \to \operatorname{Spec} B_i$ with $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique morphism $g \colon T \to \operatorname{Spec}(B_1 \otimes_A B_2)$ such that $p_i \circ g = g_i$ for $i = 1, 2$. To verify this, we apply Theorem 4.6 about morphisms into affine schemes. The morphisms g_i give maps of A-algebras $B_i \to \mathcal{O}_T(T)$. By the universal property of the tensor product, these induce a unique map of A-algebras $B_1 \otimes_A B_2 \to \mathcal{O}_T(T)$, which in turn gives the desired map $g \colon T \to \operatorname{Spec}(B_1 \otimes_A B_2)$ of schemes over $\operatorname{Spec} A$ by Theorem 4.6. By construction, this map satisfies $p_i \circ g = g_i$ for $i = 1, 2$. Finally, g is unique by the uniqueness part of Theorem 4.6 and the universal property of the tensor product. □

Products of general schemes

Here is another instance where the fiber product can be described easily.

Lemma 7.8. Let $f \colon X \to S$ be a morphism and let $V \subset S$ be an open subscheme. Then the fiber product $V \times_S X$ is given by the open subscheme $f^{-1}V$ of X.

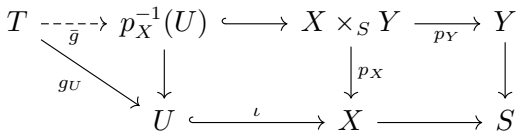
Proof By Exercise 4.11.24, any morphism $g \colon T \to X$ which takes values in the open subscheme $f^{-1}V$ factors uniquely via a morphism $T \to f^{-1}V$. □

Lemma 7.9. If $X \times_S Y$ exists and $U \subset X$ is an open subscheme, then the open subscheme $p_X^{-1}(U)$ with $p_X|_{p_X^{-1}(U)}$ and $p_Y|_{p_X^{-1}(U)}$ as projections is the fiber product $U \times_S Y$.

Proof Write $\iota \colon U \to X$ for the open embedding. Given a scheme T with morphisms $g_U \colon T \to U$ and $g_Y \colon T \to Y$ such that the compositions $T \xrightarrow{g_U} U \hookrightarrow X \to S$ and $T \xrightarrow{g_Y} Y \to S$ agree, the universal property of $X \times_S Y$ gives a unique $g \colon T \to X \times_S Y$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

with $p_X \circ g = \iota \circ g_U$ and $p_Y \circ g = g_Y$. This is pictured in the following diagram:



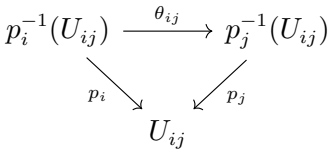
Since $p_X \circ g$ factors through U, the image of g lies in $p_X^{-1}(U)$, so g factors uniquely through a morphism $\bar{g}\colon T \to p_X^{-1}(U)$. This is precisely the universal property of $U \times_S Y$. □

The following lemma will allow us to construct fiber products in general by gluing.

Lemma 7.10. Assume that there is an open cover $\{U_i\}_{i\in I}$ of X such that $U_i \times_S Y$ exists for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open cover of $X \times_S Y$ and the projections restrict to projections.

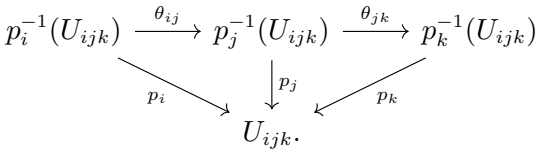
Proof Write $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ for the intersections and let $p_i\colon U_i \times_S Y \to U_i$ denote the projections.

By Lemma 7.9 the preimages $p_i^{-1}(U_{ij})$ serve as fiber products $U_{ij} \times_S Y$ with the restrictions of p_i and p_Y as projections. Similarly, $p_j^{-1}(U_{ij}) \simeq U_{ij} \times_S Y$. Hence there are *unique* isomorphisms $\theta_{ij}\colon p_i^{-1}(U_{ij}) \to p_j^{-1}(U_{ij})$ making the following triangles commute:



(7.7)

We claim that the isomorphisms θ_{ij} satisfy the three conditions of Theorem 5.4 on page 80. The two items (i) and (ii) follow readily. For (iii), note that by Lemma 7.9 the preimages $p_i^{-1}(U_{ijk})$ serve as products $U_{ijk} \times_S Y$ with the restrictions of p_i and p_Y as projections. Moreover, the restrictions of the θ_{ij}'s fit into a commutative diagram



The two small triangles commute, so the big one commutes as well, and it follows by uniqueness of the isomorphisms that $\theta_{ik} = \theta_{jk} \circ \theta_{ij}$ on $p_i^{-1}(U_{ijk})$.

From this it follows that the $U_i \times_S Y$ glue together to a scheme $X \times_S Y$. Moreover, in view of the commutative diagram (7.7) and Proposition 5.1, the p_i's glue together to a morphism $p_X\colon X \times_S Y \to X$. The projections $U_i \times_S Y \to Y$ are basically unaffected by the gluing process and glue together to a morphism $p_Y\colon X \times_S Y \to Y$.

Finally, we check that $X \times_S Y$ together with p_X and p_Y satisfy the required universal property. Let T be an S-scheme with S-morphisms $\phi_X\colon T \to X$ and $\phi_Y\colon T \to Y$ such that $f_X \circ \phi_X = f_Y \circ \phi_Y$. For each $i \in I$, the restrictions $\phi_X^{-1}(U_i) \to U_i$ and $\phi_Y\colon T \to Y$ induce a unique morphism $\phi_i\colon \phi_X^{-1}(U_i) \to U_i \times_S Y$ by the universal property of $U_i \times_S Y$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

On the overlaps $\phi_X^{-1}(U_{ij})$, the morphisms ϕ_i and ϕ_j agree by the uniqueness in the universal property. Therefore, the morphisms ϕ_i glue together to a unique morphism $\phi\colon T \to X \times_S Y$ such that $p_X \circ \phi = \phi_X$ and $p_Y \circ \phi = \phi_Y$. □

An immediate consequence is that fiber products exist when the base S is affine.

Lemma 7.11. If S is affine, then $X \times_S Y$ exists.

Proof If Y is affine, cover X by open affines U_i. Each $U_i \times_S Y$ exists by the affine case, and we may apply Lemma 7.10 above to glue them to $X \times_S Y$. In general, cover Y by affine open subschemes V_j. As we just verified, the products $X \times_S V_j$ all exist, and applying Lemma 7.10 again, we conclude that $X \times_S Y$ exists. □

For the final reduction, we will need the following lemma.

Lemma 7.12. Let $\iota\colon T \to S$ be an open embedding, and let X and Y be two T-schemes. If $X \times_T Y$ exists, then $X \times_S Y$ also exists and is canonically isomorphic to $X \times_T Y$.

Proof Let $f_X\colon X \to T$ and $f_Y\colon Y \to T$ be the structure morphisms. Then X and Y are S-schemes via the structure maps $\iota \circ f_X$ and $\iota \circ f_Y$. If Z is a scheme and $\phi_X\colon Z \to X$ and $\phi_Y\colon Z \to Y$ are morphisms such that $(\iota \circ f_X) \circ \phi_X = (\iota \circ f_Y) \circ \phi_Y$, then $f_X \circ \phi_X = f_Y \circ \phi_Y$ because ι is an open embedding (see Exercise 4.11.24). By the universal property of $X \times_T Y$, there exists a unique morphism $Z \to X \times_T Y$ making the required diagram commute. This means that $X \times_T Y$ satisfies the universal property of $X \times_S Y$. □

Finally, with Lemmas 7.11 and 7.12, we can finish the proof of Theorem 7.6.

Proof of Theorem 7.6 Let $\{S_i\}_{i\in I}$ be an open affine cover of S and let $U_i = f_X^{-1}(S_i)$ and $V_i = f_Y^{-1}(S_i)$. By Lemma 7.11 the products $U_i \times_{S_i} V_i$ all exist. By Lemma 7.12, these serve as products $U_i \times_S V_i$. In fact, $U_i \times_S V_i$ is also the fiber product $U_i \times_S Y$, since for two morphisms $\phi_{U_i}\colon T \to U_i$ and $\phi_Y\colon T \to Y$ such that $T \to U_i \to S$ and $T \to Y \to S$ agree, the morphism ϕ_Y takes values in $f_Y^{-1}(S_i) = V_i$ and hence factors through a unique map $T \to V_i$. Thus the universal properties of $U_i \times_S V_i$ and $U_i \times_S Y$ coincide. Therefore all $U_i \times_S Y$ exist. Applying Lemma 7.10 again, we see that the schemes $U_i \times_S Y$ glue to the fiber product $X \times_S Y$ and the proof is complete. □

Fiber products satisfy a few useful identities which mirror the corresponding identities for tensor products. We will postpone the proof until 7.7.

Proposition 7.13 (Basic formulas). Let X, Y, Z, T be schemes over S. Then there are canonical isomorphisms over S

(i) (Reflexivity) $X \times_S S \simeq X$.
(ii) (Symmetry) $X \times_S Y \simeq Y \times_S X$.
(iii) (Associativity) $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.
(iv) (Transitivity) If Y is a T-scheme (and T is an S-scheme), then

$$(X \times_S T) \times_T Y \simeq X \times_S Y.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

7.4 Examples

As noted in the introduction, the fiber product of schemes can exhibit unexpected behaviour in some situations, differing from what we are used to in set theory or topology. The main difference is that the underlying set is in general not the product of the underlying sets of the factors. The next few examples illustrate this.

Example 7.14. For a ring R and non-negative integers m, n, we have

$$R[x_1, \dots, x_m] \otimes_R R[y_1, \dots, y_n] \simeq R[x_1, \dots, x_m, y_1, \dots, y_n],$$

and so

$$\mathbb{A}^m_R \times_R \mathbb{A}^n_R \simeq \mathbb{A}^{m+n}_R.$$

Even when $R = \mathbb{C}$, the affine space $\mathbb{A}^{m+n}_{\mathbb{C}}$ has an underlying set which is different from the Cartesian product $\mathbb{A}^m_{\mathbb{C}} \times \mathbb{A}^n_{\mathbb{C}}$ (see Example 2.19). △

Example 7.15. Let $X \subset \mathbb{A}^m_R = \operatorname{Spec} R[x_1, \dots, x_m]$ and $Y \subset \mathbb{A}^n_R = \operatorname{Spec} R[y_1, \dots, y_n]$ be two closed subschemes, defined by the ideals $\mathfrak{a} = (f_1, \dots, f_r) \subset R[x_1, \dots, x_m]$ and $\mathfrak{b} = (g_1, \dots, g_s) \subset R[y_1, \dots, y_n]$, respectively. Then $X \times_R Y$ is given by

$$\operatorname{Spec} R[x_1, \dots, x_m, y_1, \dots, y_n]/(f_1, \dots, f_r, g_1, \dots, g_s)$$

This is the scheme version of the product in Example 1.32. △

Here are three examples of fiber products of spectra of fields:

Example 7.16. Consider the fiber product $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$. This scheme has *two* distinct closed points, even though both factors are singletons.

To see this, we compute the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, by writing $\mathbb{C} = \mathbb{R}[t]/(t^2 + 1)$:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2 + 1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2 + 1) = \mathbb{C}[t]/(t - i)(t + i) = \mathbb{C} \times \mathbb{C}.$$

Here the last equality follows from the Chinese Remainder Theorem. △

Example 7.17. The fiber product $\operatorname{Spec} \mathbb{F}_2 \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_3$ is empty. Indeed, it is the spectrum of the ring

$$\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 \simeq \mathbb{Z}/(2, 3) = 0.$$

See Exercise 7.9.1 for a generalization. △

Example 7.18. The fiber product $\operatorname{Spec} \mathbb{C}(x) \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}(y)$ is a bit surprising – it contains infinitely many points. This is because the tensor product $A = \mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ is a ring of Krull dimension 1, and it contains infinitely many maximal ideals.

To see this, we observe that A can be written as the localization $S^{-1}\mathbb{C}[x, y]$, where

$$S = \{\, p(x)q(y) \mid p(x)\, q(y) \neq 0 \,\}.$$

So $\operatorname{Spec} A$ embeds into $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$. Every prime ideal $\mathfrak{p} \subset A$ has the form $\mathfrak{p} = S^{-1}\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset \mathbb{C}[x, y]$ disjoint from S. We note that $\mathfrak{q}$ cannot be a maximal ideal: maximal ideals in $\mathbb{C}[x, y]$ are of the form $(x - a, y - b)$ with $a, b \in \mathbb{C}$, but these intersect S because $(x - a)(y - b) \in S$. Hence $\mathfrak{q}$ must have height at most 1. Since S contains all nonzero polynomials in $\mathbb{C}[x]$ and $\mathbb{C}[y]$, $\mathfrak{q}$ contains no nonzero element of

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\mathbb{C}[x]$ or of $\mathbb{C}[y]$. Hence either $\mathfrak{q} = (0)$ or $\mathfrak{q} = (f(x,y))$, where $f(x,y)$ is an irreducible polynomial which does not lie in $\mathbb{C}[x] \cup \mathbb{C}[y]$. Conversely, every such $\mathfrak{q}$ gives a point of $\operatorname{Spec} A$. Consequently, all nonzero primes in A are maximal, A has dimension 1, and $\operatorname{Spec} A$ contains infinitely many points. △

7.5 Base change

The fiber product leads to a very important technique in algebraic geometry, the ability to change base schemes. This generalizes the familiar idea of considering polynomial equations, such as $x^2 + 1 = 0$, over different rings, but in the geometric setting of schemes.

Consider a scheme X over S with structure morphism $p\colon X \to S$. If $T \to S$ is any morphism, we can form the fiber product $X_T = X \times_S T$, which is naturally a scheme over T. This fits into the Cartesian diagram

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow{\scriptstyle p_T} & & \downarrow{\scriptstyle p} \\ T & \longrightarrow & S. \end{array}$$

One says that X_T is obtained from X by *base change* along $T \to S$.

While base change is essentially a fiber product, the novelty lies in the interpretation: we start with a scheme over S and obtain a new scheme X_T by changing the base scheme to T. This philosophy turns out to be very powerful when proving statements about properties of schemes and morphisms, as we will see shortly.

Example 7.19 (Changing base field)**.** Let $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. For any field extension $k \subset K$, the base change X_K is

$$X_K = X \times_k K = \operatorname{Spec} K[x_1, \dots, x_n]/(f_1, \dots, f_r)$$

So X_K is the scheme defined by the same equations as X, but viewed over K instead of k.

For example, the base change of $X = \operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2)$ to $\mathbb{C}$ is given by

$$X_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]/(x^2 + y^2).$$

Note that X is irreducible, but $X_{\mathbb{C}}$ is not. △

Example 7.20. Suppose that X is a scheme over a field k and that $\sigma\colon k \to k$ is a field automorphism. The automorphism σ induces a morphism $\iota\colon \operatorname{Spec} k \to \operatorname{Spec} k$, and we may form the base change σX which fits into the diagram

$$\begin{array}{ccc} \sigma X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\iota} & \operatorname{Spec} k \end{array}$$

Then X and σX are isomorphic as abstract schemes, but they need not be isomorphic as k-schemes.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For a concrete example, consider $k = \mathbb{Q}(\sqrt{2})$ and the automorphism $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. We get two schemes

$$X = \operatorname{Spec} k[x, y]/(x^2 + \sqrt{2}y^2 + 1), \quad \sigma X = \operatorname{Spec} k[x, y]/(x^2 - \sqrt{2}y^2 + 1).$$

These two are non-isomorphic as k-schemes ($x^2 + \sqrt{2}y^2 + 1$ has no solutions in k, but $x^2 - \sqrt{2}y^2 + 1$ has many). △

Base change is especially important when studying properties of morphisms. In general, it is natural to ask whether a property is preserved under basic operations such as composition, products, and base change. The fact that most of the properties we will encounter are stable under these operations is very important in scheme theory, since it allows us to reduce many questions to simpler cases, for example by going to an affine open subset or by simplifying the base scheme.

To explain this formally, we say that a property of morphisms $\mathcal{P}$ is:

Stable under base change if for any morphism $f\colon X \to S$ with $\mathcal{P}$ and any morphism $g\colon T \to S$, the base change morphism $f_T\colon X \times_S T \to T$ also has $\mathcal{P}$.

Stable under composition if for any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ with $\mathcal{P}$, their composition $g \circ f\colon X \to Z$ also has $\mathcal{P}$.

Local on the target if a morphism $f\colon X \to S$ satisfies $\mathcal{P}$ whenever there exists an open cover $\{U_i\}$ of S such that each base change $f_{U_i}\colon X \times_S U_i \to U_i$ has $\mathcal{P}$.

Proposition 7.21. Closed embeddings and open embeddings are stable under base change.

Proof For open embeddings, this follows from Lemma 7.9.

Now suppose that $f\colon X \to Y$ is a closed embedding. Choose an affine open cover $\{U_i\}$ of Y as in Definition 4.20, and cover each $g^{-1}(U_i)$ by affine open subsets. So we obtain an affine open cover $\{V_j\}$ of Y' such that each V_j maps into some U_i. Fix such a $V = \operatorname{Spec} A' \subset Y'$ mapping into $U = \operatorname{Spec} A \subset Y$, and write $f^{-1}(U) = \operatorname{Spec} B$ with $A \to B$ surjective. By the formulas for the fiber product,

$$f_{Y'}^{-1}(V) \simeq f^{-1}(U) \times_U V = \operatorname{Spec}(B \otimes_A A').$$

If $B \simeq A/I$, then $B \otimes_A A' \simeq A'/IA'$. Hence $f_{Y'}^{-1}(V)$ is affine and the induced map $A' \to B \otimes_A A'$ is surjective. Therefore the cover $\{V_j\}$ satisfies Definition 4.20, and $f_{Y'}$ is a closed embedding. □

Example 7.22 (Injectivity is not stable under base change)**.** The property of being injective is not stable under base change. For instance, the morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ is injective, but its base change along itself is $\operatorname{Spec}(\mathbb{C} \times_{\mathbb{R}} \mathbb{C}) \to \operatorname{Spec}(\mathbb{C})$ which is not injective since $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ consists of two points (by Example 7.16). On the other hand, surjectivity is stable under base change, see Exercise **??**. △

Example 7.23 (Being reduced or irreducible is not stable under base change)**.** Consider the scheme $X = \operatorname{Spec}\mathbb{Z}[x, y]/(x^3 + x^2 - 2y^2)$, viewed as a scheme over $\operatorname{Spec}\mathbb{Z}$. Clearly X is integral, as $x^3 + x^2 - 2y^2$ is irreducible. However, if we take the base change via

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

the morphism $\operatorname{Spec}\mathbb{F}_2 \to \operatorname{Spec}\mathbb{Z}$, we obtain $X_{\mathbb{F}_2} = \operatorname{Spec}(\mathbb{F}_2[x,y]/(x^3+x^2))$ which is neither irreducible nor reduced. △

7.6 Scheme-theoretic fibers

The fiber product also allows us to define a scheme structure on the preimage of a point under a morphism. Let $f\colon X \to Y$ be a morphism of schemes and let $y \in Y$ be a point. Consider the morphism

$$\operatorname{Spec}\kappa(y) \longrightarrow Y$$

corresponding to the inclusion of the point y. Recall that the residue field $\kappa(y)$ is defined by $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$, and that the 'point-map' $\operatorname{Spec}\kappa(y) \to Y$ is given by the composition $\operatorname{Spec}\kappa(y) \to \operatorname{Spec}\mathcal{O}_{Y,y} \to Y$.

Motivated by Example 7.5, we define the *scheme-theoretic fiber* to be the fiber product

$$X_y = X \times_Y \operatorname{Spec}\kappa(y).$$

This scheme fits into the Cartesian diagram

$$\begin{array}{ccc} X_y = X \times_Y \operatorname{Spec}\kappa(y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec}\kappa(y) & \longrightarrow & Y. \end{array}$$

Note that X_y is a scheme over the field $\kappa(y)$. The universal property translates into the following: a morphism $g\colon T \to X$ factors via X_y if and only if $f \circ g$ factors through $\operatorname{Spec}\kappa(y) \to Y$. In particular, this means that $f \circ g$ maps every point of T to $y \in Y$.

Proposition 7.24. Let $f\colon X \to Y$ be a morphism and let $y \in Y$ be a point. Then the natural map $\iota\colon X_y \to X$ is a homeomorphism onto the topological fiber $f^{-1}(y) \subset X$.

Proof We may assume that Y is affine, say $Y = \operatorname{Spec}A$, so y corresponds to a prime ideal $\mathfrak{p}$ in A.

We first consider the case where X is also affine, say $X = \operatorname{Spec}B$ and $f\colon X \to Y$ is induced by a ring map $\phi\colon A \to B$. In this situation, Proposition 2.36 states that the fiber $f^{-1}(\mathfrak{p})$ over a point $\mathfrak{p} \in \operatorname{Spec}A$ is homeomorphic to the spectrum of the ring $(B/\mathfrak{p}B)_{\mathfrak{p}}$ via the composition $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \to \operatorname{Spec}(B_{\mathfrak{p}}) \to \operatorname{Spec}(B)$. On the other hand, standard formulas for tensor products give

$$(B/\mathfrak{p}B)_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = B \otimes_A \kappa(\mathfrak{p}).$$

Here the Zariski topology on $\operatorname{Spec}(B/\mathfrak{p}B)_{\mathfrak{p}}$ (i.e. the induced topology on $f^{-1}(\mathfrak{p})$) coincides with the Zariski topology on $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ (i.e. the topology on the scheme X_y), and so the proposition holds when X is affine.

For the general case, let U be an open and affine subset of X. By Lemma 7.9 on page 111, the fiber product $U_y = U \times_Y \operatorname{Spec}\kappa(y)$ coincides with the open subscheme $\iota^{-1}(U)$ of X_y. By the affine case, U_y maps homeomorphically onto $(f|_U)^{-1}(y) = f^{-1}(y) \cap U$ inside U. As being a homeomorphism is local on the target, ι is a homeomorphism onto $f^{-1}(y)$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 7.25. In Example 4.9, we noted that any scheme X admits a unique morphism $f\colon X \to \operatorname{Spec}\mathbb{Z}$. This means that X can be viewed as a sort of 'fibered object' over $\operatorname{Spec}\mathbb{Z}$, that is, X is the union of the schemes $X \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{F}_p)$ where p is a prime number and $X \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$. Compare this with Figure 2.5. △

Example 7.26. Let k be a field and consider the morphism

$$f\colon X = \operatorname{Spec} k[x,y,t]/(y^2 - tx) \longrightarrow \operatorname{Spec} k[t].$$

For $a \in k$, the fiber over the closed point $(t-a)$ is given by

$$X_a = \operatorname{Spec} k[x,y,t]/(y^2 - tx) \otimes_{k[t]} k[t]/(t-a) \simeq \operatorname{Spec} k[x,y]/(y^2 - ax).$$

Then X_a is integral for all $a \neq 0$ as $y^2 - ax$ is irreducible. However when $a = 0$, the fiber is

$$X_0 = \operatorname{Spec} k[x,y]/(y^2)$$

which is non-reduced (a 'double line').

This is another natural example where nilpotents show up in algebraic geometry. One thinks of the morphism $X \to \mathbb{A}^1_k$ as a family of conics $y^2 - tx = 0$ in $\mathbb{A}^2_k$ as t varies, and the double line X_0 is the limit as '$t \to 0$'. It is much more natural that this limit should be X_0 than, say, the reduced scheme $\operatorname{Spec} k[x,y]/(y)$ – it makes the degree constant in the fibers. △

Example 7.27 (Scheme-theoretic intersections)**.** If U and V are two open subschemes of a scheme X, then the fiber product $U \times_X V$ is isomorphic to the open subscheme $U \cap V \subset X$.

The fiber product can also be used to define a scheme structure on the intersection of closed subschemes. Let Y and Z be two closed subschemes of a scheme X. We define their *scheme-theoretic intersection* as the fiber product $Y \times_X Z$ of the two inclusions $Y \hookrightarrow X$ and $Z \hookrightarrow X$

When $X = \operatorname{Spec} A$ and Y and Z are given by ideals $I, J \subset A$ respectively, then

$$Y \times_X Z = \operatorname{Spec}(A/I \otimes_A A/J) = \operatorname{Spec}(A/(I+J)).$$

That is, $Y \times_X Z$ is the closed subscheme associated to the ideal $I + J$.

Using this local picture, one can prove that $Y \times_X Z$ is in general naturally a closed subscheme of X with underlying topological space given by $Y \cap Z$ in X (see Exercise 7.9.9).

More generally, if $f\colon Y \to X$ is any morphism and $Z \subset X$ is a closed subscheme, the base change $Z \times_X Y$ is a closed subscheme of Y, called the *scheme-theoretic inverse image* of Z under f; see Exercise 7.9.10. △

7.7 Yoneda's Lemma

Recall the sets $X(R)$ of R-valued points from Section 4.9. These depend functorially on the ring R. Each ring map $\phi\colon R \to S$ induces a map $X(\phi)\colon X(R) \to X(S)$ by sending a morphism $\operatorname{Spec} R \to X$ to the composition $\operatorname{Spec} S \to \operatorname{Spec} R \to X$ induced by ϕ. The scheme X therefore defines a functor

$$X\colon \mathsf{Rings} \longrightarrow \mathsf{Sets}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This functor is called the *functor of points* of X.

A morphism of schemes $f\colon X \to Y$ gives, for every ring R, a map of sets $f_R\colon X(R) \to Y(R)$, such that for every ring map $\phi\colon R \to S$, the following diagram commutes:

$$\begin{array}{ccc} X(R) & \xrightarrow{f_R} & Y(R) \\ {\scriptstyle X(\phi)}\downarrow & & \downarrow{\scriptstyle Y(\phi)} \\ X(S) & \xrightarrow{f_S} & Y(S). \end{array} \tag{7.8}$$

In categorical terms, the diagram (7.8) says precisely that the maps f_R form a *natural transformation* $f_{(-)}\colon X(-) \to Y(-)$ between the two functors (see Section B.0.1). We will see shortly that, conversely, any collection of such maps arises from a unique morphism $X \to Y$.

For many purposes it is convenient to work with the extended functor of points which considers the set of morphisms $T \to X$ from an arbitrary test scheme T into X:

$$X(-)\colon \mathsf{Sch}^{\mathrm{op}} \longrightarrow \mathsf{Sets}, \qquad T \mapsto \mathrm{Hom}_{\mathsf{Sch}}(T, X).$$

Note that $X(-)$ reverses the direction of morphisms: if $S \to T$ is a morphism, there is an induced map of sets $X(T) \to X(S)$ sending $T \to X$ to the composition $S \to T \to X$.

For a morphism of schemes $f\colon X \to Y$, the map sending $T \to X$ to the composition $f_T\colon T \to X \to Y$ defines a natural transformation of functors $\eta\colon X(-) \to Y(-)$. In clear text this means that for any morphism $\sigma\colon S \to T$ the following diagram commutes:

$$\begin{array}{ccc} X(T) & \xrightarrow{\eta_T} & Y(T) \\ {\scriptstyle (-)\circ\sigma}\downarrow & & \downarrow{\scriptstyle (-)\circ\sigma} \\ X(S) & \xrightarrow[\eta_S]{} & Y(S). \end{array}$$

If T is fixed and $\{T_i\}_{i\in I}$ is an open cover of T, then the map $X(T) \to X(T_i)$ induced by the inclusion $T_i \to T$ simply sends a morphism $s\colon T \to X$ to the restriction $s|_{T_i}$. Moreover, $X(-)$ has the following sheaf-like property: for any elements $t_i \in X(T_i)$ such that t_i and t_j map to the same element in $X(T_i \cap T_j)$, there is a unique $t \in X(T)$ with $t|_{T_i} = t_i$ for all i.

Proposition 7.28 (Yoneda's Lemma for schemes). Let X and Y be schemes. The assignments $f \mapsto \{f_T\}$ and $f \mapsto \{f_R\}$ define natural bijections among:

(i) Morphisms $f\colon X \to Y$
(ii) Natural transformations of functors $X(-) \to Y(-)$ from $\mathsf{Sch}^{\mathrm{op}}$ to Sets
(iii) Natural transformations of functors $X(-) \to Y(-)$ from Rings to Sets.

Proof The bijection between (i) and (ii) is simply a restatement of Yoneda's Lemma (Lemma B.6). The bijection between (ii) and (iii) amounts to saying that any natural transformation $X(-) \to Y(-)$ is determined by its restriction to the subcategory $\mathsf{AffSch}^{\mathrm{op}}$ and that any natural transformation on affine schemes extends uniquely to all schemes. This is basically a consequence of the results on gluing morphisms in Chapter 5.

Suppose η, η' are two natural transformations that agree on all affine schemes, i.e., $\eta_U = \eta'_U$ as maps of sets $X(U) \to Y(U)$ for every affine scheme U. Let T be any scheme and

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

let T_i be an affine cover of T. Then the map $X(T) \to X(T_i)$ induced by the inclusion $T_i \subset T$ simply sends a morphism $s\colon T \to X$ to the restriction $s|_{T_i}$. But then, since $\eta_{T_i}(s|_{T_i}) = \eta'_{T_i}(s|_{T_i})$ for all i, the two morphisms $\eta_T(s)$ and $\eta'_T(s)$ restrict to the same morphism on all T_i, so they are equal. Therefore $\eta_T = \eta'_T$ and the restriction map is injective.

For surjectivity, suppose we are given a natural transformation η defined on $\mathsf{AffSch}^{\mathrm{op}}$. Let T be a scheme and let T_i be an affine open cover of T. For each i, j cover the intersections $T_i \cap T_j$ by affine open schemes T_{ijk}. Take $s \in X(T)$ and set $t_i = \eta_{T_i}(s|_{T_i})$. Then by naturality, the morphisms t_i and t_j restrict to the same morphism on each T_{ijk} (namely the morphism $\eta_{T_{ijk}}(s|_{T_{ijk}})$). Therefore, they glue to a morphism $t\colon T \to Y$, which we define to be $\eta_T(s)$. It is straightforward to check that this defines a natural transformation. □

Thus giving a morphism $X \to Y$ is the same as giving, for every test scheme T, a map of sets $X(T) \to Y(T)$, functorial in T.

Example 7.29. The rule $r \mapsto (r^2, r^3)$ defines a map of sets $R \to R^2$ for any ring R. These assignments define a natural transformation: if $\phi\colon R \to S$ is a ring map, the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{r\mapsto(r^2,r^3)} & R^2 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi\times\phi} \\ S & \xrightarrow{s\mapsto(s^2,s^3)} & S^2 \end{array}$$

Hence we obtain a natural transformation of functors $\mathbb{A}^1(-) \to \mathbb{A}^2(-)$, corresponding to the morphism $\mathbb{A}^1 \to \mathbb{A}^2$ of Example 1.28. Specifying the morphism as $r \mapsto (r^2, r^3)$ is arguably more intuitive than specifying the corresponding ring map $\mathbb{Z}[x, y] \to \mathbb{Z}[t]$. △

There is also a relative version of Proposition 7.28 for schemes over a base scheme S: for an S-scheme X, the assignment

$$X(T) = \mathrm{Hom}_{\mathsf{Sch}/S}(T, X)$$

defines a contravariant functor from Sch/S to Sets. For two S-schemes X and Y, morphisms of S-schemes $X \to Y$ correspond bijectively to natural transformations $X(-) \to Y(-)$ of the corresponding functors. When $S = \operatorname{Spec} A$ is affine, this gives a functor $X\colon \mathsf{Alg}/A \to \mathsf{Sets}$ from A-algebras to Sets.

The functor of points provides a different way of thinking about schemes: as functors from Rings to Sets. Instead of specifying a scheme explicitly, say by explicit equations or by gluing spectra of rings, we can directly specify its functor and this will precisely pin down what scheme we are talking about. This is both natural and convenient many schemes are in the first place defined in terms of some universal property (e.g. fiber products). The viewpoint is also useful, especially for proving formal statements about schemes. For instance, it often reduces computations with schemes to calculations involving sets.

We say that a functor $F\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ is *representable* if there exists a scheme X such that F is isomorphic to $\mathrm{Hom}_{\mathsf{Sch}}(-, X)$. If such a scheme X exists, then it is uniquely determined by F, by Yoneda's Lemma.

Example 7.30 (The functor of points of $\mathbb{A}^1$)**.** The functor $F(T) = \mathcal{O}_T(T)$ is representable

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

by the affine line $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$. This follows from Theorem 4.10 on page 60, which says that giving a morphism $T \to \mathbb{A}^1_{\mathbb{Z}}$ is equivalent to giving an element of $\mathcal{O}_T(T)$.

More generally, $\mathbb{A}^n$ represents the functor $F(T) = \mathcal{O}_T(T)^n$. This is just a fancy way of saying that a morphism $T \to \mathbb{A}^n$ is the same thing as an n-tuple of regular functions. △

Example 7.31 (The functor of points of $\operatorname{Spec} A$)**.** More generally still, let A be a ring and consider the functor $F\colon \mathsf{Sch}^{op} \to \mathsf{Sets}$ defined by $F(T) = \operatorname{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_T(T))$. By Theorem 4.6, this functor is represented by $\operatorname{Spec} A$. △

As we saw in the examples of Section 7.4, the underlying set of the fiber product $X \times_S Y$ can be confusing. The points become easier to understand if we consider the R-valued points. Indeed, the universal property of the fiber product of two S-schemes X and Y translates into the equality

$$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T). \tag{7.9}$$

In particular, for a ring R, an R-point of $X \times_S Y$ is exactly a pair of R-points of X and Y with the same image in $S(R)$.

So the universal property of $X \times_S Y$ is very natural – it is essentially inherited from the universal property of fiber products of sets.

Proof of Proposition 7.13 By Yoneda's Lemma it suffices to verify the corresponding statements for sets. Let B, C, D be sets over a fixed base set A (i.e. equipped with maps to A). The following assignments give natural bijections:

$$\begin{aligned} B \times_A A &\longrightarrow B, & (b,a) &\longmapsto b,\\ B \times_A C &\longrightarrow C \times_A B, & (b,c) &\longmapsto (c,b),\\ (B \times_A C) \times_A D &\longrightarrow B \times_A (C \times_A D), & ((b,c),d) &\longmapsto (b,(c,d)). \end{aligned}$$

For transitivity, if a map $C \to A$ factors as $C \xrightarrow{g} A' \to A$, then there is a bijection

$$B \times_A C \;\to\; (B \times_A A') \times_{A'} C, \qquad (b,c) \longmapsto \big((b, g(c)),\, c\big).$$

Applying these identities to W-valued points, with $A = S(W)$, $A' = T(W)$, $B = X(W)$, $C = Y(W)$, $D = Z(W)$ and $g\colon Y(W) \to T(W)$, we obtain natural isomorphisms between the corresponding functors of points. Yoneda's Lemma then gives the canonical isomorphisms of schemes displayed above. □

Example 7.32 (Group schemes)**.** A *group scheme* is a scheme G such that the functor of points $G\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ actually takes values in the category of groups (viewed as a subcategory of Sets). Equivalently, each set $G(T)$ carries a natural group structure, functorial in T. By Yoneda's Lemma, being a group scheme is equivalent to having scheme morphisms

$$m\colon G \times G \to G, \qquad i\colon G \to G, \qquad e\colon \operatorname{Spec}\mathbb{Z} \to G$$

encoding multiplication, inversion, and the identity, which satisfy the usual group axioms (associativity, identity, and inverses). △

Example 7.33. The scheme $\mathrm{GL}_n = \operatorname{Spec}\mathbb{Z}[x_{11}, \dots, x_{nn}, \frac{1}{\det(x_{ij})}]$ is a group scheme. It represents the functor

$$\mathbf{GL}_n(T) = \{\, M \in \mathrm{Mat}_{n\times n}(\mathcal{O}_T(T)) \mid \exists N \in \mathrm{Mat}_{n\times n}(\mathcal{O}_T(T)) \text{ such that } MN = NM = \mathrm{id} \,\}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

If M and M' are $n \times n$ matrices with entries in $\mathcal{O}_T(T)$, then matrix multiplication $(M, M') \mapsto MM'$ defines a natural transformation $\mathbf{GL}_n(T) \times \mathbf{GL}_n(T) \to \mathbf{GL}_n(T)$, and so we get a morphism of schemes

$$m\colon \mathrm{GL}_n \times \mathrm{GL}_n \longrightarrow \mathrm{GL}_n,$$

which on T-points corresponds to multiplying the corresponding matrices. Likewise, taking the inverse $M \mapsto M^{-1}$ defines a natural transformation, and hence a morphism of schemes

$$i\colon \mathrm{GL}_n \longrightarrow \mathrm{GL}_n\,.$$

It would be possible to write down formulas for the ring maps defining m and i, but even checking associativity by composing these ring maps would be quite cumbersome. However, as we know that matrix multiplication is associative, Yoneda's Lemma ensures that the corresponding scheme morphisms satisfy the expected properties. △

7.8 The Segre embedding

In this section, we show that the product of two projective schemes is again projective. This is a consequence of the *Segre embedding*. On k-points, this map is defined by taking pairwise products of homogeneous coordinates:

$$\begin{aligned} \mathbb{P}^m(k) \times \mathbb{P}^n(k) &\longrightarrow \mathbb{P}^{(m+1)(n+1)-1}(k) \\ \big((a_0 : \cdots : a_m), (b_0 : \cdots : b_n)\big) &\longmapsto (\cdots : a_i b_j : \dots). \end{aligned} \tag{7.10}$$

On the right, there are $(m+1)(n+1)$ products $c_{ij} = a_i b_j$. The map is well-defined because scaling the a_i's or the b_j's will scale the products c_{ij} accordingly. Moreover, if at least one of the a_i's and one of the b_j's are nonzero, then some c_{ij} is nonzero as well. The map is also injective: if $c_{pq} = a_p b_q \neq 0$, then one can recover the original coordinates up to scaling by $a_i/a_p = c_{iq}/c_{pq}$ and $b_j/b_q = c_{pj}/c_{pq}$.

We now give the scheme analogue of this morphism.

Proposition 7.34. For any ring A, there is a closed embedding of A-schemes

$$\sigma\colon \mathbb{P}^m_A \times_A \mathbb{P}^n_A \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}_A. \tag{7.11}$$

Proof Let $R_i = A[x_0/x_i, \dots]$, $S_j = A[y_0/y_j, \dots]$ and $T_{ij} = A[t_{00}/t_{ij}, \dots]$ be the rings defining the usual affine covers. Then the $U_{ij} = \operatorname{Spec}(R_i \otimes_A S_j)$ cover $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$. Consider the ring map

$$\phi_{ij}\colon T_{ij} \longrightarrow R_i \otimes_A S_j, \qquad \frac{t_{kl}}{t_{ij}} \longmapsto \frac{x_k}{x_i} \otimes \frac{y_l}{y_j}.$$

This is surjective, as $R_i \otimes_A S_j$ is generated as an A-algebra by the products $x_k/x_i \otimes y_l/y_j$. Hence ϕ_{ij} defines a closed embedding $U_{ij} \to \operatorname{Spec} T_{ij}$.

To see that these glue, we check the overlaps, which correspond to the localizations

$$(R_i \otimes_A S_j)_{(x_p/x_i)\otimes(y_q/y_j)} = \big(R_p \otimes_A S_q\big)_{(x_i/x_p)\otimes(y_j/y_q)}, \quad (T_{ij})_{t_{pq}/t_{ij}} = (T_{pq})_{t_{ij}/t_{pq}}\,.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

In the latter localization, the identity $t_{kl}/t_{ij} = t_{kl}/t_{pq} \cdot (t_{ij}/t_{pq})^{-1}$ holds. Applying ϕ_{ij} and ϕ_{pq} shows that both maps send t_{kl}/t_{ij} to

$$\frac{x_k}{x_i} \otimes \frac{y_l}{y_j} = \left(\frac{x_k}{x_p} \otimes \frac{y_l}{y_q}\right) \cdot \left(\frac{x_i}{x_p} \otimes \frac{y_j}{y_q}\right)^{-1}$$

Therefore, the $U_{ij} \to \operatorname{Spec} T_{ij} \subset \mathbb{P}_A^{(m+1)(n+1)-1}$ glue to a morphism σ.

We also have $\sigma^{-1}(\operatorname{Spec} T_{ij}) = U_{ij}$. Indeed, restricted to $\operatorname{Spec} T_{pq}$, the open set $\operatorname{Spec} T_{ij}$ is where t_{ij}/t_{pq} is invertible, so its preimage in U_{pq} is where $(x_i/x_p) \otimes (y_j/y_q)$ is invertible, which is precisely $U_{ij} \cap U_{pq}$. It follows that σ is a closed embedding. □

Example 7.35. $\mathbb{P}^1_A \times_A \mathbb{P}^1_A$ embeds as the quadric $Q = V_+(t_{00}t_{11} - t_{01}t_{10}) \subset \mathbb{P}^3_A$. △

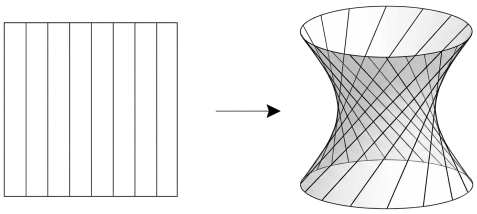


Figure 7.1 The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$

7.9 Exercises

Exercise 7.9.1. Let p and q be different prime numbers. Show the following identities:

a) $\operatorname{Spec} \mathbb{F}_p \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_q = \varnothing$.
b) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)} = \operatorname{Spec} \mathbb{Z}_{(p)}$.
c) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(q)} = \operatorname{Spec} \mathbb{Q}$.

Exercise 7.9.2. Let X and Y be schemes over S with open affine covers $\{U_i\}$ and $\{V_j\}$. Show that the family $\{U_i \times_S V_j\}_{i,j}$ is an open cover of $X \times_S Y$.

Exercise 7.9.3. Let X and Y be schemes over S. Show that there is a natural closed embedding $X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}} \to X \times_S Y$ which is a homeomorphism on the underlying topological spaces and that

$$(X \times_S Y)_{\mathrm{red}} \simeq \left(X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}}\right)_{\mathrm{red}}.$$

HINT: Reduce to the affine case and study the kernel of the surjection $A \otimes_R B \to A_{\mathrm{red}} \otimes_{R_{\mathrm{red}}} B_{\mathrm{red}}$.

Exercise 7.9.4. Let $A = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$ and let $X = \operatorname{Spec} A$. Show that the base-change $X_{\mathbb{C}}$ is isomorphic to $\mathbb{A}^1_{\mathbb{C}} - \{0\}$, but X is not isomorphic to $\mathbb{A}^1_{\mathbb{R}} - \{0\}$.

Exercise 7.9.5. Show that if B is an A-algebra, then $\mathbb{A}^n_B \simeq \mathbb{A}^n_A \times_A \operatorname{Spec} B$ and that $\mathbb{P}^n_B \simeq \mathbb{P}^n_A \times_A \operatorname{Spec} B$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 7.9.6. Let X be a scheme over a field k and let $k \to K$ be a field extension. If X_K denotes the base change to K, show that there is a natural bijection $X(K) = X_K(K)$.

Exercise 7.9.7. Consider the projective schemes $X = V_+(x^2 + y^2 + z^2)$ and $Y = V_+(x^2 + y^2 - 3z^2)$ in $\mathbb{P}^2_{\mathbb{Q}}$.

a) Show that $X(\mathbb{Q}) = Y(\mathbb{Q}) = \varnothing$.
b) Show that $X(\mathbb{Q}(\sqrt{3})) = \varnothing$, but $Y(\mathbb{Q}(\sqrt{3})) \neq \varnothing$. Deduce that X and Y are not isomorphic as $\mathbb{Q}$-schemes.
c) Show that $X \times_{\mathbb{Q}} \mathbb{C}$ and $Y \times_{\mathbb{Q}} \mathbb{C}$ are isomorphic.

Exercise 7.9.8 (Base changes are functorial)**.** If $f\colon X \to Y$ is a morphism over S, and $T \to S$ is a morphism, show that there is an induced morphism $f_T = X_T \to Y_T$ over T. Show that $p_T \circ f_T$ coincides with the natural projection map $X_T \to T$and is Cartesian:

$$\begin{array}{ccccc} X_T & \xrightarrow[f_T]{} & Y_T & \xrightarrow{p_T} & T \\ \downarrow & & \downarrow{\scriptstyle p_Y} & & \downarrow \\ X & \xrightarrow{f} & Y & \longrightarrow & S. \end{array}$$

Given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ over T, show that $(g \circ f)_T = g_T \circ f_T$.

Exercise 7.9.9. Let $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$ be closed embeddings. Show that the scheme-theoretic intersection $Y \times_X Z$ is naturally a closed subscheme of X, with underlying topological space equal to the intersection $i(Y) \cap j(Z)$ in X.

Exercise 7.9.10 (Scheme-theoretic inverse image)**.** Let $f\colon Y \to X$ be a morphism of schemes and let $Z \subset X$ be a closed subscheme. The base change $f^{-1}(Z) = Z \times_X Y$ is called the *scheme-theoretic inverse image* of Z under f.

a) Show that if $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine and Z is given by an ideal $I \subset A$, then $f^{-1}(Z) = \operatorname{Spec}(B/IB)$, where $IB \subset B$ denotes the ideal generated by the image of I.
b) Deduce that $f^{-1}(Z)$ is naturally a closed subscheme of Y whose underlying topological space is the preimage of Z under f.
c) Show that for any scheme T, the T-points of $f^{-1}(Z)$ are precisely the morphisms $g\colon T \to Y$ for which $f \circ g$ factors through Z. In particular, for a field k, the k-points of $f^{-1}(Z)$ are the k-points of Y that map into Z.

Note that the scheme-theoretic intersection of Example 7.27 is the special case where f is itself a closed embedding.

Exercise 7.9.11. Show that the polynomial $X^4 + 16$ is reducible modulo every prime number p. Describe the fibers of the morphism $\operatorname{Spec} \mathbb{Z}[X]/(X^4 + 16) \to \operatorname{Spec} \mathbb{Z}$.

Exercise 7.9.12. Let $A = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ and consider the A-algebra $B = A[u, v]/(vx - uy)$. Show that all of the fibers of the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ are isomorphic to affine lines.

Exercise 7.9.13 (Fiber product and products of fibers)**.** Let $f\colon X \to S$ and $g\colon Y \to S$ be two morphisms and let $s \in S$ be a point with 'point map' $\iota\colon \operatorname{Spec} \kappa(s) \to S$. Denote by

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$h\colon X \times_S Y \to S$ the structure map, i.e. $h = f \circ p_X = g \circ p_Y$. Show that the scheme-theoretic fiber of h is the fiber product of the scheme-theoretic fibers of f and g:

$$(X \times_S Y)_s = X_s \times_{\kappa(s)} Y_s.$$

Exercise 7.9.14. Consider the projective scheme $X = \operatorname{Proj} \mathbb{Z}[x,y,z]/(2x^2 + 2y^2 - 3z^2)$. Describe the scheme-theoretic fibers of the canonical map $\pi\colon X \to \operatorname{Spec} \mathbb{Z}$. Which ones are irreducible, reduced, integral?

Exercise 7.9.15. Show that $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \not\simeq \mathbb{P}^2_{\mathbb{C}}$.

Exercise 7.9.16. Let $X = \operatorname{Spec} \mathbb{Z}$. Compute $X(\mathbb{F}_p)$, $X(\mathbb{Q})$ and $X(\mathbb{C})$.

Exercise 7.9.17. Let $X = \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$, which by Example 7.16 has exactly two points. In this exercise, all schemes are taken over $\operatorname{Spec} \mathbb{R}$, so $Y(\mathbb{C}) = \operatorname{Hom}_{\mathsf{Sch}/\mathbb{R}}(\operatorname{Spec} \mathbb{C}, Y)$.

a) Show that $(\operatorname{Spec} \mathbb{C})(\mathbb{C})$ has two elements; $(\operatorname{Spec} \mathbb{R})(\mathbb{C})$ has one; and deduce that $X(\mathbb{C})$ has four elements.
b) Verify this directly by computing $\operatorname{Hom}_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C})$.
c) Explain why this does not contradict Proposition 4.43.
d) Why must we work over $\mathbb{R}$ rather than in Sch? HINT: Compare with Example 4.15.

Thus $X(\mathbb{C})$ is the product of the corresponding sets of $\mathbb{C}$-points, even though the underlying set of X is not the product of the underlying sets.

Exercise 7.9.18. In this exercise, all schemes and morphisms are over $\mathbb{R}$. Let $X = \operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$. Show that X has two points, but that $X(\mathbb{C}) \simeq (\operatorname{Spec} \mathbb{C})(\mathbb{C}) \times (\operatorname{Spec} \mathbb{C})(\mathbb{C})$ has four elements. Reconcile this with Proposition 4.43.

Exercise 7.9.19. Let $X = \operatorname{Spec}(\mathbb{Q}[x,y]/xy)$ and $Y = \operatorname{Spec}(\mathbb{Q}[x,y]/(x^2 + y^2))$.

a) Compute $\operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} \mathbb{Q}, X)$ and $\operatorname{Hom}_{Sch}(\operatorname{Spec} \mathbb{Q}, Y)$.
b) Show that X and Y are not isomorphic.
c) Let $X' = \operatorname{Spec}(\mathbb{Q}(i)[x,y]/xy)$ and $Y' = \operatorname{Spec}(\mathbb{Q}(i)[x,y]/(x^2+y^2))$. Show that X' and Y' are isomorphic.

Exercise 7.9.20. Let X be a scheme over a field k. Show that, for every field extension K/k, the reduction morphism $X_{\mathrm{red}} \to X$ induces a bijection $X_{\mathrm{red}}(K) \to X(K)$. This is why one has to consider $X(R)$ over *all* rings to recover X.

Exercise 7.9.21. Let k be a field, let $X = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and let $Y = \mathbb{A}^1_k$. Consider the morphisms $f_0, f_\epsilon\colon X \to Y$ corresponding to the k-algebra maps $k[t] \to k[\epsilon]/(\epsilon^2)$ given by $t \mapsto 0$ and $t \mapsto \epsilon$, respectively.

a) Show that $f_0 \neq f_\epsilon$.
b) Show that f_0 and f_ϵ agree after restriction to X_{red}. Using the previous exercise, deduce that, for every field extension K/k, the induced maps $X(K) \to Y(K)$ are equal.
c) Let $R = k[\epsilon]/(\epsilon^2)$. Find an R-point of X on which the maps $X(R) \to Y(R)$ induced by f_0 and f_ϵ take different values. Conclude that field-valued points alone do not determine morphisms of schemes.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 7.9.22 (The Frobenius morphism)**.** Let p be a prime number and let A be a ring of characteristic p. We define the *Frobenius map* $F_A\colon A \to A$ by $a \mapsto a^p$.

a) Show that F_A is a ring map and that the induced map $\operatorname{Spec} A \to \operatorname{Spec} A$ is the identity on the underlying topological space.
b) Show that if A is a local ring, then F_A is a map of local rings.
c) For a scheme X over $\mathbb{F}_p$, define the *Frobenius morphism* $F_X\colon X \to X$ by the identity on the underlying topological space and with $F_X^\sharp\colon \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$. Show that F_X is a morphism of schemes.
d) Show that F_X is natural in the sense that if $f\colon X \to Y$ is a morphism of schemes over $\mathbb{F}_p$, then $f \circ F_X = F_Y \circ f$.
e) Show that $F\colon \mathbb{A}^1_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$ is a homeomorphism but not an isomorphism.
f) Now let $k = \overline{\mathbb{F}_p}$. Show that for every $m \geqslant 1$, the fixed points of the coordinate-wise p^m-power map on k^n are precisely the points of $\mathbb{F}_{p^m}^n$. More generally, let $X_0 \subseteq \mathbb{A}^n_{\mathbb{F}_{p^m}}$ be a closed subscheme and set $X = X_0 \times_{\mathbb{F}_{p^m}} k$. Show that the coordinatewise p^m-power map restricts to a map $X(k) \to X(k)$ whose fixed points are precisely the points of $X_0(\mathbb{F}_{p^m})$.

Exercise 7.9.23 (Closed points and Galois orbits)**.** Let k_0 be a field of characteristic 0, let $k = \overline{k_0}$ be an algebraic closure of k_0, and let $G = \operatorname{Gal}(k/k_0)$. Let X be a k_0-scheme of finite type.

a) Show that G acts on the set $X(k)$ of k-valued points, and that every orbit is finite.
b) Show that there is a natural map from G-orbits of $X(k)$ to closed points of X, sending the orbit of $x\colon \operatorname{Spec} k \to X$ to the image of x in X. Show that this map is a bijection. HINT: For injectivity, two k-points with the same image differ by a k_0-automorphism of k. For surjectivity, use that closed points of a finite type k_0-scheme have finite residue field extensions.
c) Show that the k_0-rational points $X(k_0)$ correspond exactly to the G-fixed points in $X(k)$.
d) Take $k_0 = \mathbb{Q}$ and $k = \overline{\mathbb{Q}}$. For each of the following points of $\mathbb{A}^2_{\mathbb{Q}}(\overline{\mathbb{Q}})$, describe its G-orbit and the corresponding closed point of $\mathbb{A}^2_{\mathbb{Q}}$:
   (i) $(\sqrt{2}, \sqrt{2})$
   (ii) (ω, ω^2) where ω is a primitive cube root of unity
   (iii) $(\sqrt{2}, \sqrt{3})$

Exercise 7.9.24. Let R be a local ring. Show that $\mathbb{P}^n(R)$ is naturally identified with the set of tuples $(a_0, \ldots, a_n) \in R^{n+1}$ for which at least one a_i is a unit, modulo simultaneous multiplication by an element of $R^\times$. HINT: The closed point must land in some U_i; show that all other points must be contained there as well.

Exercise 7.9.25. Show that the $\mathbb{Z}$-points of $\mathbb{P}^1$ are in bijection with the set of pairs $(a, b) \in \mathbb{Z}^2$ with a, b coprime up to multiplication by -1.

Exercise 7.9.26. Let $\mathbb{F}_{p^r}$ denote the field with p^r elements.

a) Show that $\#\mathbb{A}^n(\mathbb{F}_{p^r}) = p^{rn}$.
b) Show that $\#\mathbb{P}^n(\mathbb{F}_{p^r}) = p^{rn} + p^{r(n-1)} + \cdots + p^r + 1$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 7.9.27 (A scheme is a sheaf)**.** Let X be a scheme. For a scheme T, write $X(T) = \mathrm{Hom}_{\mathrm{Sch}}(T, X)$.

a) Let $S \subset T$ be an open subscheme, with inclusion $j\colon S \hookrightarrow T$. Show that composition with j defines a natural restriction map $\rho_{T,S}\colon X(T) \to X(S)$.
b) Let $\{T_i\}_{i\in I}$ be an open cover of T, and let $s, t \in X(T)$ be such that $s|_{T_i} = t|_{T_i}$ for every $i \in I$. Show that $s = t$.
c) Let $\{T_i\}_{i\in I}$ be an open cover of T, and let $s_i \in X(T_i)$ be elements such that $s_i|_{T_i\cap T_j} = s_j|_{T_i\cap T_j}$ for every $i, j \in I$. Show that there exists a unique $s \in X(T)$ such that $s|_{T_i} = s_i$ for every $i \in I$.

Conclude that for every scheme T, the presheaf $U \mapsto \mathrm{Hom}_{\mathrm{Sch}}(U, X)$ on the open subsets of T is a sheaf of sets.

Exercise 7.9.28. Let X and Y be schemes. Let $X^{\mathrm{aff}}(-)$ denote the restriction of the functor $X(-)\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ to the subcategory of affine schemes. Define $\Phi\colon \mathrm{Hom}_{\mathsf{Sch}}(X, Y) \to \mathrm{Hom}(X^{\mathrm{aff}}(-), Y^{\mathrm{aff}}(-))$ by sending f to $f \circ (-)$.

a) Let $\eta\colon X^{\mathrm{aff}}(-) \to Y^{\mathrm{aff}}(-)$ be a natural transformation. Let U_i be an affine cover of X and define $f_i = \eta_{U_i}(U_i \hookrightarrow X)$, where $U_i \hookrightarrow X$ is the inclusion. Show that the morphisms $f_i\colon U_i \to Y$ glue to a morphism $f\colon X \to Y$. Define $\Psi(\eta) = f$.
b) Show that $\Phi \circ \Psi = \mathrm{id}$, i.e., f induces η. HINT: Cover T by affine opens T_α such that each T_α map into some $U_{i(\alpha)}$. Show that η and $f \circ (-)$ agree over each T_α.
c) Applying the construction of Ψ to $\eta = \Phi(f)$, show that $\Psi \circ \Phi = \mathrm{id}$. Conclude that $\mathrm{Hom}_{\mathsf{Sch}}(X, Y) = \mathrm{Hom}(X^{\mathrm{aff}}(-), Y^{\mathrm{aff}}(-))$.

Exercise 7.9.29. Let $f, g\colon X \to Y$ be morphisms of schemes. Show that $f = g$ if the induced maps $f_T, g_T\colon X(T) \to Y(T)$ agree for every scheme T.

Exercise 7.9.30 (Natural operations on rings are polynomials)**.** Let $m, n \geqslant 1$. Suppose that for every ring R there is a map $\eta_R\colon R^n \to R^m$, and that these maps are natural with respect to ring homomorphisms. Show that there are unique polynomials $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$ such that $\eta_R(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$ for every ring R and every $(a_1, \ldots, a_n) \in R^n$.

HINT: Consider $\mathbb{Z}[x_1, \ldots, x_n] \to R$.

Exercise 7.9.31. Show that $\mathbb{P}^1_A \times_A \mathbb{P}^2_A$ embeds as the closed subscheme of $\mathbb{P}^5_A$ defined by the 2×2-minors of the matrix

$$\begin{pmatrix} t_{00} & t_{01} & t_{02} \\ t_{10} & t_{11} & t_{12} \end{pmatrix}.$$

Exercise 7.9.32.
a) Let $n > 0$ be an integer. Find a scheme which represents the functor $F(X) = \{\, s \in \mathcal{O}_X(X) \mid s^n = 0 \,\}$.
b) Show that the functor defined by $F(X) = \{s \in \mathcal{O}_X(X) \mid s \text{ is nilpotent}\}$ is not representable. HINT: On $X = \coprod_{n\geqslant 1} \operatorname{Spec} k[t]/(t^n)$, t is locally nilpotent, but not nilpotent.

Exercise 7.9.33. Find a ring R and a scheme X with an open cover $X = U \cup V$ such that $X(R) \neq U(R) \cup V(R)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 7.9.34. Show that the functor $T \mapsto \mathrm{GL}_n(\mathcal{O}_T(T))$ is represented by the scheme $\mathbf{GL}_n = \operatorname{Spec} \mathbb{Z}[t_{ij}, \det(t_{ij})^{-1}]$. In particular, $\mathbb{A}^1 - \{0\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ represents the functor $F(T) = \Gamma(T, \mathcal{O}_T)^\times$.

Exercise 7.9.35. For each integer $n \geqslant 1$, the *group of n-th roots of 1* is defined by

$$\mu_n = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]/(t^n - 1).$$

Show that μ_n represents the functor $F(T) = \{\, a \in \mathcal{O}_T(T) \mid a^n = 1 \,\}$.

Exercise 7.9.36 (Group actions)**.** Let G be a group scheme. We say that G *acts* on a scheme X if there is a morphism $a\colon G \times X \to X$ satisfying the identity and associativity axioms. Equivalently, for every ring R, the map a_R defines an action of the group $G(R)$ on $X(R)$ in a way compatible with ring maps.

Let $G = \mathrm{GL}_{n+1}$. Show that G acts on $\mathbb{A}^{n+1}$. Show that this induces an action of G on $\mathbb{P}^n$.

Exercise 7.9.37. Let k be a field with $\operatorname{char} k \neq 2$.

a) Show that $X = \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ is an integral scheme.
b) Consider the base change along $\operatorname{Spec} k[[x, y]] \to \mathbb{A}^2_k$. Show that $\operatorname{Spec} k[[x, y]]/(y^2 - x^2 - x^3)$ is reducible. Explain the geometric picture.

Exercise 7.9.38 (Quotients)**.** Let G be a finite group acting by automorphisms on a locally ringed space X. Define Y as the quotient space X/G and let $\pi\colon X \to Y$ be the quotient map. Define a presheaf of rings on Y by $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}U)^G$ (the ring of invariants of $\mathcal{O}_X(\pi^{-1}U)$). Show that $\mathcal{O}_Y$ is a sheaf of rings and that $(Y, \mathcal{O}_Y)$ is a locally ringed space.

Exercise 7.9.39 ($\mathbb{P}^n$ as a quotient space)**.** Let $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ and $\mathbb{A}^{n+1} = \operatorname{Spec} \mathbb{Z}[x_0, \ldots, x_n]$. Let $m\colon \mathbb{G}_m \times \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ be the scalar multiplication morphism, i.e., the morphism which acts on R-valued points as $R^\times \times R^{n+1} \to R^{n+1}$ and $(t, (x_0, \ldots, x_n)) \mapsto (tx_0, \ldots, tx_n)$.

a) Let $U = \mathbb{A}^{n+1} - V(x_0, \ldots, x_n)$. Show that m restricts to an action $\mu\colon \mathbb{G}_m \times U \to U$.
b) Let $\pi\colon U \to \mathbb{P}^n$ denote the quotient morphism. Show that $\pi \circ \mu = \pi \circ \mathrm{pr}_2$ as morphisms $\mathbb{G}_m \times U \to \mathbb{P}^n$. In other words, π is *$\mathbb{G}_m$-equivariant for the trivial action on $\mathbb{P}^n$*.
c) Show that $\mathbb{P}^n$ is a *categorical quotient* of U by $\mathbb{G}_m$ via π: for any scheme Z with trivial $\mathbb{G}_m$-action and any $\mathbb{G}_m$-invariant morphism $f\colon U \to Z$ (i.e., $f \circ \mu = f \circ \mathrm{pr}_2$), there exists a unique morphism $\overline{f}\colon \mathbb{P}^n \to Z$ such that $f = \overline{f} \circ \pi$. HINT: Work on the standard affine charts $D_+(x_i) \subset \mathbb{P}^n$.

Exercise 7.9.40. A property of morphisms $\mathcal{P}$ is *stable under products* if for any $f\colon X \to Y$ and $f'\colon X' \to Y'$ with $\mathcal{P}$, their product $f \times f'\colon X \times_{\mathbb{Z}} X' \to Y \times_{\mathbb{Z}} Y'$ also has $\mathcal{P}$. Show that if $\mathcal{P}$ is stable under base change and stable under composition, then it is stable under products.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

8

Finiteness properties and dimension

In this chapter we survey some basic geometric properties of schemes. Many of these properties are *finiteness conditions*, such as being Noetherian or being of finite type over a base scheme. These conditions hold for most schemes encountered in practice and exclude many pathological examples. Towards the end of the chapter, we also discuss dimension theory for schemes.

8.1 Covering intersections by distinguished opens

A common theme in scheme theory is to reduce geometric questions to the affine case, where they can be translated into statements in commutative algebra. In many situations this strategy works well, but it requires some care. Even when a scheme is affine, an open subset of it need not be affine, as we saw in Example 4.18 on page 63. A related, more serious issue is that the intersection of two affine open sets need not be affine in general. The following example illustrates this.

Example 8.1 (The affine plane with two origins)**.** Let X be the scheme obtained by gluing together two copies U_1, U_2 of the affine plane $\mathbb{A}^2_k$ along the open set $U_{12} = \mathbb{A}^2_k - V(x, y)$. Then, inside X, we have $U_1 \cap U_2 \simeq \mathbb{A}^2_k - V(x, y)$, which is not affine by Example 4.18 on page 63. △

In such situations, one has to cover the intersections by smaller affine open subsets. The following is a basic, but very useful lemma which will help us to construct such covers.

Lemma 8.2 (Doubly distinguished covers). Let X be a scheme.

(i) If $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are two affine open subschemes of X, then for every point $P \in U \cap V$ there exists an open neighbourhood $W \subset U \cap V$ of P that is a distinguished open in both U and V.

(ii) If $\{U_i\}_{i\in I}$ is an affine cover of X and $U = \operatorname{Spec} A$ is any affine open in X, then there exist elements $f_1, \dots, f_r \in A$ such that the distinguished opens $D(f_1), \dots, D(f_r)$ cover U, and for each j the open set $D(f_j)$ is a distinguished open in both U and some $U_{i(j)}$.

Proof (i): Let $P \in U \cap V$. Since the distinguished open sets form a basis for $V = \operatorname{Spec} B$, we may find a $g \in B$ such that $P \in D(g) \subset U \cap V$. As a distinguished open set in a distinguished open set is a distinguished open, we may replace V by $D(g)$, and hence assume that $V \subset U$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The inclusion $\iota\colon V \to U$ is induced by a ring map $\phi\colon A \to B$. Choose $a \in A$ such that $P \in D(a) \subset V$ and set $W = D(a) \subset U$. By Proposition 2.25, we have $W = \iota^{-1}(D(a)) = D(\phi(a))$, so W is distinguished in both U and V.

(ii): Since $\{U_i\}_{i\in I}$ covers X, for each $P \in U$, there exists U_i such that $P \in U_i$. By part (i), there exists an open subset $W_P \subset U \cap U_i$ containing P that is distinguished in both U and U_i. Since $U = \operatorname{Spec} A$ is quasi-compact, we may reduce this to a finite subcover, and we obtain finitely many such open sets $D(f_1), \dots, D(f_r)$ for $f_1, \dots, f_r \in A$ which cover U, and each $D(f_j)$ is distinguished in some $U_{i(j)}$. □

We will use the above lemma repeatedly when proving properties of schemes and morphisms of schemes. Its role is to reduce statements about arbitrary affine open subsets to statements about the members of a fixed affine open cover.

The basic principle is as follows. Suppose we want to prove that some property holds for every affine open subset of X. Assume that the property (i) is preserved under going to distinguished open subsets and (ii) can be verified on finite covers by distinguished opens. Then it suffices to verify it on the members of a single affine open cover of X. Indeed, if $U \subset X$ is any affine open subset, the lemma gives a finite cover of U by open subsets each distinguished in both U and some member of the chosen cover; by (i) the property holds on each of them, and by (ii) on U. In practice, this reduces many arguments to concrete computations with localizations.

8.2 Schemes of finite type

Building on the analogy with affine varieties, which arise from finitely generated k-algebras, we can extend the notion of 'finite generation' to schemes over a general ring A.

Recall that an A-algebra B is called *finitely generated*, or *of finite type* over A, if there exists a finite set of elements $x_1, \dots, x_r \in B$ such that every element in B can be expressed as a polynomial in the x_i with coefficients in A. Equivalently, B is isomorphic to a quotient of a polynomial ring $A[x_1, \dots, x_n]$.

Definition 8.3. An A-scheme X is of *finite type over A* if it admits a finite open cover $\{U_i\}_{i=1}^r$ where each $U_i = \operatorname{Spec} B_i$ and each B_i is a finitely generated A-algebra.

Note that there are two 'finiteness conditions' present here: the affine cover should be finite, and each A-algebra should be finitely generated.

Example 8.4. Any scheme of the form

$$X = \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_r)$$

is of finite type over A. △

Example 8.5. Projective n-space $\mathbb{P}^n_A$ is of finite type over A. Indeed, it is covered by finitely many affine open subschemes isomorphic to $\mathbb{A}^n_A$, and $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \dots, x_n]$ is of finite type over A. △

Every affine scheme of finite type over A is of the form in Example 8.4. This follows from the next proposition:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 8.6. Let X be a scheme of finite type over A. Then for any affine open $U \subset X$, the ring $\mathcal{O}_X(U)$ is a finitely generated A-algebra.

Proof Write $U = \operatorname{Spec} B$, and let $U_i = \operatorname{Spec} B_i$ be the affine covering of X, where each B_i is a finitely generated A-algebra. By Lemma 8.2, we may refine the cover $\{U \cap U_i\}$ of U to a finite cover by distinguished open sets $D(f_1), \ldots, D(f_r)$ such that each $D(f_j)$ is distinguished in some $U_{i(j)}$, say $D(f_j) = \operatorname{Spec}((B_{i(j)})_{h_j})$. As $B_{i(j)}$ is a finitely generated A-algebra, the localization $(B_{i(j)})_{h_j} \simeq B_{f_j}$ is finitely generated as well. We want to prove that B itself is finitely generated as an A-algebra. This follows from Lemma 8.7(i) below. □

Lemma 8.7. Let R be a ring and let $g_1, \ldots, g_r \in R$ generate the unit ideal.

(i) If R is an algebra over a ring A, and R_{g_i} is a finitely generated A-algebra for all i, then R is also a finitely generated A-algebra.

(ii) If M is an R-module such that each M_{g_i} is a finitely generated R_{g_i}-module, then M is also a finitely generated R-module.

Proof (i): Let $c_1, \ldots, c_r \in R$ be such that $\sum c_i g_i = 1$ and choose $x_1, \ldots, x_s \in R$ such that for each i, R_{g_i} is generated as an A-algebra by $x_1/1, \ldots, x_s/1$ and $1/g_i$. Let $R' \subset R$ be the A-subalgebra generated by all the c_i's, all the g_j's and all the x_k's. We claim that $R = R'$.

Take any $y \in R$. We may write $y/1 = y_i/g_i^{n_i} \in R_{g_i}$ where $y_i \in A[x_1, \ldots, x_s] \subset R$ and $n_i \in \mathbb{N}$. This implies that we may find an N such that $yg_i^N \in A[x_1, \ldots, x_s] \subset R'$ for all i. Now, raising both sides of the equation $\sum c_i g_i = 1$ to a sufficiently large power, and expanding, we see that there is a relation of the form $\sum_i p_i g_i^N = 1$ where each p_i is a polynomial in the c_i's and g_j's (hence $p_i \in R'$). Since $yg_i^N \in R'$, we obtain

$$y = y \cdot 1 = \sum_{i=1}^{r} p_i \cdot yg_i^N \in R'$$

Hence $R = R'$, and so R is finitely generated as an A-algebra.

(ii): Pick $c_1, \ldots, c_r \in R$ as above. For each $i = 1, \ldots, r$, choose finitely many elements of M whose images generate M_{g_i} as an R_{g_i}-module. Let $m_1, \ldots, m_N \in M$ be the union of all these elements and let $M' \subset M$ be the submodule they generate. We claim that $M = M'$.

Take any $m \in M$. By assumption, $M_{g_i} = M'_{g_i}$ for each i, so there exists an $N \in \mathbb{N}$ such that $g_i^N m \in M'$ for every i. Writing $1 = \sum p_i g_i^N$ for some $p_i \in R$ as before, we obtain

$$m = 1 \cdot m = \sum_{i=1}^{r} p_i \cdot g_i^N m \in M'$$

Hence $M = M'$, and M is finitely generated as an R-module. □

More generally, a morphism of schemes $f\colon X \to S$ is of *finite type* if for every affine open set $V = \operatorname{Spec} A \subset S$, the inverse image $f^{-1}(V)$ is of finite type over A.

Although this definition requires the condition to hold for every affine open in S, the following proposition shows that it suffices to verify it for the opens in some affine cover.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 8.8. Let $f\colon X \to S$ be a morphism. If there exists an affine cover $V_i = \operatorname{Spec} A_i$ of S such that for each i, the scheme $f^{-1}(V_i) \to V_i$ is of finite type, then f is of finite type.

Proof Recall that if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism induced by a ring map $\phi\colon A \to B$ and $D(g) \subset \operatorname{Spec} A$ is a distinguished open, then $f^{-1}D(g) = D(\phi(g)) = \operatorname{Spec}(B_{\phi(g)})$ is a distinguished open set in $\operatorname{Spec} B$.

Let $V = \operatorname{Spec} A \subset S$ be any affine open subscheme. We need to show that $f^{-1}(V)$ is of finite type over A. Pick elements $g_1, \dots, g_r \in A$ such that for each j, $D(g_j)$ is distinguished in both V and some $V_{i(j)}$. Write $D(g_j) = D(h_j) \subset V_{i(j)}$, where $h_j \in A_{i(j)}$. Choose a finite affine cover $\{\operatorname{Spec} B_{jk}\}_k$ of $f^{-1}(V_{i(j)})$, where each B_{jk} is a finitely generated $A_{i(j)}$-algebra. Restricting this cover to $f^{-1}(D(h_j))$ gives a finite affine cover by spectra of localizations $(B_{jk})_{h_j}$, each of which is a finitely generated algebra over $(A_{i(j)})_{h_j} \simeq A_{g_j}$, and hence over A. Taking the union of these covers over $j = 1, \dots, r$ gives a finite affine cover of $f^{-1}(V)$ by spectra of finitely generated A-algebras. Hence f is of finite type. □

Example 8.9. The morphism $\coprod_{i=1}^{\infty} \mathbb{A}^1_k \to \mathbb{A}^1_k$ which is the identity on each component is not of finite type, even though its restriction to each component is of finite type. △

Example 8.10. Any closed embedding is of finite type, by definition. △

Example 8.11. An open embedding need not be of finite type. For instance, the open embedding $\bigcup_{i=1}^{\infty} D(x_i) \to \operatorname{Spec} k[x_1, x_2, \dots]$ is not of finite type, because the scheme on the left is not quasi-compact (and so cannot be covered by finitely many affine subschemes).

This example is however somewhat atypical. If U is quasi-compact, then an open embedding $U \hookrightarrow X$ is of finite type (see Exercise 8.11.47). △

8.3 Affine morphisms and finite morphisms

A morphism $f\colon X \to Y$ is called *affine* if Y admits an affine cover $V_i = \operatorname{Spec} A_i$ such that each $f^{-1}(V_i) = \operatorname{Spec} B_i$ is affine. The morphism f is *finite* if we can furthermore choose the cover so that each B_i is a finitely generated A_i-module.

Even though the names *finite type* and *finite* are similar, the two notions are very different. To say that a scheme X is of finite type over a field k means it can be covered by open affine subschemes of the form $\operatorname{Spec} k[x_1, \dots, x_r]/\mathfrak{a}$. On the other hand, to say X is finite over $\operatorname{Spec} k$ means that $X = \operatorname{Spec} A$ is affine and A is a k-algebra which is finite-dimensional as a k-vector space. This implies that X is a finite set. So for instance, $\mathbb{A}^1_k$ is of finite type over k, not finite: $k[t]$ is finitely generated as a k-algebra but not as a k-vector space.

Example 8.12. The morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is finite; $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Q}$ is not. △

Example 8.13. For a polynomial $p(x) \in k[x]$ of degree $n \geqslant 1$, the morphism

$$\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k, \qquad x \longmapsto p(x)$$

is finite. Indeed, it is induced by the ring map $k[y] \to k[x]$ sending $y \mapsto p(x)$. As a $k[y]$-module, $k[x]$ decomposes as

$$k[x] = k[y][x]/(y - p(x)) \simeq k[y] \oplus k[y]x \oplus \cdots \oplus k[y]x^{n-1}. \qquad \triangle$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 8.14 (Open embeddings)**.** The open embedding $\operatorname{Spec} A_g \hookrightarrow \operatorname{Spec} A$ of a distinguished open subscheme is affine and of finite type, but typically not finite. For instance, the morphism $\operatorname{Spec} k[x]_x \to \operatorname{Spec} k[x]$ is not finite, as $k[x]_x$ is not a finitely generated $k[x]$-module. △

Example 8.15 (Closed embeddings)**.** Any closed embedding is finite, by definition. △

An important fact about finite morphisms is that they have finite fibers. Thus the algebraic condition of being a finite module is related to the geometric condition of having finite fibers.

Proposition 8.16. Let $f\colon X \to Y$ be a finite morphism. Then
(i) f is a closed map.
(ii) Each fiber $f^{-1}(y)$ is a finite discrete set.

Proof (i): As f is affine and being a closed map can be checked locally on the target, we reduce to the affine case $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and f is induced by a ring map $\phi\colon A \to B$. We claim that $f(V(\mathfrak{b})) = V(\phi^{-1}\mathfrak{b})$ for any ideal $\mathfrak{b} \subset B$. The containment $f(V(\mathfrak{b})) \subset V(\phi^{-1}\mathfrak{b})$ is clear: if $\mathfrak{q} \supset \mathfrak{b}$, then $\phi^{-1}(\mathfrak{q}) \supset \phi^{-1}(\mathfrak{b})$. Conversely, the induced map $A/\phi^{-1}\mathfrak{b} \to B/\mathfrak{b}$ is injective and finite, hence an integral extension of rings. Theorem C.21(i) then shows that the induced map $\operatorname{Spec}(B/\mathfrak{b}) \to \operatorname{Spec}(A/\phi^{-1}\mathfrak{b})$ is surjective. Hence $f(V(\mathfrak{b})) = V(\phi^{-1}\mathfrak{b})$ and f is closed.

(ii): Since any $y \in Y$ is contained in some affine open and f is affine, we again reduce $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and f is induced by $A \to B$. Let $y \in Y$ correspond to a prime ideal $\mathfrak{p} \subset A$. Then the fiber is given by $f^{-1}(y) = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. Since B is a finitely generated A-module, $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = B \otimes_A \kappa(\mathfrak{p})$ is a finite-dimensional $\kappa(\mathfrak{p})$-vector space. In particular, $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is an Artinian ring and so its spectrum $X_y = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ is finite and discrete. □

In particular, if f is finite and $f(X) \subset Y$ is dense, then f is surjective.

Example 8.17. Consider a hyperelliptic curve and the morphism

$$f\colon \operatorname{Spec} k[x,y]/(y^2 - p(x)) \longrightarrow \operatorname{Spec} k[x].$$

Then f is finite, as $k[x,y]/(y^2 - p(x)) \simeq k[x] \oplus k[x]y$ is finite as a $k[x]$-module.

On the other hand, the morphism

$$f\colon \operatorname{Spec} k[x,y]/(xy^2 - p(x)) \longrightarrow \operatorname{Spec} k[x]$$

is not finite. Indeed, it is dominant, because it is induced by an injective ring map. If $p(0) \neq 0$, then the origin $0 \in \mathbb{A}^1(k)$ does not belong to its image, and so f is not surjective. If $p(0) = 0$, then f does not have finite fibers: $V(x)$ is a component of $\operatorname{Spec} k[x,y]/(xy^2 - p(x))$ which maps to a point. △

Having finite fibers is not in general enough to be finite (open embeddings). However, for morphisms between *projective* schemes over an algebraically closed field, it turns out to be enough (see Proposition 15.15).

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

8.4 Noetherian schemes

The Noetherian property for rings has the following geometric analogue, which is one of the most important finiteness conditions for schemes.

Definition 8.18. A scheme X is *Noetherian* if it has a finite open cover $\{U_i\}_{i=1}^r$ where $U_i = \operatorname{Spec} A_i$ where each A_i is a Noetherian ring.

It is important to note that the cover is required to be finite. This in particular implies that a Noetherian scheme is quasi-compact, as it can be covered by finitely many affine schemes, each of which is quasi-compact.

Proposition 8.19. Let X be a scheme. The following are equivalent:
(i) X is Noetherian.
(ii) X is quasi-compact and for every affine open $U \subset X$, the ring $\mathcal{O}_X(U)$ is Noetherian.

Proof (ii) $\Rightarrow$ (i): Let $\{U_i\}_{i\in I}$ be any affine cover and write $U_i = \operatorname{Spec} A_i$. By assumption, each A_i is Noetherian, and by quasi-compactness, we may reduce the covering to a finite one, so X is a Noetherian scheme.

(i) $\Rightarrow$ (ii): Let $\{U_i\}_{i=1}^r$ be an affine cover of X, where $U_i = \operatorname{Spec} A_i$ and each A_i is a Noetherian ring. Since X is covered by finitely many affine open sets, it is quasi-compact. It remains to show that for every affine open $U = \operatorname{Spec} A \subset X$, the ring A is Noetherian.

By Lemma 8.2, we can choose finitely many $f_1, \ldots, f_r \in A$ generating the unit ideal such that the opens $D(f_1), \ldots, D(f_r)$ cover U and each $D(f_j)$ is also a distinguished open in some $U_{i(j)} = \operatorname{Spec}(A_{i(j)})$. That is, $D(f_j) = \operatorname{Spec}(A_{f_j}) = \operatorname{Spec}((A_{i(j)})_{g_j})$ for some $g_j \in A_{i(j)}$. As any localization of $A_{i(j)}$ is Noetherian, we see that A_{f_j} is a Noetherian ring as well.

To conclude, it suffices to show that every ideal $\mathfrak{a} \subset A$ is finitely generated. But this follows by applying Lemma 8.7(ii) to $M = \mathfrak{a}$: since A_{f_i} is Noetherian, each localized ideal $\mathfrak{a}A_{f_i}$ is finitely generated, and hence so is $\mathfrak{a}$. Hence A is Noetherian, as required. □

Corollary 8.20. The scheme $\operatorname{Spec} A$ is Noetherian if and only if A is a Noetherian ring.

The property of being Noetherian is preserved under finite type morphisms. In particular, any scheme of finite type over a Noetherian ring is Noetherian.

Proposition 8.21. Let $f\colon X \to Y$ be a morphism of finite type and suppose Y is Noetherian. Then X is Noetherian.

Proof Since Y is Noetherian, it can be covered by finitely many affine opens of the form $V = \operatorname{Spec} A$, where each A is a Noetherian ring. As f is of finite type, each $f^{-1}V$ can be covered by finitely many spectra $\operatorname{Spec} B_i$ where each B_i is a finitely generated A-algebra. By Hilbert's Basis theorem (Theorem C.24), each B_i is Noetherian. Therefore, X is covered by finitely many Noetherian affine schemes and is therefore Noetherian. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 8.22. Let X be a Noetherian scheme. Then any open or closed subscheme of X is also Noetherian.

Proof Let $\{U_i\}_{i\in I}$ be a finite affine cover with $U_i = \operatorname{Spec} A_i$ with A_i Noetherian. It suffices to prove that if $Y \subset X$ is a closed or open subscheme, then $Y \cap U_i$ is Noetherian. In particular, since $Y \cap U_i$ is closed or open subscheme of an affine scheme, it suffices to consider the case where $X = \operatorname{Spec} A$ and A is Noetherian.

When Y is an open subscheme, write $X - Y = V(\mathfrak{a})$. Since A is Noetherian, the ideal $\mathfrak{a}$ is generated by finitely many elements $g_1, \dots, g_n \in A$, and $Y = \bigcup_{i=1}^n \operatorname{Spec} A_{g_i}$. Each localization A_{g_i} is Noetherian, and hence Y is Noetherian.

When Y is a closed subscheme, we may assume $Y = \operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. If A is Noetherian, then so is $A/\mathfrak{a}$, and hence $Y = \operatorname{Spec}(A/\mathfrak{a})$ is also Noetherian. □

Example 8.23. $X = \operatorname{Spec} \mathbb{Q}$ is Noetherian and is of finite type over $\mathbb{Q}$, but not over $\mathbb{Z}$. △

Another convenient property of Noetherian schemes is that the underlying topological space is Noetherian (that is, descending chains of closed subsets stabilize, see page 8).

Proposition 8.24. Let X be a Noetherian scheme. Then
(i) The topological space of X is Noetherian.
(ii) Any closed subset $Y \subset X$ admits a decomposition, unique up to ordering, into its distinct irreducible components

$$Y = Y_1 \cup \cdots \cup Y_r.$$

Proof (i): By definition, X may be covered by a finite number of open affine subsets. A descending chain stabilizes if and only if the intersection with each of those open sets stabilizes, so we reduce the proof to showing the proposition for $X = \operatorname{Spec} A$ where A is a Noetherian ring, in which case the statement is clear.

(ii): See Proposition 1.15. □

Example 8.25. For a field k, the disjoint union $X = \coprod_{i=1}^{\infty} \operatorname{Spec} k$ is not Noetherian (it is not even quasi-compact). △

Example 8.26. The scheme $X = \operatorname{Spec}\left(\prod_{i=1}^{\infty} k\right)$ is affine, hence quasi-compact. However it is not Noetherian, because the ring is not Noetherian. In fact, the set of prime ideals in infinite products of fields is remarkably complicated (see Exercise 8.11.29). △

The topological space $\operatorname{Spec} A$ can be Noetherian even without A being Noetherian: the condition is equivalent to the weaker condition that ascending chains of *radical ideals* eventually stabilize, and there are many rings which satisfy this without being Noetherian. Here is an example:

Example 8.27. Consider the polynomial ring $k[x_1, x_2, x_3, \dots]$ and the maximal ideal $\mathfrak{m} = (x_1, x_2, \dots)$. The ring

$$A = k[x_1, x_2, x_3, \dots]/\mathfrak{m}^2$$

has only one prime ideal, the maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{m}^2$. Therefore, $\operatorname{Spec} A$ consists of a

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

single point and is Noetherian as a topological space. The ring A, however, is not Noetherian, as $\overline{\mathfrak{m}}$ requires infinitely many generators, namely the images of all the x_i's. △

Most rings encountered in practice are Noetherian. Nevertheless, non-Noetherian schemes do appear, for example when considering normalizations of schemes or certain fiber products.

Example 8.28. Let $A \subset \mathbb{C}$ be the ring of algebraic integers, i.e., the integral closure of $\mathbb{Z}$ in $\mathbb{C}$. That is, A consists of the complex numbers z satisfying a polynomial relation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0, \qquad a_i \in \mathbb{Z}.$$

This ring is not Noetherian. For instance, $(2^{1/2}) \subset (2^{1/4}) \subset (2^{1/8}) \subset \cdots$ is an ascending chain which does not stabilize. △

Example 8.29. The tensor product $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ is not Noetherian. △

8.5 Dimension

Recall that the *Krull dimension* of a ring A is defined as the supremum of all integers n for which there exists a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

where each inclusion is strict.

This has the following geometric counterpart:

Definition 8.30. The *dimension* of a topological space X is defined as the supremum of all integers n for which there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n \tag{8.1}$$

of distinct irreducible closed subsets of X. If no such finite upper bound exists, we say that $\dim X = \infty$.

For a scheme X, we define $\dim X$ to be the dimension of the underlying topological space.

This definition is formulated for any topological space, but it mostly makes sense for spaces with 'Zariski-like' topologies such as the ones found in algebraic geometry. For instance, $\mathbb{R}^n$ with the standard topology is 0-dimensional according to the above definition, as the only irreducible closed subsets are single points.

Proposition 8.31. The dimension of $\operatorname{Spec} A$ equals the Krull dimension of A, that is,

$$\dim(\operatorname{Spec} A) = \dim A.$$

Proof This follows from the one-to-one inclusion-reversing correspondence between prime ideals $\mathfrak{p} \subset A$ and irreducible closed subsets $V(\mathfrak{p})$ of $\operatorname{Spec} A$. □

Example 8.32. A field has Krull dimension 0, since the only prime ideal is (0). △

Example 8.33. The ring $\mathbb{Z}$ has dimension 1. All maximal chains of prime ideals in $\mathbb{Z}$ have the form $(0) \subset (p)$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 8.34. The affine line $\mathbb{A}^1_k$ has dimension 1. The maximal chains of prime ideals in $k[x]$ are of the form $(0) \subset (f(x))$, where $f(x)$ is an irreducible polynomial. △

Example 8.35. The affine line $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x]$ has dimension 2. For example, there are maximal chains of the form $(0) \subset (p) \subset (p, f(x))$, where p is a prime and the image of $f(x) \in \mathbb{Z}[x]$ modulo p is irreducible. △

Example 8.36. The following schemes are 0-dimensional:

$$\operatorname{Spec} \mathbb{Z}/60, \qquad \operatorname{Spec} \mathbb{C}[x]/(x^n), \qquad \operatorname{Spec} \mathbb{C}[x, y]/(x^2, xy, y^3),$$

More generally, the spectrum of an Artinian ring has dimension 0. If A is Noetherian, then A has dimension zero if and only if A is Artinian. △

Most questions and computations involving the dimension of schemes reduce to the affine case, thanks to the following result.

Lemma 8.37. Let X be a scheme.

(i) If $S \subset X$ is any subset, then $\dim S \leqslant \dim X$.

(ii) If $\{U_i\}_{i \in I}$ is an open cover of X, then

$$\dim X = \sup_{i \in I} \dim U_i.$$

(iii) Assume that $\dim X < \infty$. If $Y \subset X$ is a closed irreducible subset and $\dim Y = \dim X$, then Y is an irreducible component of X.

Proof (i): Any irreducible subset of S has an irreducible closure in X, and different irreducible closed subsets have different closures. Hence a chain in S gives a chain in X.

(ii): By (i), we have $\dim U_i \leqslant \dim X$ for all i, so $\sup_i \dim U_i \leqslant \dim X$.

For the opposite inequality, let $Z_0 \subset \cdots \subset Z_n$ be a chain of distinct irreducible closed subsets of X, and pick an open set U from the covering such that $Z_0 \cap U \neq \varnothing$. We obtain a chain

$$Z_0 \cap U \subset Z_1 \cap U \subset \cdots \subset Z_n \cap U.$$

Here each $Z_j \cap U$ is a closed irreducible subset of U (any nonempty open subset of an irreducible space is irreducible). Also, since $Z_j = \overline{Z_j \cap U}$, the intersections $Z_j \cap U$ are distinct.

Therefore, for every chain in X of length n, there is an open set U_i with $\dim U_i \geqslant n$. Taking the supremum over all such n, we obtain $\dim X \leqslant \sup_i \dim U_i$, as required.

(iii): Assume $\dim X < \infty$ and let $Y \subset X$ be a closed irreducible subset with $\dim Y = \dim X$. If Y is not an irreducible component, it would be contained in a strictly larger irreducible closed subset Z. Appending Z to a maximal chain in Y would show that $\dim X > \dim Y$, a contradiction. □

Example 8.38. Let $\mathbb{A}^3_k = \operatorname{Spec} k[x, y, z]$, and consider the subscheme $X = V(xy, xz)$. Then as $(xy, xz) = (x) \cap (y, z)$, we see that X has two components, $V(x)$ and $V(y, z)$. The dimension of X is the maximum of the dimensions of its irreducible components, so $\dim X = 2$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

There is a large body of commutative algebra literature devoted to dimension theory (see for example Atiyah and MacDonald (2018) and Eisenbud (2013)). Many statements in this setting are formulated for local rings, which is sufficient for our purposes, because

$$\dim A = \sup_{\mathfrak{p} \text{ prime}} \dim A_{\mathfrak{p}} = \sup_{\mathfrak{m} \text{ maximal}} \dim A_{\mathfrak{m}}. \tag{8.2}$$

Geometrically, this equality means that the dimension of a scheme may be computed locally by looking in neighborhoods of its closed points.

Dimension theory also works best under certain finiteness conditions. Many intuitive statements hold only for Noetherian rings, and in the non-Noetherian setting one quickly runs into pathological examples. The class of rings where dimension theory is easiest and most intuitive is the finitely generated k-algebras over a field k.

We begin by showing that affine spaces have their expected dimensions.

Proposition 8.39.
(i) $\dim \mathbb{A}^n_k = n$.
(ii) $\dim \mathbb{A}^n_{\mathbb{Z}} = n + 1$.

This result, combined with Lemma 8.37, implies that any scheme of finite type over a field has finite dimension.

While it is intuitive that affine n-space $\mathbb{A}^n_k$ should have dimension n, proving this is not so easy. What is clear is that $\dim \mathbb{A}^n_k \geqslant n$, since in the polynomial ring $k[x_1, \ldots, x_n]$ we have the chain of prime ideals

$$(0) \subset (x_1) \subset \cdots \subset (x_1, \ldots, x_{n-1}) \subset (x_1, \ldots, x_n), \tag{8.3}$$

which has length n. However, the reverse inequality requires more work.

To prove this, we will use two important algebraic facts which are very powerful when combined. The first is that dimension is preserved under finite morphisms.

Proposition 8.40. Let $f\colon X \to Y$ be a finite morphism. Then

$$\dim X \leqslant \dim Y.$$

If f is moreover dominant, then $\dim X = \dim Y$.

Proof First suppose that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, and let $\phi\colon A \to B$ be the finite ring map inducing f. Let $\mathfrak{a} = \operatorname{Ker} \phi$. Then f factors as

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec}(A/\mathfrak{a}) = V(\mathfrak{a}) \longrightarrow \operatorname{Spec} A.$$

Here $A/\mathfrak{a} \subset B$ is an integral ring extension, so Theorem C.21 shows that $\operatorname{Spec} B \to \operatorname{Spec}(A/\mathfrak{a})$ is surjective and that $\dim B = \dim(A/\mathfrak{a})$. Hence $\dim X = \dim V(\mathfrak{a}) \leqslant \dim Y$.

For general X and Y, choose an affine open cover $\{V_i\}_{i\in I}$ of Y. Since f is finite, the preimages $f^{-1}(V_i)$ are affine and form an open cover of X. Applying the affine case and (ii) of Lemma 8.37 gives $\dim X \leqslant \dim Y$.

If f is dominant, then it is surjective because finite morphisms are closed by Proposition 8.16. So each restriction $f^{-1}(V_i) \to V_i$ is surjective. In the affine argument above this means

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

that $V(\mathfrak{a}) = \operatorname{Spec} A$, so $\dim(A/\mathfrak{a}) = \dim A$. Therefore $\dim f^{-1}(V_i) = \dim V_i$ for every i, and Lemma 8.37 gives $\dim X = \dim Y$. □

The second ingredient is Noether's Normalization Lemma. Let k be a field and let A be a finitely generated k-algebra. Then there exist elements $y_1, \dots, y_d \in A$ such that $B = k[y_1, \dots, y_d]$ is a polynomial ring and $B \subset A$ makes A into a finitely generated B-module. Equivalently, we may present A as

$$A = B[z_1, \dots, z_r]/\mathfrak{a} \tag{8.4}$$

where $\mathfrak{a}$ contains, for each $i = 1, \dots, r$, a polynomial of the form

$$z_i^{m_i} + b_{i,m_i-1} z_i^{m_i-1} + \cdots + b_{i,0}, \qquad b_{i,j} \in B.$$

In particular, since A is integral over B, Theorem C.21 gives $\dim A = \dim B = d$.

Proposition 8.41. Let k be a field. Then $\dim k[x_1, \dots, x_n] = n$.

Proof We proceed by induction on n. For $n = 0$, $k[x_1, \dots, x_n] = k$ is a field, which has dimension 0.

Assume $\dim k[x_1, \dots, x_{n-1}] = n - 1$. Suppose for a contradiction that there is a chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n+1}$$

of length $n + 1$. Since the polynomial ring is an integral domain, the zero ideal is prime, so we may assume that $\mathfrak{p}_0 = (0)$.

Choose $f \in \mathfrak{p}_1$ with $f \neq 0$. After a change of variables (Lemma C.28), we may assume that f is monic in x_n, hence $B = k[x_1, \dots, x_n]/(f)$ is integral over $k[x_1, \dots, x_{n-1}]$. Then the chain $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n+1}$ induces a chain of length n in B, so $\dim B \geqslant n$. However, Theorem C.21 and the induction hypothesis give $\dim B = \dim k[x_1, \dots, x_{n-1}] = n - 1$. This is a contradiction, so no such chain exists and $\dim k[x_1, \dots, x_n] = n$. □

More generally, if A is Noetherian, then

$$\dim A[x] = \dim A + 1. \tag{8.5}$$

If A is not Noetherian and has finite dimension d, then $\dim A[x]$ can take any value between $d + 1$ and $2d + 1$. See Exercise 8.11.53 for a proof that $\dim \mathbb{A}^n_{\mathbb{Z}} = n + 1$.

Example 8.42. The ring $\mathbb{C}[x, y]/(y^3 - x^7 - x^2)$ has dimension 1, since it is integral over the subring $\mathbb{C}[x]$ and integral extensions preserve dimension. △

In fact, the above example generalizes.

Proposition 8.43. Let $X = V(f) \subset \mathbb{A}^n_k$ be a hypersurface defined by a nonconstant polynomial $f \in k[x_1, \dots, x_n]$. Then $\dim X = n - 1$.

Proof After a change of variables, we may assume that f is monic in the variable x_n. Then $A = k[x_1, \dots, x_n]/(f)$ is integral over $k[x_1, \dots, x_{n-1}]$. By Theorem C.21, integral extensions preserve dimension, and hence X has dimension $n - 1$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

These results have the following useful consequence for general integral schemes of finite type over a field.

Proposition 8.44. Let X be an integral scheme of finite type over a field k. Then for every nonempty open set $U \subset X$, we have

$$\dim U = \dim X.$$

Proof Choose a finite affine cover of X. By Lemma 8.37, one member V of this cover satisfies $\dim V = \dim X$. Since X is irreducible, $V \cap U$ is nonempty. Hence, after replacing X by V and U by $V \cap U$, we may assume that $X = \operatorname{Spec} A$ is affine.

Choose a polynomial subring $B \subset A$ as in (8.4) and consider the morphism

$$\pi\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec} B \simeq \mathbb{A}^d_k.$$

Let $Z = X - U$. Since X is irreducible and $U \neq \varnothing$, we have $Z \neq X$. Write $Z = V(I)$ for a nonzero ideal $I \subset A$. As π is finite, Proposition 8.16 gives $\pi(Z) = V(I \cap B)$. Note that $I \cap B \neq (0)$. Indeed, if $0 \neq a \in I$ satisfies a minimal monic relation $a^n + b_{n-1}a^{n-1} + \cdots + b_0$ over B, then $b_0 \in (a) \cap B \subset I \cap B$. Hence $\pi(Z) \neq \operatorname{Spec} B$. It follows that the open set $V = \operatorname{Spec} B - \pi(Z)$ is non-empty, and by construction $\pi^{-1}(V) \subset U$. Choose a distinguished open $D(g) \subset V$ with $g \neq 0$ in B. Then

$$\pi^{-1}(D(g)) = \operatorname{Spec} A_g \subset U,$$

so $\dim U \geqslant \dim(\operatorname{Spec} A_g)$. As A_g is integral over B_g, Theorem C.21 gives

$$\dim(\operatorname{Spec} A_g) = \dim(\operatorname{Spec} B_g).$$

It remains to compute $\dim(\operatorname{Spec} B_g)$. Note that

$$B_g \;\simeq\; k[y_1, \ldots, y_d, t]/(gt - 1).$$

This is a hypersurface in $\mathbb{A}^{d+1}_k$ of dimension $(d+1) - 1 = d$ by Proposition 8.43. Therefore $\dim(\operatorname{Spec} A_g) = d$, and we obtain $d = \dim(\operatorname{Spec} A_g) \leqslant \dim U \leqslant \dim X = d$, so $\dim U = \dim X$. □

Example 8.45. In general, it is not true that a dense open subset $U \subset X$ has the same dimension as X, even if X is an integral scheme. For instance, for $X = \operatorname{Spec} k[x]_{(x)}$, the open set $U = \{\xi\}$ consisting of the generic point is dense in X, but it has dimension 0, whereas X has dimension 1. △

In the setting of above proposition, the dimension of A can be computed by considering any localization A_f with $f \neq 0$. This is useful in computations, as the localizations often simplify the defining equations.

Example 8.46. Here are two ways to compute the dimension of the ring

$$A = \mathbb{C}[x, y, z, w]/(xz - y^2, xw - yz, yw - z^2).$$

Consider the open set $D(y) \cap D(z) \subset \operatorname{Spec} A$. Since A is an integral domain, this subset is nonempty. It corresponds to the localization A_{yz}, in which both y and z are invertible. In this ring, the relations $xz = y^2$ and $yw = z^2$ give $x = y^2 z^{-1}$ and $w = y^{-1} z^2$. Substituting

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

these into the remaining relation gives $xw - yz = (y^2z^{-1})(y^{-1}z^2) - yz = 0$, so no further relations remain. Therefore

$$A_{yz} \simeq \mathbb{C}[y^{\pm 1}, z^{\pm 1}],$$

which has dimension 2. Therefore, A has dimension 2 as well.

Alternatively, let $B \subset A$ be the subring generated by x and w. Then A is integral over B. Indeed, in A we have

$$y^3 = y(xz) = x(yz) = x(xw) = x^2w$$

and similarly $z^3 = xw^2$. Moreover, x and w are algebraically independent in A, because the composition $\mathbb{C}[x, w] \to A \to A/(y, z) \simeq \mathbb{C}[x, w]$ is the identity. By Theorem C.21, we find $\dim A = \dim \mathbb{C}[x, w] = 2$.

The second method has the advantage that we do not need to know in advance that A is an integral domain. △

A consequence of Noether's Normalization Lemma is that the dimension of a finitely generated k-algebra domain A is equal to the *transcendence degree* of the fraction field $K(A)$, that is, the maximal number of elements in $K(A)$ which are algebraically independent over k. In particular, this means that for an integral scheme X of finite type over k, the dimension of X depends only on the function field:

$$\dim X = \text{tr.deg}_k K(X). \tag{8.6}$$

In practice, however, most dimension computations use Noether's Normalization Lemma. One algorithm is as follows. Choose an affine open $U = \operatorname{Spec} A$ and find a polynomial subring $B = k[y_1, \ldots, y_d] \subset A$; then X has dimension equal to d. The polynomial ring B can in principle be found as follows. Write $A = k[x_1, \ldots, x_n]/I$. If $I \neq (0)$, pick a nonzero polynomial $f \in I$ and perform a suitable change of variables so that f is monic with respect to one of the variables, say x_n (see Lemma C.28). Then x_n is integral over $A' = k[x_1, \ldots, x_{n-1}]/(I \cap k[x_1, \ldots, x_{n-1}])$. One repeats the same procedure inductively with A' until one arrives at a polynomial ring over which A is integral.

8.6 Products of finite type schemes

Theorem 8.47. Let X and Y be schemes of finite type over an algebraically closed field k. Then

(i) $X \times_k Y$ is of finite type over k.
(ii) The closed points of $X \times_k Y$ are exactly the k-points and $(X \times_k Y)(k) = X(k) \times Y(k)$.
(iii) if X and Y are reduced, then so is $X \times_k Y$.
(iv) if X and Y are integral, then so is $X \times_k Y$ and

$$\dim(X \times_k Y) = \dim X + \dim Y. \tag{8.7}$$

Proof All statements can be reduced to the affine case. Write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, with A, B finitely generated k-algebras. Then $X \times_k Y = \operatorname{Spec}(A \otimes_k B)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(i): The tensor product $A \otimes_k B$ is again a finitely generated k-algebra (Example 7.15), hence $X \times_k Y$ is of finite type.

(ii): Since k is algebraically closed, and $X \times Y$ is of finite type over k, the closed points of $X \times_k Y$ are precisely the k-points (by the Nullstellensatz). The formula $(X \times_k Y)(k) = X(k) \times Y(k)$ follows by the universal property of the fiber product.

(iii): We need to show that $A \otimes_k B$ is reduced. Let $f = \sum_i a_i \otimes b_i \in A \otimes_k B$ be nilpotent. Replacing the a_i if necessary, we may assume that the a_i are k-linearly independent. For each maximal ideal $\mathfrak{m} \subset B$, consider the k-algebra map

$$\phi\colon A \otimes_k B \longrightarrow A \otimes_k B/\mathfrak{m} \simeq A \otimes_k k = A$$

induced by $B \to B/\mathfrak{m} = k$. Since A is reduced, $\phi(f) = \sum_i a_i \bar{b_i} = 0$ in A. This forces $\bar{b_i} = 0$, hence $b_i \in \mathfrak{m}$, because the a_i are k-linearly independent. Since this happens for every $\mathfrak{m}$, we see that $b_i \in \bigcap_{\mathfrak{m}} \mathfrak{m}$. However, as B is reduced, the Nullstellensatz (Exercise 1.6.10) shows that $\bigcap_{\mathfrak{m}} \mathfrak{m} = \sqrt{(0)} = (0)$, so $f = 0$. Hence $A \otimes_k B$ is reduced.

(iv): Suppose A and B are integral domains. We need to show that $A \otimes_k B$ is an integral domain. Suppose that $f = \sum a_i \otimes b_i$ and $g = \sum c_i \otimes d_i$ are two elements such that $fg = 0$. As in (iii), we may assume that the a_i's and the c_i's are linearly independent over k. Then $\phi(fg) = \phi(f)\phi(g) = 0$ in A. Since A is a domain, either $\phi(f) = 0$ or $\phi(g) = 0$, so we see that either $b_i \in \mathfrak{m}$ for all i or $d_j \in \mathfrak{m}$ for all j. In either case, if $\mathfrak{b}$ and $\mathfrak{d}$ denote the ideal generated by the b_i and d_j respectively, then $\mathfrak{b} \cap \mathfrak{d} \subset \mathfrak{m}$. Since this holds for every $\mathfrak{m}$, we obtain $\mathfrak{b} \cap \mathfrak{d} = (0)$ and hence $V(\mathfrak{b}) \cup V(\mathfrak{d}) = V(0) = \operatorname{Spec} B$. As B is an integral domain, this implies that either $\mathfrak{b} = (0)$ or $\mathfrak{d} = (0)$, which means that either $f = 0$ or $g = 0$.

For the dimension formula (8.12), we may by Noether's Normalization Lemma choose polynomial subrings such that $k[x_1, \dots, x_m] \subset A$ and $k[y_1, \dots, y_n] \subset B$ are both finite. Then $A \otimes_k B$ is finite over

$$k[x_1, \dots, x_m] \otimes_k k[y_1, \dots, y_n] = k[x_1, \dots, x_m, y_1, \dots, y_n],$$

and Theorem C.21 gives $\dim(A \otimes_k B) = m + n = \dim X + \dim Y$, as required. □

8.7 Codimension

In addition to dimension, there is a relative notion called *codimension*, which is defined for irreducible closed subsets.

Definition 8.48 (Codimension)**.** Let $Y \subset X$ be an irreducible closed subset of X. The *codimension* of Y in X, denoted $\operatorname{codim}(Y, X)$ or simply $\operatorname{codim} Y$, is the supremum of all integers n for which there exists a chain

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of X.

When $X = \operatorname{Spec} A$, the one-to-one correspondence between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals in A shows that the codimension of the subset $V(\mathfrak{p})$ is equal to the height of $\mathfrak{p}$, that is, the supremum of the lengths of chains of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r = \mathfrak{p}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Equivalently,

$$\operatorname{codim} V(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}). \tag{8.8}$$

This extends to arbitrary schemes:

Proposition 8.49. Let X be a scheme and $P \in X$ be a point. Let $Y = \overline{\{P\}}$. Then

$$\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,P}. \tag{8.9}$$

Proof Given a chain $Y \subset Y_1 \subset \cdots \subset Y_n$, the generic points $\xi_1, \ldots, \xi_n$ of the Y_i's will be contained in any affine open neighbourhood of P. We may therefore assume that X is affine, and in that case (8.9) reduces to (8.8). □

In light of this, one sometimes talks about the *codimension of a point*, by which we mean the codimension of its closure in X.

Any chain of irreducible closed subsets of Y can be extended to a chain in X by appending a chain from Y up to X. This leads to the basic inequality

$$\dim Y + \operatorname{codim}(Y, X) \leqslant \dim X. \tag{8.10}$$

This need not be an equality in general (see Example 8.53), but it holds for all irreducible schemes of finite type over a field. In that setting we have

$$\operatorname{codim}(Y, X) = \dim X - \dim Y,$$

which explains the name 'codimension'. This is a consequence of the following algebraic fact.

Proposition 8.50. Let A be a finitely generated integral k-algebra and $\mathfrak{p} \subset A$ a prime ideal. Then

$$\dim(A) = \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}). \tag{8.11}$$

Proof The dimension formula (8.11) is a consequence of the fact that finitely generated domains over a field are equidimensional—indeed, equicodimensional—and catenary. Catenarity means that for two prime ideals $\mathfrak{q} \subset \mathfrak{p}$, all maximal chains of prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$ have the same length, while equicodimensionality means that all maximal ideals have height $\dim A$. See (Eisenbud, 2013, pages 456–457) for a proof. These are non-trivial facts in commutative algebra, and the dimension formula does not hold for general rings (see Example 8.53). □

Proposition 8.51. If $X \subset \mathbb{A}^n_k$ is an integral closed subscheme of dimension $n-1$, then $X = V(f)$ for some nonconstant polynomial f.

Proof Write $X = V(\mathfrak{p})$ for a prime ideal $\mathfrak{p} \subset k[x_1, \ldots, x_n]$. Proposition 8.50 gives $\operatorname{ht}(\mathfrak{p}) = 1$. Choose a nonzero element $g \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, some irreducible factor f of g lies in $\mathfrak{p}$. Then (f) is prime, and Propositions 8.43 and 8.50 show that it has height 1. Since $(f) \subset \mathfrak{p}$, we conclude that $(f) = \mathfrak{p}$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The following theorem is very important in dimension theory.

Theorem 8.52 (Krull's Principal Ideal Theorem). Let A be a Noetherian ring, and let $f \in A$ be a nonunit. Then every prime ideal $\mathfrak{p}$ which is minimal among those containing f has height at most 1. If f is also a nonzerodivisor, then every such prime has height 1.

Geometrically, this theorem says that every irreducible component of $V(f)$ has codimension at most 1 in $\operatorname{Spec} A$. The codimension can be 0 only when $V(f)$ contains an irreducible component of $\operatorname{Spec} A$, equivalently when f lies in a minimal prime of A. If f is a nonzerodivisor, this does not occur, so every irreducible component of $V(f)$ has codimension exactly 1.

When A is a finitely generated domain over a field, codimension equals the difference of the dimensions, so in that setting a nonzero nonunit f cuts the dimension down by exactly 1, that is, every irreducible component of $V(f)$ has dimension $\dim A - 1$.

Example 8.53. Let $R = \mathbb{C}[x]_{(x)}$ and $\mathfrak{n} = (tx-1) \subset R[t]$. Then $R[t]/(tx-1) \simeq \mathbb{C}(x)$ is a field, so $\mathfrak{n}$ is a maximal ideal in $R[t]$. The element $tx-1$ is neither a unit nor a zerodivisor in $R[t]$, so by Krull's Principal Ideal Theorem (Theorem 8.52), the ideal $(tx-1)$ has height 1.

Setting $X = \operatorname{Spec} R[t]$ and $Y = V(\mathfrak{n})$, we find that $\dim X = 2$, $\dim Y = 0$, and $\operatorname{codim}(Y, X) = 1$. Hence $\dim Y + \operatorname{codim}(Y, X) < \dim X$ in this example. △

By an inductive argument, one obtains the following generalization.

Corollary 8.54. Let A be a Noetherian ring and let $\mathfrak{a} = (f_1, \ldots, f_r) \subset A$. If $\mathfrak{p} \subset A$ is a prime ideal which is minimal among those containing $\mathfrak{a}$, then $\operatorname{ht} \mathfrak{p} \leqslant r$.

Proof See (Eisenbud, 2013, Theorem 10.2). □

The following example shows that adding a generator need not increase the codimension.

Example 8.55. In $k[x, y, z, w]$, consider the ideal

$$\mathfrak{a} = (y^2 - xz, xw - yz, z^2 - yw).$$

Then $\mathfrak{a}$ is a prime ideal (see Exercise 1.6.13). While $\mathfrak{a}$ requires three generators, the codimension of $V(\mathfrak{a})$ is actually 2. If one tries to apply Corollary 8.54 inductively, one finds that $z^2 - yw$ becomes a zerodivisor modulo the first two generators, by the following relation

$$x(z^2 - yw) = (-z)(y^2 - xz) + (-y)(xw - yz).$$

Therefore, adding $z^2 - yw$ does not increase the codimension. △

Krull's Principal Ideal Theorem says that the codimension of the zero set can increase by *at most* 1 when adding a new generator, and by exactly 1 when the equation is a nonzerodivisor. To cut out a subscheme of codimension r by r equations, we want to apply Krull's Principal Ideal Theorem r times, imposing one equation at a time and requiring at each stage that the new equation be a nonzerodivisor on what is left. Such sequences of equations are called regular sequences.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Definition 8.56. Let A be a ring and let M be an A-module. A sequence of elements $x_1, \dots, x_r \in A$ is called an *M-regular sequence* if
(i) x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ for each $i = 1, \dots, r$; and
(ii) $M/(x_1, \dots, x_r)M \neq 0$.
An A-regular sequence is simply called a *regular sequence*, and an ideal $\mathfrak{a} \subset A$ is said to be generated by a regular sequence if $\mathfrak{a} = (x_1, \dots, x_r)$ for some regular sequence $x_1, \dots, x_r$.

Condition (i) for $i = 1$ says that x_1 is a nonzerodivisor on M; condition (ii) rules out the degenerate case $(x_1, \dots, x_r)M = M$, which for A local and $x_i \in \mathfrak{m}$ is automatic by Nakayama's lemma.

The definition depends on the order of the elements (see Exercise C.12.1). However, if A is a *Noetherian local ring* and the x_i lie in the maximal ideal $\mathfrak{m}$, then any permutation of a regular sequence remains a regular sequence (Exercise C.12.2).

Example 8.57. The variables $x_1, \dots, x_n$ form a regular sequence in the polynomial ring $A = k[x_1, \dots, x_n]$. Indeed, the quotient $A/(x_1, \dots, x_i)$ is the polynomial ring $k[x_{i+1}, \dots, x_n]$, an integral domain in which the image of x_{i+1} is nonzero, hence a nonzerodivisor; and $A/(x_1, \dots, x_n) = k \neq 0$. △

Example 8.58. Let A be a ring and $f \in A$ a nonzerodivisor which is not a unit. Then f is a regular sequence of length 1, and $V(f) \subset \operatorname{Spec} A$ is the associated hypersurface. If A is a UFD and f, g are nonzero nonunits with no common irreducible factor such that $(f, g) \neq A$, then f, g is a regular sequence, and $V(f, g)$ has codimension 2. △

Example 8.59. In $A = k[x, y]/(xy)$ the element x is a zerodivisor, since $xy = 0$ while $y \neq 0$; so x alone is not a regular sequence. This reflects the fact that $V(x)$ is one of the two coordinate axes making up $\operatorname{Spec} A$, hence contains an irreducible component of $\operatorname{Spec} A$ and does not cut the dimension down at all. The element $x - y$, on the other hand, is a nonzerodivisor, and $A/(x - y) \simeq k[x]/(x^2)$ is a fat point at the origin. △

The point of the definition is that a regular sequence cuts the codimension exactly: unlike in Example 8.55, each successive element increases the codimension by 1.

Proposition 8.60. Let A be a Noetherian ring and let $x_1, \dots, x_r$ be a regular sequence in A. Then every prime ideal $\mathfrak{p}$ minimal over $(x_1, \dots, x_r)$ has height exactly r. In particular $\operatorname{ht}(x_1, \dots, x_r) = r$, and $V(x_1, \dots, x_r) \subset \operatorname{Spec} A$ has pure codimension r.

Proof The inequality $\operatorname{ht} \mathfrak{p} \leqslant r$ is Corollary 8.54. For the reverse inequality we induct on r, the case $r = 0$ being trivial. Let $\mathfrak{p}$ be minimal over $\mathfrak{a} = (x_1, \dots, x_r)$ and put $\mathfrak{b} = (x_1, \dots, x_{r-1})$. Then $\mathfrak{p}$ is minimal over $\mathfrak{b} + (x_r)$, so $\mathfrak{p}/\mathfrak{b}$ is a minimal prime over the image of x_r in $A/\mathfrak{b}$. Since x_r is a nonzerodivisor on $A/\mathfrak{b}$, it lies in no minimal prime of $A/\mathfrak{b}$, so $\mathfrak{p}/\mathfrak{b}$ is not a minimal prime of $A/\mathfrak{b}$ and therefore strictly contains some prime $\mathfrak{q}/\mathfrak{b}$ with $\mathfrak{q}$ minimal over $\mathfrak{b}$. By induction $\operatorname{ht} \mathfrak{q} = r - 1$, so $\operatorname{ht} \mathfrak{p} \geqslant r$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Corollary 8.61. Let A be a finitely generated domain over a field k and let $f_1, \dots, f_r$ be a regular sequence in A. Then every irreducible component of $V(f_1, \dots, f_r)$ has dimension $\dim A - r$; in particular

$$\dim A/(f_1, \dots, f_r) = \dim A - r.$$

Proof Combine Proposition 8.60 with the dimension formula of Proposition 8.50. □

Two converses fail. First, r elements can generate an ideal of height strictly less than r: the ideal $\mathfrak{a} = (y^2 - xz,\ xw - yz,\ z^2 - yw)$ of Example 8.55, the cone over the twisted cubic curve, has height 2 but needs three generators, so by Proposition 8.60 no reordering of them is a regular sequence. Second, an ideal of height r generated by r elements need not be generated by a regular sequence. For instance, in $A = k[x, y]/(x^2, xy)$ the ideal (y) has height 1, but y is a zerodivisor, since $xy = 0$ while $x \neq 0$; the culprit is the embedded point of $\operatorname{Spec} A$ at the origin. If we assume that A is *Cohen–Macaulay*, this second converse does hold, and the condition $\operatorname{ht}(x_1, \dots, x_r) = r$ is equivalent to being a regular sequence; see Section **??**.

Exercise 8.7.1 (Regular sequences and ordering)**.** Let $A = k[x, y, z]$. Show that x, $y(1-x)$, $z(1-x)$ form a regular sequence, but that $y(1-x)$, $z(1-x)$, x do not. Describe what happens in terms of planes in $\mathbb{A}^3_k$. What happens if we consider these elements in the localization at (x, y, z)?

Exercise 8.7.2 (Permutability)**.** Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. Show that if $x_1, \dots, x_r \in \mathfrak{m}$ is an M-regular sequence, then so is $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ for every permutation σ of $\{1, \dots, r\}$. HINT: Reduce to the case of two elements $x, y \in \mathfrak{m}$.

8.8 Products of finite type schemes

Theorem 8.62. Let X and Y be schemes of finite type over an algebraically closed field k. Then

(i) $X \times_k Y$ is of finite type over k.
(ii) The closed points of $X \times_k Y$ are exactly the k-points and $(X \times_k Y)(k) = X(k) \times Y(k)$.
(iii) if X and Y are reduced, then so is $X \times_k Y$.
(iv) if X and Y are integral, then so is $X \times_k Y$ and

$$\dim(X \times_k Y) = \dim X + \dim Y. \tag{8.12}$$

Proof All statements can be reduced to the affine case. Write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, with A, B finitely generated k-algebras. Then $X \times_k Y = \operatorname{Spec}(A \otimes_k B)$.

(i): The tensor product $A \otimes_k B$ is again a finitely generated k-algebra (Example 7.15), hence $X \times_k Y$ is of finite type.

(ii): Since k is algebraically closed, and $X \times Y$ is of finite type over k, the closed points

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

of $X \times_k Y$ are precisely the k-points (by the Nullstellensatz). The formula $(X \times_k Y)(k) = X(k) \times Y(k)$ follows by the universal property of the fiber product.

(iii): We need to show that $A \otimes_k B$ is reduced. Let $f = \sum_i a_i \otimes b_i \in A \otimes_k B$ be nilpotent. Replacing the a_i if necessary, we may assume that the a_i are k-linearly independent. For each maximal ideal $\mathfrak{m} \subset B$, consider the k-algebra map

$$\phi\colon A \otimes_k B \longrightarrow A \otimes_k B/\mathfrak{m} \simeq A \otimes_k k = A$$

induced by $B \to B/\mathfrak{m} = k$. Since A is reduced, $\phi(f) = \sum_i a_i \bar{b_i} = 0$ in A. This forces $\bar{b_i} = 0$, hence $b_i \in \mathfrak{m}$, because the a_i are k-linearly independent. Since this happens for every $\mathfrak{m}$, we see that $b_i \in \bigcap_{\mathfrak{m}} \mathfrak{m}$. However, as B is reduced, the Nullstellensatz (Exercise 1.6.10) shows that $\bigcap_{\mathfrak{m}} \mathfrak{m} = \sqrt{(0)} = (0)$, so $f = 0$. Hence $A \otimes_k B$ is reduced.

(iv): Suppose A and B are integral domains. We need to show that $A \otimes_k B$ is an integral domain. Suppose that $f = \sum a_i \otimes b_i$ and $g = \sum c_i \otimes d_i$ are two elements such that $fg = 0$. As in (iii), we may assume that the a_i's and the c_i's are linearly independent over k. Then $\phi(fg) = \phi(f)\phi(g) = 0$ in A. Since A is a domain, either $\phi(f) = 0$ or $\phi(g) = 0$, so we see that either $b_i \in \mathfrak{m}$ for all i or $d_j \in \mathfrak{m}$ for all j. In either case, if $\mathfrak{b}$ and $\mathfrak{d}$ denote the ideal generated by the b_i and d_j respectively, then $\mathfrak{b} \cap \mathfrak{d} \subset \mathfrak{m}$. Since this holds for every $\mathfrak{m}$, we obtain $\mathfrak{b} \cap \mathfrak{d} = (0)$ and hence $V(\mathfrak{b}) \cup V(\mathfrak{d}) = V(0) = \operatorname{Spec} B$. As B is an integral domain, this implies that either $\mathfrak{b} = (0)$ or $\mathfrak{d} = (0)$, which means that either $f = 0$ or $g = 0$.

For the dimension formula (8.12), we may by Noether's Normalization Lemma choose polynomial subrings such that $k[x_1, \ldots, x_m] \subset A$ and $k[y_1, \ldots, y_n] \subset B$ are both finite. Then $A \otimes_k B$ is finite over

$$k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] = k[x_1, \ldots, x_m, y_1, \ldots, y_n],$$

and Theorem C.21 gives $\dim(A \otimes_k B) = m + n = \dim X + \dim Y$, as required. □

8.9 Dimensions of intersections

Krull's Principal Ideal Theorem leads to the following important bounds on the dimension of intersections in affine space $\mathbb{A}^n_k$ and projective space $\mathbb{P}^n_k$.

Proposition 8.63. Let X and Y be affine varieties in $\mathbb{A}^n_k$, where k is algebraically closed. Then each component Z of $X \cap Y$ satisfies

$$\dim Z \geqslant \dim X + \dim Y - n.$$

Proof The main observation is that $X \times_k Y$ is an integral closed subscheme of $\mathbb{A}^n_k \times_k \mathbb{A}^n_k \simeq \mathbb{A}^{2n}_k$ of dimension $\dim X + \dim Y$, by Theorem 8.62. Moreover, $X \cap Y$ is isomorphic to $\Delta \cap (X \times_k Y)$, where $\Delta = V(x_1 - y_1, \ldots, x_n - y_n)$ is the diagonal in $\mathbb{A}^{2n}_k$. Thus $X \cap Y$ is cut out in $X \times_k Y$ by n equations. The generalized form of Krull's Principal Ideal Theorem (Corollary 8.54) and the codimension formula therefore show that, for each component Z of $X \cap Y$,

$$\dim Z \geqslant \dim X + \dim Y - n. \qquad \square$$

Example 8.64. The inequality in Proposition 8.63 does not hold for subvarieties of general

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

varieties. For instance, consider the 3-dimensional quadric $Q = V(xw - yz) \subset \mathbb{A}^4_k$. This quadric contains two planes $X = V(x, y)$ and $Y = V(z, w)$ which intersect only at the origin, so the left-hand side is 0 whereas the right-hand side is $2 + 2 - 3 = 1$. △

In the projective case, we get a cleaner statement.

Proposition 8.65. Let X and Y be projective varieties in $\mathbb{P}^n_k$, where k is algebraically closed. If $\dim X + \dim Y \geqslant n$, then $X \cap Y$ is non-empty and each component Z of $X \cap Y$ satisfies

$$\dim Z \geqslant \dim X + \dim Y - n.$$

Proof If $\dim X + \dim Y \geqslant n$, then the affine cones $C(X), C(Y) \subset \mathbb{A}^{n+1}_k$ satisfy $\dim C(X) + \dim C(Y) \geqslant n + 2$. Any component W of $C(X) \cap C(Y)$ satisfies $\dim W \geqslant \dim X + \dim Y - n + 1 \geqslant 1$, so $C(X) \cap C(Y)$ contains a point other than the origin and hence $X \cap Y$ is nonempty. Moreover, if Z is a component of $X \cap Y$, then $C(Z)$ is a component of $C(X) \cap C(Y)$ and $\dim C(Z) = \dim Z + 1$. The affine case therefore gives $\dim Z \geqslant \dim X + \dim Y - n$. □

Example 8.66. Any two surfaces in $\mathbb{P}^4_k$ intersect. △

8.10 Dimensions of the fibers of a morphism

Given a morphism $f\colon X \to Y$, it is useful to know how the dimensions of the fibers $f^{-1}(y)$ vary with the point y. If f is dominant, one expects the dimension of a fiber to be related to the *relative dimension* $\dim X - \dim Y$. The next proposition says that the dimension of $f^{-1}(y)$ is at least this number.

Proposition 8.67. Let $f\colon X \to Y$ be a dominant morphism between integral schemes of finite type over a field k. Then for every closed point $y \in Y$ in the image of f, and for every irreducible component Z of the fiber $f^{-1}(y)$, one has

$$\dim Z \geqslant \dim X - \dim Y. \tag{8.13}$$

Proof Choose an affine open neighbourhood $U \subset Y$ of y and an affine open neighbourhood $V \subset X$ of the generic point of Z such that $f(V) \subset U$. The open subset $Z \cap V$ has the same dimension as Z and is an irreducible component of the fiber of $f|_V$ over y. Replacing X by V, Y by U, and Z by $Z \cap V$ therefore does not change their dimensions. Thus we reduce to considering $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, with A and B integral finitely generated k-algebras.

We first consider the case $Y = \mathbb{A}^n_k = \operatorname{Spec} k[t_1, \dots, t_n]$. Let $\mathfrak{m} \subset k[t_1, \dots, t_n]$ be the maximal ideal corresponding to y. The scheme-theoretic fiber is $X_y = \operatorname{Spec}(B/\mathfrak{m}B)$. If Z is an irreducible component of X_y, then $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ minimal over $\mathfrak{m}B$. The regular local ring $k[t_1, \dots, t_n]_{\mathfrak{m}}$ has dimension n, so its maximal ideal is generated by n elements. Consequently, $\mathfrak{m}B_{\mathfrak{p}}$ is generated by n elements, and $\mathfrak{p}B_{\mathfrak{p}}$ is minimal over this ideal. By the generalized form of Krull's Principal Ideal Theorem (Corollary 8.54), $\operatorname{ht}(\mathfrak{p}) \leqslant n$, or equivalently, $\operatorname{codim}(Z, X) = \dim B_{\mathfrak{p}} \leqslant n$. By the codimension formula (Proposition 8.50),

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

we obtain

$$\dim Z \geqslant \dim X - n = \dim X - \dim \mathbb{A}^n_k.$$

For the general case, we use Noether's Normalization Lemma to find a finite dominant morphism $p\colon Y \to \mathbb{A}^n_k$, where $n = \dim Y$. Consider the composition $h = p \circ f\colon X \to \mathbb{A}^n_k$, and set $z = p(y)$. Since p is finite, the fiber $p^{-1}(z)$ is a finite discrete set of closed points. Therefore $h^{-1}(z) = f^{-1}\big(p^{-1}(z)\big)$ is a finite disjoint union of the fibers $f^{-1}(y')$ with $y' \in p^{-1}(z)$. In particular, the irreducible component $Z \subset f^{-1}(y)$ is also an irreducible component of $h^{-1}(z)$. Applying the special case already proved to the morphism $h\colon X \to \mathbb{A}^n_k$, we obtain the desired inequality (8.13). □

8.11 Exercises

Exercise 8.11.1. Let k be a field and let X, Y be schemes of finite type over k. For two points $x \in X$ and $y \in Y$, suppose we are given an isomorphism of k-algebras $h\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Show that there exist open sets $U \subset X$ and $V \subset Y$ with $x \in U$ and $y \in V$ and an isomorphism $f\colon U \to V$ such that $f(x) = y$ and $f^\sharp_x = h$.

Exercise 8.11.2. Let X be a Noetherian scheme and let $f \in \mathcal{O}_X(X)$. Assume that f is a nonzerodivisor in every local ring $\mathcal{O}_{X,x}$. Show that every irreducible component of the zero set of f has codimension 1 in X.

Exercise 8.11.3 (The Jouanolou trick)**.** Let k be an algebraically closed field and let $n \geqslant 1$. Consider the closed subset $\Gamma \subset \mathbb{P}^n \times_k \mathbb{P}^n$ given by the equation $x_0y_0 + \cdots + x_ny_n = 0$.

a) Show that Γ is irreducible.
b) Show that $\mathbb{P}^n \times \mathbb{P}^n - \Gamma$ is affine.
c) Show that the fibers of the first projection $\pi\colon \mathbb{P}^n_k \times \mathbb{P}^n_k - \Gamma \to \mathbb{P}^n_k$ are affine spaces $\mathbb{A}^n_k$.
d) Show that π is Zariski locally trivial, i.e., there is an affine covering U_i of $\mathbb{P}^n_k$ such that $\pi^{-1}(U_i) \simeq U_i \times \mathbb{A}^n_k$.
e) Deduce that for any projective variety Y, there is an affine variety X and a morphism $X \to Y$ which is a Zariski locally trivial $\mathbb{A}^n_k$-bundle.

Exercise 8.11.4 (The Jouanolou trick II)**.** Let k be an algebraically closed field, and let $V \subset \mathbb{A}^{(n+1)^2}$ be the closed subscheme defined by $\operatorname{rank}(A) \leqslant 1$ and $\operatorname{Tr}(A) = 1$.

a) Show that V is an affine variety and that there is a morphism $\pi\colon V \to \mathbb{P}^n_k$ given on k-points by $A \mapsto \operatorname{Im}(A)$.
b) Show that every fiber of π over a k-point is isomorphic to $\mathbb{A}^n_k$.
c) Show that π is Zariski locally trivial. That is, there is an affine open covering U_i of $\mathbb{P}^n_k$ such that $\pi^{-1}(U_i) \simeq U_i \times \mathbb{A}^n_k$ over U_i.
d) Deduce that for every projective variety Y, there is an affine variety X and a Zariski locally trivial affine-space bundle $X \to Y$.

Exercise 8.11.5 (Graph of a morphism II)**.** Let X and Y be varieties over a field k. Let $f\colon X \dashrightarrow Y$ be a rational map, defined over a maximal open set $U \subset X$. Define Γ_f to be the closure of the graph $\Gamma_f \subset U \times_k Y$ in $X \times_k Y$, and give it the reduced scheme structure.

a) Show that Γ_f is a variety.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

b) Show that the first projection $\Gamma_f \to X$ is a birational map.
c) Show that f extends to a morphism $\overline{f}\colon X \to Y$ if and only if $\Gamma_f \to X$ is an isomorphism.

Exercise 8.11.6. Let X be a variety over k. Show that X is geometrically irreducible if and only if for each finite separable extension $k \to L$, X_L is irreducible.

Exercise 8.11.7 (Generic structure theorem for morphisms of finite type)**.**

a) Let $A \subset B$ be integral domains, and assume that B is a finitely generated A-algebra. Show that there exists an element $a \in A$ and elements $x_1, \dots, x_n \in B$, algebraically independent over $K(A)$, such that B_a is finite as a module over $A_a[x_1, \dots, x_n]$. HINT: Write $B = A[y_1, \dots, y_r]$. Find elements $x_1, \dots, x_n \in B' = K(A)[y_1, \dots, y_r]$ such that B' is finite over $K(A)[x_1, \dots, x_n]$. Then let a be the product of all denominators appearing in integral relations for the y_i over $K(A)[x_1, \dots, x_n]$.
b) Let X and Y be integral schemes, and let $f\colon X \to Y$ be a dominant morphism of finite type. Show that there exist open affine subsets $U \subset Y$ and $V \subset X$ such that $f(V) \subset U$ and such that $f|_V$ factors as

$$V \xrightarrow{\ q\ } U \times \mathbb{A}^n \xrightarrow{\ p\ } U$$

where q is finite, p is the first projection, and $n = \operatorname{trdeg}_{K(Y)} K(X)$.

If X and Y are affine, show moreover that V may be chosen to be the inverse image of a distinguished open subset of Y.

Exercise 8.11.8 (Dimension of generic fibers)**.** Let X and Y be varieties over k and let $f\colon X \to Y$ be a dominant morphism. Show that there is an open dense subset $U \subset Y$ so that for all closed points $y \in Y$ and all irreducible components Z of X_y, we have $\dim Z = \dim X - \dim Y$. HINT: Use Exercise 8.11.7.

Exercise 8.11.9 (General fibers of generically finite morphisms)**.** Let $f\colon X \to Y$ be a generically finite morphism of varieties over an algebraically closed field k. Suppose that $K(X)/K(Y)$ is separable, and set $d = [K(X) : K(Y)]$. Prove that there is a dense open subset $U \subset Y$ such that, for every k-point $y \in U$, the scheme-theoretic fiber X_y consists of d distinct reduced points, as follows:

a) After shrinking Y, reduce to $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$, where $A \subset B$ are integral domains and B is finite over A.
b) Put $K = K(A)$ and $L = K(B)$. Choose a primitive element $\alpha \in L$, and let $p(t) \in K[t]$ be its minimal polynomial. Show that $\deg p = d$ and $\Delta = \operatorname{disc}(p) \neq 0$. HINT: Use the Primitive Element Theorem.
c) After localizing A and B, show that we may assume $p(t) \in A[t]$ and $B = A[\alpha] \simeq A[t]/(p(t))$. HINT: Clear denominators, then localize so that $B/A[\alpha]$ vanishes.
d) Show that $U = D(\Delta) \subset \operatorname{Spec} A$ has the required property.
e) Show that the conclusion can fail if k is not algebraically closed. HINT: $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

f) Show that the conclusion can fail if $K(X)/K(Y)$ is not separable, by considering the Frobenius morphism $F\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ over $k = \overline{\mathbb{F}_p}$. Show that in this case, every fiber over a k-point is non-reduced of multiplicity p.

Exercise 8.11.10. Let X be a smooth projective variety over k. Show that the following are equivalent:

(i) X is geometrically integral.
(ii) X is geometrically connected.
(iii) The natural map $k \to \mathcal{O}_X(X)$ is an isomorphism.

Exercise 8.11.11 (Semicontinuity of the fiber dimension)**.** Let X and Y be varieties over k and let $f\colon X \to Y$ be a surjective morphism. Then for all integers r the set

$$W_r = \{\, y \in Y \mid \dim X_y \geqslant r \,\}$$

is closed in Y. HINT: Induction on $\dim Y$. If U is the open set of Exercise 8.11.7, consider components above $Y - U$.

Exercise 8.11.12. Let k be an infinite field and let X be a variety over k. Show that if X is rational, then the k-points $X(k)$ are dense in X.

Exercise 8.11.13. Let X be the conic curve $X = V_+(x_0^2 + x_1^2 - 7x_2^2)$ in $\mathbb{P}^2_{\mathbb{Q}}$.

a) Show that $X(\mathbb{Q}) = \varnothing$.
b) Show that X is not rational, but geometrically rational.

Exercise 8.11.14 (Elliptic curves are irrational)**.** Let k be a field of characteristic not equal to 2 and let $X \subset \mathbb{A}^2_k$ be the curve defined by the equation $y^2 = x(x-1)(x+1)$. In this exercise, we will show that X is not rational.

a) Write $K = k(x)$ and $L = K(y) = K(X)$. Show that for any valuation $\nu\colon L^\times \to \mathbb{Z}$, the number $\nu(x)$ is even. HINT: Consider the cases $\nu(x) < 0$ and $\nu(x) \geqslant 0$ separately.
b) Show that x is not a square in L. HINT: If $x = (a + by)^2$, then expand using the equation $y^2 = x(x-1)(x+1)$.
c) Show that $k(t)$ has the following property: if $g \in k(t)$ is an element such that $\nu(g)$ is even for every valuation $\nu\colon k(t)^\times \to \mathbb{Z}$, then g is a square.
d) Conclude that $L \not\simeq k(t)$ and that X is not rational.

Exercise 8.11.15. View $\mathbb{P}^3_{\mathbb{C}}$ with homogeneous coordinates a, b, c, d as the projective space parameterizing complex cubic polynomials $aX^3 + bX^2 + cX + d$.

a) Show that the closed set of cubics with a double root is a surface of degree 4. Show that this surface is rational.
b) Show that the locus of cubic polynomials having a triple root is a curve isomorphic to the twisted cubic curve.
c) Using *b*), show that there exist $s, t \in \mathbb{C}$ such that $X^3 + sX^2 + tX + (s^5 + t^5 + 1)$ has a triple root.

Exercise 8.11.16 (An irreducibility criterion)**.** Let k be algebraically closed. Let $\pi\colon X \to Y$ be a proper morphism of k-varieties, with Y irreducible. Assume that all fibers X_y over

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

closed points $y \in Y$ are nonempty, irreducible, and of the same dimension d. We will prove that X is irreducible of dimension $\dim Y + d$.

a) Show that at least one component X_0 of X satisfies $\pi(X_0) = Y$.
b) Let $\pi_0 = \pi|_{X_0} \colon X_0 \to Y$. Show that there is a dense open set $U \subset Y$ such that for closed points $y \in U$, the fiber $\pi_0^{-1}(y)$ is irreducible of dimension d inside X_0. Deduce that $\pi_0^{-1}(y) = \pi^{-1}(y)$ for all closed $y \in U$.
c) Show that X_0 is the only component which dominates Y. HINT: If X_1 satisfies $\pi(X_1) = Y$, define an open set U' as in *b*) and consider $\pi^{-1}(U \cap U')$.
d) Show that there is no component X_1 such that $Z = \pi(X_1) \subsetneq Y$ is a proper subset. HINT: Consider the morphism $\pi^{-1}(Z) \to Z$ and apply induction on $\dim Z$.
e) Conclude that X is irreducible of dimension $\dim Y + d$.

Exercise 8.11.17. Find a connected scheme X with a disconnected open subset $U \subset X$.

Exercise 8.11.18. Let X be the disjoint union of an infinite set of distinct closed points of the affine line $\mathbb{A}^1_{\mathbb{C}}$. Show that the natural morphism $X \to \mathbb{A}^1_{\mathbb{C}}$ induced by the inclusions of these points is dominant.

Exercise 8.11.19. Consider the morphism

$$f \colon \operatorname{Spec} k[x,y,z]/(x - yz) \longrightarrow \operatorname{Spec} k[x,y]$$

Describe the fibers and determine whether it is finite.

Exercise 8.11.20. Show that the morphism $\operatorname{Spec} \mathbb{C}[x,b,c]/(x^2 + bx + c) \to \operatorname{Spec} \mathbb{C}[b,c]$ is finite and describe its fibers.

What about the morphism $\operatorname{Spec} \mathbb{C}[x,a,b,c]/(ax^2 + bx + c) \to \operatorname{Spec} \mathbb{C}[a,b,c]$?

Exercise 8.11.21. Let $Y \subset X$ be a closed subset of an affine variety X. Suppose that every component of Y has the same dimension $\dim Y$. Set $r = \dim X - \dim Y$. Show that there exist $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ such that

(i) Y is a union of components of $V(f_1, \ldots, f_r)$;
(ii) Each component of $V(f_1, \ldots, f_r)$ has dimension $\dim X - r$.

Exercise 8.11.22. Let k be a field. Show that:

a) Any integral closed subscheme of $\mathbb{A}^n_k$ of dimension $n - 1$ is a hypersurface.
b) Any integral closed subscheme of $\mathbb{P}^n_k$ of dimension $n - 1$ is a hypersurface.

Exercise 8.11.23. Show that if X and Y are of finite type over a scheme S, then $X \times_S Y$ is also of finite type over S.

Exercise 8.11.24 (Standard form of finite morphisms)**.** Let A be a ring and let $P(t) \in A[t]$ be a monic polynomial. Show that $\operatorname{Spec}(A[t]/(P(t))) \to \operatorname{Spec}(A)$ is finite.

More generally, if $f \colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is finite, show that there exist monic polynomials $P_i(t_i) \in A[t_i]$ such that, with $C = A[t_1, \ldots, t_n]/(P_1(t_1), \ldots, P_n(t_n))$, the morphism f factors as $\operatorname{Spec}(B) \hookrightarrow \operatorname{Spec}(C) \to \operatorname{Spec}(A)$, with $\operatorname{Spec}(B) \hookrightarrow \operatorname{Spec}(C)$ a closed embedding.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 8.11.25. Let $f\colon X \to Y$ be a finite morphism. Let $Z_1 \subset Z_2$ be closed subsets of X where Z_2 is irreducible. Show that if $Z_1 \neq Z_2$, then $f(Z_1) \subsetneq f(Z_2)$.

Exercise 8.11.26.
a) Find an affine scheme with an open subset which is not quasi-compact.
b) Let U, V be quasi-compact open subsets of $\operatorname{Spec} A$ for some ring A. Show that $U \cap V$ is also quasi-compact.

Exercise 8.11.27 (A 1-dimensional non-Noetherian domain)**.** Let R be the subring of $\mathbb{C}(x, y)$ consisting of rational functions $f(x, y)$ that are defined and constant along the y-axis. The elements of R, when written in lowest terms, have a denominator not divisible by x, and $f(0, y)$, which is then meaningful, is constant.
a) Show that the ideals $\mathfrak{a}_r = (x, xy^{-1}, \dots, xy^{-r})$ with $r \in \mathbb{N}$ form an ascending chain that does not stabilize. Conclude that R is not Noetherian.
b) Show that R is local with the set $\mathfrak{m}$ of elements $f \in R$ that vanish along the y-axis as the maximal ideal.
c) Prove that there are no other primes than $\mathfrak{m}$ and (0) in R. HINT: Show first that any element in R is of the form $x^i y^j \alpha$ where $i \geqslant 0$, $j \in \mathbb{Z}$ and α is a unit in R.

Exercise 8.11.28. For a Noetherian scheme X, the ring $\mathcal{O}_X(X)$ may fail to be Noetherian in general. Consider $\mathbb{P}^3_k$ with homogeneous coordinates x_0, x_1, x_2, x_3. Let $H_0 = V_+(x_0)$ and $H_1 = V_+(x_1)$ and let $L = V_+(x_0, x_2) \subset H_0$ be a line on H_0 different from $H_0 \cap H_1$. Let $Y = H_0 \cup H_1$ and $X = Y - L$.
a) Show that X is Noetherian.
b) Show that $U = H_0 - L$ is an open set of X and that $\mathcal{O}_X(U) = k[x, y]$ where $x = \frac{x_1}{x_2}$ and $y = \frac{x_3}{x_2}$.
c) Show that $V = H_1 - H_1 \cap L$ is an open set of X and that $\mathcal{O}_X(V) = k$
d) Show that $\mathcal{O}_X(X) = k + xk[x, y] \subset k[x, y]$.
e) Show that the ring in *d*) is not Noetherian.

Exercise 8.11.29. Let $R = \prod_{i=1}^{\infty} k$ and $X = \operatorname{Spec} R$. For each i, let $e_i \in R$ be the element with 1 in the i-th slot and 0 elsewhere. Let $\mathfrak{m}_i = \operatorname{Ker}(\pi_i : R \to k) = \{(a_j) \in R : a_i = 0\}$.
a) Show that $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$.
b) Show that the localization R_{e_i} is isomorphic to k and that $D(e_i) = \{\mathfrak{m}_i\}$. Conclude that the set $M = \{\mathfrak{m}_1, \mathfrak{m}_2, \dots\}$ is an infinite subset of X consisting of isolated points. Explain why X cannot equal M.
c) Show that $\bigcap_{i \geqslant 1} \mathfrak{m}_i = (0)$ and that M is dense in X. Show that X is not discrete.
d) Show that a prime $\mathfrak{p}$ either equals some $\mathfrak{m}_i$ or contains every e_i. In particular,

$$Y = V\big((e_1, e_2, \dots)\big) = \{\mathfrak{p} \in X : e_i \in \mathfrak{p} \text{ for all } i\}$$

is a nonempty closed subset consisting precisely of the 'non-coordinate' points.
e) For $f = (f_i) \in R$, prove that $D(f) \cap M = \{\mathfrak{m}_i : f_i \neq 0\}$. Conclude that every open neighborhood of a point of Y contains infinitely many of the $\mathfrak{m}_i$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 8.11.30. Let $f\colon X \to S$ be a morphism of finite type and let Y be a closed subscheme of X. Show that the induced morphism $Y \to S$ is of finite type.

Exercise 8.11.31 (The ring of eventually constant sequences)**.** Consider the subring A of $\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ consisting of sequences $(e_i)_{i\geqslant 1}$ which are eventually constant, that is, sequences with $e_i = e_{i+1}$ for $i \gg 0$.

a) Show that all elements of A are idempotents and conclude that every prime ideal is maximal. HINT: the only idempotents in a domain are 0 and 1.
b) Let $\mathfrak{m}_n$ denote the ideal generated by $1 - a_n$ where $a_n = (0, \dots, 0, 1, 0, \dots)$ with a '1' in the n-th factor. Show that $\mathfrak{m}_n$ is a maximal ideal.
c) Show that $D(a_n) = \{\mathfrak{m}_n\}$ and conclude that the one-point set $\{\mathfrak{m}_n\}$ is both open and closed in $\operatorname{Spec} A$.
d) Let $\mathfrak{m}_\infty$ denote the ideal consisting of sequences which are eventually zero, i.e., $e_i = 0$ for all $i \gg 0$. Show that $\mathfrak{m}_\infty$ is a maximal ideal. HINT: Consider the 'limit map' $A \to \mathbb{Z}/2$.
e) Show that A is not Noetherian. HINT: Show that $\mathfrak{m}_\infty$ is not finitely generated.
f) Show that these are all the prime ideals of A, i.e., that $\operatorname{Spec} A = \{\, \mathfrak{m}_i \mid i \in \mathbb{N} \,\} \cup \{\mathfrak{m}_\infty\}$. HINT: Consider the cases $a_i \notin \mathfrak{m}$ for some i and $a_i \in \mathfrak{m}$ for all i separately. Use the identity $a_i(1 - a_i) = 0$.
g) Show that $\operatorname{Spec} A$ is homeomorphic to the set $\{\, \frac{1}{n} \mid n \in \mathbb{N} \,\} \cup \{0\}$ (with the standard topology).

Exercise 8.11.32. Let $R = \prod_{i\in\mathbb{N}} \mathbb{F}_2$ and let $A = R/I$ where I is the ideal $I = \bigoplus_{i\in\mathbb{N}} \mathbb{F}_2 \subset R$. Show that A is reduced and $\dim A = 0$, but it is not a field. Show that A has the following curious property: for every nonzero $f \in A$, there is an isomorphism of rings $A_f \simeq A$. HINT: f is represented by a sequence $(e_i)_{i\in\mathbb{N}}$. The set $\{\, i \in \mathbb{N} \mid e_i \neq 0 \,\}$ is infinite, hence in bijection with $\mathbb{N}$.

Exercise 8.11.33. For each of the following rings A, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type.

$$\mathbb{Z}[i],\ \mathbb{Z}[\tfrac{1}{p}],\ \mathbb{Z}_{(p)},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}[x].$$

Exercise 8.11.34.
a) Show that there exist surjective morphisms $X \to Y$ with Y integral and such that all fibers X_y are irreducible, without X being irreducible.
b) Let k be an algebraically closed field. Show that there exist morphisms $X \to \mathbb{A}^1_k$ with X integral, the generic fiber X_η non-empty and integral, but no closed fiber integral.
c) Show that there exists a morphism $X \to \mathbb{A}^1_{\mathbb{Q}}$ with X integral and with infinitely many irreducible and infinitely many reducible closed fibers. What happens for the geometric closed fibers?
d) Show that there exists a morphism $f\colon X \to Y$ with X and Y integral whose geometric generic fiber is not reduced.

Exercise 8.11.35. Find an example of a morphism of schemes which is surjective, of finite type, and has finite fibers, but not finite.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 8.11.36. Show that the morphism $f\colon \mathbb{A}^1_k \to \mathbb{P}^1_k$ defined by $u \mapsto (u : 1 - u^2)$ is surjective and has finite fibers, but not finite.

Exercise 8.11.37 (Nagata's example)**.** Let $R = k[x_1, x_2, \dots]$ be the polynomial ring in countably many variables. Choose a partition $\mathbb{N} = \coprod_{i \geqslant 1} J_i$ into finite nonempty sets such that $|J_i| \to \infty$ as $i \to \infty$. For example, one may take J_i to be the set of positive integers whose binary expansion has length i.

For each i, let $\mathfrak{n}_i = (x_j \mid j \in J_i) \subset R$, and let $S = \bigcap_i (R - \mathfrak{n}_i)$, the set of elements of R that do not lie in any $\mathfrak{n}_i$. Set $A = S^{-1}R$, and write $\mathfrak{m}_i = \mathfrak{n}_i A$. Also let $K_i = k(x_j \mid j \notin J_i)$, let $P_i = K_i[x_j \mid j \in J_i]$, and let $\mathfrak{a}_i = (x_j \mid j \in J_i) \subset P_i$.

The goal is to show that A is Noetherian but has infinite Krull dimension.

a) Show that $R_{\mathfrak{n}_i} \simeq (P_i)_{\mathfrak{a}_i}$. HINT: In $R_{\mathfrak{n}_i}$, every nonzero polynomial in the variables x_j with $j \notin J_i$ becomes invertible.
b) Show that $A_{\mathfrak{m}_i} \simeq R_{\mathfrak{n}_i}$. Deduce that each local ring $A_{\mathfrak{m}_i}$ is Noetherian and has dimension $|J_i|$. Conclude that $\dim A = \infty$.
c) Show that every element of A lies in only finitely many of the ideals $\mathfrak{m}_i$.
d) Deduce that every nonzero ideal of A is contained in only finitely many of the $\mathfrak{m}_i$.
e) Prove that A is Noetherian. HINT: For a nonzero ideal $\mathfrak{a} \subset A$, $\mathfrak{a}$ is contained in finitely many maximal ideals $\mathfrak{m}_{i_1}, \dots, \mathfrak{m}_{i_r}$. Apply Lemma 8.7 to show that $\mathfrak{a}$ is finitely generated.

Exercise 8.11.38. Let X be a quasi-compact scheme. Show that for any $x \in X$, there exists a closed point in $\overline{\{x\}} \subset X$. HINT: Every ring has a maximal ideal.

Exercise 8.11.39. Show that the formula $\mathcal{O}_{X_{\mathrm{red}}}(U) = \mathcal{O}_X(U)_{\mathrm{red}}$ does not hold in general for non-affine open sets in X. HINT: Consider $\coprod_{n \geqslant 1} (\operatorname{Spec} \mathbb{Q}[x]/x^n)$.

Exercise 8.11.40 (Criterion for affineness)**.** Show that a scheme is affine if and only if there is a finite set of elements $f_1, \dots, f_n \in \mathcal{O}_X(X)$ such that each of the open sets $X_{f_i} = \{x \in X | f_i(x) \neq 0\}$ are affine, and $f_1, \dots, f_n$ generate the unit ideal. HINT: Show that $X = \bigcup X_{f_i}$. Glue the morphisms $X_{f_i} \to \operatorname{Spec} \mathcal{O}_X(X)$ to a morphism $X \to \operatorname{Spec} \mathcal{O}_X(X)$.

Exercise 8.11.41. Let X be a scheme. Show that X_{red} is affine if and only if X is affine. HINT: Apply the criterion for affineness above.

Exercise 8.11.42. Let X be a scheme of finite type over an algebraically closed field k of characteristic $p > 0$. Show that the Frobenius morphism $F\colon X \to X$ is finite.

Exercise 8.11.43. Let X be a scheme and let $x \in X$ be a point. Show that x is a closed point if and only if the corresponding morphism $\operatorname{Spec} \kappa(x) \to X$ is finite.

Exercise 8.11.44. Compute the dimension of $X = V(\mathfrak{a}) \subset \mathbb{A}^3_k$ where where $\mathfrak{a}$ is the ideal $(x^2, xz, yz, z^2 - z)$ in $k[x, y, z]$. Describe X geometrically.

Exercise 8.11.45. Let $A = \mathbb{C}[[t]]$ be the ring of formal power series over $\mathbb{C}$, with fraction

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

field $K = \mathbb{C}((t))$. By the Newton–Puiseux theorem, an algebraic closure $\overline{K}$ of K is

$$\overline{K} = \bigcup_{n \geqslant 1} \mathbb{C}((t^{1/n})).$$

a) Show that the integral closure of A in $\overline{K}$ is given by $\overline{A} = \bigcup_{n \geqslant 1} \mathbb{C}[[t^{1/n}]]$..
b) Show that $\overline{A}$ is not Noetherian.
c) Show that $\operatorname{Spec} \overline{A}$ consists of only two points. HINT: Show that $\operatorname{Spec} \overline{A}$ is obtained as a limit of a system

$$\cdots \to \operatorname{Spec} \mathbb{C}[[t^{1/4}]] \to \operatorname{Spec} \mathbb{C}[[t^{1/2}]] \to \operatorname{Spec} \mathbb{C}[[t]]$$

where each map is a homeomorphism.

Exercise 8.11.46 (Properties of Noetherian spaces)**.**

a) Show that any Noetherian topological space T is quasi-compact. HINT: If $\{U_i\}_{i \in I}$ is a cover, start with any U_{i_1}, and pick U_{i_2} so that $U_{i_1} \subsetneq U_{i_1} \cup U_{i_2}$, and so on.
b) Show that if T is Noetherian, then every subset $S \subset T$ is also Noetherian. HINT: The closed subsets of S are of the form $W \cap S$ where $W \subset T$ is closed.

Exercise 8.11.47. Let X be a scheme and let $\iota \colon U \to X$ be an open embedding. Show that if U is quasi-compact, then ι is of finite type. Deduce that if X is Noetherian, then any open embedding into X is of finite type.

Exercise 8.11.48. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Show that the following are equivalent:

(i) $\dim R = 0$ (i.e. the only prime ideal of R is the maximal ideal $\mathfrak{m}$).
(ii) $\mathfrak{m} = \sqrt{0}$ (the maximal ideal is the nilradical).
(iii) $\mathfrak{m}$ is nilpotent.
(iv) R is Artinian, i.e., every descending chain of ideals stabilizes.

Deduce more generally that a Noetherian ring has dimension 0 if and only if it is Artinian.

Exercise 8.11.49. Consider the morphism $f \colon \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{Q}}$ defined by the inclusion $\mathbb{Q}[x] \to \mathbb{R}[x]$. Compute the images of the points $(x - \sqrt{2})$ and $(x - \pi)$. Is the morphism dominant, surjective, finite?

Exercise 8.11.50. Let X be a scheme of finite type over a field k. For a closed point $P \in X$, we define the *degree* of P as the degree of the field extension $[\kappa(P) : k]$.

a) Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$. Show that if P is the closed point defined by an irreducible polynomial $f(x) \in k[x]$, then $\deg(P) = \deg(f)$.
b) Let $f \colon X \to Y$ be a morphism of schemes of finite type over k, and let $P \in X$ be a closed point with image $Q = f(P)$. Show that Q is closed and that there is a natural inclusion of fields $\kappa(Q) \hookrightarrow \kappa(P)$, and deduce the degree relation

$$\deg(P) \;=\; [\kappa(P) : \kappa(Q)] \cdot \deg(Q).$$

c) Let K/k be a field extension, and set $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. If $P \in X$ is a closed point such that $\kappa(P)/k$ is separable, describe the set of closed points of X_K lying above P, and show that their degrees over K sum to $\deg(P)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 8.11.51. Consider $\mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ and $X = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal $(x^2, xz, yz, z^2 - z)$. Find a decomposition of X into irreducible subsets and compute $\dim X$.

Exercise 8.11.52. Let X be a scheme. Show that $\dim X$ is equal to the supremum of the dimensions of the open affine subschemes of X.

Exercise 8.11.53. Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ and let $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_s$ be a chain of prime ideals in R. Show that $s \leqslant n+1$ as follows:

a) If $\mathfrak{p}_s \cap \mathbb{Z} = (0)$, show that $s \leqslant n$ by considering the chain in $\mathbb{Q}[x_1, \ldots, x_n]$.
b) If $\mathfrak{p}_s \cap \mathbb{Z} = (p)$, i.e., $p \in \mathfrak{p}_s$ for some prime number p, show that the chain can be modified to produce a chain $(0) = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_s$ with $p \in \mathfrak{q}_1$. Conclude that $\dim R \leqslant n+1$ by considering the corresponding chain in $(\mathbb{Z}/p)[x_1, \ldots, x_n]$.

Exercise 8.11.54. We say that a morphism $f\colon X \to Y$ is *integral* if it is affine and there is an affine open cover $\{V_i\}$ of Y such that $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(f^{-1}V_i)$ is an integral ring extension for every i. Show that if $f\colon X \to Y$ is integral, then $\dim X \leqslant \dim Y$. If f is moreover surjective, then $\dim X = \dim Y$.

Exercise 8.11.55 (Constructible sets)**.** A subset E of a topological space X is *locally closed* if it is the intersection of an open and a closed set. When X is Noetherian, a *constructible set* is defined to be a finite union of locally closed sets.

a) Show that finite unions and finite intersections of constructible sets are constructible.
b) Show that a subset which is constructible in a closed subspace, is constructible in the surrounding space.
c) Consider the morphism $f\colon \mathbb{A}^2_k \to \mathbb{A}^2_k$ defined by $(x,y) \mapsto (xy, y)$. Compute the image of f and show that it is neither open nor closed in $\mathbb{A}^2_k$. However, show that it is constructible.

Exercise 8.11.56. Let X be a scheme and $x \in X$ a point. One says that a point $y \in X$ is a *specialization* of x if $y \in \bar{x}$, and that y is a generalization of x if $x \in \bar{y}$.

One says that a subset $E \subset X$ is *closed under specialization* if specializations of points in E belong to E. Likewise, E is said to be *closed under generalization* if generalizations of points in E belong to E.

a) Show that E is closed under specializations if and only if the complement $X - E$ is closed under generalizations.
b) Show that E is closed under specialization if and only if $\overline{\{x\}} \subset E$ for every $x \in E$.
c) Show that closed sets are closed under specialization and that open sets are closed under generalization.
d) Let X be an irreducible Noetherian space and let $E \subset X$ be a dense constructible subset. Show that E contains a nonempty open subset of X.
e) Show that in a Noetherian scheme, a constructible subset E is closed if it is closed under specialization, and that it is open if it is closed under generalization.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

f) Give an example showing that the Noetherian hypothesis is necessary. HINT: Consider the spectrum in Exercise 8.11.31.

Exercise 8.11.57 (Chevalley's theorem)**.** The aim of this exercise is to prove *Chevalley's theorem*: if X and Y are Noetherian schemes and $f\colon X \to Y$ is a morphism of finite type, then the image $f(X)$ is constructible.

a) Consider the collection of subsets of Y given by

$$\Sigma = \{Z \subset Y \mid Z \text{ is closed and } f(f^{-1}(Z)) \text{ is not constructible}\}.$$

Show that if $f(X)$ is not constructible, then Σ contains a minimal element.

b) Reduce to proving the theorem for the case f is dominant and Y is reduced.

c) Show that under the assumptions of *b*), there is an open non-empty subset $U \subset f(X)$.

d) By writing $f(X) = U \cup \bigcup_{i=1}^{r} f(f^{-1}(Z_i))$, where the Z_i are the irreducible components of $Y - U$, conclude that $f(X)$ is constructible.

e) Generalize the statement to the following: Let X and Y be Noetherian schemes and let $f\colon X \to Y$ be a morphism of finite type. For each constructible subset $E \subset X$ the image $f(E)$ is constructible.

Exercise 8.11.58 (The Nullstellensatz again)**.** Let k be a field and let $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ be a maximal ideal. Write $L = k[x_1, \ldots, x_n]/\mathfrak{m}$. The aim of this exercise is to give a new proof of Lemma 1.7, saying that L is a finite extension of k.

a) Use Chevalley's theorem to show that the ideal $\mathfrak{p} = \mathfrak{m} \cap k[x_n]$ is nonzero in $k[x_n]$. HINT: If $\mathfrak{p} = (0)$, then the extension $k[x_n] \to k[x_1, \ldots, x_n]/\mathfrak{m}$ induces a morphism to $\mathbb{A}^1_k$.

b) Deduce that $\mathfrak{p}$ is a maximal ideal and that $k' = k[x_n]/\mathfrak{p}$ is a finite extension of k which is contained in L.

c) Construct a surjection $k'[x_1, \ldots, x_{n-1}] \to L$ and conclude by induction that L is a finite extension of k.

The next few exercises are about *flat morphisms*. Recall that an A-module M is *flat* if the functor $(-) \otimes_A M$ is exact. We say that a morphism $f\colon X \to S$ is *flat at* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{S,f(x)}$-module. A morphism f is called *flat* if it is flat at every point of X.

Exercise 8.11.59. Let A be a ring.

a) Show that every free A-module is flat.

b) Show that $A[x_1, \ldots, x_n]$ is flat over A for every $n \geqslant 0$. Deduce that $\mathbb{A}^n_S \to S$ is flat for every scheme S.

c) Show that any localization $S^{-1}A$ is flat over A. Deduce that open embeddings are flat.

d) Show that $M = k[x]/(x)$ is not flat as a $k[x]$-module. Deduce that closed embeddings are in general not flat.

e) Let $f(T) \in A[T]$ be monic of degree n. Show that $A[T]/(f)$ is a free A-module of rank n, and conclude that $\operatorname{Spec}(A[T]/(f)) \to \operatorname{Spec} A$ is finite and flat.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 8.11.60. Let $f\colon X \to S$ be a morphism.

a) Show that if $U = \operatorname{Spec} B \subset X$ and $V = \operatorname{Spec} A \subset S$ with $f(U) \subset V$, then $f|_U\colon U \to V$ is flat if and only if B is flat as an A-module.
b) Show that flatness is local on the source and on the target.
c) Show that compositions of flat morphisms are flat.
d) Show that flat morphisms are stable under base change.
e) Show that flat morphisms are stable under products.

Exercise 8.11.61. Let A be a DVR with local parameter t and let M be an A-module.

a) Show that if M is flat over A, then multiplication by t on M is injective.
b) Prove that the following are equivalent:
   (i) M is flat over A;
   (ii) M is torsion-free;
c) Using the structure theorem for finitely generated modules over a PID, show that if M is finitely generated, then M is flat if and only if it is free.
d) Let B be an A-algebra. Show that $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if t is not a zero-divisor in B. HINT: Every module is the filtered union of its finitely generated submodules.
e) Decide whether $\operatorname{Spec}(k[x,t]/(x^2-t)) \to \operatorname{Spec} k[t]$ and $\operatorname{Spec}(k[x,t]/(tx)) \to \operatorname{Spec} k[t]$ are flat.

Exercise 8.11.62 (Flat morphisms and fiber dimensions)**.** Let k be a field. In this exercise, we will show that if $f\colon X \to S$ is a dominant flat morphism of integral k-schemes of finite type, then every irreducible component of every closed fiber has dimension $\dim X - \dim S$.

a) Let $s \in S$ be a closed point, and let Z be an irreducible component of the fiber X_s with generic point ξ. Using the proposition from the main text on dimensions of fibers of dominant morphisms, show that $\dim Z \geqslant \dim X - \dim S$.
b) Show that flat ring maps $A \to B$ satisfy the Going-Down property: if $\mathfrak{p}_0 \subset \mathfrak{p}_1$ are prime ideals of A, and $\mathfrak{q}_1$ is a prime ideal of B lying over $\mathfrak{p}_1$, then there exists $\mathfrak{q}_0 \subset \mathfrak{q}_1$ lying over $\mathfrak{p}_0$.
c) Applying *b*) to the flat ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,\xi}$, show that $\dim \mathcal{O}_{X,\xi} \geqslant \dim S$, and deduce that $\dim Z \leqslant \dim X - \dim S$. Conclude that every irreducible component of every closed fiber of f has dimension exactly $\dim X - \dim S$. In particular, the dimensions of the closed fibers are constant.
d) Show that $\operatorname{Spec} k[t,x]/(x^2 - t) \to \operatorname{Spec} k[t]$ is flat and that every closed fiber has dimension 0. Show that $\operatorname{Spec} k[t,x]/(tx) \to \operatorname{Spec} k[t]$ is not flat, and describe the fibers. Show that $\operatorname{Spec} k[t,x]/(x^2,tx) \to \operatorname{Spec} k[t]$ is not flat although every closed fiber has dimension 0.
e) Show directly that $\pi\colon \mathbb{A}^2_k \to \mathbb{A}^2_k$, $(x,y) \mapsto (x,xy)$ is not flat. Explain this in terms of the fiber over the origin.

Exercise 8.11.63. Let $f\colon X \to S$ be a flat morphism.

a) Show that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of quasi-coherent sheaves on S, then $0 \to f^*\mathcal{F}' \to f^*\mathcal{F} \to f^*\mathcal{F}'' \to 0$ is exact on X.
b) Let $Z \subset S$ be a closed subscheme with ideal sheaf $\mathcal{I}$. Show that the natural

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

map $f^*\mathcal{I} \to \mathcal{O}_X$ identifies $f^*\mathcal{I}$ with the ideal sheaf of the closed subscheme $X \times_S Z \subset X$.

Exercise 8.11.64 (Flat base change)**.** Let A be a ring and let B be a flat A-algebra. Let X be a quasi-compact separated A-scheme and let X_B denote the base change to B. Show that for any quasi-coherent sheaf $\mathcal{F}$, there is a natural isomorphism

$$\Gamma(X_B, \mathcal{F}_B) \simeq \Gamma(X, \mathcal{F}) \otimes_A B \tag{8.14}$$

where $\mathcal{F}_B = q^*\mathcal{F}$ and $q\colon X_B \to X$ is the morphism obtained by base change. HINT: Tensor the sheaf exact sequence. Show that (8.14) may fail for non-flat A-algebras.

Exercise 8.11.65 (Morphisms and maps on local rings)**.**

a) Let A be a Noetherian ring and let $\varphi\colon A \to B$ be a finite type A-algebra. Let $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.
   (i) If $A_\mathfrak{p} \to B_\mathfrak{q}$ is surjective, show that there are $s \in A - \mathfrak{p}$ and $g \in B - \mathfrak{q}$ such that $A_s \to B_{sg}$ is surjective.
   (ii) If $A_\mathfrak{p} \to B_\mathfrak{q}$ is an isomorphism, show that s and g can be chosen so that $A_s \to B_{sg}$ is an isomorphism.

   Hint. Write $B = A[b_1, \dots, b_n]$. Express $b_i = a_i/s_i$ in $B_\mathfrak{q}$ with $a_i \in A$ and $s_i \in A - \mathfrak{p}$. For the second part, first make the map surjective and then use that its kernel is finitely generated and vanishes after localization at $\mathfrak{p}$.

b) Let $f\colon X \to Y$ be a morphism of finite type between Noetherian schemes. Suppose that $f_P^\sharp\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is an isomorphism for every $P \in X$. Show that f is a local isomorphism: for every $P \in X$ there are open neighborhoods $P \in U \subset X$ and $f(P) \in V \subset Y$ such that $f|_U\colon U \to V$ is an isomorphism.

c) Show that a local isomorphism which is injective on points is an open embedding. Show that this does not hold in general without the injectivity hypothesis.

d) Suppose instead that $f_P^\sharp$ is surjective for every $P \in X$. Show that f is locally a closed embedding: for every $P \in X$ there are open neighborhoods $P \in U \subset X$ and $f(P) \in V \subset Y$ such that $f|_U\colon U \to V$ is a closed embedding.

e) Suppose that X and Y are integral, and let η be the generic point of X. Show that the following conditions are equivalent:
   (i) f is dominant;
   (ii) $f_\eta^\sharp\colon \mathcal{O}_{Y,f(\eta)} \to K(X)$ is injective;
   (iii) $f_P^\sharp$ is injective for every $P \in X$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

9

More on sheaves

In this chapter we develop the theory of sheaves in more detail. We will see that most of the familiar algebraic constructions, such as forming kernels, cokernels and images, direct sums and products, have analogues for sheaves. In short, the category of sheaves on a scheme behaves very much like the category of modules over a ring.

9.1 Kernels and images

For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we define its kernel sheaf as follows:

Definition 9.1. The *kernel* $\operatorname{Ker}\phi$ of ϕ is the subsheaf of $\mathcal{F}$ defined by

$$(\operatorname{Ker}\phi)(U) = \operatorname{Ker}\phi_U$$

for each open $U \subset X$. In other words, $(\operatorname{Ker}\phi)(U)$ consists of the sections in $\mathcal{F}(U)$ that are mapped to zero by $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

The restriction maps of $\mathcal{F}$ induce restriction maps of $\operatorname{Ker}\phi$. Indeed, if $s \in \mathcal{F}(U)$ maps to zero by ϕ_U, then for any $V \subset U$,

$$\phi_V(s|_V) = \phi_U(s)|_V = 0$$

and hence $s|_V \in (\operatorname{Ker}\phi)(V)$.

The Locality axiom for $\operatorname{Ker}\phi$ follows automatically from the Locality axiom for $\mathcal{F}$. For the Gluing axiom, let $\{U_i\}_{i\in I}$ be an open cover of an open set U and let $s_i \in (\operatorname{Ker}\phi)(U_i)$ be sections that agree on the overlaps. As $\mathcal{F}$ is a sheaf, we may glue together the s_i's to a section $s \in \mathcal{F}(U)$. Now we have $\phi_U(s)|_{U_i} = \phi_U(s|_{U_i}) = \phi_U(s_i) = 0$ for all i. By the Locality axiom for $\mathcal{G}$, it follows that $\phi_U(s) = 0$, and hence $s \in (\operatorname{Ker}\phi)(U)$.

Lemma 9.2. For each point $x \in X$, one has $(\operatorname{Ker}\phi)_x = \operatorname{Ker}\phi_x$.

Proof To see the inclusion $(\operatorname{Ker}\phi)_x \subset \operatorname{Ker}\phi_x$, let $s_x \in (\operatorname{Ker}\phi)_x$. By definition of the stalk, we may find an open set V and $s \in \operatorname{Ker}\phi_V$ so that s_x is the germ of s at x. As $\phi_V(s) = 0$, we also have $\phi_x(s_x) = (\phi_V(s))_x = 0$ in $\mathcal{G}_x$, and hence $s_x \in \operatorname{Ker}\phi_x$.

Conversely, let $s_x \in \operatorname{Ker}\phi_x$ and let $s \in \mathcal{F}(V)$ be a section representing s_x. As $\phi_x(s_x) = 0$, we have $(\phi_V(s))_x = 0$. This means that there exists some open set $W \subset V$ so that $\phi_V(s)|_W = 0$, and hence $s|_W \in (\operatorname{Ker}\phi)(W)$. But this means that $s_x \in (\operatorname{Ker}\phi)_x$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Defining the *image* of a map $\phi\colon \mathcal{F} \to \mathcal{G}$ between sheaves is more subtle than defining the kernel. One can define the *image presheaf* by setting

$$(\operatorname{Im}_p \phi)(U) = \big\{\phi_U(s) \in \mathcal{G}(U) \;\big|\; s \in \mathcal{F}(U)\big\}. \tag{9.1}$$

The restriction maps of $\mathcal{G}$ induce restriction maps of $\operatorname{Im}_p\phi$, as ϕ is compatible with restrictions, but it will not be a sheaf in general. The issue arises from the Gluing axiom: given an open cover U_i of U and sections of the form $t_i = \phi_{U_i}(s_i)$, we can glue the t_i to a section $t \in \mathcal{G}(U)$. However, there is no guarantee that t is of the form $\phi(s)$ for some $s \in \mathcal{F}(U)$. For that, the sections s_i would need to agree on the overlaps $U_i \cap U_j$, and there is no reason why they should. Example 9.4 below illustrates this.

To define the image sheaf, we need to add in all sections that can be obtained by gluing together local sections of the form $\phi_{U_i}(s_i)$ as above. In other words, we include the sections of $\mathcal{G}(U)$ which are 'locally images of ϕ'. For later applications, we allow $\mathcal{F}$ to be simply a presheaf.

Definition 9.3. For a map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, we define the *image sheaf* $\operatorname{Im}\phi$ by

$$(\operatorname{Im}\phi)(U) = \left\{ t \in \mathcal{G}(U) \;\middle|\; \begin{array}{l}\text{there is a cover } U_i \text{ of } U \text{ and sections} \\ s_i \in \mathcal{F}(U_i) \text{ such that } t|_{U_i} = \phi_{U_i}(s_i)\end{array} \right\}.$$

The restriction maps of $\mathcal{G}$ induce restriction maps of $\operatorname{Im}\phi$. The Locality axiom holds for free because $\mathcal{G}$ is a sheaf. As for the Gluing axiom, suppose we are given an open cover $\{U_i\}_{i\in I}$ of an open set U and sections $t_i \in (\operatorname{Im}\phi)(U_i)$ that agree on the overlaps. Since $\mathcal{G}$ is a sheaf, the t_i's glue together to a section $t \in \mathcal{G}(U)$, and t is by construction locally an image because each t_i is. The sheaf $\operatorname{Im}\phi$ is therefore a subsheaf of $\mathcal{G}$: it is the smallest subsheaf of $\mathcal{G}$ containing the images of ϕ.

Unlike the situation for kernels, $(\operatorname{Im}\phi)(U)$ is not always equal to $\operatorname{Im}\phi_U$. We always have the inclusion $\operatorname{Im}\phi_U \subset (\operatorname{Im}\phi)(U)$, as any section of the form $\phi(s)$ clearly lies in $\operatorname{Im}\phi$, but there may also be additional elements of $(\operatorname{Im}\phi)(U)$ which are only locally images of ϕ.

Example 9.4. Let $X = \operatorname{Spec} A$, where $A = k[x, y]$ and consider the ideal $I = (xy)$. The quotient map $A \to A/I$ induces a morphism of sheaves $\phi\colon \widetilde{A} \to \widetilde{A/I}$. We will consider this map over the open set $U = D(x) \cup D(y)$.

Over the distinguished opens, we have

$$\begin{aligned}\Gamma(D(x), \widetilde{A/I}) &= k[x, y]_x/(xy) = k[x]_x \\ \Gamma(D(y), \widetilde{A/I}) &= k[x, y]_y/(xy) = k[y]_y \\ \Gamma(D(xy), \widetilde{A/I}) &= k[x, y]_{xy}/(xy) = 0.\end{aligned}$$

A section of $\Gamma(U, \widetilde{A/I})$ is a pair agreeing on the intersection, we obtain

$$\Gamma(U, \widetilde{A/I}) = k[x]_x \times k[y]_y.$$

Now consider the section s corresponding to $(x^{-1}, 0) \in k[x]_x \times k[y]_y$. Then s is locally in

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

the image of ϕ: over $D(x)$, it is the image of x^{-1} via $\phi_{D(x)}\colon k[x,y]_x \to k[x]_x$, and over $D(y)$ it is the image of 0 via $\phi_{D(y)}\colon k[x,y]_y \to k[y]_y$.

On the other hand, by Example 3.22, $\Gamma(U,\widetilde{A}) = k[x,y]$. Therefore, s is not in the image of ϕ_U, because there is no element of $k[x,y]$ that maps to x^{-1} in $\widetilde{k[x]}_x$. Therefore, the image presheaf is not a sheaf. In fact, the image sheaf equals $\operatorname{Im}\phi = \widetilde{A/I}$. △

Example 9.5. There is one situation when the formula $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ in fact holds for every U, namely when each $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

To see this, let $t \in (\operatorname{Im}\phi)(U)$ be a section and let $s_i \in \mathcal{F}(U_i)$ be elements so that $t|_{U_i} = \phi(s_i)$ for every i. Then $s_i|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j}$ map to the same element in $\mathcal{G}(U_i \cap U_j)$ for every i and j, so by injectivity, this forces $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Therefore, the s_i glue to an element $s \in \mathcal{F}(U)$. We must have $\phi_U(s) = t$, because $\phi_U(s)|_{U_i} = \phi_{U_i}(s|_{U_i}) = \phi_{U_i}(s_i) = t|_{U_i}$ for all i. Hence $t = \phi(s)$ and so $t \in \operatorname{Im}\phi_U$. △

In general, the image sheaf behaves well when it comes to stalks:

Lemma 9.6. For each $x \in X$, we have $(\operatorname{Im}\phi)_x = \operatorname{Im}\phi_x$.

Proof Let $t_x \in \operatorname{Im}\phi_x$ and pick an $s_x \in \mathcal{F}_x$ with $\phi_x(s_x) = t_x$. Choose sections $s \in \mathcal{F}(V)$ and $t \in \mathcal{G}(V)$ representing s_x and t_x respectively, for some neighbourhood V of x. As $\phi_x(s_x) = t_x$, we have $\phi_V(s) = t$ after possibly shrinking V. This implies that $t \in (\operatorname{Im}\phi)(V)$, and hence $t_x \in (\operatorname{Im}\phi)_x$.

Conversely, let $t_x \in (\operatorname{Im}\phi)_x$. By definition, there exists an open set U and a section $t \in (\operatorname{Im}\phi)(U)$ so that t_x is the germ of t at x. As $t \in (\operatorname{Im}\phi)(U)$, there exists an open cover $\{U_i\}_{i\in I}$ of U and sections $s_i \in \mathcal{F}(U_i)$ such that $t|_{U_i} = \phi_{U_i}(s_i)$ for each i. Choosing an index i such that $x \in U_i$, we have

$$t_x = (t|_{U_i})_x = (\phi_{U_i}(s_i))_x = \phi_x((s_i)_x).$$

This means that $t_x \in \operatorname{Im}\phi_x$. □

9.2 Injective and surjective maps of sheaves

We say that a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if $\operatorname{Ker}\phi = 0$, and that ϕ is *surjective* if $\operatorname{Im}\phi = \mathcal{G}$.

The next two lemmas say that being injective or surjective can be verified by checking the induced maps on stalks.

Lemma 9.7. For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, the following are equivalent:
(i) ϕ is injective.
(ii) $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \subset X$.
(iii) $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.

This is a consequence of the formula $(\operatorname{Ker}\phi)(U) = \operatorname{Ker}\phi_U$, and the fact that a sheaf is zero if and only if its stalks are all zero.

When it comes to surjectivity, the formula $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ does not hold in general,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

and in fact the item (ii) above does not hold with 'injective' replaced by 'surjective'. Indeed, surjectivity of ϕ means that for every open set $U \subset X$, and $t \in \mathcal{G}(U)$, there exists an open cover $\{U_i\}$ of U and elements $s_i \in \mathcal{F}(U_i)$ so that $\phi(s_i) = t|_{U_i}$. The best we can say is:

Lemma 9.8. For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, the following are equivalent:
(i) ϕ is surjective.
(ii) $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.

This follows by applying the following lemma to $\operatorname{Im}\phi$ and $\mathcal{G}$:

Lemma 9.9. Two subsheaves $\mathcal{H}, \mathcal{G}$ of a sheaf $\mathcal{F}$ are equal if and only if $\mathcal{H}_x = \mathcal{G}_x$ (as subgroups of $\mathcal{F}_x$) for all $x \in X$.

Proof The 'only if'-direction is trivial. For the 'if' direction, let $U \subset X$ be an open set and let $s \in \mathcal{G}(U)$ be a section. By assumption, for each $x \in X$, s_x lies in $\mathcal{G}_x = \mathcal{H}_x$. This means that for each $x \in U$, there exists a neighbourhood U_x of x and a section $t^x \in \mathcal{H}(U_x)$ such that t^x represents the germ s_x. As the $\{U_x\}_{x\in U}$ form a cover of U, and the sections t^x and t^y agree over $U_x \cap U_y$ (they are both germs of s), they glue together to a section $t \in \mathcal{H}(U)$. By construction s and t have the same germs at every point $x \in U$, so by the Locality axiom, $s = t$. This shows that $\mathcal{G}(U) \subset \mathcal{H}(U)$ for every open set U, so $\mathcal{G}$ is a subsheaf of $\mathcal{H}$. By symmetry, we also have $\mathcal{H} \subset \mathcal{G}$ and hence $\mathcal{H} = \mathcal{G}$. □

For a map $\phi\colon \mathcal{F} \to \mathcal{G}$ to be an *isomorphism* (that is, it admits an inverse map $\psi\colon \mathcal{G} \to \mathcal{F}$), we have the following clean statement.

Proposition 9.10. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then the following four conditions are equivalent.
(i) The map ϕ is an isomorphism.
(ii) For every $x \in X$, the map on stalks $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.
(iii) One has $\operatorname{Ker}\phi = 0$ and $\operatorname{Im}\phi = \mathcal{G}$.
(iv) For all open subsets $U \subset X$ the map on sections $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

Proof (i) ⇒ (ii). This implication is clear.

(ii) ⇒ (iii). $\operatorname{Ker}\phi = 0$ follows by Lemma 9.2 and $\operatorname{Im}\phi = \mathcal{G}$ follows by Lemma 9.6 and Lemma 9.9.

(iii) ⇒ (iv). As $\operatorname{Ker}\phi = 0$, ϕ is injective. In that case, taking images commutes with taking sections (Example 9.5), so we have $\operatorname{Im}\phi_U = (\operatorname{Im}\phi)(U)$. But by assumption, $\operatorname{Im}\phi = \mathcal{G}$, so ϕ_U is also surjective.

(iv) ⇒ (i). If ϕ_U is an isomorphism for every U, the inverse maps $\psi_U = \phi_U^{-1}$ define an inverse morphism $\psi\colon \mathcal{G} \to \mathcal{F}$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

9.3 Exact sequences

A sequence of maps of sheaves

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \tag{9.2}$$

is said to be *exact* if $\operatorname{Im}\phi = \operatorname{Ker}\psi$ as subsheaves of $\mathcal{G}$.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0 \tag{9.3}$$

which is exact at every term. Equivalently, this means that ϕ is injective, ψ is surjective and $\operatorname{Im}\phi = \operatorname{Ker}\psi$.

Exactness for a sequence of sheaves is a purely local condition; the sequence (9.2) is exact if and only if for each $x \in X$ the sequence induced on stalks

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \tag{9.4}$$

is exact. This follows from Lemma 9.9 applied to $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$.

Proposition 9.11 (Taking sections is left exact). Given a short exact sequence as in (9.3), then for each open subset $U \subset X$, the following sequence is exact:

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

Proof As ϕ is injective, we have that ϕ_U is injective, and also that $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ by Example 9.5. By definition, we have $\operatorname{Ker}\psi_U = (\operatorname{Ker}\psi)(U)$. Combining these, we obtain

$$\operatorname{Im}\phi_U = (\operatorname{Im}\phi)(U) = (\operatorname{Ker}\psi)(U) = \operatorname{Ker}\psi_U,$$

and hence the above sequence is exact. □

One way of phrasing Proposition 9.11 is to say that taking sections over an open set U is a *left exact functor*. This functor, however, is not right exact in general. This failure of exactness is a fundamental problem in algebraic geometry. We will explore this in more detail in Chapter 20 where we discuss cohomology.

Example 9.12 ($M \mapsto \widetilde{M}$ is exact)**.** Consider $X = \operatorname{Spec} A$ and an exact sequence of A-modules $M' \to M \to M''$. Then the corresponding sequence of sheaves is exact:

$$\widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \tag{9.5}$$

Indeed, for any point $\mathfrak{p} \in \operatorname{Spec} A$, the induced sequence on stalks is $M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$ which is simply the localization of $M' \to M \to M''$ at $\mathfrak{p}$. As localization is exact, this sequence is exact for every stalk. Therefore, the functor $M \mapsto \widetilde{M}$ is exact. △

Example 9.13. Consider again Example 9.4, with $A = k[x, y]$ and $I = (xy)$. There is an exact sequence of sheaves on $X = \operatorname{Spec} A$

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{A} \longrightarrow \widetilde{A/I} \longrightarrow 0.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This is exact by Example 9.12. On the other hand, over the open set $U = D(x) \cup D(y)$, the sequence takes the form

$$0 \longrightarrow (xy) \longrightarrow k[x,y] \longrightarrow k[x]_x \times k[y]_y,$$

which is not surjective at the right. △

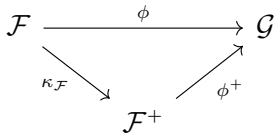
9.4 The sheaf associated to a presheaf

Essentially any construction for abelian groups, such as forming kernels, cokernels, direct sums, and so on have natural analogues for sheaves. For these constructions, one typically first defines a naive presheaf and then checks whether it satisfies the two sheaf axioms. This works in some cases (e.g., for the kernel sheaf), but it can fail in others (e.g., the image presheaf). Fortunately, there is a canonical procedure to produce a sheaf $\mathcal{F}^+$ from a given presheaf $\mathcal{F}$, called the *sheafification* of $\mathcal{F}$.

The sheaf $\mathcal{F}^+$ can be thought of as the sheaf which best approximates $\mathcal{F}$. This is formulated precisely in terms of the following universal property: $\mathcal{F}^+$ comes equipped with a morphism of presheaves $\kappa\colon \mathcal{F} \to \mathcal{F}^+$, such that any presheaf morphism $\mathcal{F} \to \mathcal{G}$ into a sheaf $\mathcal{G}$ factors uniquely through $\kappa_{\mathcal{F}}$ via a sheaf morphism $\mathcal{F}^+ \to \mathcal{G}$.

Proposition 9.14 (Sheaf associated to a presheaf). Let $\mathcal{F}$ be a presheaf on X. There exists a sheaf $\mathcal{F}^+$ and a morphism of presheaves $\kappa_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}^+$ satisfying:

(i) (Universal property) For any morphism of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique sheaf morphism $\phi^+\colon \mathcal{F}^+ \to \mathcal{G}$ making the following diagram commute:



In other words, if $\mathcal{G}$ is a sheaf, there is a natural bijection

$$\mathrm{Hom}_{\mathsf{PAb}(X)}(\mathcal{F},\mathcal{G}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Ab}(X)}(\mathcal{F}^+,\mathcal{G}), \tag{9.6}$$

where on the left-hand side $\mathcal{G}$ is considered as a presheaf. This property characterizes $\mathcal{F}^+$ up to a unique isomorphism.

(ii) (Stalks) The morphism $\kappa_{\mathcal{F}}$ induces isomorphisms $\mathcal{F}_x \simeq \mathcal{F}^+_x$ for all $x \in X$.

If $\mathcal{F}$ is already a sheaf, then $\kappa_{\mathcal{F}}$ is an isomorphism. This can be seen either via the universal property, or the fact that $\kappa_{\mathcal{F}}$ induces an isomorphism on stalks.

As in the construction of a sheaf from a sheaf on a basis, the details of the explicit construction will not be terribly important. Essentially all arguments involving $\mathcal{F}^+$ in this book rely mainly on the two properties in the Proposition 9.14. This is a good illustration of the principle: understand objects by their properties, not their explicit construction.

What prevents a presheaf $\mathcal{F}$ from being a sheaf is of course the failure of one of the sheaf axioms:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(i) There are nonzero sections which are locally zero, i.e., become zero when restricted to the open sets of a covering. These are sections $s \in \mathcal{F}(U)$ so that $s_x = 0$ for every $x \in U$.
(ii) There are collections of local sections which are compatible over the overlaps, but do not glue to a global section of $\mathcal{F}$.

In light of this, it is clear what the sheafification should do: mod out by the locally zero sections in the item (i) and add in all the missing sections in item (ii).

Explicitly, for an open set $U \subset X$, the sections of $\mathcal{F}^+(U)$ are collections of germs[1] $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ that are *locally induced by sections of* $\mathcal{F}$. That is, for every $x \in U$, there is an open set $V \subset U$ containing x and a section $t \in \mathcal{F}(V)$ such that $s_y = t_y$ (the germ of t) for every $y \in V$. In other words,

$$\mathcal{F}^+(U) = \left\{ (s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x \;\middle|\; (s_x) \text{ is locally induced by sections of } \mathcal{F} \right\}.$$

An element of $\mathcal{F}^+(U)$ is given by a collection of germs s_x that vary 'continuously' in x. Near every point, they are simultaneously represented by a section of $\mathcal{F}$.

The projection maps $\prod_{x\in U} \mathcal{F}_x \to \prod_{x\in V} \mathcal{F}_x$ give restriction maps $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$ for each $U \supset V$, making $\mathcal{F}^+$ into a presheaf.

Lemma 9.15. $\mathcal{F}^+$ is a sheaf.

Proof Locality: if $\{U_i\}$ is an open cover of U, and $t = (s_x)_{x\in U}$ is a section of $\mathcal{F}^+(U)$ such that $t|_{U_i} = 0$ for each i, then $s_x = 0$ for every $x \in U_i$. Hence, if $t|_{U_i} = 0$ for all i, it follows that $t = 0$ in $\mathcal{F}^+(U)$.

Gluing: Given an open cover $\{U_i\}$ of U and sections $t_i = (s_x^i)_{x\in U_i}$ of $\mathcal{F}^+$ over U_i matching on the intersections $U_i \cap U_j$, we define $t = (s_x)_{x\in U}$ by $s_x = s_x^i$ for any i such that $x \in U_i$. It is clear that $t|_{U_i} = t_i$. Moreover, t is locally induced by sections of $\mathcal{F}$ because the t_i are, so it belongs to $\mathcal{F}^+(U)$. □

There is a canonical map of presheaves

$$\kappa_{\mathcal{F}} \colon \mathcal{F} \longrightarrow \mathcal{F}^+$$

that sends a section $s \in \mathcal{F}(U)$ to the sequence of all its germs $(s_x)_{x\in U}$. Its kernel consists exactly of the sections that are locally zero (that is, $s_x = 0$ for all $x \in U$).

For a morphism of presheaves $\phi \colon \mathcal{F} \to \mathcal{G}$, we define a map of sheaves $\phi^+ \colon \mathcal{F}^+ \to \mathcal{G}^+$ over an open set U using the product of all the stalk maps $\mathcal{F}_x \to \mathcal{G}_x$ with $x \in U$. In other words, ϕ_U^+ sends $(s_x)_{x\in U}$ to $(\phi_x(s_x))_{x\in U}$. This makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow{\scriptstyle \kappa_{\mathcal{F}}} & & \downarrow{\scriptstyle \kappa_{\mathcal{G}}} \\ \mathcal{F}^+ & \xrightarrow{\phi^+} & \mathcal{G}^+ \end{array} \tag{9.7}$$

It is not hard to check that $\mathrm{id}_{\mathcal{F}}^+ = \mathrm{id}_{\mathcal{F}^+}$ and that $(\psi \circ \phi)^+ = \psi^+ \circ \phi^+$ for two composable

[1] The notation is not ideal: s_x is a germ at x, while at the same time, x is used as an index.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

morphisms between presheaves on X, so that $\mathcal{F} \mapsto \mathcal{F}^+$ is a functor from the category of presheaves on X to the category of sheaves on X.

Proof of Proposition 9.14 We begin with item (ii). We first show that the stalk map is injective. Let $t_x \in \mathcal{F}_x$, and represent it as the germ of a section $t \in \mathcal{F}(V)$ defined in some neighbourhood of x. Then over V, the map $\kappa_{\mathcal{F}}$ sends t to the collection of germs $(t_y)_{y\in V}$. Hence, if t maps to zero in $\mathcal{F}^+(V)$, then in particular $t_x = 0$.

For surjectivity, take an element in $(\mathcal{F}^+)_x$ and represent it by a section $t \in \mathcal{F}^+(V)$ over some neighbourhood V of x. Taking a smaller open set V if necessary, we may assume that t is induced from a section of $\mathcal{F}$, that is, there is a section $s \in \mathcal{F}(V)$ such that for each $y \in V$, the y-th component of t is equal to the germ s_y. But then over V, $\kappa_{\mathcal{F}}$ sends s to t, so the germ $s_x \in \mathcal{F}_x$ maps to $t_x \in (\mathcal{F}^+)_x$.

(i): If $\mathcal{G}$ is a sheaf, then $\kappa_{\mathcal{G}}$ is an isomorphism by (ii). The bijection (9.6) follows formally from this. The map from left to right sends a presheaf map $\phi\colon \mathcal{F} \to \mathcal{G}$ to the composition $\mathcal{F}^+ \to \mathcal{G}^+ \simeq \mathcal{G}$ defined by $\kappa_{\mathcal{G}}^{-1} \circ \phi^+$. The inverse map sends $\psi\colon \mathcal{F}^+ \to \mathcal{G}$ to the composition $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{G}$ defined by $\psi \circ \kappa_{\mathcal{F}}$. These assignments are indeed inverses: starting with $\phi\colon \mathcal{F} \to \mathcal{G}$, the composition $\mathcal{F} \to \mathcal{F}^+ \to \mathcal{G}^+ \to \mathcal{G}$ coincides with ϕ by the commutative diagram (9.7). Likewise, given $\psi\colon \mathcal{F}^+ \to \mathcal{G}$, the composition $\mathcal{F}^+ \to \mathcal{F}^{++} \to \mathcal{G}^+ \to \mathcal{G}$ coincides with ψ, by applying (9.7) to ψ^+. This gives the required bijection. □

Example 9.16. A presheaf $\mathcal{F}$ which is contained in a sheaf $\mathcal{G}$ is particularly easy to sheafify. The sheafification $\mathcal{F}^+$ equals the image sheaf of the inclusion map $\mathcal{F} \to \mathcal{G}$. In other words, the sections in $\mathcal{F}^+(U)$ are the sections $s \in \mathcal{G}(U)$ such that $s|_{U_i} \in \mathcal{F}(U_i)$ for some open cover $\{U_i\}_{i\in I}$ of U. △

Example 9.17 (The structure sheaf)**.** Let $X = \operatorname{Spec} A$ for a ring A. For an open set $U \subset X$, define $S(U) = \{\, s \in A \mid D(s) \supset U \,\}$ as the set of elements that do not vanish anywhere on U. As in Section 3.5, we consider the assignment $U \mapsto S(U)^{-1}A$. This defines a presheaf on X, but it is not always a sheaf (see Exercise 3.7.15). The associated sheaf is exactly the structure sheaf $\mathcal{O}_X$. To see this, write $\mathcal{O}'_X$ for this associated sheaf and note that there are natural maps $S(U)^{-1}A \to \mathcal{O}'_X(U)$, given by the universal property of sheafification. These maps define a morphism of $\mathscr{B}$-sheaves (where as usual, $\mathscr{B}$ is the basis of distinguished opens), and hence a morphism of sheaves $\mathcal{O}_X \to \mathcal{O}'_X$. This is an isomorphism, because it induces an isomorphism on stalks. △

Example 9.18. Let $X = \mathbb{A}^1_{\mathbb{C}}$ and let P, Q be two distinct points. Consider the presheaf $\mathcal{F}$ defined by

$$\mathcal{F}(U) = \Big\{ f \in \mathcal{O}_X(U) \mid f(P) = f(Q) \Big\}.$$

Then $\mathcal{F}$ is not a sheaf. The reason is that $\mathcal{F}|_U = \mathcal{O}_U$ for any open set U which contains P but not Q (as the condition $f(P) = f(Q)$ becomes redundant there). The same holds for open sets containing Q but not P. Therefore, if we take any $f \in \mathcal{O}_X(X)$ such that $f(P) \neq f(Q)$, we can restrict it to the open sets $U = \mathbb{A}^1_{\mathbb{C}} - \{P\}$ and $V = \mathbb{A}^1_{\mathbb{C}} - \{Q\}$, which form an open cover of X. The restrictions $f|_U$ and $f|_V$ give two sections of $\mathcal{F}(U) = \mathcal{O}_X(U)$ and $\mathcal{F}(V) = \mathcal{O}_X(V)$, which are equal to $f|_{U\cap V}$, over $U \cap V$, but the element they glue to, namely f, is not an element of $\mathcal{F}(X)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Note that $\mathcal{F}$ is a subpresheaf of $\mathcal{O}_X$. The above argument shows that every section of $\mathcal{O}_X$ locally lies in $\mathcal{F}$, so that when we sheafify, we get $\mathcal{F}^+ = \mathcal{O}_X$. $\triangle$

9.5 Cokernels and quotients

One of the main reasons to introduce sheafification is to be able to define cokernels and quotient sheaves. For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we define the *cokernel* $\operatorname{Coker}\phi$ to be the sheaf associated to the presheaf

$$(\operatorname{Coker}_p\phi)(U) = \mathcal{G}(U)/\operatorname{Im}\phi(U).$$

Similarly, for a subsheaf $\mathcal{G} \subset \mathcal{F}$ of a sheaf $\mathcal{G}$, the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ is the sheaf associated to the presheaf

$$(\mathcal{F}/\mathcal{G})_p(U) = \mathcal{F}(U)/\mathcal{G}(U).$$

In other words, $\mathcal{F}/\mathcal{G}$ is the cokernel of the inclusion map $\mathcal{G} \to \mathcal{F}$.

Note that for each open set U, the sections of the cokernel presheaf are simply given by the cokernel of $\mathcal{F}(U) \to \mathcal{G}(U)$. By composing the quotient map $\mathcal{G}(U) \to (\operatorname{Coker}_p\phi)(U)$ with the sheaffication map $\operatorname{Coker}_p\phi \to \operatorname{Coker}\phi$ we obtain a morphism of sheaves $\mathcal{G} \to \operatorname{Coker}\phi$. It fits into the exact sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \operatorname{Coker}\phi \longrightarrow 0. \tag{9.8}$$

Example 9.19. Continuing Example 9.13, applying $M \mapsto \widetilde{M}$ to the sequence $0 \to I \to A \to A/I \to 0$, we see that the quotient sheaf $\widetilde{A}/\widetilde{I}$ identifies with the sheaf $\widetilde{A/I}$. Even in this basic example it is necessary to sheafify, as the quotient presheaf takes the value

$$\widetilde{A}(U)/\widetilde{I}(U) = k[x,y]/(xy)$$

over $U = D(x) \cup D(y)$, whereas $\Gamma(U, \widetilde{A/I}) = k[x]_x \times k[y]_y$. $\triangle$

9.6 The inverse image sheaf

For a morphism $f\colon X \to Y$ and a sheaf $\mathcal{G}$ on Y, we can define the *inverse image sheaf* $f^{-1}\mathcal{G}$, which is a sheaf on X. This sheaf has a universal property which is closely related to the pushforward functor f_*: for any sheaf $\mathcal{F}$ on X, the sheaf morphisms $f^{-1}\mathcal{G} \to \mathcal{F}$ are in one-to-one correspondence with sheaf morphisms $\mathcal{G} \to f_*\mathcal{F}$.

The basic intuition is that a local section of $f^{-1}\mathcal{G}$ near a point $P \in X$ should come from a local section of $\mathcal{G}$ near the point $f(P) \in Y$. At first sight, one might try to define something like $(f^{-1}\mathcal{G})(U) = \mathcal{G}(f(U))$ for an open set $U \subset X$. However, this does not work in general because $f(U)$ may not be open in Y.

The remedy is to imitate the definition of stalks. Instead of requiring a section to be defined exactly on $f(U)$, we allow it to be defined on some open neighbourhood of $f(U)$ and identify two such sections if they agree on a smaller neighbourhood of $f(U)$.

Precisely, for an open set $U \subset X$, we define

$$(f_p^{-1}\mathcal{G})(U) = \coprod_{V \supset f(U)} \mathcal{G}(V)/\sim. \tag{9.9}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

where the disjoint union runs over all open sets $V \subset Y$ containing $f(U)$. An element is represented by a pair (s, V), where $s \in \mathcal{G}(V)$, and two such pairs (s, V) and (s', V') are defined to be equivalent if there is an open set W with $f(U) \subset W \subset V \cap V'$ such that $s|_W = s'|_W$.

The restriction maps are defined in the natural way. If $U' \subset U$, then any open set V containing $f(U)$ also contains $f(U')$. Hence a pair (s, V) representing an element in $(f_p^{-1}\mathcal{G})(U)$ also determines an element in $(f_p^{-1}\mathcal{G})(U')$.

A different way of expressing (9.9) is as a direct limit

$$(f_p^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V), \tag{9.10}$$

where V runs over open sets in Y containing $f(U)$.

This construction closely parallels the definition of a stalk. Indeed, any section of $(f_p^{-1}\mathcal{G})(U)$ is represented by a section of $\mathcal{G}(V)$ over some open neighbourhood V of $f(U)$, and two such representatives define the same element precisely when they agree on a smaller neighbourhood of $f(U)$.

Unfortunately, the presheaf (9.9) is not a sheaf in general, so we take the associated sheaf:

Definition 9.20. Let $f\colon X \to Y$ be a continuous map and let $\mathcal{G}$ be a presheaf on Y. The *inverse image* $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf $f_p^{-1}\mathcal{G}$.

The sections of $f^{-1}\mathcal{G}$ can be explicitly described as follows: for an open set $U \subset X$, the group $(f^{-1}\mathcal{G})(U)$ is the set of sequences $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{G}_{f(x)}$ such that every point of U has a neighbourhood on which the family comes from an actual section of $\mathcal{G}$. More precisely, for every $x \in U$, there exists an open neighbourhood $U' \subset U$ of x, an open neighbourhood $V \subset Y$ of $f(x)$ with $f(U') \subset V$, and a section $t \in \mathcal{G}(V)$ such that $s_y = t_{f(y)}$ for every $y \in U'$. Informally, sections of $f^{-1}\mathcal{G}$ are locally induced by sections of $\mathcal{G}$ on Y. In particular, if $\mathcal{G}$ is a sheaf of groups, rings, k-vector spaces, etc, then $f^{-1}\mathcal{G}$ will be a sheaf of the same type.

Given a morphism of sheaves on Y, $\phi\colon \mathcal{G} \to \mathcal{H}$, we obtain a collection of maps $\phi_V\colon \mathcal{G}(V) \to \mathcal{H}(V)$ for open sets V containing $f(U)$. These maps are compatible with restrictions, so they induce a map of presheaves $f_p^{-1}\mathcal{G} \to f_p^{-1}\mathcal{H}$ and consequently a map of sheaves $f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$. The inverse image therefore defines a functor from sheaves on Y to sheaves on X.

Although the pushforward $f_*\mathcal{F}$ is easier to define and the sections are more intuitively understood, the inverse image $f^{-1}\mathcal{G}$ has better properties. For instance, the stalks are easy to compute:

Proposition 9.21. For each $x \in X$, there is a natural isomorphism $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.

Proof Since sheafification preserves stalks, it is enough to show that $(f_p^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.

A germ in $(f_p^{-1}\mathcal{G})_x$ may be represented by some pair (s, V), where V is an open neighbourhood of $f(x)$ and $s \in \mathcal{G}(V)$. We define a map $(f_p^{-1}\mathcal{G})_x \to \mathcal{G}_{f(x)}$ by $[(s, V)] \mapsto s_{f(x)}$. It is straightforward to check that this is well-defined.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Surjectivity is clear: any germ $t_{f(x)} \in \mathcal{G}_{f(x)}$ is represented by some $t \in \mathcal{G}(V)$, and (t, V) defines a germ in $(f_p^{-1}\mathcal{G})_x$ mapping to $t_{f(x)}$.

For injectivity, suppose $[(s, V)]$ and $[(s', V')]$ map to the same element. Then $s_{f(x)} = s'_{f(x)}$, so for some open neighbourhood $W \subset V \cap V'$ of $f(x)$ we have $s|_W = s'|_W$. But then (s, V) and (s', V') determine the same germ in $(f_p^{-1}\mathcal{G})_x$. □

Note that there is no such simple description of the stalks of a pushforward. Also, unlike the pushforward, the inverse image functor is exact:

Corollary 9.22. If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence on Y, then

$$0 \longrightarrow f^{-1}\mathcal{G}' \longrightarrow f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}'' \longrightarrow 0$$

is an exact sequence of sheaves on X.

Proof This follows from Proposition 9.21, since a sequence of sheaves is exact if and only if it is exact on stalks. □

Example 9.23 (Restriction to an open set)**.** Let $\iota\colon U \to X$ be the inclusion of an open set in X. Then for a sheaf $\mathcal{F}$ on X, we have

$$\iota^{-1}\mathcal{F} = \mathcal{F}|_U.$$

Indeed, if $V \subset U$ is an open set, then V is also open in X, and $\varinjlim_{W \supset \iota(V)} \mathcal{F}(W)$ simply evaluates to $\mathcal{F}(V)$. The inverse presheaf is a sheaf in this case. △

Example 9.24 (General restrictions)**.** If $Z \subset X$ is an arbitrary subset, then the naive restriction of sections does not directly give a sheaf on Z, because an open subset $V \subset Z$ will typically not be open in X. The sections of the inverse image sheaf are instead determined by sections of $\mathcal{F}(U)$ as U runs over the open sets in X containing Z.

In particular, if $Z = \{x\}$, we recognize the definition of the stalk, and $\iota^{-1}\mathcal{F} = \mathcal{F}_x$, as a constant sheaf on $\{x\}$. △

Example 9.25. The presheaf defined by (9.9) is not in general a sheaf. For instance, if $f\colon X \to Y$ is the constant map with image $y \in Y$, then for each open set $U \subset X$, the direct limit in (9.9) will be the stalk $\mathcal{G}_y$. Therefore, (9.9) defines the constant presheaf with value $\mathcal{G}_y$, and as we observed in Example 3.7 on page 41, this is not always a sheaf.

For an explicit example, we can let $X = \operatorname{Spec} k \sqcup \operatorname{Spec} k$, $Y = \operatorname{Spec} k$ and $f\colon X \to \operatorname{Spec} k$. Then $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$, which is the constant sheaf with value k, whereas $f_p^{-1}\mathcal{O}_Y$ is the constant presheaf on k. △

Here is the aforementioned universal property of the inverse image sheaf.

Theorem 9.26. Let $f\colon X \to Y$ be a morphism and let $\mathcal{G}$ be a sheaf on Y. Then there are functorial bijections

$$\operatorname{Hom}_{\mathsf{Ab}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_{\mathsf{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F}) \tag{9.11}$$

for each sheaf $\mathcal{F}$ on X.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof As $\mathcal{F}$ is a sheaf, the universal property of sheafification tells us that there is a one-to-one correspondence between sheaf maps $f^{-1}\mathcal{G} \to \mathcal{F}$ and maps of presheaves $f_p^{-1}\mathcal{G} \to \mathcal{F}$. Therefore, it suffices to establish a bijection between the set of presheaf maps $f_p^{-1}\mathcal{G} \to \mathcal{F}$ and maps of sheaves $\mathcal{G} \to f_*\mathcal{F}$.

To give a map of presheaves $\phi\colon f_p^{-1}\mathcal{G} \to \mathcal{F}$ is equivalent to specifying, for each open $U \subset X$, a map

$$\phi_U\colon (f_p^{-1}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction in U. By the universal property of the direct limit, giving ϕ_U is equivalent to giving, for every $V \supset f(U)$, a map

$$\phi_{U,V}\colon \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

such that whenever $V' \subset V$ (with $V' \supset f(U)$) the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\phi_{U,V}} & \mathcal{F}(U) \\ \downarrow & \nearrow_{\phi_{U,V'}} & \\ \mathcal{G}(V') & & \end{array} \tag{9.12}$$

Moreover, saying that ϕ_U is compatible with restrictions in U means that for any $U' \subset U$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\phi_{U,V}} & \mathcal{F}(U) \\ & \searrow^{\phi_{U',V}} & \downarrow \\ & & \mathcal{F}(U') \end{array} \tag{9.13}$$

In particular, the collection $\phi_{U,V}$ is uniquely determined by the maps of the form $\phi_{f^{-1}V,V}$.

The diagrams (9.12) and (9.13) imply that the following diagram commutes (both compositions are given by $\phi_{f^{-1}V',V}$):

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\phi_{f^{-1}V,V}} & \mathcal{F}(f^{-1}V) \\ \downarrow & & \downarrow \\ \mathcal{G}(V') & \xrightarrow{\phi_{f^{-1}V',V'}} & \mathcal{F}(f^{-1}V'). \end{array} \tag{9.14}$$

In other words, the $\phi_{f^{-1}V,V}$ define a map of sheaves $\phi\colon \mathcal{G} \to f_*\mathcal{F}$. Conversely, any morphism of sheaves $\phi\colon \mathcal{G} \to f_*\mathcal{F}$ arises from a unique compatible collection of maps $\phi_{f^{-1}V,V}$, which by what we said above determines a unique collection $\phi_{U,V}$ coming from a map of presheaves $\phi\colon f_p^{-1}\mathcal{G} \to \mathcal{F}$. □

Given a map of sheaves $\phi\colon \mathcal{G} \to f_*\mathcal{F}$, one can understand the induced map $\phi^\sharp\colon f^{-1}\mathcal{G} \to \mathcal{F}$ on stalks as follows. For $P \in X$, identify $(f^{-1}\mathcal{G})_P$ with $\mathcal{G}_{f(P)}$. If $t_{f(P)} \in \mathcal{G}_{f(P)}$ is represented by a section $t \in \mathcal{G}(V)$ on an open neighbourhood V of $f(P)$, then $\phi_P^\sharp$ sends $t_{f(P)}$ to the germ of the section $\phi_V(t) \in \mathcal{F}(f^{-1}V)$ at P.

Despite the name, the functors f_* and f^{-1} are not inverses: in general neither $f_*f^{-1}\mathcal{G} \simeq \mathcal{G}$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

nor $f^{-1}f_*\mathcal{F} \simeq \mathcal{F}$ holds. However, there is a relationship between them. Applying the bijection (9.11) to the identity maps $f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$ and $f_*\mathcal{F} \to f_*\mathcal{F}$, we obtain sheaf maps

$$\eta\colon \mathcal{G} \longrightarrow f_*f^{-1}\mathcal{G}, \qquad \varepsilon\colon f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}.$$

These can be understood more explicitly as follows. For an open set in X of the form $f^{-1}V$, the direct limit (9.10) is identified with $\mathcal{G}(V)$. Therefore, a section of $\mathcal{G}(V)$ defines a section of $f_p^{-1}\mathcal{G}$ over $f^{-1}(V)$ and hence a section of $f^{-1}\mathcal{G}(V)$. In this way, η is induced by the composition

$$\mathcal{G}(V) \longrightarrow f_*f_p^{-1}\mathcal{G}(V) \longrightarrow f_*f^{-1}\mathcal{G}(V).$$

In particular, taking $V = Y$, gives a natural map $\Gamma(Y, \mathcal{G}) \to \Gamma(X, f^{-1}\mathcal{G})$.

Likewise, to define ε, let $U \subset X$ be open. Taking the direct limit of the restriction maps $\mathcal{F}(f^{-1}V) \to \mathcal{F}(U)$ over open sets V with $V \supset f(U)$, we get a map of presheaves $f_p^{-1}f_*\mathcal{F} \to \mathcal{F}$. Taking the associated map of sheaves, we get ε.

9.7 Exercises

Exercise 9.7.1. Let $\mathcal{F}$ be a presheaf on a scheme X and let $\mathcal{F}^+$ be the sheaf associated to $\mathcal{F}$. Show that there is a natural map $\mathcal{F}(X) \to \mathcal{F}^+(X)$. Find examples where this fails to be injective or surjective.

Exercise 9.7.2. Define a presheaf on a topological space X by $\mathcal{F}(X) = \mathbb{Z}$ and $\mathcal{F}(U) = 0$ for all other open sets. Show that $\mathcal{F}$ is a presheaf and describe the associated sheaf $\mathcal{F}^+$.

Exercise 9.7.3. Suppose $\phi\colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves with the property that there is a covering $\{U_i\}$ of X so that each ϕ_{U_i} is injective. Must ϕ be injective? If so, prove it, if not, give a counterexample. What about ϕ surjective?

Exercise 9.7.4. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves on X and let $\mathscr{B}$ be a basis for the topology on X. Show that ϕ is injective (resp. isomorphism) if and only if ϕ_V is, for every $V \in \mathscr{B}$. What happens for surjectivity?

Exercise 9.7.5 (Extension by zero)**.** Let X be a topological space, let $j\colon U \hookrightarrow X$ be the inclusion of an open subset, and let $\mathcal{G}$ be a sheaf of abelian groups on U. Define $j_!\mathcal{G}$ to be the sheaf associated to the presheaf

$$(j_{p!}\mathcal{G})(V) = \begin{cases} \mathcal{G}(V), & V \subset U, \\ 0, & V \not\subset U. \end{cases}$$

a) Show that this presheaf is not necessarily a sheaf. For example, take $X = \mathbb{R}$, $U = (0, 1)$, and $\mathcal{G} = \underline{\mathbb{Z}}_U$. Use the open set $V = (-1, 1/2)$ and the cover $V = (-1, 0) \cup (0, 1/2)$.
b) Show that $(j_!\mathcal{G})_x \simeq \mathcal{G}_x$ for $x \in U$, and $(j_!\mathcal{G})_x = 0$ for $x \notin U$.
c) View $j_!\mathcal{G}$ as a subsheaf of $j_*\mathcal{G}$. Show that for an open $V \subset X$, a section of $(j_*\mathcal{G})(V) = \mathcal{G}(V \cap U)$ lies in $(j_!\mathcal{G})(V)$ exactly when it is zero near every point of $V - U$.
d) Show that $j_!$ is left adjoint to j^{-1}: for any sheaf $\mathcal{H}$ on X there is a natural isomorphism $\mathrm{Hom}_X(j_!\mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}_U(\mathcal{G}, j^{-1}\mathcal{H})$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

e) Let $U = X - Z$, where $i\colon Z \hookrightarrow X$ is the closed complement. For a sheaf $\mathcal{F}$ on X, show that there is an exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0,$$

where $\mathcal{F}|_U = j^{-1}\mathcal{F}$ and $\mathcal{F}|_Z = i^{-1}\mathcal{F}$. HINT: Check exactness on stalks.

f) Let $X = \mathbb{A}^n_k$, let $Z = \{0\}$, let $U = X - Z$, and take $\mathcal{F} = \mathcal{O}_X$, considered as a sheaf of abelian groups. Describe the exact sequence above on stalks and compute $\Gamma(X, j_!\mathcal{O}_U)$. Explain what changes between $n = 1$ and $n \geqslant 2$.

Exercise 9.7.6. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms.

a) Show that $(g \circ f)_* = g_* \circ f_*$

b) Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. HINT: There are sheafifications involved. Define a map from one side to the other and show that it induces an isomorphism on stalks.

Exercise 9.7.7. Let X be a scheme and let $\mathcal{F}$ be a presheaf on X. Show that sheafification commutes with restriction, i.e., $\mathcal{F}^+|_U = (\mathcal{F}|_U)^+$. HINT: Use the universal property and check the stalks.

Exercise 9.7.8. Let $\mathcal{F}$ be a presheaf of rings on X. Show that the associated sheaf $\mathcal{F}^+$ is a sheaf of rings.

Exercise 9.7.9. Let $X = \{x, y, z\}$ be a topological space with three elements and open sets $\varnothing, X, V = \{x\}, U_1 = \{x, y\}, U_2 = \{x, z\}$. Let $\mathcal{F}$ be the constant presheaf on X.

a) Show that $\mathcal{F}$ is a sheaf.

b) Let $\mathcal{G}$ be the presheaf defined by $\mathcal{G}(\varnothing) = 0, \mathcal{G}(X) = \mathbb{Z}\oplus\mathbb{Z}$, and $\mathcal{G}(V), \mathcal{G}(U_1)$, $\mathcal{G}(U_2)$ all equal to $\mathbb{Z}$, and let the restriction maps be $\rho_{U_1,V} = \rho_{U_2,V} = \mathrm{id}_{\mathbb{Z}}$ and let $\mathcal{G}(X) \to \mathcal{G}(U_i)$ be the i-th projection, for $i = 1, 2$. Show that $\mathcal{G}$ is a sheaf.

c) Consider the map $\phi\colon \mathcal{F} \to \mathcal{G}$ defined by $\phi_X\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $x \mapsto (x, x)$, and all the other maps equal to the identity map. Show that ϕ is a map of sheaves.

d) Compute the values of the image presheaf and conclude that it is not a sheaf.

Exercise 9.7.10. Show that if $f\colon X \to Y$ is a morphism, and A_Y is a constant sheaf on Y, then $f^{-1}A_Y$ is a constant sheaf on X.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the map $t \mapsto t^2$ (in the Euclidean topology). Compute all stalks of the pushforward of the constant sheaf $f_*\mathbb{Z}$. Deduce that $f_*\mathbb{Z}$ is not a constant sheaf.

Exercise 9.7.11. Show that the natural map $\Gamma(Y, \mathcal{G}) \longrightarrow \Gamma(X, f^{-1}\mathcal{G})$ need not be injective or surjective in general, even in the basic case in Example 9.23.

Exercise 9.7.12. Let $\iota\colon \operatorname{Spec} k \to \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ be the inclusion of a k-point in $\mathbb{A}^2_k$. Compute $\iota^{-1}\mathcal{O}_{\mathbb{A}^2_k}$ and study the morphisms $\mathcal{O}_{\mathbb{A}^2_k} \to \iota_*\iota^{-1}\mathcal{O}_{\mathbb{A}^2_k}$ and $\iota^{-1}\iota_*\mathcal{O}_{\operatorname{Spec} k} \to \mathcal{O}_{\operatorname{Spec} k}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

10

Quasi-coherent sheaves

10.1 Sheaves of modules

A module over a ring is an additive abelian group equipped with a multiplicative action of the ring. Informally, we can multiply elements of the module by elements of the ring. Similarly, an $\mathcal{O}_X$*-module* is a sheaf of abelian groups whose sections can be multiplied by sections of $\mathcal{O}_X$. Concretely, this means that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for every open set $U \subset X$, and for every inclusion $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear in the sense that for all $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$,

$$(a \cdot s)|_V = a|_V \cdot s|_V.$$

We will sometimes also talk about presheaves of $\mathcal{O}_X$-modules, defined in the same way.

A *map of* $\mathcal{O}_X$*-modules* or an $\mathcal{O}_X$*-linear map* is a map of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ between two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ such that for each open U the map $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. The $\mathcal{O}_X$-modules on a scheme X form a category, which we denote by $\mathsf{Mod}_{\mathcal{O}_X}$. We write $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the additive subgroup of $\mathrm{Hom}_{\mathsf{Ab}(X)}(\mathcal{F}, \mathcal{G})$ consisting of the sheaf maps $\mathcal{F} \to \mathcal{G}$ which are $\mathcal{O}_X$-linear.

If $\mathcal{F}$ is an $\mathcal{O}_X$-module and $x \in X$ is a point, the stalk $\mathcal{F}_x$ is naturally a module over the local ring $\mathcal{O}_{X,x}$. Moreover, for a map of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{F} \to \mathcal{G}$, the induced map $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is $\mathcal{O}_{X,x}$-linear.

Most of the familiar constructions for modules over a ring have analogues for $\mathcal{O}_X$-modules. For example, an $\mathcal{O}_X$*-submodule* is a subsheaf $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-submodule for every open set U, or more formally, the inclusion $\mathcal{G} \hookrightarrow \mathcal{F}$ is a map of $\mathcal{O}_X$-modules. An ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is a submodule of $\mathcal{O}_X$, that is, each $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal.

In some constructions one must first define an object as a presheaf and then sheafify. We will use the term presheaf of $\mathcal{O}_X$-modules in the evident analogous sense. By Exercise 10.13.2, the sheaf associated to a presheaf of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. In particular, the quotient sheaf $\mathcal{F}/\mathcal{G}$, which is the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$, is naturally an $\mathcal{O}_X$-module.

Example 10.1 (Pushforwards)**.** For a morphism $f\colon X \to Y$ and an $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward $f_*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module via the map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Explicitly, for a section $s \in f_*\mathcal{F}(V)$ and $a \in \mathcal{O}_Y(V)$, we define $a \cdot s \in f_*\mathcal{F}(V)$ to be the section $f^\sharp(a) \cdot s \in \mathcal{F}(f^{-1}V)$. △

Example 10.2. Let $\iota\colon Y \to X$ be a closed embedding. Then the kernel $\mathcal{I}$ of the map

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_Y$ is an ideal sheaf of $\mathcal{O}_X$, and there is a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0.$$

We also see that $\iota_*\mathcal{O}_Y \simeq \mathcal{O}_X/\mathcal{I}$ as $\mathcal{O}_X$-modules. △

Example 10.3 (Gluing)**.** If $\mathcal{F}$ is a sheaf obtained by gluing together sheaves $\mathcal{F}_i$ defined on a cover $\mathcal{U} = \{U_i\}_{i\in I}$, and each $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module, then $\mathcal{F}$ is an $\mathcal{O}_X$-module. Indeed, for $V \subset X$ open, $s \in \mathcal{F}(V)$, and $a \in \mathcal{O}_X(V)$, the product $a \cdot s$ is defined by gluing the local products $a|_{U_i\cap V} \cdot s|_{U_i\cap V}$ in $\mathcal{F}_i(U_i \cap V)$. △

Example 10.4. Write $\mathbb{P}^1_k$ for the projective line over a field k, and consider the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ from Section 6.3, that is, $\mathcal{O}_{\mathbb{P}^1_k}(n)$ is the sheaf obtained by gluing $\mathcal{O}_{U_0}$ to $\mathcal{O}_{U_1}$ using the isomorphism $\mathcal{O}_{U_0}|_{U_0\cap U_1} \to \mathcal{O}_{U_1}|_{U_0\cap U_1}$ on $U_0 \cap U_1 = \operatorname{Spec} k[u, u^{-1}]$ given by multiplication by u^{-n}. Then $\mathcal{O}_{\mathbb{P}^1_k}(n)$ is an $\mathcal{O}_{\mathbb{P}^1_k}$-module by Example 10.3. △

10.2 The sheaf associated to a module

The most important examples of $\mathcal{O}_X$-modules are the sheaves of the form $\widetilde{M}$, which we introduced in Section **??**. Let us recall the construction. If A is a ring, and M is an A-module, the sheaf $\widetilde{M}$ is the sheaf extending the following sheaf on the basis of distinguished open sets $D(f) \subset \operatorname{Spec} A$:

$$\widetilde{M}(D(f)) = M_f.$$

The restriction maps are the canonical localization maps, which are described as follows: when $D(fg) \subset D(f)$, the localization map $M_f \to M_{fg}$ sends m/f^n to $mg^n/(fg)^n$.

It is almost immediate that $\widetilde{M}$ is an $\mathcal{O}_X$-module. Over a distinguished open set $U = D(f)$, the group $\widetilde{M}(D(f)) = M_f$ is a module over A_f, and if $U \subset X$ is any open subset, we may cover it by distinguished open sets $D(f)$ and define an $\mathcal{O}_X(U)$-module structure on $\widetilde{M}(U)$ using the sheaf sequence. In the same way, one checks that the restriction maps are $\mathcal{O}_X(U)$-linear.

Proposition 10.5. Let $X = \operatorname{Spec} A$ be an affine scheme. For an A-module M and an $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \xrightarrow{\simeq} \operatorname{Hom}_A(M, \mathcal{F}(X))$$

that sends $\phi\colon \widetilde{M} \to \mathcal{F}$ to $\phi_X\colon M \to \mathcal{F}(X)$. It is functorial in both M and $\mathcal{F}$.

Proof Let $f \in A$, and consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi_X} & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ M_f & \xrightarrow{\phi_{D(f)}} & \mathcal{F}(D(f)) \end{array}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

where the vertical maps are restriction maps. This gives the following relation:

$$\phi_{D(f)}(m/1) = \phi_X(m)|_{D(f)}.$$

Note that $\mathcal{F}(D(f))$ is an A_f-module, because $\mathcal{F}$ is an $\mathcal{O}_X$-module. Therefore, in the localizations at f, we have the following relation

$$\phi_{D(f)}(m/f^n) = \phi_X(m)|_{D(f)} \cdot f^{-n}, \tag{10.1}$$

where $m/f^n \in M_f$. This means that the maps $\phi_{D(f)}$ are completely determined by $\phi_X : M \to \mathcal{F}(X)$. By Proposition 3.8, the map of sheaves ϕ is completely determined once it is specified over the $D(f)$'s. Hence ϕ is determined by ϕ_X, and the map in the proposition is injective.

For the surjectivity, suppose we are given a map of A-modules $\alpha\colon M \to \mathcal{F}(X)$. As usual, to define a map $\widetilde{M} \to \mathcal{F}$ it suffices to tell what it does to sections over the distinguished open sets $D(f)$. Inspired by (10.1), we define $\alpha_{D(f)}$ by

$$\alpha_{D(f)}(m/f^n) = \alpha(m)|_{D(f)} \cdot f^{-n}.$$

(Note that $\alpha(m)|_{D(f)}$ is a section of $\mathcal{F}(D(f))$, so the multiplication makes sense because $\mathcal{F}$ is an $\mathcal{O}_X$-module). Hence $\alpha_{D(f)}$ is simply the composition of the two maps of A_f-modules

$$M_f \xrightarrow{\alpha_f} \mathcal{F}(X)_f \longrightarrow \mathcal{F}(D(f)),$$

where the right-hand map is induced from the restriction map $\mathcal{F}(X) \to \mathcal{F}(D(f))$ by localization (note that $\mathcal{F}(D(f))$ is an A_f-module). This is compatible with the restriction maps, so we get a well-defined map of sheaves $\phi\colon \widetilde{M} \to \mathcal{F}$. Taking $f = 1$, we see that we recover α from ϕ on global sections.

The statement about the functoriality also follows from formula (10.1). The details are left to the reader. □

If we apply Proposition 10.5 to $M = \mathcal{F}(X)$ and consider the preimage of the identity map $\mathcal{F}(X) \to \mathcal{F}(X)$, we obtain the following corollary:

Corollary 10.6. For each $\mathcal{O}_X$-module $\mathcal{F}$ on an affine scheme X, there is a unique $\mathcal{O}_X$-linear map

$$\beta\colon \widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F} \tag{10.2}$$

that induces the identity on the spaces of global sections. The map β is functorial in $\mathcal{F}$.

In concrete terms, over a distinguished open set $D(f)$, the map β sends a section s/f^n where $s \in \mathcal{F}(X)$, to the section $s|_{D(f)} \cdot f^{-n}$. Here f^{-n} is regarded as a section of $\mathcal{O}_X(D(f)) = A_f$ and the product $s|_{D(f)} \cdot f^{-n}$ is well-defined because $\mathcal{F}$ is an $\mathcal{O}_X$-module.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Corollary 10.7. For A-modules M and N, the map $\alpha \mapsto \widetilde{\alpha}$ defines an A-linear isomorphism

$$\mathrm{Hom}_A(M,N) = \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}),$$

with inverse $\phi \mapsto \phi_X$.

Proof Apply Proposition 10.5 with $\mathcal{F} = \widetilde{N}$. □

In particular, the functor $M \mapsto \widetilde{M}$ is fully faithful. Hence it establishes an equivalence between the category Mod_A of A-modules and a subcategory of Mod_X. This subcategory is usually a strict subcategory; most $\mathcal{O}_X$-modules are not the form $\widetilde{M}$.

10.3 Quasi-coherent sheaves

A quasi-coherent sheaf is an $\mathcal{O}_X$-module which is locally of the form $\widetilde{M}$ for a module M.

Definition 10.8. Let X be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *quasi-coherent* if there exists an open affine cover $\{U_i\}_{i\in I}$ of X, with $U_i = \operatorname{Spec} A_i$, and for each i, an A_i-module M_i such that $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ as $\mathcal{O}_{U_i}$-modules.

In the definition, it is enough that $\mathcal{F}$ is of tilde-type on some affine cover. The next result implies that this condition in fact forces $\mathcal{F}$ to be of tilde-type when restricted to every affine open.

Theorem 10.9. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for every affine open subset $U = \operatorname{Spec} A$ of X, there exists an A-module M such that $\mathcal{F}|_U \simeq \widetilde{M}$ as $\mathcal{O}_U$-modules.

Proof Let $U \subset X$ be an affine open set. We need to show that the canonical map

$$\beta\colon \widetilde{\mathcal{F}(U)} \longrightarrow \mathcal{F}|_U \tag{10.3}$$

is an isomorphism. It suffices to check this over every distinguished open set $D(f) \subset U$. This reduces to showing that the map

$$\mathcal{F}(U)_f \longrightarrow \mathcal{F}(D(f)) \tag{10.4}$$

which sends s/f^n to $s|_{D(f)} \cdot f^{-n}$ is an isomorphism.

By the assumption on the covering U_i, the restrictions $\mathcal{F}|_{U_i}$ is of 'tilde type'. The covering $U \cap U_i$ forms an open cover of U. Using Lemma 8.2, we may cover each $U \cap U_i$ by affine open sets which are distinguished in both U and U_i. Doing so, we obtain an affine cover of U of open sets $D(g_1), \ldots, D(g_r)$ such that $\mathcal{F}|_{D(g_i)}$ is of tilde type over each $D(g_i)$.

As $\mathcal{F}|_{D(g_i)}$ is of tilde type on $D(g_i)$ for each i, the canonical maps

$$\beta_{\mathcal{F}|_{D(g_i)}}\colon \widetilde{\mathcal{F}(D(g_i))} \longrightarrow \mathcal{F}|_{D(g_i)} \tag{10.5}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

are isomorphisms. In particular, over each open set $D(f) \cap D(g_i) = D(fg_i)$, the map

$$\mathcal{F}(D(g_i))_f \longrightarrow \mathcal{F}(D(fg_i)) \tag{10.6}$$

which sends s/f^n with $s \in \mathcal{F}(D(g_i))$ to $s|_{D(fg_i)} \cdot f^{-n}$, is an isomorphism. For each i and j, the restriction of $\mathcal{F}$ to $D(g_ig_j)$ is also of tilde-type, so we also have the isomorphisms

$$\mathcal{F}(D(g_ig_j))_f \longrightarrow \mathcal{F}(D(fg_ig_j)) \tag{10.7}$$

Consider the sheaf exact sequence for the cover $D(g_1), \ldots, D(g_r)$:

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \textstyle\bigoplus_{i=1}^r \mathcal{F}(D(g_i)) \longrightarrow \bigoplus_{i,j=1}^r \mathcal{F}(D(g_ig_j)). \tag{10.8}$$

There is a similar sequence for the cover of $D(f)$ given by $D(fg_1), \ldots, D(fg_r)$. These sequences give the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}(U)_f & \longrightarrow & \bigoplus_{i=1}^r \mathcal{F}(D(g_i))_f & \longrightarrow & \bigoplus_{i,j=1}^r \mathcal{F}(D(g_ig_j))_f \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}(D(f)) & \longrightarrow & \bigoplus_{i=1}^r \mathcal{F}(D(fg_i)) & \longrightarrow & \bigoplus_{i,j=1}^r \mathcal{F}(D(fg_ig_j))
\end{array}$$

The top row in the diagram is the localization of (10.8) with respect to f, so both rows are exact. The three vertical maps are direct sums of the appropriate β-maps. By (10.6) and (10.7), the two vertical maps to the right are isomorphisms. By the 5-lemma, the left-most vertical map, which equals (10.4) is also an isomorphism, as we wanted to show. □

We write QCoh_X for the category of quasi-coherent sheaves on X, that is, the full subcategory of $\mathsf{Mod}_{\mathcal{O}_X}$ where the objects are quasi-coherent $\mathcal{O}_X$-modules (and whose morphisms are morphisms of $\mathcal{O}_X$-modules).

For an affine scheme $X = \operatorname{Spec} A$, the assignment $M \mapsto \widetilde{M}$ defines a functor from A-modules to QCoh_X. By the previous theorem, every quasi-coherent sheaf is of the form $\widetilde{M}$, and the module can be recovered by $M = \mathcal{F}(X)$.

Theorem 10.10. Let $X = \operatorname{Spec} A$ be an affine scheme. Then the functor

$$\widetilde{(-)} \colon \mathsf{Mod}_A \longrightarrow \mathsf{QCoh}_X$$

is an equivalence of categories. The global sections functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ defines a quasi-inverse.

Note that we always have an equality of A-modules $\Gamma(X, \widetilde{M}) = M$. On the other hand, the functorial map $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ from Corollary 10.6 is an isomorphism rather than an equality (hence the term *quasi-inverse* in the theorem).

By their link to modules, the quasi-coherent sheaves have very nice properties on affine schemes. For instance, the global section functor preserves exact sequences:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Corollary 10.11. Let $X = \operatorname{Spec} A$ and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules. Then the sequence

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0 \tag{10.9}$$

is also exact.

Proof Since the global section functor is left exact, we need only show that (10.9) is right exact. Let $C = \operatorname{Coker}(\mathcal{F}(X) \to \mathcal{F}''(X))$ and consider the exact sequence $\mathcal{F}(X) \to \mathcal{F}''(X) \to C \to 0$. Applying the exact functor $M \mapsto \widetilde{M}$, we get an exact sequence $\widetilde{\mathcal{F}(X)} \to \widetilde{\mathcal{F}''(X)} \to \widetilde{C} \to 0$. Since $\mathcal{F}$ and $\mathcal{F}''$ are quasi-coherent, the first map $\widetilde{\mathcal{F}(X)} \to \widetilde{\mathcal{F}''(X)}$ identifies with $\mathcal{F} \to \mathcal{F}''$. Since this is surjective by assumption, $\widetilde{C} = 0$. Therefore, $C = \Gamma(X, \widetilde{C}) = 0$ and $\mathcal{F}(X) \to \mathcal{F}''(X)$ is surjective. □

For a quasi-coherent sheaf, the restriction maps are closely related to localization. For each affine open $U \subset X$ and distinguished open $D(g) \subset U$, the natural map

$$\mathcal{F}(U)_g \longrightarrow \mathcal{F}(D(g)), \qquad s/g^n \longmapsto s|_{D(g)} \cdot g^{-n} \tag{10.10}$$

is an isomorphism. This means that the sections over $D(g)$ are completely determined by the sections over U.

This is one sense in which the sections of a quasi-coherent sheaf fit together coherently. Inside an affine open set U, the sections over smaller distinguished open subsets $D(g) \subset U$ are obtained from the sections over U by localization. On the other hand, Exercise 10.13.11 shows that the sections of a general $\mathcal{O}_X$-module can vary rather wildly.

In fact, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if the localization maps (10.10) are isomorphisms for every affine U and every $D(g) \subset U$. This follows from the next proposition. We include the case where $\mathcal{F}$ is only assumed to be a $\mathcal{O}_X$-module *presheaf*, since this will be useful in contructing quasi-coherent sheaves later on.

Proposition 10.12. Let X be a scheme and let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. Suppose that for every open affine $U = \operatorname{Spec} A \subset X$ and every $g \in A$, the natural map $\mathcal{F}(U)_g \to \mathcal{F}(D(g))$ is an isomorphism. Then the sheafification $\mathcal{F}^+$ is naturally a quasi-coherent $\mathcal{O}_X$-module, and $\mathcal{F}(U) \simeq \mathcal{F}^+(U)$ for every affine open set $U \subset X$.

Proof Let $U = \operatorname{Spec} A \subset X$ be an affine open and let $M = \mathcal{F}(U)$, viewed as an A-module. For each distinguished open $D(g) \subset U$, the assumed isomorphism gives $\widetilde{M}(D(g)) \simeq \mathcal{F}(D(g))$. Composing with the canonical map $\mathcal{F} \to \mathcal{F}^+$, we obtain maps

$$\widetilde{M}(D(g)) \longrightarrow \mathcal{F}^+(D(g)).$$

These maps are compatible with restrictions to smaller distinguished opens, so they define a morphism of $\mathscr{B}$-sheaves on the basis $\mathscr{B}$ of distinguished opens in U. Therefore they induce a morphism of sheaves $\widetilde{M} \to \mathcal{F}^+\big|_U$. This map is an isomorphism, because it is an isomorphism on stalks. Indeed, for $\mathfrak{p} \in U$, the stalk map is the composition

$$M_{\mathfrak{p}} = \varinjlim_{g \notin \mathfrak{p}} M_g \longrightarrow \varinjlim_{g \notin \mathfrak{p}} \mathcal{F}(D(g)) = \mathcal{F}_{\mathfrak{p}} \longrightarrow \mathcal{F}^+_{\mathfrak{p}}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The first map is an isomorphism by assumption, and the second because sheafification preserves stalks. Therefore, $\mathcal{F}^+|_U \simeq \widetilde{M}$ and $\mathcal{F}^+$ is quasi-coherent on X. Finally, taking sections over U, we obtain $\mathcal{F}^+(U) = \Gamma(U, \widetilde{M}) = M = \mathcal{F}(U)$. □

Example 10.13 (Quasi-coherent sheaves on $\mathbb{P}^1$)**.** Consider the projective line $\mathbb{P}^1_k$ over k, with the usual covering by $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$, glued together along their common open set $\operatorname{Spec} k[u, u^{-1}]$.

The sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$, as defined in Section 6.3, are all quasi-coherent, because they are isomorphic to the structure sheaf $\mathcal{O}_{U_i}$ when restricted to the standard opens.

More generally, we can describe any quasi-coherent sheaf on $\mathbb{P}^1_k$ as follows. A quasi-coherent sheaf on $\mathbb{P}^1_k$ is given by a triple (M_0, M_1, τ), where M_0 is a module over $\mathcal{O}_X(U_0) = k[u]$, where M_1 is a module over $\mathcal{O}_X(U_1) = k[u^{-1}]$ and where

$$\tau\colon M_0 \otimes_{k[u]} k[u, u^{-1}] \longrightarrow M_1 \otimes_{k[u^{-1}]} k[u, u^{-1}].$$

is an isomorphism of modules over $k[u, u^{-1}]$.

In terms of this description, the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ are given by the triples $M_0 = k[u]$, $M_1 = k[u^{-1}]$ and $\tau\colon k[u, u^{-1}] \to k[u, u^{-1}]$ is multiplication by u^{-n}.

△

Example 10.14 (Discrete valuation rings)**.** Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. The scheme $X = \operatorname{Spec} A$ consists of two points: the closed point s, corresponding to $\mathfrak{m}$, and the generic point ξ, corresponding to (0). Its nonempty open sets are $\{\xi\}$ and $X = \{\xi, s\}$. It follows that any $\mathcal{O}_X$-module $\mathcal{F}$ can be described by the following data:

(i) an A-module $M = \mathcal{F}(X)$,
(ii) a K-vector space $N = \mathcal{F}(\{\xi\})$,
(iii) an A-linear restriction map $\rho\colon M \to N$ where N is viewed as an A-module via $A \subset K$.

Indeed, any such triple (M, N, ρ) determines an $\mathcal{O}_X$-module $\mathcal{F}$ by setting $\mathcal{F}(X) = M$, $\mathcal{F}(\{\xi\}) = N$, $\mathcal{F}(\varnothing) = 0$, with the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{\xi\})$ given by ρ.

Note that $\rho\colon M \to N$ can be arbitrary. It need not be injective nor surjective, and it may even be the zero map (in that case M and N can be completely arbitrary A- modules).

In this setting, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if the induced map $\rho \otimes \mathrm{id}_K\colon M \otimes_A K \to N$ is an isomorphism of K-vector spaces. To see this, note that if $\mathcal{F}$ is quasi-coherent, then it is isomorphic to $\widetilde{M}$. This forces $N = \mathcal{F}(\{\xi\}) = M_{(0)} = M \otimes_A K$ and ρ to induce an isomorphism after tensoring with K. Conversely, if $\rho \otimes \mathrm{id}_K\colon M \otimes_A K \to N$ is an isomorphism, then $\mathcal{F}$ is isomorphic to $\widetilde{M}$, with $\mathcal{F}(\{\xi\}) = M_{(0)} = M \otimes_A K$, so it is quasi-coherent. △

10.4 Kernels, images and cokernels

Most of the constructions for modules over a ring have analogues for $\mathcal{O}_X$-modules and quasi-coherent sheaves. For instance, given a map of $\mathcal{O}_X$-modules $\phi\colon \mathcal{F} \to \mathcal{G}$, the kernel, image and cokernel of ϕ have natural $\mathcal{O}_X$-module structures. Here it is clear that the image

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

and cokernel *presheaves* have $\mathcal{O}_X$-module structures, and then Exercise 10.13.2 shows that also the associated sheaves are $\mathcal{O}_X$-modules.

If $\phi\colon \mathcal{F} \to \mathcal{G}$ is a map between quasi-coherent sheaves, then $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$ and $\operatorname{Coker}\phi$ are also quasi-coherent. To see this, we may restrict to an open affine $U = \operatorname{Spec} A$. On U, the restriction of ϕ is of the form $\widetilde{\alpha}$ for some map of A-modules $\alpha\colon M \to N$. Since $\widetilde{(-)}$ is an equivalence of categories, we have:

$$\operatorname{Ker}\phi = \widetilde{\operatorname{Ker}\alpha}, \qquad \operatorname{Im}\phi = \widetilde{\operatorname{Im}\alpha}, \qquad \operatorname{Coker}\phi = \widetilde{\operatorname{Coker}\alpha}.$$

In particular, the global sections over an affine open set U are computed as follows:

$$\Gamma(U, \operatorname{Ker}\phi) = \operatorname{Ker}\alpha_U, \qquad \Gamma(U, \operatorname{Im}\phi) = \operatorname{Im}\alpha_U, \qquad \Gamma(U, \operatorname{Coker}\phi) = \operatorname{Coker}\alpha_U.$$

Likewise, if $\mathcal{G} \subset \mathcal{F}$ is a subsheaf of an $\mathcal{O}_X$-module $\mathcal{F}$, then the quotient sheaf $\mathcal{F}/\mathcal{G}$ is naturally an $\mathcal{O}_X$-module. If $\mathcal{F}$ and $\mathcal{G}$ are both quasi-coherent, then so is $\mathcal{F}/\mathcal{G}$ and for every affine open subset $U \subset X$, we have

$$\Gamma(U, \mathcal{F}/\mathcal{G}) = \mathcal{F}(U)/\mathcal{G}(U).$$

Note that this formula does not hold for arbitrary open subsets (see Example 9.19).

Example 10.15. On $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$, the map $\mathcal{O}_{\mathbb{A}^2_k}^{\oplus 2} \to \mathcal{O}_{\mathbb{A}^2_k}$ defined by $(f,g) \mapsto xf + yg$ is a map of $\mathcal{O}_{\mathbb{A}^2_k}$-modules. The cokernel is given by $\mathcal{O}_{\mathbb{A}^2_k}/\mathcal{I}$, where $\mathcal{I} = \widetilde{(x,y)}$ is the ideal sheaf of the origin $P \in \mathbb{A}^2_k$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}^2_k} \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{A}^2_k} \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}/\mathcal{I} \longrightarrow 0,$$

where the first map sends $f \in \mathcal{O}_X(U)$ to $(-yf, xf)$. △

10.5 Direct sums and products

For a finite collection of sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_n$, their direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ is defined by

$$\Gamma(U, \bigoplus_{i=1}^n \mathcal{F}_i) = \bigoplus_{i=1}^n \mathcal{F}_i(U).$$

This is a sheaf with restriction maps defined componentwise. Locality holds because if $s = (s_1, \ldots, s_n) \in \bigoplus_{i=1}^n \mathcal{F}_i(U)$ restricts to 0 on a covering, then all $s_1 = \cdots = s_n = 0$ by Locality for the $\mathcal{F}_i$'s. Likewise, given local sections matching on the overlaps, one can glue componentwise.

More generally, for an arbitrary collection of sheaves $\{\mathcal{F}_i\}_{i\in I}$, we define the direct sum $\bigoplus_{i\in I} \mathcal{F}_i$ to be the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I} \mathcal{F}_i(U)$. If the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, the direct sum $\bigoplus_{i\in I} \mathcal{F}_i$ is naturally an $\mathcal{O}_X$-module, with multiplication being defined componentwise. We usually write $\mathcal{O}_X^n$ for the direct sum of n copies of $\mathcal{O}_X$.

Proposition 10.16. If $\{\mathcal{F}_i\}_{i\in I}$ is a collection of quasi-coherent sheaves, then the direct sum $\bigoplus_{i\in I} \mathcal{F}_i$ is quasi-coherent. If $X = \operatorname{Spec} A$, then for all A-modules M_i, we have

$$\widetilde{\bigoplus_{i\in I} M_i} \simeq \bigoplus_{i\in I} \widetilde{M}_i. \tag{10.11}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof As quasi-coherence can be checked locally on affine opens, it suffices to prove the formula (10.11). We may assume $X = \operatorname{Spec} A$.

Let $S(U) = \bigoplus_{i\in I} \widetilde{M}_i(U)$ be the direct sum presheaf. As direct sums commute with localization, there is, for each affine open $U \subset X$ and $g \in \mathcal{O}_X(U)$, an isomorphism

$$S(U)_g = \left(\bigoplus_{i\in I} \widetilde{M}_i(U)\right)_g \simeq \bigoplus_{i\in I} \widetilde{M}_i(U)_g \simeq \bigoplus_{i\in I} \widetilde{M}_i(D(g)) = S(D(g)).$$

By Proposition 10.12, S^+ is quasi-coherent and $\bigoplus_{i\in I} \widetilde{M}_i = S^+ \simeq \widetilde{S^+(X)} = \widetilde{\bigoplus_{i\in I} M_i}$. □

For a collection of sheaves $\{\mathcal{F}_i\}_{i\in I}$, we define the *direct product* $\prod_{i\in I} \mathcal{F}_i$ by:

$$\Gamma\left(U, \prod_{i\in I} \mathcal{F}_i\right) = \prod_{i\in I} \mathcal{F}_i(U). \tag{10.12}$$

This is a sheaf with componentwise restriction maps. Moreover, if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, $\prod_{i\in I} \mathcal{F}_i$ is naturally an $\mathcal{O}_X$-module.

If $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are quasi-coherent sheaves, then the product $\prod_{i=1}^n \mathcal{F}_i$ is also quasi-coherent. In fact, it is isomorphic to the direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$. Moreover, if $X = \operatorname{Spec} A$, then

$$\Big(\prod_{i=1}^n M_i\Big)^{\sim} = \prod_{i=1}^n \widetilde{M}_i$$

For an infinite collection of sheaves, however, it can happen that the direct product is not quasi-coherent. The reason is that direct products do not commute with localization in general (see Exercise 10.13.6).

10.6 Tensor products

Given two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we define their *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ as the sheaf associated to the presheaf

$$T(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \tag{10.13}$$

When the context is clear, we may simply write $\mathcal{F} \otimes \mathcal{G}$ for brevity.

More generally, we can define the tensor product of any finite number of $\mathcal{O}_X$-modules. In particular, for an integer $n \in \mathbb{N}$, we let $\mathcal{F}^{\otimes n}$ denote the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}$.

In the definition of the tensor product it necessary to sheafify, as the presheaf T is not always a sheaf (see Example 10.18 below). Fortunately, the following proposition allows us to understand the sections of tensor products of quasi-coherent sheaves over affine open subsets.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 10.17. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasi-coherent and for any open affine subset $U \subset X$, there is a canonical isomorphism

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \tag{10.14}$$

In particular, for $X = \operatorname{Spec} A$ and A-modules M and N,

$$\widetilde{M \otimes_A N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}. \tag{10.15}$$

Proof Let $T(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ be the tensor product presheaf. As in Proposition 10.16, we show that T satisfies the condition of Proposition 10.12. The main point is that tensor products commute with localization. For each $U = \operatorname{Spec} A$ and $g \in A$, we have

$$\begin{aligned} T(U)_g &= \big(\mathcal{F}(U) \otimes_A \mathcal{G}(U)\big)_g = \mathcal{F}(U)_g \otimes_{A_g} \mathcal{G}(U)_g \\ &= \mathcal{F}(D(g)) \otimes_{\mathcal{O}_X(D(g))} \mathcal{G}(D(g)) = T(D(g)). \end{aligned}$$

It follows that $T^+ = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasi-coherent and $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ for every affine open subset $U \subset X$.

Finally, if $X = \operatorname{Spec} A$, then $\Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) = M \otimes_A N$. Since $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ is quasi-coherent, $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = \widetilde{M \otimes_A N}$. □

Example 10.18. Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1_k}(1)$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^1_k}(-1)$. Then the global sections of the tensor product presheaf (10.13) are given by

$$\mathcal{F}(\mathbb{P}^1_k) \otimes_k \mathcal{G}(\mathbb{P}^1_k) = k^2 \otimes_k 0 = 0.$$

However, there is an isomorphism of sheaves

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{O}_{\mathbb{P}^1_k}(1) \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \mathcal{O}_{\mathbb{P}^1_k}(-1) \simeq \mathcal{O}_{\mathbb{P}^1_k},$$

so the tensor product sheaf has global sections equal to k. This means that the tensor product presheaf is not a sheaf. △

Example 10.19. Let $A = k[x, y, z]$, and $X = \mathbb{A}^3_k = \operatorname{Spec} A$. Consider the A-modules $M = k[x, y, z]/(x)$ and $N = k[x, y, z]/(y)$. Then

$$M \otimes_A N = k[x, y, z]/(x, y) \simeq k[z]. \tag{10.16}$$

Let $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. If we let $U = \mathbb{A}^3_k - \{(0, 0, 0)\}$, then arguing as in Example 4.18, we find that $\mathcal{F}(U) = M$ and $\mathcal{G}(U) = N$, so

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \simeq k[z].$$

On the other hand, by (10.16), $(\mathcal{F} \otimes_{\mathcal{O}_X(U)} \mathcal{G})(U) = \Gamma(U, \widetilde{k[z]}) \simeq k[z, z^{-1}]$. △

10.7 The Hom-sheaf

Recall that we defined the Hom-sheaf of two sheaves $\mathcal{F}$ and $\mathcal{G}$ as the sheaf

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}_{\mathsf{Ab}(U)}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we define the Hom-sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$, by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U). \tag{10.17}$$

where the right-hand side means the set of $\mathcal{O}_U$-linear maps.

The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is naturally an $\mathcal{O}_X$-module, but it is not always quasi-coherent, even if both $\mathcal{F}$ and $\mathcal{G}$ are. This is rooted in the fact that Hom does not commute with localization in general.

Example 10.20. Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$, and consider the modules $M = \bigoplus_{i\in\mathbb{N}} k[x]$ and $N = k[x]$. Then $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$ is not quasi-coherent. To see this, note that its global sections satisfy

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})(X) = \mathrm{Hom}_{k[x]}(M,N) = \prod_{i\in\mathbb{N}} k[x].$$

On the other hand, over the open set $D(x) \subset \mathbb{A}^1_k$, we have

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})(D(x)) = \mathrm{Hom}_{k[x]_x}(M_x,N_x) = \prod_{i\in\mathbb{N}} k[x]_x.$$

If $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$ were quasi-coherent, then the latter group would be isomorphic to the localization $(\prod_{i\in\mathbb{N}} k[x])_x$. However, this is not the case, since elements of $(\prod_{i\in\mathbb{N}} k[x])_x$ have bounded denominators (of the form p/x^r where $p \in \prod_{i\in\mathbb{N}} k[x]$), whereas $\prod_{i\in\mathbb{N}} k[x,x^{-1}]$ contains elements with unbounded negative exponents, such as $(1, x^{-1}, x^{-2}, \dots)$. △

10.8 Finiteness conditions for $\mathcal{O}_X$-modules

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $U \subset X$ be an open set. Given sections $s_i \in \mathcal{F}(U)$ for $i \in I$, there is an $\mathcal{O}_U$-linear map

$$\bigoplus_{i\in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U, \qquad (a_i)_{i\in I} \longmapsto \sum_{i\in I} a_i s_i.$$

We say that the sections s_i *generate* $\mathcal{F}|_U$ if this morphism is surjective. Since surjectivity can be checked on stalks, this is equivalent to asking that the germs $(s_i)_P$ generate the $\mathcal{O}_{X,P}$-module $\mathcal{F}_P$ for every point $P \in U$.

If there are global sections $s_i \in \mathcal{F}(X)$ that generate $\mathcal{F}$, then we say that $\mathcal{F}$ is *globally generated*. When $X = \operatorname{Spec} A$ is affine and $\mathcal{F} = \widetilde{M}$, the sections corresponding to elements $m_i \in M$ generate $\mathcal{F}$ if and only if the elements m_i generate the A-module M. In particular, every quasi-coherent sheaf on an affine scheme is globally generated by some set of global sections.

An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be of *finite type* if for every point $P \in X$, there is an open neighbourhood $U \subset X$ of P such that $\mathcal{F}|_U$ is generated by finitely many sections. Equivalently, there is a surjection $\mathcal{O}_U^{\oplus r} \twoheadrightarrow \mathcal{F}|_U$ for some $r \geqslant 0$. If X is Noetherian, we say that an $\mathcal{O}_X$-module is *coherent* if it is quasi-coherent and of finite type.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 10.21. Let X be a scheme and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:

(i) $\mathcal{F}$ is of finite type.

(ii) There is an affine covering $U_i = \operatorname{Spec} A_i$ of X and finitely generated A_i-modules M_i such that $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ for every i.

(iii) For every affine open subset $U = \operatorname{Spec} A$ of X, there is a finitely generated A-module M such that $\mathcal{F}|_U \simeq \widetilde{M}$.

Proof (i)⇒ (ii): Suppose that $\mathcal{F}$ is of finite type. For every point $P \in X$, choose an affine open neighbourhood $U = \operatorname{Spec} A$ of P such that $\mathcal{F}|_U$ is generated by finitely many sections. Since $\mathcal{F}$ is quasi-coherent, we have $\mathcal{F}|_U \simeq \widetilde{\mathcal{F}(U)}$. The finite set of generators gives a surjection $\mathcal{O}_U^{\oplus r} \twoheadrightarrow \mathcal{F}|_U$. Taking global sections over the affine scheme U, we get a surjection $A^{\oplus r} \twoheadrightarrow \mathcal{F}(U)$, because global sections are exact on quasi-coherent sheaves over affine schemes. Thus $\mathcal{F}(U)$ is a finitely generated A-module. Therefore (ii) holds.

(ii)⇒ (iii): Let $U = \operatorname{Spec} A$ be an affine open subset of X. Choose finitely many distinguished opens $D(f_1), \ldots, D(f_r)$ covering U, such that each $D(f_j)$ is contained in one of the affine opens $U_i = \operatorname{Spec} A_i$ from the given covering. Since $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ with M_i finitely generated, it follows that each $\mathcal{F}(D(f_j))$ is a finitely generated A_{f_j}-module. Since $\mathcal{F}$ is quasi-coherent on $U = \operatorname{Spec} A$, we have $\mathcal{F}(D(f_j)) \simeq \mathcal{F}(U)_{f_j}$. So each localization $\mathcal{F}(U)_{f_j}$ is a finitely generated A_{f_j}-module. By Proposition 8.7, the A-module $\mathcal{F}(U)$ is finitely generated. Since $\mathcal{F}$ is quasi-coherent, $\mathcal{F}|_U \simeq \widetilde{\mathcal{F}(U)}$, so (iii) holds.

(iii)⇒ (i). Let $P \in X$, and choose an affine open neighbourhood $U = \operatorname{Spec} A$ of P. By assumption, $\mathcal{F}|_U \simeq \widetilde{M}$ for some finitely generated A-module M. If $m_1, \ldots, m_r$ generate M, then the corresponding sections of $\widetilde{M}$ generate $\widetilde{M}$ over U. Hence $\mathcal{F}|_U$ is generated by finitely many sections. Therefore $\mathcal{F}$ is of finite type. □

Example 10.22. If $X = \operatorname{Spec} A$, then an $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type if and only if $\mathcal{F} \simeq \widetilde{M}$ for some finitely generated A-module. △

Example 10.23. Let $X = \operatorname{Spec} \mathbb{Z}$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, then $\mathcal{F} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module M, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^{\oplus r} \oplus T$, where T is a finite direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$. This implies that

$$\mathcal{F} = \mathcal{O}_X^{\oplus r} \oplus \mathcal{T} \tag{10.18}$$

where $\mathcal{T}$ is a sheaf having stalks $\mathcal{T}_p = 0$ for all but finitely many p and $\mathcal{T}_{(0)} = 0$. $\mathcal{T}$ is called a *torsion sheaf* (see Exercise 10.13.30). △

Example 10.24. The argument of the previous example in fact applies over any PID A: for $X = \operatorname{Spec} A$, any $\mathcal{O}_X$-module of finite type sheaf must have the form $\widetilde{M}$ for $M = A^{\oplus r} \oplus T$ where T is a finitely generated torsion module. In particular, any finite type sheaf on the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ decomposes as $\mathcal{F} = \mathcal{O}_X^{\oplus r} \oplus \mathcal{T}$ where $\mathcal{T}$ is a torsion sheaf. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 10.25. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on a Noetherian scheme X, and assume that $\mathcal{F}$ is coherent.

(i) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is quasi-coherent.

(ii) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is coherent if $\mathcal{G}$ is coherent.

(iii) For every affine open subset $U = \operatorname{Spec} A$ of X,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U = \operatorname{Hom}_A(\mathcal{F}(U),\mathcal{G}(U))\,\widetilde{}\,. \tag{10.19}$$

(iv) The stalk at a point $x \in X$ is given by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x = \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x). \tag{10.20}$$

Proof It suffices to prove the formula on affine open subsets. Let $U = \operatorname{Spec} A$ be affine. Since X is Noetherian and $\mathcal{F}$ is coherent, we may write $\mathcal{F}|_U = \widetilde{M}$ for some finitely generated A-module M. Since A is Noetherian, M is finitely presented. Also write $\mathcal{G}|_U = \widetilde{N}$ for the A-module $N = \mathcal{G}(U)$. In this situation, $\operatorname{Hom}(M,-)$ commutes with localization (Proposition C.7), so for every $g \in A$, there is an isomorphism of A_g-modules

$$\operatorname{Hom}_A(M,N)_g = \operatorname{Hom}_{A_g}(M_g,N_g), \tag{10.21}$$

These isomorphisms are compatible with further localization. Hence they identify the restrictions of the two sheaves $\operatorname{Hom}_A(M,N)\,\widetilde{}$ and $\mathcal{H}om_{\mathcal{O}_U}(\widetilde{M},\widetilde{N})$ on all distinguished open subsets of U. Therefore

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U) = \operatorname{Hom}_A(M,N)\,\widetilde{}\,.$$

This shows that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is quasi-coherent and the formula (10.19).

If $\mathcal{G}$ is coherent, then N is a finitely generated A-module. Since A is Noetherian and M is finitely generated, $\operatorname{Hom}_A(M,N)$ is finitely generated. Therefore, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is coherent by the formula above.

Finally, using (10.19) to compute stalks, gives the formula (10.20). □

Remark 10.26. There is a more general definition of 'coherence' for $\mathcal{O}_X$-modules on arbitrary schemes. Specifically, an $\mathcal{O}_X$-module $\mathcal{F}$ is defined to be coherent if it is of finite type, and for every open $U \subset X$ and every morphism of $\mathcal{O}_U$-modules $\phi\colon \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$, the kernel of ϕ is of finite type. The main reason for introducing this stronger notion of coherence, is that the coherent sheaves form an abelian category even in the non-Noetherian setting. The downside is that the coherence condition is very difficult to check in general, and in fact, for some schemes, even affine ones, the structure sheaf $\mathcal{O}_X$ itself is not coherent.

In this book, we will be interested in coherent modules exclusively in the Noetherian setting, and here the notion is equivalent to being of finite type.

10.9 Pushforwards

For a morphism of schemes $f\colon X \to Y$, and an $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward sheaf $f_*\mathcal{F}$ defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$, is an $\mathcal{O}_Y$-module (see Example 10.1). In this section, we explore when the pushforward $f_*\mathcal{F}$ of a quasi-coherent sheaf is again quasi-coherent.

We begin by considering the affine case. Consider a morphism $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

induced by a ring map $\phi\colon A \to B$. Via ϕ, any B-module M can also be considered as a module over A; we write M_A for this A-module. This works well with respect to localization: if $g \in A$, then there is an isomorphism $M_{\phi(g)} = (M_A)_g$ as A_g-modules.

Proposition 10.27. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism and let M be a B-module. Then

$$f_*\widetilde{M} = \widetilde{M_A}. \tag{10.22}$$

Proof By Proposition 2.25, we have $f^{-1}D(g) = D(\phi(g))$. This gives the equalities

$$(f_*\widetilde{M})(D(g)) = \widetilde{M}(f^{-1}D(g)) = \widetilde{M}(D(\phi(g))) = M_{\phi(g)} = (M_A)_g,$$

where the last equality holds by the above paragraph. These identifications are compatible with the restriction maps of the sheaves involved, so we conclude that $f_*\widetilde{M} = \widetilde{M_A}$. □

Example 10.28. For a morphism $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$, we have $f_*\mathcal{O}_{\operatorname{Spec} B} = \widetilde{B}$, where B is considered as an A-module via the corresponding ring map $\phi\colon A \to B$. Moreover, the canonical map $f^\sharp\colon \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_{\operatorname{Spec} B}$ is exactly the sheaf map associated to $A \to B$. For $g \in A$, the map $f^\sharp_{D(g)}$ is given by the localized map $A_g \to B_{\phi(g)}$. △

The pushforward of a quasi-coherent sheaf is not always quasi-coherent, but it is for most morphisms encountered in practice. The following theorem gives a sharp criterion for when this holds.

Theorem 10.29 (Quasi-coherence of pushforwards). Let $f\colon X \to Y$ be a morphism. Assume that Y admits an affine open cover $\mathcal{V}$ such that for every $V \in \mathcal{V}$, the inverse image $f^{-1}(V)$ can be written as a finite union of affine open subsets $U_1, \dots, U_r$, and for all i, j, the intersection $U_i \cap U_j$ is again a finite union of affine open subsets. Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward $f_*\mathcal{F}$ is quasi-coherent on Y.

Proof To show that $f_*\mathcal{F}$ is quasi-coherent, it suffices to show that it is quasi-coherent when restricted to an affine cover. Therefore, we may assume that $Y = \operatorname{Spec} A$ is one of the affine open sets in $\mathcal{V}$.

Let $M = \Gamma(Y, f_*\mathcal{F}) = \mathcal{F}(X)$. Note that $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module, so we may view M as an A-module. We claim that the natural map

$$\widetilde{M} \longrightarrow f_*\mathcal{F} \tag{10.23}$$

which sends s/g^n with $s \in \mathcal{F}(X)$ to $s|_{f^{-1}D(g)} \cdot g^{-n}$, is an isomorphism. It suffices to check this over the distinguished open subsets $D(g) \subset Y$ for $g \in A$.

By our assumptions, we may cover X by finitely many affine open subsets $U_i = \operatorname{Spec} B_i$, and then cover the intersections $U_i \cap U_j$ by finitely many affines $U_{ijk} = \operatorname{Spec} B_{ijk}$.

Let $\phi_i\colon A \to B_i$ and $\phi_{ijk}\colon A \to B_{ijk}$ be the ring maps that induce the morphisms $U_i \to Y$ and $U_{ijk} \to Y$. We also consider the distinguished open sets $U_{i,g} = D(\phi_i(g)) \subset U_i$ and $U_{ijk,g} = D(\phi_{ijk}(g)) \subset U_{ijk}$.

With these definitions, a section of $(f_*\mathcal{F})(Y) = \mathcal{F}(X)$ is given by a collection of elements $s_i \in \mathcal{F}(U_i)$ that agree over the various intersections U_{ijk}. Equivalently, the following

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

sequence is exact:

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j,k} \mathcal{F}(U_{ijk}). \tag{10.24}$$

There is a similar sequence for the cover of $f^{-1}D(g)$ given by $U_{i,g}$, for $i = 1, \ldots, r$. These sequences fit into the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_g & \longrightarrow & \prod_i \mathcal{F}(U_i)_g & \longrightarrow & \prod_{i,j,k} \mathcal{F}(U_{ijk})_g \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(f^{-1}D(g)) & \longrightarrow & \prod_i \mathcal{F}(U_{i,g}) & \longrightarrow & \prod_{i,j,k} \mathcal{F}(U_{ijk,g}) \end{array}$$

The top row in the diagram is the localization of (10.24) with respect to g, so it is exact. The three vertical maps are induced by the restriction maps. As U_i and U_{ijk} are all affine and $\mathcal{F}$ is quasi-coherent, the second and third vertical maps are isomorphisms. By the 5-lemma, the left-most vertical map, $\mathcal{F}(X)_g \to \mathcal{F}(f^{-1}D(g))$, which is exactly the map $\Gamma(D(g), \widetilde{M}) \to \Gamma(D(g), f_*\mathcal{F})$, is an isomorphism as well. □

The conditions in the theorem may seem technical, but they are quite straightforward to verify in most situations. Here is one example:

Corollary 10.30. Let $f\colon X \to Y$ be a morphism with X Noetherian and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

Proof Let $V \subset Y$ be any affine open subset. Since X is Noetherian, the open set $f^{-1}(V)$ is quasi-compact and therefore admits a finite affine open cover $\{U_1, \ldots, U_r\}$. Similarly, each intersection $U_i \cap U_j$ is quasi-compact, so admits a finite affine cover. Hence the conditions of Theorem 10.29 are satisfied and $f_*\mathcal{F}$ is quasi-coherent. □

Example 10.31. The pushforward of a coherent sheaf is not always coherent. Consider the structure morphism $f\colon \mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$. Then $f_*\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}$ is quasi-coherent, but not of finite type on $\operatorname{Spec} \mathbb{C}$, because it corresponds to the $\mathbb{C}$-module $\mathbb{C}[x]$, which is not finitely generated. △

The following example shows that even $f_*\mathcal{O}_X$ can fail to be quasi-coherent in general.

Example 10.32. Let $X = \coprod_{i\in\mathbb{N}} \mathbb{A}^1_k$ be the disjoint union of countably infinitely many copies of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, and let $f\colon X \to \mathbb{A}^1_k$ be the morphism that is the identity on each component. Then $f_*\mathcal{O}_X$ is not quasi-coherent. As in Example 10.20, the reason is that localization does not commute with infinite products. Indeed, we have

$$\Gamma(\mathbb{A}^1_k, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i\in\mathbb{N}} k[x], \tag{10.25}$$

whereas on $D(x) \subset \mathbb{A}^1_k$ one has

$$\Gamma(D(x), f_*\mathcal{O}_X) = \Gamma(f^{-1}D(x), \mathcal{O}_X) = \prod_{i\in\mathbb{N}} k[x, x^{-1}]. \tag{10.26}$$

As explained in Example 10.20, the module (10.26) is not a localization of (10.25), so $f_*\mathcal{O}_X$ is not quasi-coherent. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

10.10 Pullbacks

Let $f\colon X \to Y$ be a morphism of schemes. Recall that we defined the pushforward functor which produces an $\mathcal{O}_Y$-module $f_*\mathcal{F}$ from an $\mathcal{O}_X$-module $\mathcal{F}$. There is an opposite operation, called the *pullback*, which produces an $\mathcal{O}_X$-module $f^*\mathcal{G}$ on X from an $\mathcal{O}_Y$-module $\mathcal{G}$ on Y.

The sheaf $f^*\mathcal{G}$ satisfies a universal property similar to that of $f^{-1}\mathcal{G}$. Namely, maps of $\mathcal{O}_X$-modules $f^*\mathcal{G} \to \mathcal{F}$ are in one-to-one correspondence with maps of $\mathcal{O}_Y$-modules $\mathcal{G} \to f_*\mathcal{F}$. The precise statement is the following theorem:

Theorem 10.33. Let $f\colon X \to Y$ be a map of schemes and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then there exists an $\mathcal{O}_X$-module $f^*\mathcal{G}$ on X along with functorial bijections

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \tag{10.27}$$

for each $\mathcal{O}_X$-module $\mathcal{F}$. The sheaf $f^*\mathcal{G}$ is unique up to isomorphism.

As f^* is a left adjoint to f_*, and left adjoints are unique up to isomorphism, the pullback $f^*\mathcal{G}$ is determined up to isomorphism by the bijections (10.27). We will therefore refer to a sheaf satisfying this condition as the *pullback of $\mathcal{G}$ by f*.

Pullbacks of quasi-coherent sheaves on affines

We begin by studying the pullback in the most important special case, namely for quasi-coherent sheaves via morphisms of affine schemes.

Suppose $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and $f\colon X \to Y$ is induced by a ring map $\phi\colon A \to B$. Consider a quasi-coherent sheaf of the form $\mathcal{G} = \widetilde{N}$ on Y, where N is an A-module. As B is an A-algebra, the tensor product $N \otimes_A B$ is naturally a B-module, and we define the pullback of $\mathcal{G}$ by the formula

$$f^*\widetilde{N} = \widetilde{N \otimes_A B}. \tag{10.28}$$

This is also functorial. Indeed, any map of quasi-coherent sheaves $\widetilde{N} \to \widetilde{N'}$ is induced by a map of A-modules $N \to N'$. This in turn induces a map of B-modules $N \otimes_A B \to N' \otimes_A B$ and consequently a map of $\mathcal{O}_X$-modules $f^*\widetilde{N} \to f^*\widetilde{N'}$.

In this case, the bijections in (10.27) come from the formal properties of Hom and the tensor product. More precisely, we recall the following natural bijection, which holds for all A-modules N and B-modules M:

$$\operatorname{Hom}_B(N \otimes_A B, M) = \operatorname{Hom}_A(N, M_A) \tag{10.29}$$

This bijection sends a B-linear map ϕ on the left-hand side to the A-linear map $N \to M_A$ given by $n \mapsto \phi(n \otimes 1)$. This map is functorial in M and N. (See Exercise 10.13.25.)

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Using this, and Proposition 10.5, we have for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{\mathcal{G}(Y)\otimes_A B},\mathcal{F})\\ &= \operatorname{Hom}_A(\mathcal{G}(Y)\otimes_A B,\mathcal{F}(X))\\ &= \operatorname{Hom}_A(\mathcal{G}(Y),\mathcal{F}(X)_A)\\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\widetilde{\mathcal{G}(Y)},f_*\mathcal{F})\\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G},f_*\mathcal{F}).\end{aligned}$$

The fact that these isomorphisms are functorial in $\mathcal{F}$ and $\mathcal{G}$ follows from the functoriality of the isomorphism in (10.29) and the isomorphism β in Proposition 10.5.

General pullbacks

We will now define the pullback for an arbitrary $\mathcal{O}_Y$-module $\mathcal{G}$ for a general morphism of schemes $f\colon X\to Y$. To do this, we recall the inverse image sheaf in Section 9.6. Given a sheaf $\mathcal{G}$ on Y, the inverse image sheaf $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$f_p^{-1}\mathcal{G}(U) = \varinjlim_{V\supset f(U)} \mathcal{G}(V).$$

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, each $\mathcal{G}(V)$ is an $\mathcal{O}_Y(V)$-module for each $V\supset f(U)$. This means that $f_p^{-1}\mathcal{G}(U)$ is naturally a module over the ring $f_p^{-1}\mathcal{O}_Y(U)$. Explicitly, over an open set $U\subset X$, we can represent elements $f_p^{-1}\mathcal{O}_Y(U)$ and $f_p^{-1}\mathcal{G}(U)$ by $a\in\mathcal{O}_Y(V)$ and $s\in\mathcal{G}(V)$ for some open set $V\subset Y$ containing $f(U)$, and the product $a\cdot s\in\mathcal{G}(V)$ is well-defined, and its class defines an element in $f_p^{-1}\mathcal{G}(U)$.

Also the ring $\mathcal{O}_X(U)$ becomes a module over the ring $f_p^{-1}\mathcal{O}_Y(U)$ via the ring map $f_p^{-1}\mathcal{O}_Y(U)\to\mathcal{O}_X(U)$ induced by $(a,V)\mapsto f^\sharp(a)|_U$. We then define the presheaf $f_p^*\mathcal{G}$ by

$$(f_p^*\mathcal{G})(U) = f_p^{-1}\mathcal{G}(U)\otimes_{f_p^{-1}\mathcal{O}_Y(U)}\mathcal{O}_X(U)$$

Note that this is now an $\mathcal{O}_X(U)$-module. The pullback $f^*\mathcal{G}$ is then defined to be the associated sheaf. It is an $\mathcal{O}_X$-module by Exercise 10.13.2.

While this definition of $f^*\mathcal{G}$ is general and natural, it is not very practical for concrete computations. The next step will be to show that $f^*\mathcal{G}$ satisfies the universal property (10.27). This will imply that when $\mathcal{G}$ is quasi-coherent, the sheaf $f^*\mathcal{G}$ restricts to the pullback sheaf defined earlier for quasi-coherent sheaves. It is this description that is used in most computations.

Proof of Theorem 10.33 By Theorem 9.26, we have a natural isomorphism

$$\operatorname{Hom}_{\mathsf{PAb}(X)}(f_p^{-1}\mathcal{G},\mathcal{F})\simeq\operatorname{Hom}_{\mathsf{Ab}(Y)}(\mathcal{G},f_*\mathcal{F}).$$

Under this bijection, $\mathcal{O}_Y$-linear maps $\mathcal{G}\to f_*\mathcal{F}$ on the right correspond to $f_p^{-1}\mathcal{O}_Y$-linear maps of presheaves $f_p^{-1}\mathcal{G}\to\mathcal{F}$ on the left. Furthermore, by the usual change-of-rings formula (10.29), these are in natural bijection with the set of $\mathcal{O}_X$-linear maps of presheaves $f_p^{-1}\mathcal{G}\otimes_{f_p^{-1}\mathcal{O}_Y}\mathcal{O}_X\to\mathcal{F}$. Finally, by sheafification, these are in turn in one-to-one correspondence with $f^{-1}\mathcal{O}_Y$-linear maps $f^{-1}\mathcal{G}\to\mathcal{F}$, as required. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 10.34 (Restrictions)**.** Let $i\colon U \to X$ be the inclusion of an open subset and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the pullback is given by the restriction $i^*\mathcal{F} = \mathcal{F}|_U$, which is naturally an $\mathcal{O}_U$-module. △

Proposition 10.35. Let $f\colon X \to Y$ be a morphism. Then:
(i) $f^*\mathcal{O}_Y = \mathcal{O}_X$.
(ii) f^* is a right-exact functor.
(iii) If $g\colon W \to X$ is a morphism, then $(f \circ g)^* = g^* \circ f^*$.
(iv) For any two $\mathcal{O}_Y$-modules $\mathcal{G}$ and $\mathcal{H}$, we have

$$f^*(\mathcal{G} \oplus \mathcal{H}) = f^*\mathcal{G} \oplus f^*\mathcal{H} \quad \text{and} \quad f^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) = f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{H}.$$

(v) For $x \in X$, we have $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$.

Proof (i): To show that $f^*\mathcal{O}_Y = \mathcal{O}_X$, it suffices by Yoneda's Lemma to show that $\mathcal{O}_X$ satisfies the required universal property (10.27). This follows from the identities

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) &= \mathcal{F}(X) = f_*\mathcal{F}(Y) \\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, f_*\mathcal{F}).\end{aligned}$$

(iii) and (iv): These also follow from the universal property: the sheaf $f^*(g^*\mathcal{G})$ is an $\mathcal{O}_X$-module, and for any $\mathcal{O}_X$-module $\mathcal{F}$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(f^*(g^*\mathcal{G}), \mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_Y}(g^*\mathcal{G}, f_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, g_*f_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, (g \circ f)_*\mathcal{F}).\end{aligned}$$

Here all equalities are canonical bijections. Therefore $f^*(g^*\mathcal{G})$ and $(g \circ f)^*\mathcal{G}$ are isomorphic. Similar arguments show the identities in (iv).

(v): The description of the stalks of $f^*\mathcal{G}$ follows because $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$, and $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{X,x}$, so that $(f^*\mathcal{G})_x = (\mathcal{G}_{f(x)}) \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$. Here we have also used the fact that sheafification preserves stalks.

(ii): If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of $\mathcal{O}_Y$-modules, then the induced sequence obtained by applying f^* can be obtained by applying $f_p^{-1}(-)$ (which is exact), then $(-) \otimes_{f_p^{-1}\mathcal{O}_Y} \mathcal{O}_X$ (which is right-exact) and then sheafifying (which is exact). □

Proposition 10.36. Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then $f^*\mathcal{G}$ is quasi-coherent on X and for each pair of affine open subsets $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, we have

$$f^*\mathcal{G}|_U = \mathcal{G}(V) \widetilde{\otimes_{\mathcal{O}_Y(V)}} \mathcal{O}_X(U) \tag{10.30}$$

Proof This follows formally from the universal property. Let $i\colon U \to X$, $j\colon V \to Y$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

denote the inclusions and write $g = f|_U$. Then we have bijections

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_U}(g^*j^*\mathcal{G}, \mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_V}(j^*\mathcal{G}, g_*\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, j_*g_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*i_*\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, i_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_U}(i^*f^*\mathcal{G}, \mathcal{F}).\end{aligned}$$

This shows that $i^*f^*\mathcal{G} \simeq g^*j^*\mathcal{G}$. By Example 10.34, this means that $f^*\mathcal{G}|_U = (f|_U)^*(\mathcal{G}|_V)$. On the other hand, by the affine case, the pullback $(f|_U)^*(\mathcal{G}|_V)$ is isomorphic to the sheaf to the right in (10.30), as required. □

Example 10.37. Consider $X = \operatorname{Spec} k[x, y]$, $Y = \operatorname{Spec} k[u, v]$, and the morphism

$$f\colon \operatorname{Spec} k[x, y] \longrightarrow \operatorname{Spec} k[u, v], \qquad (u, v) \mapsto (x^2, y^2)$$

The map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ acts on sections by composition: it sends a local section $a(u, v)/b(u, v)$ to $a(x^2, y^2)/b(x^2, y^2)$.

In this example, there is no natural $\mathcal{O}_X$-structure on $f^{-1}\mathcal{O}_Y$: the element x^2 defines a section of $f^{-1}\mathcal{O}_Y$ over $D(x)$, but multiplying this with $y/x \in \mathcal{O}_X(D(x))$ gives xy, which is not of the form $a(x^2, y^2)/b(x^2, y^2)$. This is the reason why one considers the tensor product with $\mathcal{O}_X$ in the definition of $f^*\mathcal{G}$. △

While the inverse image functor f^{-1} is exact, the pullback f^* is only right-exact. The reason is that tensoring by a module is only a right-exact operation in general.

Example 10.38. Consider the ideal sheaf sequence of the origin $P \in \mathbb{A}^1_k = \operatorname{Spec} k[t]$:

$$0 \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_{\mathbb{A}^1_k} \longrightarrow \iota_*\mathcal{O}_P \longrightarrow 0. \tag{10.31}$$

where $\iota\colon \operatorname{Spec} \kappa(P) \to \mathbb{A}^1_k$ is the closed embedding. This sequence is obtained by applying tilde to the sequence

$$0 \longrightarrow k[t] \xrightarrow{\ t\ } k[t] \longrightarrow k[t]/(t) \longrightarrow 0.$$

Applying ι^* corresponds to tensoring by $\kappa(P) = k[t]/(t)$. Applying this to (10.31) we obtain the tilde of the following sequence of $\kappa(P)$-modules

$$0 \longrightarrow k[t] \otimes_{k[t]} \kappa(P) \xrightarrow{t\otimes\mathrm{id}} k[t] \otimes_{k[t]} \kappa(P) \longrightarrow k[t]/(t) \otimes_{k[t]} \kappa(P) \longrightarrow 0.$$

However the latter sequence is not exact because the map $t \otimes \mathrm{id}$ is the zero map. △

Pulling back sections

For a morphism $f\colon X \to Y$ and a quasi-coherent sheaf $\mathcal{G}$ on Y, one can also pull back sections of $\mathcal{G}$ to sections of $f^*\mathcal{G}$. By the universal property (10.27) there are canonical maps

$$\eta\colon \mathcal{G} \longrightarrow f_*f^*\mathcal{G}, \qquad \varepsilon\colon f^*f_*\mathcal{F} \longrightarrow \mathcal{F}. \tag{10.32}$$

Here η is map of $\mathcal{O}_Y$-modules and ε is a map of $\mathcal{O}_X$-modules. These are obtained by applying (10.27) to the two identity maps $f^*\mathcal{G} \to f^*\mathcal{G}$ and $f_*\mathcal{F} \to f_*\mathcal{F}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For a section $s \in \mathcal{G}(V)$, we define the *pullback* to be the section

$$f^*(s) = \eta(s) \in \Gamma(f^{-1}V, f^*\mathcal{G}).$$

In the special case when $\mathcal{G} = \mathcal{O}_Y$, we have $f^*\mathcal{O}_Y = \mathcal{O}_X$ and the pullback of $s \in \mathcal{O}_Y(V)$ is simply given by $f^*(s) = f^\sharp(s) \in \mathcal{O}_X(f^{-1}V)$.

In the affine setting, the maps η and ε are described as follows:

Example 10.39. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be induced by a ring map $A \to B$, and $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. Then $\eta\colon \widetilde{N} \to f_*f^*\widetilde{N}$ corresponds to the map of A-modules

$$N \to (N \otimes_A B)_A, \qquad n \mapsto (n \otimes 1).$$

Likewise, $\varepsilon\colon f^*f_*\widetilde{M} \to \widetilde{M}$ corresponds to the map of B-modules

$$M_A \otimes_A B \longrightarrow M, \qquad m \otimes b \longmapsto bm$$

For $s \in \mathcal{G}(Y) = N$, the pullback $f^*(s) \in (f^*\mathcal{G})(X) = N \otimes_A B$ is given by $s \otimes 1$.

This example is helpful for computing pullbacks $f^*(s)$ for general morphisms: cover V and $f^{-1}V$ by affine schemes, then the affine pullbacks glue to a section $f^*(s)$. △

Example 10.40. Let X be a scheme over a field k and let $f\colon X \to \operatorname{Spec} k$ be the structure morphism. Then for an $\mathcal{O}_X$-module $\mathcal{F}$, we have $f_*\mathcal{F} = \mathcal{F}(X)$ (the constant sheaf on $\operatorname{Spec} k$), and $f^{-1}f_*\mathcal{F} = \mathcal{F}(X)$ is the constant sheaf on X. The canonical map $f^*f_*\mathcal{F} \to \mathcal{F}$ is given by the 'multiplication map'

$$\mathcal{F}(X) \otimes_k \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$$

induced by $s \otimes f \mapsto f \cdot s$. △

10.11 Relative Spec

Let X be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-*algebras*. This means that $\mathcal{A}$ is a quasi-coherent sheaf and for each open set $U \subset X$, the ring $\mathcal{A}(U)$ is an algebra over the ring $\mathcal{O}_X(U)$. These algebra structures are required to be compatible with the restrictions: for all $V \subset U$, the diagram

$$\begin{array}{ccc} \mathcal{A}(U) & \longrightarrow & \mathcal{A}(V) \\ \uparrow & & \uparrow \\ \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(V) \end{array}$$

commutes, where all the arrows are ring maps.

The main example is $\mathcal{A} = f_*\mathcal{O}_Y$ for a morphism $f\colon Y \to X$. This is an $\mathcal{O}_X$-algebra via the map $f^\sharp\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$.

If $U \subset X$ is an affine open, we can form the affine scheme $\operatorname{Spec} \mathcal{A}(U)$ and the morphism

$$\pi_U\colon \operatorname{Spec} \mathcal{A}(U) \longrightarrow U$$

induced by the ring map $\mathcal{O}_X(U) \to \mathcal{A}(U)$. Our goal is to glue these together to a scheme $\mathbf{Spec}\, \mathcal{A}$ equipped with a morphism

$$\pi\colon \mathbf{Spec}\, \mathcal{A} \longrightarrow X$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

such that for each affine $U \subset X$, $\pi^{-1}(U) \simeq \operatorname{Spec} \mathcal{A}(U)$ and $\pi^{-1}(U) \to U$ coincides with the morphism π_U. The scheme $\mathbf{Spec}\, \mathcal{A}$ is called the *relative spectrum* of $\mathcal{A}$.

To see how the gluing works, let $V \subset U$ be two affine opens. The restriction map $\mathcal{A}(U) \to \mathcal{A}(V)$ induces a morphism $\sigma_{V,U} \colon \operatorname{Spec} \mathcal{A}(V) \to \operatorname{Spec} \mathcal{A}(U)$ which fits into the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{A}(V) & \xrightarrow{\sigma_{V,U}} & \operatorname{Spec} \mathcal{A}(U) \\ \big\downarrow{\scriptstyle \pi_V} & & \big\downarrow{\scriptstyle \pi_U} \\ V & \hookrightarrow & U \end{array} \tag{10.33}$$

The diagram is actually Cartesian because $\mathcal{A}$ is quasi-coherent and the restriction map induces

$$\mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \simeq \mathcal{A}(V) \tag{10.34}$$

This means that $\sigma_{V,U}$ is an isomorphism between $\operatorname{Spec} \mathcal{A}(V)$ and the open set $\pi_U^{-1}(V)$ of $\operatorname{Spec} \mathcal{A}(U)$.

Now let $\{U_i\}_{i\in I}$ be an affine cover of X. If $W \subset U_i \cap U_j$ is an affine open, then quasi-coherence of $\mathcal{A}$ gives canonical isomorphisms

$$\mathcal{A}(U_j) \otimes_{\mathcal{O}_X(U_j)} \mathcal{O}_X(W) \xrightarrow{\simeq} \mathcal{A}(W) \xleftarrow{\simeq} \mathcal{A}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(W). \tag{10.35}$$

Applying Spec to the composition from left to right gives us an isomorphism $\tau_{ij}^W \colon \pi_i^{-1}(W) \simeq \pi_j^{-1}(W)$. These are compatible with restrictions to smaller affine opens, so they glue to an isomorphism

$$\tau_{ij} \colon \pi_i^{-1}(U_i \cap U_j) \longrightarrow \pi_j^{-1}(U_i \cap U_j).$$

These isomorphisms satisfy the cocycle condition. Indeed, for any open affine $W \subset U_i \cap U_j \cap U_k$, the isomorphisms τ_{ij}^W, τ_{jk}^W and τ_{ik}^W arise from an isomorphism with $\mathcal{A}(W)$ as in (10.35), so $\tau_{jk}^W \circ \tau_{ij}^W = \tau_{ik}^W$. Since this happens for any affine W, we conclude that $\tau_{jk} \circ \tau_{ij} = \tau_{ik}$ on $U_i \cap U_j \cap U_k$. Therefore, the schemes $\operatorname{Spec} \mathcal{A}(U_i)$ glue to a scheme $\mathbf{Spec}\, \mathcal{A}$. The morphisms $\operatorname{Spec} \mathcal{A}(U_i) \to U_i$ are also compatible with the gluing and define a morphism $\pi \colon \mathbf{Spec}\, \mathcal{A} \to X$.

By construction, we also have a canonical identification

$$\pi_* \mathcal{O}_{\mathbf{Spec}\, \mathcal{A}} = \mathcal{A}.$$

Indeed, on each U_i, $\Gamma(U_i, \pi_* \mathcal{O}_{\mathbf{Spec}\, \mathcal{A}}) = \Gamma(\pi_i^{-1} U_i, \mathcal{O}_{\mathbf{Spec}\, \mathcal{A}}) = \mathcal{A}(U_i)$.

If $\phi \colon \mathcal{A} \to \mathcal{B}$ is a map of $\mathcal{O}_X$-algebras (that is, the map $\mathcal{A}(U) \to \mathcal{B}(U)$ is a map of $\mathcal{O}_X(U)$-algebras for every open U), the induced morphisms $\operatorname{Spec}(\mathcal{B}(U)) \to \operatorname{Spec}(\mathcal{A}(U))$ glue to a morphism $\mathbf{Spec}(\mathcal{B}) \to \mathbf{Spec}(\mathcal{A})$ over X. We have therefore defined a contravariant functor from the category of $\mathcal{O}_X$-algebras to the category of X-schemes.

Example 10.41. It follows from the definitions that $\mathbf{Spec}\, \mathcal{O}_X = X$. More generally,

$$\mathbf{Spec}\, \mathcal{O}_X[x_1, \dots, x_n] = \mathbb{A}^n_X = X \times_{\mathbb{Z}} \mathbb{A}^n,$$

with structure map $\mathbb{A}^n_X \to X$. △

Recall that a morphism $f \colon X \to Y$ is said to be *affine* if there is an open affine cover $\{V_i\}_{i\in I}$ of Y such that $f^{-1}(V_i)$ is affine for every i. If this holds, the pushforward $f_* \mathcal{O}_X$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is quasi-coherent by Theorem 10.29, so it makes sense to form $\mathbf{Spec}(f_*\mathcal{O}_X)$. There is a canonical morphism

$$X \longrightarrow \mathbf{Spec}(f_*\mathcal{O}_X), \tag{10.36}$$

constructed by gluing together the canonical maps $f^{-1}V \to \operatorname{Spec}\mathcal{O}_X(f^{-1}V)$ for $V \subset Y$ open affine (Corollary 4.8).

Proposition 10.42. Let $f\colon X \to Y$ be a morphism. The following are equivalent:
(i) f is affine.
(ii) For every affine open $V \subset Y$, the preimage $f^{-1}(V)$ is affine.
(iii) $f_*\mathcal{O}_X$ is quasi-coherent and the canonical map $X \to \mathbf{Spec}(f_*\mathcal{O}_X)$ is an isomorphism.

Proof (ii)⇒(i) is immediate.

(i)⇒(iii): Let $\mathcal{A} = f_*\mathcal{O}_X$. Let $V = \operatorname{Spec} A$ be one of the V_i and write $f^{-1}V = \operatorname{Spec} B$, so that $f^{-1}V \to V$ is induced by a ring map $\phi\colon A \to B$. For each $g \in A$, we have $f^{-1}(D(g)) = D(\phi(g))$ in $\operatorname{Spec} B$ and therefore

$$\mathcal{A}(D(g)) = \Gamma(f^{-1}D(g), \mathcal{O}_X) = \Gamma(D(\phi(g)), \mathcal{O}_X) = B_{\phi(g)} = \mathcal{A}(V)_g.$$

So $\mathcal{A}$ is quasi-coherent. The canonical morphism $X \to \mathbf{Spec}(f_*\mathcal{O}_X)$ restricts over each V_i to the morphism $f^{-1}(V_i) \to \operatorname{Spec}(\mathcal{O}_X(f^{-1}V_i))$, which is an isomorphism because $f^{-1}V_i$ is affine. Therefore the global map is also an isomorphism.

(iii)⇒(ii): If $X \simeq \mathbf{Spec}(f_*\mathcal{O}_X)$ over Y, then for every affine open $V \subset Y$, we have $f^{-1}(V) \simeq \pi^{-1}(V) \simeq \operatorname{Spec}(\Gamma(f^{-1}V, \mathcal{O}_X))$, which shows that f is affine. □

Corollary 10.43. Let $f\colon X \to Y$ be a finite morphism. Then for every affine open $V = \operatorname{Spec} A \subset Y$, the preimage $f^{-1}(V) = \operatorname{Spec} B$ is affine with B a finitely generated A-module.

Proof By definition, finite morphisms are affine. Therefore, if $V = \operatorname{Spec} A$, then $f^{-1}(V) = \operatorname{Spec} B$ for some A-algebra B. Finally, B is finitely generated as an A-module by Proposition 8.7 (ii). □

Proposition 10.44. Affine morphisms, finite morphisms and closed embeddings are stable under base change and composition, and they are local on the target.

Proof We prove this for affine morphisms. The two other cases are similar.

Stability under base change is proved exactly as in Proposition 7.21.

For composition, let $f\colon X \to Y$ and $g\colon Y \to Z$ be affine morphisms and let $W \subset Z$ be an affine open. Then $g^{-1}(W)$ is an affine open of Y, and hence $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is affine as well, using Proposition 8.7 twice. Such W cover Z, so $g \circ f$ is affine.

Finally, suppose $\{W_j\}$ is an open cover of Y such that each restriction $f^{-1}(W_j) \to W_j$ is affine. If $V \subset W_j$ is an affine open, then $f^{-1}(V)$ is affine by Proposition 8.7 applied to $f^{-1}(W_j) \to W_j$. These V form an affine open cover of Y with affine preimages, so f is affine. Hence being affine is local on the target. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

10.12 Closed subschemes and ideal sheaves

As an application of the theory of quasi-coherent sheaves, we now give a more systematic treatment of closed subschemes. As we discussed in Section 4.6, the prototype example of a closed subscheme is the affine subscheme $\operatorname{Spec} A/I \to \operatorname{Spec} A$ defined by an ideal $I \subset A$. The general definition will involve *ideal sheaves* $\mathcal{I} \subset \mathcal{O}_X$ rather than ideals.

To explain the correspondence, let $\iota\colon Y \to X$ be a closed embedding. By definition, there is an open affine cover $U_i = \operatorname{Spec} A_i$ of X such that (i) $\iota^{-1}(U_i)$ is affine and (ii) $\mathcal{O}_X(U_i) \to \mathcal{O}_Y(\iota^{-1}U_i)$ is surjective. Here (i) implies that ι is affine, so by Theorem 10.29, the pushforward $\iota_*\mathcal{O}_Y$ is quasi-coherent. The condition (ii) shows that $\iota^\sharp$ is surjective. Indeed, over each U_i it is the tilde of the map $\mathcal{O}_X(U_i) \to \mathcal{O}_Y(\iota^{-1}U_i)$, so it is surjective on stalks.

If we set $\mathcal{I} = \operatorname{Ker}(\iota^\sharp)$, we obtain the *ideal sheaf sequence*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0. \tag{10.37}$$

Since $\iota_*\mathcal{O}_Y$ is quasi-coherent, so is $\mathcal{I}$, being the kernel of a map of quasi-coherent sheaves. Thus every closed subscheme determines a quasi-coherent ideal sheaf $\mathcal{I}$.

Over each affine open $U = \operatorname{Spec} A$, if $I = \mathcal{I}(U)$, then $\mathcal{I}|_U = \widetilde{I}$ and 10.37 becomes

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{A} \longrightarrow \widetilde{A/I} \longrightarrow 0. \tag{10.38}$$

Conversely, given a quasi-coherent ideal sheaf $\mathcal{I}$, we may form $Y = \mathbf{Spec}(\mathcal{O}_X/\mathcal{I})$ and the morphism $\iota\colon Y \to X$ which is induced by $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$. On an affine open $U = \operatorname{Spec} A$, with $I = \mathcal{I}(U)$, this morphism is simply the closed embedding $\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$.

Theorem 10.45. Let X be a scheme. Then the map

$$\Big\{\text{closed subschemes } \iota\colon Y \to X\Big\} \longrightarrow \Big\{\text{quasi-coherent ideal sheaves } \mathcal{I} \subset \mathcal{O}_X\Big\}$$

sending $\iota\colon Y \to X$ to $\mathcal{I} = \operatorname{Ker}(\mathcal{O}_X \to \iota_*\mathcal{O}_Y)$ is a bijection.

Proof For injectivity, let $\iota\colon Y \to X$ be a closed embedding and let $\mathcal{I} = \operatorname{Ker}(\iota^\sharp)$. By Proposition 10.42, the morphism ι is affine, and $Y \to \mathbf{Spec}(\iota_*\mathcal{O}_Y)$ is an isomorphism over X. But $\iota_*\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$, so the X-scheme Y is determined by $\mathcal{I}$. Hence the map $\iota \mapsto \mathcal{I}$ is injective.

For surjectivity, let $\mathcal{I}$ be a quasi-coherent ideal sheaf and let $Y = \mathbf{Spec}(\mathcal{O}_X/\mathcal{I})$ with structure morphism $\pi\colon Y \to X$. For affine open $U \subset X$, the restriction of π to $\pi^{-1}(U)$ identifies with

$$\operatorname{Spec}(\mathcal{O}_X(U)/\mathcal{I}(U)) \longrightarrow \operatorname{Spec}(\mathcal{O}_X(U))$$

In particular, π is a closed embedding. Moreover, since $\pi_*\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$, we have $\operatorname{Ker}(\mathcal{O}_X \to \pi_*\mathcal{O}_Y) = \operatorname{Ker}(\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}) = \mathcal{I}$. Therefore, the closed embedding π maps to the ideal sheaf $\mathcal{I}$ and we get surjectivity. □

Corollary 10.46. The assignment $I \mapsto \operatorname{Spec}(A/I)$ gives a one-to-one correspondence between the set of ideals of a ring A and the set of closed subschemes of $\operatorname{Spec} A$.

Working with ideal sheaves rather than morphisms into X is both more natural and more

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

convenient in practice. It also removes the ambiguity of thinking of a closed subscheme as an equivalence class of morphisms.

Example 10.47. If Y and Z are two closed subschemes defined by ideal sheaves $\mathcal{I}$ and $\mathcal{J}$ respectively, then the scheme-theoretic intersection is defined by the ideal sheaf $\mathcal{I} + \mathcal{J} \subset \mathcal{O}_X$. △

Example 10.48 (Reduced scheme structure on a closed subset)**.** Let $Z \subset X$ be a closed subset. For each open set $U \subset X$, define

$$\mathcal{I}(U) = \{\, s \in \mathcal{O}_X(U) \mid s(P) = 0 \text{ for all } P \in Z \cap U \,\}, \tag{10.39}$$

Then $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$. We claim that $\mathcal{I}$ is quasi-coherent.

Let $U = \operatorname{Spec} A$ be an affine open set of X and write $Z \cap U = V(\mathfrak{a})$ for a (unique) radical ideal $\mathfrak{a} \subset A$. Then $f \in A$ satisfies $f(\mathfrak{p}) = 0$ if and only if $f \in \mathfrak{p}$, so

$$\mathcal{I}(U) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \sqrt{\mathfrak{a}} = \mathfrak{a}.$$

Likewise, for $g \in A$, we have $Z \cap D(g) = V(\mathfrak{a}) \cap D(g) = V(\mathfrak{a}A_g) \subset A_g$, so

$$\mathcal{I}(D(g)) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a}A_g)} \mathfrak{p} = \sqrt{\mathfrak{a}A_g} = \mathfrak{a}_g.$$

Therefore, $\mathcal{I}|_U \simeq \widetilde{\mathfrak{a}}$, and $\mathcal{I}$ is quasi-coherent.

Note that each ideal $\mathcal{I}(U)$ is a radical ideal. This means that the corresponding subscheme is reduced. Hence for any closed subset $Z \subset X$ there is a reduced closed subscheme with underlying set Z.

In particular, for $Z = X$, we obtain the reduced scheme X_{red} of Exercise 4.11.23. △

As a side-remark, if $\mathcal{I} \subset \mathcal{O}_X$ is any ideal sheaf, the quotient $\mathcal{O}_X/\mathcal{I}$ is always the structure sheaf of a locally ringed space (see Exercise 10.13.39). It is the quasi-coherence of $\mathcal{I}$ which guarantees that this is a scheme.

Example 10.49 (An ideal sheaf which is not quasi-coherent)**.** Let $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ and let $P \in \mathbb{A}^1_k$ be the origin, corresponding to the ideal (t). Define an ideal sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \begin{cases} \mathcal{O}_{\mathbb{A}^1_k}(U), & P \notin U, \\ 0, & P \in U. \end{cases}$$

On the level of stalks, we have $\mathcal{I}_Q = \mathcal{O}_{\mathbb{A}^1_k, Q}$ for any $Q \neq P$, and $\mathcal{I}_P = 0$. This means that the support of $\mathcal{O}_{\mathbb{A}^1_k}/\mathcal{I}$ is the one-point set $\{P\}$ and $\mathcal{O}_{\mathbb{A}^1_k}/\mathcal{I}$ is identified with the skyscraper sheaf at P with value $\mathcal{O}_{\mathbb{A}^1_k, P}$. The corresponding locally ringed space $(P, \mathcal{O}_{\mathbb{A}^1_k, P})$ is not a scheme: if it were, it would be affine (a single point in a scheme is contained in some affine open), hence isomorphic to $\operatorname{Spec}(\mathcal{O}_{\mathbb{A}^1_k, P})$. However, as $\mathcal{O}_{\mathbb{A}^1_k, P} = k[t]_{(t)}$, the latter consists of *two* points (the generic point and the closed point).

In this example, the ideal sheaf $\mathcal{I}$ is not quasi-coherent, as $\mathcal{I}(X) = 0$, while

$$\mathcal{I}(D(t)) = \mathcal{O}_X(D(t)) = k[t]_t \neq \mathcal{I}(X)_t = 0.$$

See Exercise 10.13.37 for a related example. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

10.13 Exercises

Exercise 10.13.1. For each of the schemes below, describe the $\mathcal{O}_X$-modules on X.

a) X is the scheme obtained by gluing $\operatorname{Spec}\mathbb{Z}_{(2)}$ and $\operatorname{Spec}\mathbb{Z}_{(3)}$ along their common open subscheme $\operatorname{Spec}\mathbb{Q}$.
b) X is the scheme obtained by gluing two copies of $\operatorname{Spec}\mathbb{Z}_{(2)}$ along $\operatorname{Spec}\mathbb{Q}$.
c) X is the scheme obtained by gluing $\operatorname{Spec}\mathbb{Z}_{(p_i)}$ for all primes p_i, along $\operatorname{Spec}\mathbb{Q}$.

Exercise 10.13.2. Suppose that $\mathcal{F}$ is a *presheaf* of $\mathcal{O}_X$-modules (i.e. a presheaf satisfying the usual $\mathcal{O}_X$-module axioms). Show that the associated sheaf $\mathcal{F}^+$ is naturally an $\mathcal{O}_X$-module.

Exercise 10.13.3. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Show that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)$.

Exercise 10.13.4. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves.

a) Show that $\operatorname{Ker}\phi$ satisfies the following universal property: Any map of sheaves $\nu : \mathcal{H} \to \mathcal{F}$ such that $\nu \circ \phi = 0$ factors via a unique map $\overline{\nu} : \mathcal{H} \to \operatorname{Ker}\phi$.
b) Show that $\operatorname{Im}\phi$ satisfies the following universal property: Given a map of sheaves $\alpha : \mathcal{F} \to \mathcal{H}$ and $\beta : \mathcal{H} \to \mathcal{G}$ such that $\beta \circ \alpha = \phi$, there is a unique morphism $t : \mathcal{H} \to \operatorname{Im}\phi$ factoring β.
c) Show that $\operatorname{Coker}\phi$ satisfies the following universal property: Given a map $\psi\colon \mathcal{G} \to \mathcal{H}$ with $\psi \circ \phi = 0$, there is a unique map $t : \operatorname{Coker}\phi \to \mathcal{H}$ factoring ψ.
d) Show that if ϕ is a map of $\mathcal{O}_X$-modules, the corresponding universal properties in a),b) and c) hold in the category of $\mathcal{O}_X$-modules as well.
e) Show that a sequence of $\mathcal{O}_X$-modules is exact if and only if it is exact as a sequence of sheaves.

HINT: For b), use the explicit description of $\operatorname{Im}\phi$. For c), the universal property of sheafification may be helpful.

Exercise 10.13.5 (General direct sums)**.** Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of sheaves. We define the direct sum $\bigoplus_{i\in I}\mathcal{F}_i$ as the sheaf associated to the presheaf

$$\big(\bigoplus_{i\in I}\mathcal{F}_i\big)_p(U) = \bigoplus_{i\in I}\mathcal{F}_i(U). \tag{10.40}$$

a) Show that if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, then $\bigoplus_{i\in I}\mathcal{F}_i$ is an $\mathcal{O}_X$-module.
b) Show that the direct sum has canonical inclusions $\epsilon_i\colon \mathcal{F}_i \to \bigoplus_i \mathcal{F}_i$, which have the following universal property: for any family of maps $\eta_i\colon \mathcal{F}_i \to \mathcal{G}$ there is a unique map $\eta\colon \bigoplus_i \mathcal{F}_i \to \mathcal{G}$ such that $\eta_i = \eta \circ \epsilon_i$.
c) Show that if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, then the sheaves $\bigoplus_{i\in I}\mathcal{F}_i$, and the maps η_i and ϵ_i are maps of $\mathcal{O}_X$-modules.
d) Show that if $X = \operatorname{Spec} A$, then for all A-modules M_i, we have $\widetilde{\bigoplus_{i\in I} M_i} \simeq \bigoplus_{i\in I}\widetilde{M}_i$.
e) Show that if the $\mathcal{F}_i$ are quasi-coherent, then $\bigoplus_{i\in I}\mathcal{F}_i$ is again quasi-coherent

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

and for each open *affine* $U \subset X$,

$$\left(\bigoplus_{i\in I} \mathcal{F}_i\right)(U) = \bigoplus_{i\in I} \mathcal{F}_i(U).$$

f) Let $X = \coprod_{n=1}^{\infty} \operatorname{Spec} \mathbb{C}$ be the disjoint union of countably many copies of $\operatorname{Spec} \mathbb{C}$. For each $n \in \mathbb{N}$, let $\iota_n \colon p_n = \operatorname{Spec} \mathbb{C} \to X$ be the open embedding of the n-th copy of $\operatorname{Spec} \mathbb{C}$ and let $\mathcal{F}_n = \iota_{n*}\mathbb{C}$ the skyscraper sheaf at p_n. Consider the direct sum presheaf $\mathcal{S} = \bigoplus_{n=1}^{\infty} \mathcal{F}_n$. Show that $\mathcal{S}(X) \neq \bigoplus_{n=1}^{\infty} \mathcal{F}_n(X)$. Deduce that $\mathcal{S}$ is not a sheaf.

g) Let $\{\mathcal{F}_i\}$ be a family of sheaves on X and $U \subset X$ an open set. If U is quasi-compact, show that $(\bigoplus_i \mathcal{F}_i)(U) = \bigoplus_i \mathcal{F}_i(U)$.

h) Conclude that if X is Noetherian, then the presheaf defined in (10.40) is a sheaf.

Exercise 10.13.6 (Direct products)**.** Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of sheaves.

a) Show that the direct product presheaf $\prod_{i\in I} \mathcal{F}_i$ defined in the text is a sheaf.

b) Show that $\prod_{i\in I} \mathcal{F}_i$ is an $\mathcal{O}_X$-module if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules.

c) Show that the direct product has canonical projections $\pi_i \colon \prod_i \mathcal{F}_i \to \mathcal{F}_i$ having the universal property: for any collection of maps $\epsilon_i \colon \mathcal{G} \to \mathcal{F}_i$ there is a map $\eta \colon \mathcal{G} \to \prod_i \mathcal{F}_i$ such that $\pi_i \circ \eta = \epsilon_i$.

d) Let $X = \operatorname{Spec} A$ and let $\{M_i\}_{i\in I}$ be a collection of A-modules. Show that $\widetilde{\prod_i M_i}$ satisfies the universal property of $\prod_{i\in I} \widetilde{M_i}$ in the subcategory QCoh_X.

e) Let $X = \operatorname{Spec} \mathbb{Z}$ and $M_i = \mathbb{Z}$ for $i \in \mathbb{N}$. Show that $\prod_{i\in\mathbb{N}} \widetilde{M_i}$ and $\widetilde{\prod_{i\in\mathbb{N}} M_i}$ are not isomorphic. Deduce that a direct product of quasi-coherent sheaves need not be quasi-coherent in general. HINT: Infinite direct products do not commute with localization.

Exercise 10.13.7 (Hom is left-exact)**.** Show that a sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ of $\mathcal{O}_X$-modules is exact if and only if for all open subsets $U \subset X$ and for all $\mathcal{O}_U$-modules $\mathcal{G}$, the sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}'|_U) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}|_U) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}, \mathcal{F}''|_U)$$

of $\mathcal{O}_X(U)$-modules is exact. Likewise, show that $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of $\mathcal{O}_X$-modules is exact if and only if for all open subsets $U \subset X$ and all $\mathcal{O}_U$-modules $\mathcal{G}$, the sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}''|_U, \mathcal{G}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}'|_U, \mathcal{G})$$

of $\mathcal{O}_X(U)$-modules is exact. In particular, the Hom-functor is left exact in both arguments.

Exercise 10.13.8 (Sums and intersections of submodules)**.** Let $\mathcal{G}$ be an $\mathcal{O}_X$-module and let $\{F_i\}_{i\in I}$ be a collection of $\mathcal{O}_X$-submodules of $\mathcal{G}$.

a) We define the sum $\sum \mathcal{F}_i$ as the sheaf associated to $U \mapsto \sum_i \mathcal{F}_i(U)$. Show that $\sum \mathcal{F}_i$ is an $\mathcal{O}_X$-module which coincides with the image of the canonical map $\bigoplus_{i\in I} \mathcal{F}_i \to \mathcal{G}$. Show that the stalk of $\sum \mathcal{F}_i$ at a point $x \in X$ is given by the sums of the stalks $\sum_i \mathcal{F}_{i,x}$ inside $\mathcal{G}_x$.

b) We define the intersection $\bigcap_i \mathcal{F}_i$ as the sheaf associated to $U \mapsto \bigcap_i \mathcal{F}_i(U)$. Show that $\bigcap_i \mathcal{F}_i$ is naturally an $\mathcal{O}_X$-module and that it coincides with the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

kernel of the canonical map $\mathcal{G} \to \prod_{i\in I} \mathcal{G}/\mathcal{F}_i$. Show that when I is finite, $(\bigcap_i \mathcal{F}_i)_x = \bigcap_i \mathcal{F}_{i,x}$ inside $\mathcal{G}_x$.

c) Let $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ and consider the family of ideals (x^n) in $k[x]$. Let $\mathcal{J}_n = \widetilde{(x^n)}$. Show that $\mathcal{I} = \bigcap_{n\geqslant 0} \mathcal{J}_n$ defines an ideal sheaf on $\mathbb{A}^1_k$. Show that $\mathcal{I}(\mathbb{A}^1_k) = 0$ but $\mathcal{I}(D(x)) \neq 0$. Deduce that $\mathcal{I}$ is not quasi-coherent.

Exercise 10.13.9. Show that for an affine morphism $f\colon X \to Y$, the pushforward $f_*\mathcal{O}_X$ is of finite type if and only if f is of finite type.

Exercise 10.13.10. Let X be a scheme. A directed system of sheaves is a family of sheaves $\{\mathcal{F}_i\}_{i\in I}$ indexed by a directed set I, together with morphisms of sheaves $\phi_{ij}\colon \mathcal{F}_i \to \mathcal{F}_j$ for each $i \leqslant j$ such that (i) ϕ_{ii} is the identity map on F_i; and (ii) For $i \leqslant j \leqslant k$, we have $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$. We define the direct limit sheaf as the sheaf associated to the presheaf

$$(\varinjlim \mathcal{F}_i)(U) = \varinjlim \mathcal{F}_i(U). \tag{10.41}$$

a) Show that there are sheaf maps $\phi_i\colon \mathcal{F}_i \to \varinjlim \mathcal{F}_i$ satisfying the following compatibility condition: For all $i \leqslant j$, we have $\phi_j \circ \phi_{ij} = \phi_i$.

b) Show that $\varinjlim \mathcal{F}_i$ satisfies the following universal property: if there exists another sheaf $\mathcal{G}$ with morphisms $\psi_i\colon \mathcal{F}_i \to \mathcal{G}$ satisfying the compatibility condition, then there exists a unique morphism of sheaves $u\colon \mathcal{F} \to \mathcal{G}$ such that $u \circ \phi_i = \psi_i$ for all $i \in I$.

c) Show that if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, then $\varinjlim \mathcal{F}_i$ is naturally an $\mathcal{O}_X$-module.

d) Show that if each $\mathcal{F}_i$ is quasi-coherent, then $\varinjlim \mathcal{F}_i$ is quasi-coherent, and:
   (i) For each affine open set $U \subset X$, $(\varinjlim \mathcal{F}_i)(U) = \varinjlim \mathcal{F}_i(U)$.
   (ii) For each $x \in X$, we have $(\varinjlim \mathcal{F}_i)_x = \varinjlim (\mathcal{F}_i)_x$.

Exercise 10.13.11 (Godement sheaves)**.** Given any collection of abelian groups $\{A_x\}_{x\in X}$ indexed by the points x of X, we can define a sheaf $\mathcal{A}$ by

$$\mathcal{A}(U) = \prod_{x\in U} A_x,$$

and whose restriction maps to smaller open subsets are just the projections onto the corresponding smaller products.

a) Show that $\mathcal{A}$ is a sheaf.

b) If we assume that each A_x be a module over the local ring $\mathcal{O}_{X,x}$, show that sheaf $\mathcal{A}$ becomes an $\mathcal{O}_X$-module.

c) For a presheaf $\mathcal{F}$ consider the sheaf $\Pi(\mathcal{F})$ defined by $\Pi(\mathcal{F})(U) = \prod_{x\in U} \mathcal{F}_x$. This is sometimes called the 'sheaf of discontinuous sections'. Show that there is a canonical map of presheaves $\sigma : \mathcal{F} \to \Pi(\mathcal{F})$ so that $\mathcal{F}^+ = \operatorname{Im}\sigma$.

Exercise 10.13.12 (Stalks of the Hom-sheaf)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules.

a) For $x \in X$, show that there is a natural map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x) \tag{10.42}$$

Show that this is an isomorphism if X is Noetherian and $\mathcal{F}$ is coherent (for any $\mathcal{O}_X$-module $\mathcal{G}$).

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

b) Show that the map (10.42) may fail to be surjective. HINT: Let $\mathcal{F}$ be a skyscraper sheaf and $\mathcal{G}$ a constant sheaf; then $\mathcal{H}om(F, G)$ is the zero-sheaf.
c) Show that the map (10.42) may fail to be injective. HINT: Let $U = X - x$ and let $\mathcal{F} = \mathcal{G} = i_!\mathbb{Z}$, where $i : U \to X$ is the inclusion; then $\mathcal{F}_x = \mathcal{G}_x = 0$.

Exercise 10.13.13. If $s \in \mathcal{F}(U)$, we define the *support of* s as the subset of points $x \in X$ such that $s_x \in \mathcal{F}_x$ is nonzero.

a) Show that the support of a section $s \in \mathcal{F}(U)$ is closed in U.
b) If $s, t \in \mathcal{F}(U)$, show that $\operatorname{Supp}(st) \subset \operatorname{Supp}(s) \cap \operatorname{Supp}(t)$ and that $\operatorname{Supp}(s+t) \subset \operatorname{Supp}(s) \cup \operatorname{Supp}(t)$.
c) Show that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_X$-modules, then $\operatorname{Supp}(\phi(s)) \subset \operatorname{Supp}(s)$.

Exercise 10.13.14 (Support of a sheaf)**.** For a sheaf $\mathcal{F}$ on a space X, we define the *support* as the subset of points $x \in X$ such that the stalk $\mathcal{F}_x$ is nonzero.

a) Consider $X = \operatorname{Spec} A$ and M some finitely generated A-module. Show that
$$\operatorname{Supp}(\widetilde{M}) = \operatorname{Supp}(M) = \{\, \mathfrak{p} \mid M_{\mathfrak{p}} \neq 0 \,\} = V(\operatorname{Ann}(M))$$
where $\operatorname{Ann}(M)$ is the annihilator ideal $\operatorname{Ann}(M) = \{\, a \in A \mid a \cdot m = 0 \text{ for all } m \in M \,\}$.
b) Show that if $\mathfrak{a} \subset A$ is an ideal, then $\operatorname{Supp}(\widetilde{A/\mathfrak{a}}) = V(\mathfrak{a})$.
c) Show that on a scheme X, if $\mathcal{F}$ is a quasi-coherent sheaf of finite type, then $\operatorname{Supp}(\mathcal{F})$ is closed.
d) Let $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$, and consider the $\mathbb{C}[t]$-module $M = \bigoplus_{i \in \mathbb{N}} \mathbb{C}[t]/(t-i)$. Compute $\operatorname{Supp}(\widetilde{M})$ and show that it is not closed.
e) Show that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-algebras, then $\operatorname{Supp}(\mathcal{F})$ is closed. HINT: A ring is the zero ring if and only if $1 = 0$.

Exercise 10.13.15 (Subsheaf with support)**.** Let $Z \subset X$ be a closed subset and $\mathcal{F}$ a sheaf on X. For an open set $U \subset X$, define $\Gamma_{Z \cap U}(U, \mathcal{F})$ to be the subgroup of $\mathcal{F}(U)$ consisting of sections s with $\operatorname{Supp}(s) \subset Z \cap U$. Show that this is a sheaf. This is usually denoted by $\mathcal{H}^0_Z(\mathcal{F})$. Show that the functor $\mathcal{F} \mapsto \Gamma_{Z \cap U}(U, \mathcal{F})$ is left exact.

Exercise 10.13.16. Let X be the nodal cubic curve and consider the normalization morphism $f\colon \mathbb{A}^1_k \to X$ of Example A.46, which identifies the two points 0 and 1 in $\mathbb{A}^1_k$.

a) Show that the presheaf $f_p^{-1}\mathcal{O}_X$ consists of rational functions g on $\mathbb{A}^1_k$ such that $g(0) = g(1)$. Show that this is not a sheaf.
b) Compute $f^{-1}\mathcal{O}_X$ and $f^*\mathcal{O}_X$.

Exercise 10.13.17. Let $X = \operatorname{Spec} A$ be an affine scheme and let
$$\cdots \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \cdots$$
be an exact sequence of quasi-coherent sheaves. Show that
$$\cdots \longrightarrow \mathcal{F}_{i-1}(X) \longrightarrow \mathcal{F}_i(X) \longrightarrow \mathcal{F}_{i+1}(X) \longrightarrow \cdots$$
is also exact.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 10.13.18. Let k be a field. Show that

(i) The skyscraper sheaf of k on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ at the origin 0 is quasi-coherent.

(ii) The skyscraper sheaf of $k(t)$ on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ at the origin 0 is *not* quasi-coherent. HINT: Consider sections over $U = D(t)$.

Exercise 10.13.19. Let $f\colon \mathbb{A}^2_k = \operatorname{Spec} k[x,y] \to \operatorname{Spec} k$ be the structure morphism.

a) Show that $f^{-1}\mathcal{O}_{\operatorname{Spec} k}$ is a constant sheaf with value k and describe the morphism $\mathcal{O}_{\operatorname{Spec} k} \to f_* f^{-1}\mathcal{O}_{\operatorname{Spec} k}$.

b) Show that $f_*\mathcal{O}_{\mathbb{A}^2_k}$ is the constant sheaf with value $k[x,y]$ on $\operatorname{Spec} k$ and $f^{-1}f_*\mathcal{O}_{\mathbb{A}^2_k}$ is the constant sheaf on $\mathbb{A}^2_k$ with value $k[x,y]$.

c) Describe the map $f^{-1}f_*\mathcal{O}_{\mathbb{A}^2_k} \to \mathcal{O}_{\mathbb{A}^2_k}$

Exercise 10.13.20. Let $f\colon \mathbb{P}^1_k \to \operatorname{Spec} k$ be the structure morphism and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1_k}(-1)$. Compute $f_*\mathcal{F}$ and show that $f^{-1}f_*\mathcal{F} \neq \mathcal{F}$.

Exercise 10.13.21. Let $X = \operatorname{Spec} \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Show that the sheaf $\mathcal{F} = \widetilde{M}$ is nonzero but satisfies $\mathcal{F}(P) = 0$ for every $P \in X$.

Exercise 10.13.22. Let X be a scheme.

a) Show that the direct sum of two $\mathcal{O}_X$-modules of finite type is again of finite type.

b) Show that the '2-out-of-3'-property holds for $\mathcal{O}_X$-module of finite type. That is, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, and if two of $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are of finite type, then so is the third.

c) Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of $\mathcal{O}_X$-module of finite type. Show that $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$ and $\operatorname{Coker}\phi$ are all of finite type. HINT: Apply Exercise 10.13.22.

Exercise 10.13.23. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules and let $x \in X$ be a point. Show that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

Exercise 10.13.24. Let $f\colon X \to Y$ be a morphism and let $\mathcal{G}$ a be a quasi-coherent $\mathcal{O}_Y$-module. Show that the stalk of $f^*\mathcal{G}$ at $x \in X$ is given by $\mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$.

Exercise 10.13.25. Show that the map in (10.29) is a bijection, and that the isomorphism is functorial in M and N.

Exercise 10.13.26. Show directly that the pullback $f^*\mathcal{G}$ can be constructed by gluing together sheaves of the form $\widetilde{M \otimes_A B}$. HINT: Cover X by affines that map into affines in Y. The adjunction with f_* and Yoneda's Lemma might make the gluing easier.

Exercise 10.13.27. Let $f\colon X \to Y$ be a morphism and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.

a) Show that $f_*\mathcal{F}$ need not be of finite type in general.

b) Show that if f is finite, then $f_*\mathcal{F}$ is a finite type $\mathcal{O}_Y$-module. In particular, this holds for closed embeddings.

Exercise 10.13.28. Let $f\colon X \to Y$ be a morphism and let $\mathcal{J}$ be an ideal sheaf of $\mathcal{O}_Y$.

a) Show that there is a map $f^*\mathcal{J} \to \mathcal{O}_X$

b) Give an example to show that $f^*\mathcal{J}$ need not be an ideal of $\mathcal{O}_X$ in general.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

c) Let $\mathcal{I} \subset \mathcal{O}_X$ denote the image of $f^*\mathcal{J}$ in $\mathcal{O}_X$. Show that $\mathcal{I}$ is an ideal sheaf and that $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) = f^{-1}\operatorname{Supp}(\mathcal{O}_Y/\mathcal{J})$.

Exercise 10.13.29. Consider the affine plane $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ over a field k and let $P \in \mathbb{A}^2_k$ be the closed point corresponding to the origin. Define the sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \begin{cases} 0 & P \in U \\ \mathcal{O}_X(U) & P \notin U \end{cases}$$

a) Show that $\mathcal{I}$ is an ideal sheaf and that $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is a closed subset of X.
b) Show that $\mathcal{I}$ is not quasi-coherent. In fact, show that $\mathcal{I}$ is not even locally generated by sections. HINT: The idea is that any section of $\mathcal{I}$ is 0 in any neighbourhood of P, but $\mathcal{F}$ is not 0 when restricted to any neighbourhood of P.
c) Show directly that $\mathcal{I}$ is not quasi-coherent by showing that $\mathcal{I}(X) = 0$, but $\mathcal{I} \neq 0$.
d) Show that there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_{\mathbb{A}^2_k,P} \longrightarrow 0$$

where $i_*\mathcal{O}_{\mathbb{A}^2_k,P}$ is the skyscraper sheaf on P.
e) Show that $i_*\mathcal{O}_{\mathbb{A}^2_k,P}$ is an $\mathcal{O}_X$-module of finite type, but not quasi-coherent.

Exercise 10.13.30 (Torsion sheaves)**.** Let X be an integral scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Define for each open set $U \subset X$, a subgroup $T(U) \subset \mathcal{F}(U)$ consisting of all the elements $m \in \mathcal{F}(U)$ such that the germ m_x is torsion in $\mathcal{F}_x$ for all $x \in X$, i.e., $a_x \cdot m_x = 0$ for some nonzero $a_x \in \mathcal{O}_{X,x}$.

a) Show that T is a subsheaf of $\mathcal{F}$. Also, show that T is quasi-coherent. T is called the *torsion subsheaf of* $\mathcal{F}$. Another notation for it is $\mathcal{F}_{\text{tors}}$.
b) Let $\mathscr{K}$ denote the constant sheaf on $K = K(X)$. Define a map of sheaves $\nu\colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{K}$. such that $T = \operatorname{Ker}\nu$.
c) A sheaf is called *torsion free if* $\mathcal{F}_{\text{tors}} = 0$. Show that the quotient $\mathcal{F}/T$ is always torsion free, i.e., $(\mathcal{F}/T)_{\text{tors}} = 0$.
d) Show that any locally free sheaf is torsion free.

Exercise 10.13.31. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $U \subset X$ be an open set. Show that the restriction $\mathcal{F}|_U$ is also quasi-coherent.

Exercise 10.13.32. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Show that $\mathcal{F}$ is quasi-coherent if and only if for any pair $V \subset U$ open affine subsets the natural map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V) \tag{10.43}$$

that sends $s \otimes g$ to $gs|_V$, is an isomorphism.

Exercise 10.13.33. Find a morphism $f\colon X \to \operatorname{Spec}\mathbb{Z}$ such that $f_*\mathcal{O}_X$ is not quasi-coherent sheaf on $\operatorname{Spec}\mathbb{Z}$.

Exercise 10.13.34. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent sheaves. Show that ϕ is injective (resp. surjective) if and only if $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp. surjective) for each affine open $U \subset X$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 10.13.35. Show that a quasi-coherent sheaf $\mathcal{F}$ on a scheme X is of finite type if and only if there exists an affine open cover $\{U_i\}_{i\in I}$ with $U_i = \operatorname{Spec} A_i$ such that for each i, $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ for some finitely generated A_i-module. HINT: See the proof of Proposition 8.19.

Exercise 10.13.36 (The 2-out-of-3 property)**.** Let X be a scheme and consider a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$.

a) Show that if $\mathcal{F}$ is quasi-coherent, then $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$ is exact for each affine open $U \subset X$.
b) Show that if two of the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are quasi-coherent, then the third is quasi-coherent as well.

Exercise 10.13.37 (A non-quasi-coherent ideal sheaf)**.** Let $X = \operatorname{Spec} A$ where $A = k[t]_{(t)}$. Let ξ be the generic point and let s be the closed point. Define a presheaf $\mathcal{I}$ by $\mathcal{I}(\{\xi\}) = \mathcal{I}(\varnothing) = 0$ and $\mathcal{I}(X) = A$.

a) Show that $\mathcal{I}$ is an ideal sheaf.
b) Show that the quotient presheaf $\mathcal{O}_X/\mathcal{I}$ is a sheaf and compute the sections of $\mathcal{O}_X/\mathcal{I}$ over $\varnothing$, $\{\xi\}$ and X.
c) Show that the support of $\mathcal{O}_X/\mathcal{I}$ is equal to $\{\xi\}$.
d) Does $\mathcal{I}$ correspond to a closed subscheme?
e) Show directly that $\mathcal{I}$ is not quasi-coherent.

Exercise 10.13.38 (Universal property of $\mathbf{Spec}\,\mathcal{A}$)**.** Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Write $\pi\colon \mathbf{Spec}\,\mathcal{A} \to S$ for the structure morphism.

a) Let $f\colon X \to S$ be an S-scheme and let $\phi\colon \mathcal{A} \to f_*\mathcal{O}_X$ be a map of $\mathcal{O}_S$-algebras. Show that ϕ induces a unique S-morphism of S-schemes $g\colon X \to \mathbf{Spec}\,\mathcal{A}$ with $\pi \circ g = f$. HINT: Work locally on S and glue using quasi-coherence.
b) Conversely, let $g\colon X \to \mathbf{Spec}\,\mathcal{A}$ be an S-morphism and set $f = \pi \circ g$. Show that the composition

$$\mathcal{A} \simeq \pi_*\mathcal{O}_{\mathbf{Spec}\,\mathcal{A}} \xrightarrow{\pi_*(g^\sharp)} \pi_* g_* \mathcal{O}_X = f_*\mathcal{O}_X$$

is a morphism of $\mathcal{O}_S$-algebras.
c) Show that these assignments are inverse to each other, giving a natural bijection

$$\operatorname{Hom}_{\mathsf{Sch}/S}(X, \mathbf{Spec}\,\mathcal{A}) = \operatorname{Hom}_{\mathcal{O}_S-\mathrm{alg}}(\mathcal{A}, f_*\mathcal{O}_X)$$

HINT: Cover S by affines and use $\operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) = \operatorname{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_X(X))$.

Exercise 10.13.39. Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf.

a) Show that $Z = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is closed in X.
b) Show that $\mathcal{O}_Z := \mathcal{O}_X/\mathcal{I}$ is naturally a sheaf on Z and that $(Z, \mathcal{O}_Z)$ is a locally ringed space.
c) Assume that $\mathcal{I}$ is quasi-coherent. Show that for an affine open set $U = \operatorname{Spec} A$, we have $\mathcal{O}_X/\mathcal{I}|_U \simeq \widetilde{A/I}$ where $I = \mathcal{I}(U)$.
d) Show that $(Z, \mathcal{O}_Z)$ is a scheme if and only if $\mathcal{I}$ is quasi-coherent.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

e) Show that this construction gives the same subscheme as the $V(\mathcal{I})$ constructed in the text.

Exercise 10.13.40 (Scheme-theoretic image)**.** Let $f\colon X \to Y$ be a morphism.

a) Let $i\colon Z \hookrightarrow Y$ be a closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$, and let $h\colon W \to Y$ be a morphism of schemes. Show that h factors through Z if and only if $\mathcal{I} \subset \operatorname{Ker}(h^\sharp)$. Equivalently, $h(W) \subset Z$ as sets and $h^\sharp$ factors through $\mathcal{O}_Y/\mathcal{I}$.

b) The *scheme-theoretic image* of f, if it exists, is the closed subscheme $Z \hookrightarrow Y$ through which f factors and which is minimal with this property: whenever f factors through a closed subscheme $Z' \hookrightarrow Y$, the closed embedding $Z \hookrightarrow Y$ factors through Z'. Show that the scheme-theoretic image is unique if it exists.

c) Show that if $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is induced by $\phi\colon A \to B$, then the scheme-theoretic image is $\operatorname{Spec}(A/\operatorname{Ker}\phi) \hookrightarrow \operatorname{Spec} A$.

d) More generally, suppose $X = U_1 \cup \cdots \cup U_r$ is a finite affine open cover, with $U_i = \operatorname{Spec} B_i$, and $Y = \operatorname{Spec} A$. Show that the scheme-theoretic image of f is $\operatorname{Spec}(A/I)$, where $I = \operatorname{Ker}(A \to \prod_i B_i)$.

From now on, we assume that $f\colon X \to Y$ is quasi-compact (inverse images of quasi-compact subsets are quasi-compact). In particular, this holds whenever X is Noetherian.

e) Show that the ideal sheaf $\mathcal{I}_f = \operatorname{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent on Y and show that it defines the scheme-theoretic image Z.

f) Show that formation of the scheme-theoretic image commutes with restriction to an open $V \subset Y$: the image of $f^{-1}(V) \to V$ is the intersection of the image of f with V.

g) Show that the underlying topological space of Z is $\overline{f(X)}$.

h) Suppose that X is reduced. Show that the scheme-theoretic image Z is simply $\overline{f(X)}$ with its reduced scheme structure.

i) Consider the morphism

$$g\colon \coprod_{r\geqslant 1} \operatorname{Spec}(\mathbb{C}[x]/(x^r)) \longrightarrow \operatorname{Spec}\mathbb{C}[x].$$

Show that the ideal sheaf $\mathcal{I}_g$ is not quasi-coherent. Show that $\operatorname{Spec}\mathbb{C}[x]$ itself is the scheme-theoretic image. Compare this with the closure of the set-theoretic image.

j) A morphism $f\colon X \to Y$ is said to be *scheme-theoretically dominant* if its scheme-theoretic image is all of Y. Show that, for quasi-compact f, this is equivalent to $\mathcal{O}_Y \to f_*\mathcal{O}_X$ being injective.

Exercise 10.13.41. Let $\iota\colon Y \to X$ be a morphism. Show that the following are equivalent:

(i) ι is a closed embedding.

(ii) For every affine $U = \operatorname{Spec} A \subset X$, $\iota^{-1}(U) = \operatorname{Spec} B$ is affine and the induced ring map $A \to B$ is surjective.

(iii) ι is a homeomorphism onto a closed subset in X and the map of sheaves $\mathcal{O}_X \to \iota_*\mathcal{O}_Y$ is surjective.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 10.13.42. Prove that a morphism $f\colon X \to Y$ is a closed embedding if and only if f induces a homeomorphism from X onto a closed subset of Y, and for each $x \in X$, the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective. HINT: Consider the stalks of $f_*\mathcal{O}_X$. You need to use the fact that f is a homeomorphism onto $f(X)$.

Exercise 10.13.43. Let A be a ring and let $\{M_i\}_{i\in I}$, $\{N_j\}_{j\in J}$ and M, N be A-modules.

a) Show that there are natural isomorphisms

$$\operatorname{Hom}_A\big(\bigoplus_{i\in I} M_i, N\big) = \prod_{i\in I} \operatorname{Hom}_A(M_i, N)$$

and

$$\operatorname{Hom}_A\big(M, \prod_{j\in J} N_j\big) = \prod_{j\in J} \operatorname{Hom}_A(M, N_j).$$

Show that if M is finitely generated, then

$$\operatorname{Hom}_A\big(M, \bigoplus_{j\in J} N_j\big) = \bigoplus_{j\in J} \operatorname{Hom}_A(M, N_j).$$

b) Let $X = \operatorname{Spec}\mathbb{Z}$ and consider the $\mathbb{Z}$-modules $M = \bigoplus_{\mathbb{N}} \mathbb{Z}$ and $N = \mathbb{Z}$. Use the identities above to show that $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ is not quasi-coherent in general.

Exercise 10.13.44. Let X be the blow-up of the origin. With coordinates (x, y) on $\mathbb{A}^2_k$ and $(u : v)$ on $\mathbb{P}^1_k$, show that X embeds as a closed subscheme $V(xv - yu) \subset \mathbb{A}^2_k \times_k \mathbb{P}^1_k$.

Exercise 10.13.45. Let $X \subset \mathbb{P}^m_A$ and $Y \subset \mathbb{P}^n_A$ be two projective A-schemes. Show that $X \times_A Y$ is projective over A. HINT: The composition of two closed embeddings is a closed embedding.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

11

Locally free sheaves

Many sheaves that arise in algebraic geometry look locally like finite direct sums of the structure sheaf. Because of this 'freeness' property, these sheaves are in many respects the nicest examples of sheaves on a scheme and the easiest to work with. They are also the counterpart to vector bundles in topology.

11.1 Locally free sheaves

Let X be a scheme. We say that an $\mathcal{O}_X$-module $\mathcal{E}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$ and it is *locally free* if there is an open cover $\{U_i\}_{i\in I}$ of X such that the restriction $\mathcal{E}|_{U_i}$ is free for every i. The cover $\{U_i\}$ is called a 'trivializing cover', and a locally free sheaf which is globally free is sometimes said to be *trivial*.

If $P \in X$ is a point, say contained in U_i, the *rank* of $\mathcal{E}$ at P is the number of copies of $\mathcal{O}_{U_i}$ needed to express $\mathcal{E}|_{U_i}$ as a direct sum of $\mathcal{O}_{U_i}$'s. The rank may be finite or infinite, but we will almost exclusively concern ourselves with the case of finite rank. A locally free sheaf $\mathcal{E}$ is also allowed to have different ranks at different points of X, but the rank will be constant on each connected component of X. If $\mathcal{E}$ has rank equal to r at every point of X, we say that $\mathcal{E}$ is locally free of rank r and write $\operatorname{rank}(\mathcal{E}) = r$.

The next proposition summarizes some of the basic properties of locally free sheaves.

Proposition 11.1. Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be two locally free $\mathcal{O}_X$-modules of rank e and f respectively. Then

(i) The direct sum $\mathcal{E} \oplus \mathcal{F}$ is locally free of rank $e + f$.
(ii) The tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free of rank $e \cdot f$.
(iii) The Hom sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is locally free of rank $e \cdot f$.
(iv) The dual sheaf $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O})$ is locally free of rank e. The double dual $(\mathcal{E}^\vee)^\vee$ is canonically isomorphic to $\mathcal{E}$. Moreover, there is a canonical isomorphism
$$\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}). \tag{11.1}$$
(v) For every $p \geqslant 0$, the sheaf $\bigwedge^p \mathcal{E}$ is locally free of rank $\binom{e}{p}$. In particular, $\det(\mathcal{E}) = \bigwedge^e \mathcal{E}$ is an invertible sheaf.

Proof All statements about local freeness and ranks are local on X, so we may work on an affine open set $U = \operatorname{Spec} A$ where $\mathcal{E}|_U \simeq \widetilde{E}$ and $\mathcal{F}|_U \simeq \widetilde{F}$ are trivial, with E and F finite free A-modules of ranks e and f. On such an open set, the sheaves in question are the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

sheaves associated to the A-modules $E \oplus F$, $E \otimes_A F$, $\mathrm{Hom}_A(E,F)$ and $\bigwedge^p E$, and each of these modules are free of the ranks indicated in the proposition. This proves (i), (ii), (iii), and (v).

It remains to prove (iv). If M is a finite free A-module, then $M^\vee = \mathrm{Hom}_A(M,A)$ is finite free of the same rank. Moreover, for finite free A-modules M and N, there is a canonical isomorphism

$$M^\vee \otimes_A N \longrightarrow \mathrm{Hom}_A(M,N), \qquad \phi \otimes n \longmapsto \big(m \mapsto \phi(m)n\big). \tag{11.2}$$

These isomorphisms are compatible with localization and therefore glue to a global canonical isomorphism $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$. □

Example 11.2. On $X = \mathbb{P}^1_A$, the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(m)$ are invertible sheaves, as they are constructed by gluing together two copies of $\mathcal{O}_{\mathbb{A}^1_A}$ over the standard covering of $\mathbb{P}^1_A$. These sheaves are non-trivial for all $m \neq 0$.

The isomorphism $\mathcal{O}_{\mathbb{P}^1_A}(m) \otimes \mathcal{O}_{\mathbb{P}^1_A}(-m) \simeq \mathcal{O}_{\mathbb{P}^1_A}$ shows that $\mathcal{O}_{\mathbb{P}^1_A}(-m)$ can be identified with the dual of $\mathcal{O}_{\mathbb{P}^1_A}(m)$. △

Example 11.3 (The integers)**.** Let $X = \mathrm{Spec}\,\mathbb{Z}$. If $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_X$-module of finite type, then $\mathcal{E} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module M, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^r \oplus T$, where T is a finite direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$. If $\mathcal{E}$ in addition is required to be locally free, it must hold that $T = 0$ (otherwise, if p is a prime factor of an n appearing in one of the summands of T, the stalk at (p) will not be free). Hence $\mathcal{E} = \widetilde{\mathbb{Z}^r} = \mathcal{O}_X^r$, and we conclude that every locally free sheaf on $\mathrm{Spec}\,\mathbb{Z}$ is trivial. △

Example 11.4 (The affine line)**.** The argument of the previous example in fact applies over any principal ideal domain A. Every $\mathcal{O}_X$-module of finite type on $X = \mathrm{Spec}\,A$ is of the form $\widetilde{M}$ for $M = A^r \oplus T$ where T is a finitely generated torsion module. If we require $\widetilde{M}$ to be locally free, the torsion part must be zero, so $\widetilde{M} = \widetilde{A^r} = \mathcal{O}_X^r$. In particular, this applies to locally free sheaves on the affine line $\mathbb{A}^1_k = \mathrm{Spec}\,k[x]$ over a field k. △

Example 11.5 (The tangent bundle of the 2-sphere)**.** Let $A = \mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$ and consider the A-linear map

$$\phi\colon A^3 \longrightarrow A, \quad (a,b,c) \mapsto ax + by + cz.$$

The kernel $M = \mathrm{Ker}(\phi)$ defines a quasi-coherent sheaf $\widetilde{M}$ which is locally free of rank 2.

To see this, note that M consists of triples $(a,b,c) \in A^3$ satisfying $ax + by + cz = 0$. After inverting x, any such triple is uniquely determined by b and c as $a = -x^{-1}(by + cz)$. This gives an isomorphism $M_x \simeq A_x^2$, and hence $\widetilde{M}|_{D(x)} \simeq \mathcal{O}^2_{D(x)}$. The same argument applies to y and z. The open sets $D(x), D(y), D(z)$ cover X, since $x^2+y^2+z^2 = 1$ in A, so $\widetilde{M}$ is locally free.

While $\widetilde{M}$ is locally trivial, it is in fact globally non-trivial, that is, M is not a free A-module. This is not so easy to prove: it reflects the famous topological fact that the tangent bundle of S^2 does not admit a nowhere-vanishing section. In contrast, if we replace $\mathbb{R}$ by $\mathbb{C}$ in this example, the corresponding sheaf *is* trivial. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

11.2 Transition functions

Let $\mathcal{E}$ be a locally free sheaf of rank r. By definition, there is an open cover $\{U_i\}_{i\in I}$ and, for each $i \in I$, an isomorphism of $\mathcal{O}_{U_i}$-modules

$$\phi_i \colon \mathcal{O}_{U_i}^{\oplus r} \longrightarrow \mathcal{E}|_{U_i}.$$

Write $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Over U_{ij}, the two trivializations are related by an isomorphism

$$g_{ij} = \phi_j^{-1} \circ \phi_i \colon \mathcal{O}_{U_{ij}}^{\oplus r} \longrightarrow \mathcal{O}_{U_{ij}}^{\oplus r}.$$

These isomorphisms are called the transition functions of $\mathcal{E}$ with respect to the chosen trivializations. The map g_{ij} converts coordinates with respect to the i-trivialization into coordinates with respect to the j-trivialization. By construction, we have $g_{ii} = \mathrm{id}$ and $g_{ji} = g_{ij}^{-1}$. Moreover, on triple overlaps, they satisfy the cocycle condition

$$g_{ik} = g_{jk} \circ g_{ij}.$$

Indeed, $\phi_k^{-1} \circ \phi_i = (\phi_k^{-1} \circ \phi_j) \circ (\phi_j^{-1} \circ \phi_i)$.

After choosing the standard basis of $\mathcal{O}_{U_{ij}}^{\oplus r}$, each g_{ij} is represented by an invertible $r \times r$ matrix with entries in $\mathcal{O}_X(U_{ij})$ and these matrices satisfy $g_{ik} = g_{jk}g_{ij}$ on U_{ijk}.

Conversely, it follows from the Gluing Lemma for sheaves that if we have an open cover $\{U_i\}_{i\in I}$ and isomorphisms

$$g_{ij} \colon \mathcal{O}_{U_{ij}}^{\oplus r} \longrightarrow \mathcal{O}_{U_{ij}}^{\oplus r}$$

satisfying the cocycle condition $g_{ik} = g_{jk} \circ g_{ij}$ on triple overlaps, then the sheaves $\mathcal{O}_{U_i}^{\oplus r}$ glue to a locally free sheaf of rank r.

The transition functions defining $\mathcal{E}$ are not unique. If we replace ϕ_i by $\phi_i' = \phi_i \circ h_i$, where $h_i \colon \mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus r}$ is an isomorphism, then the new transition functions are $g_{ij}' = h_j^{-1} g_{ij} h_i$. Furthermore, $\mathcal{E}$ could admit trivializations by a different open cover. This is therefore not an ideal way of *classifying* locally free sheaves and in practice can be difficult to decide when two constructions yield isomorphic sheaves.

We can also read off triviality from the transition functions. With respect to the fixed cover $\{U_i\}$, the sheaf $\mathcal{E}$ is trivial if and only if there exist isomorphisms $h_i \colon \mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus r}$ such that $g_{ij} = h_j h_i^{-1}$ on U_{ij} (see Exercise 11.2.1).

Exercise 11.2.1. Let $\mathcal{E}$ be a locally free sheaf of rank r, defined by transition functions $g_{ij} \in \mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j))$. Show that $\mathcal{E} \simeq \mathcal{O}_X^{\oplus r}$ if and only if there exist isomorphisms $h_i \colon \mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus r}$ such that $g_{ij} = h_j h_i^{-1}$ on $U_i \cap U_j$.

Example 11.6. Suppose $\mathcal{E}$ and $\mathcal{F}$ are locally free of ranks r and s, respectively. After refining the cover, we may assume that they are trivialized on the same open sets $\{U_i\}$. If g_{ij} and h_{ij} are the transition functions of $\mathcal{E}$ and $\mathcal{F}$, then $\mathcal{E} \oplus \mathcal{F}$ is obtained by gluing the sheaves $\mathcal{O}_{U_i}^{\oplus r} \oplus \mathcal{O}_{U_i}^{\oplus s}$ with transition functions

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ij} \end{pmatrix}.$$

△

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 11.7. Let $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the third-power map, given by $(x_0 : x_1) \mapsto (x_0^3 : x_1^3)$. We claim that

$$f_*\mathcal{O}_{\mathbb{P}^1_k} \simeq \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)^{\oplus 2}.$$

Write $V_0 = \operatorname{Spec} k[v]$ and $V_1 = \operatorname{Spec} k[v^{-1}]$ for the standard affine charts on the target, where $v = y_1/y_0$. Similarly, write $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$ on the source, where $u = x_1/x_0$. Over V_0, the morphism is induced by $k[v] \to k[u]$, $v \mapsto u^3$. Hence $f_*\mathcal{O}_{\mathbb{P}^1_k}|_{V_0}$ is associated to $k[u]$, viewed as a $k[v]$-module. Since

$$k[u] = k[v][u]/(u^3 - v) \simeq k[v] \oplus k[v]u \oplus k[v]u^2,$$

it is free with basis $e_0 = 1$, $e_1 = u$, and $e_2 = u^2$.

Over V_1, the morphism is induced by $k[v^{-1}] \to k[u^{-1}]$, $v^{-1} \mapsto u^{-3}$. Thus $f_*\mathcal{O}_{\mathbb{P}^1_k}|_{V_1}$ is associated to $k[u^{-1}]$, viewed as a $k[v^{-1}]$-module, and this module is free with basis $f_0 = 1$, $f_1 = u^{-2}$, and $f_2 = u^{-1}$.

On the overlap $V_0 \cap V_1 = \operatorname{Spec} k[v, v^{-1}]$, we have $u^3 = v$. Hence $e_0 = f_0$, while $e_1 = u = vu^{-2} = vf_1$ and $e_2 = u^2 = vu^{-1} = vf_2$. Therefore the transition matrix from the basis (e_0, e_1, e_2) to the basis (f_0, f_1, f_2) is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{pmatrix}.$$

This is precisely the gluing data for $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)^{\oplus 2}$, so the claim follows. △

11.3 Invertible sheaves and the Picard group

Among the locally free sheaves, the ones of rank 1 are particularly important. These are called *invertible sheaves*. By definition, an $\mathcal{O}_X$-module L is invertible whenever there exists an open cover $\{U_i\}_{i\in I}$ and isomorphisms

$$\phi_i\colon \mathcal{O}_{U_i} \longrightarrow L|_{U_i}.$$

The section $s_i = \phi_i(1) \in L(U_i)$ is called a *local generator* for L over U_i. With the notation of the previous section, the transition functions of L are units $g_{ij} \in \mathcal{O}_X(U_{ij})^\times$, which relate the local generators by $s_i = g_{ij}s_j$ on U_{ij}.

Changing the local generators changes the transition functions in the expected way. If $s_i' = u_i s_i$, where $u_i \in \mathcal{O}_X(U_i)^\times$, then $g_{ij}' = u_i g_{ij} u_j^{-1}$.

Example 11.8. Consider $\mathbb{P}^n_A$ with the standard cover by the open sets $U_i = D_+(x_i)$. For each integer m, the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(m)$ is free on U_i with generator x_i^m. On $U_i \cap U_j$ one has

$$x_i^m = \left(\frac{x_i}{x_j}\right)^m x_j^m,$$

so the transition functions are

$$g_{ij} = \left(\frac{x_i}{x_j}\right)^m.$$

Note that the ordering of i and j matters: $g_{ij} = (x_j/x_i)^m$ defines the sheaf $\mathcal{O}(-m)$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

As a special case of Proposition 11.1, we have:

Proposition 11.9. Let X be a scheme and let L and M be invertible sheaves. Then:
(i) $L \otimes_{\mathcal{O}_X} M$ is an invertible sheaf.
(ii) $\mathit{Hom}_{\mathcal{O}_X}(L, M)$ is invertible. In particular, $\mathit{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$ is invertible, and

$$\mathit{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X) \otimes_{\mathcal{O}_X} M \simeq \mathit{Hom}_{\mathcal{O}_X}(L, M). \tag{11.3}$$

This implies that tensor product gives a binary operation on the set of isomorphism classes of invertible sheaves. The sheaf $\mathcal{O}_X$ serves as the identity, since $L \otimes_{\mathcal{O}_X} \mathcal{O}_X \simeq L$. For an invertible sheaf L, we write $L^{-1} = \mathit{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$. By the proposition, L^{-1} is again invertible, and it serves as an inverse to L under tensor product. In particular, this is the reason for the term *invertible*.

Definition 11.10. For a scheme X, the *Picard group* $\operatorname{Pic}(X)$ of X is the group of isomorphism classes of invertible sheaves on X under the tensor product.

Note that it is the set of isomorphism classes of invertible sheaves that form a group, not the invertible sheaves themselves. That is, $L \otimes_{\mathcal{O}_X} L^{-1}$ is isomorphic, but strictly speaking, not *equal* to $\mathcal{O}_X$. Note also that $\operatorname{Pic}(X)$ is an abelian group, because $L \otimes_{\mathcal{O}_X} M$ is canonically isomorphic to $M \otimes_{\mathcal{O}_X} L$.

Example 11.11 ($\operatorname{Spec}\mathbb{Z}$)**.** By Example 11.3, any invertible sheaf on $\operatorname{Spec}\mathbb{Z}$ is trivial, so

$$\operatorname{Pic}(\operatorname{Spec}\mathbb{Z}) = 0.$$

On the other hand, $\operatorname{Pic}(\mathbb{Z}[\sqrt{-5}]) \neq 0$, as we will show in Proposition A.42. △

Example 11.12 (The affine line)**.** By Example 10.24, any invertible sheaf on $\mathbb{A}^1_k$ is trivial, so

$$\operatorname{Pic}(\mathbb{A}^1_k) = 0.$$

We will prove more generally that $\operatorname{Pic}(\mathbb{A}^n_k) = 0$ for any $n \geqslant 0$ in Chapter 19. △

Example 11.13 (The projective line)**.** Every invertible sheaf L on $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(m)$ for a unique $m \in \mathbb{Z}$. This follows from the classification of quasi-coherent sheaves on $\mathbb{P}^1_k$ of Example 10.13. Since any invertible sheaf on $\mathbb{A}^1_k$ is trivial, L is defined by $M_0 = k[x]$, $M_1 = k[x^{-1}]$ and an isomorphism

$$\tau\colon k[x, x^{-1}] \longrightarrow k[x, x^{-1}]$$

Such an isomorphism is multiplication by a unit of $k[x, x^{-1}]$, and every unit has the form cx^m with $c \in k^\times$ and $m \in \mathbb{Z}$. After rescaling one trivialization, we may assume that the transition function is x^m. This is precisely the transition function of $\mathcal{O}_{\mathbb{P}^1_k}(m)$, so $L \simeq \mathcal{O}_{\mathbb{P}^1_k}(m)$.

If $\mathcal{O}_{\mathbb{P}^1_k}(m) \simeq \mathcal{O}_{\mathbb{P}^1_k}(m')$, then also $\mathcal{O}_{\mathbb{P}^1_k}(m + a) \simeq \mathcal{O}_{\mathbb{P}^1_k}(m' + a)$ for $a \gg 0$. Since these sheaves have different numbers of global sections unless $m = m'$, we get $m = m'$. Therefore the assignment $m \mapsto [\mathcal{O}_{\mathbb{P}^1_k}(m)]$ defines an isomorphism

$$\operatorname{Pic}(\mathbb{P}^1_k) \simeq \mathbb{Z}.$$

In Chapter 19, we will show more generally that $\operatorname{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ for any $n \geqslant 0$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

11.4 Fibers and local freeness

Let $\mathcal{E}$ be a coherent sheaf on the scheme X and let $P \in X$. The *fiber of $\mathcal{E}$ at P* is the $\kappa(P)$-vector space

$$\mathcal{E}(P) = \mathcal{E}_P \otimes_{\mathcal{O}_{X,P}} \kappa(P) \simeq \mathcal{E}_P/\mathfrak{m}_P\mathcal{E}_P.$$

If $U \subset X$ is an open subset containing P and $s \in \Gamma(U, \mathcal{E})$, we denote by $s(P)$ the image of the germ $s_P \in \mathcal{E}_P$ in $\mathcal{E}(P)$.

Geometrically, $\mathcal{E}(P)$ is the $\kappa(P)$-vector space corresponding to the pullback $\iota^*\mathcal{E}$ via the canonical morphism $\iota\colon \operatorname{Spec}\kappa(P) \to X$.

The *rank* of $\mathcal{E}$ at P is defined by

$$\operatorname{rank}_P(\mathcal{E}) = \dim_{\kappa(P)} \mathcal{E}(P).$$

If X is integral with generic point ξ, we call $\operatorname{rank}(\mathcal{E}) = \operatorname{rank}_\xi(\mathcal{E}) = \dim_{K(X)} \mathcal{E}_\xi$ the *rank* of $\mathcal{E}$.

If $\mathcal{E}$ is locally free of rank r, then $\mathcal{E}_P \simeq \mathcal{O}_{X,P}^r$ and $\mathcal{E}(P) \simeq \kappa(P)^r$ for every P, so $\operatorname{rank}_P(\mathcal{E}) = r$. That is, the rank takes the value r at every point. The main goal of this section is to give converses to this and, more importantly, to identify the maximal open set on which a coherent sheaf is locally free.

The rank at P has a second useful interpretation: it is the minimal number of generators of the stalk $\mathcal{E}_P$. This follows by applying the following lemma to $A = \mathcal{O}_{X,P}$.

Lemma 11.14. Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ, and let M be a finitely generated A-module. For every $i \geqslant 0$, the following are equivalent:
(i) M can be generated by i elements as an A-module.
(ii) $\dim_\kappa(M \otimes_A \kappa) \leqslant i$.
Consequently, the minimal number of generators of M equals $\dim_\kappa(M \otimes_A \kappa)$.

Proof Any A-generating set $m_1, \dots, m_i$ of M reduces modulo $\mathfrak{m}$ to a κ-spanning set of $M/\mathfrak{m}M \simeq M \otimes_A \kappa$, giving (i)⇒(ii). Conversely, assume $\dim_\kappa(M \otimes_A \kappa) \leqslant i$ and let $m_1, \dots, m_i \in M$ be elements whose images span $M/\mathfrak{m}M$. Let $N \subset M$ be the submodule they generate. Then $M = N + \mathfrak{m}M$, so $\mathfrak{m} \cdot (M/N) = M/N$. Since M/N is finitely generated, Nakayama's lemma gives $M/N = 0$, i.e. $M = N$. □

Fitting ideals

Let A be a ring and let M be a finitely presented A-module. Choose a presentation

$$A^m \xrightarrow{\varphi} A^n \to M \to 0.$$

For each $i \geqslant 0$, the *i-th Fitting ideal* $\operatorname{Fitt}_i(M) \subset A$ is the ideal generated by the $(n-i) \times (n-i)$-minors of a matrix for φ. By convention, the 0×0 minor is 1, so $\operatorname{Fitt}_i(M) = A$ for $i \geqslant n$. At the other extreme, $\operatorname{Fitt}_{n-1}(M)$ is the ideal generated by the entries of φ. The Fitting ideals form an increasing chain

$$\operatorname{Fitt}_0(M) \subset \operatorname{Fitt}_1(M) \subset \cdots \subset \operatorname{Fitt}_{n-1}(M) \subset \operatorname{Fitt}_n(M) = A.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Fitting ideals give a convenient way to locate the open subset where a coherent sheaf is locally free.

A basic fact is that $\mathrm{Fitt}_i(M)$ does not depend on the chosen presentation.

Theorem 11.15. For each i, the ideal $\mathrm{Fitt}_i(M)$ is independent of the chosen presentation of M.

We first record a simple matrix fact.

Lemma 11.16. Let X be a matrix over A. If P and Q are matrices over A such that PXQ is defined, then every $k \times k$ minor of PXQ lies in the ideal generated by the $k \times k$ minors of X. If P and Q are invertible, then the $k \times k$ minors of PXQ generate the same ideal as the $k \times k$ minors of X.

Proof The columns of XQ are A-linear combinations of the columns of X. Since the determinant is multilinear, every $k \times k$ minor of XQ is an A-linear combination of $k \times k$ minors of X. Similarly, the rows of PX are A-linear combinations of the rows of X, so every $k \times k$ minor of PX lies in the ideal generated by the $k \times k$ minors of X. Applying these two observations gives the first assertion. When P and Q are invertible, then $X = P^{-1}(PXQ)Q^{-1}$. Applying the first assertion again gives the reverse containment of ideals, so the two ideals are equal. □

Proof of Theorem 11.15 Fix a surjection $A^n \to M$ and let K be its kernel. Giving a presentation inducing this surjection is the same as choosing finitely many generators of K. Suppose two such choices give maps $\varphi\colon A^m \to A^n$ and $\psi\colon A^{m'} \to A^n$. The columns of the matrices for φ and ψ generate the same submodule $K \subset A^n$. Hence there are matrices P and Q such that $\psi = \varphi Q$ and $\varphi = \psi P$. Lemma 11.16 gives both containments between the ideals generated by the $(n-i) \times (n-i)$ minors of φ and of ψ. Hence, for a fixed surjection $A^n \to M$, the ideal $\mathrm{Fitt}_i(M)$ is independent of the chosen generators of K.

Lemma 11.16 also shows that changing bases in the free modules does not affect the ideal. A change of basis in the source or target replaces a matrix for φ by φU or $V\varphi$, where U and V are invertible. The lemma then shows that the ideals generated by the relevant minors are unchanged.

It remains to compare different generating sets for M. Suppose $x_1, \ldots, x_n$ generate M, and adjoin one redundant generator y. Write $y = a_1x_1 + \cdots + a_nx_n$. If φ is a matrix for the original presentation, then, after choosing compatible bases, a matrix for the enlarged presentation is

$$\begin{pmatrix} \varphi & -a \\ 0 & 1 \end{pmatrix}.$$

where $a = (a_1, \ldots, a_n)^t$. Let $k = n - i$. The old presentation gives the ideal generated by the $k \times k$ minors of φ, while the enlarged presentation gives the ideal generated by the $(k+1) \times (k+1)$ minors of this block matrix.

These two ideals are equal. A $(k+1) \times (k+1)$ minor of the block matrix has four possible forms. If it uses the last row and the last column, it is, up to sign, a $k \times k$ minor of φ. If it uses the last row but not the last column, it is zero. If it uses the last column but not the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

last row, expansion along that column writes it as an A-linear combination of $k \times k$ minors of φ. If it uses neither the last row nor the last column, it is a $(k+1) \times (k+1)$ minor of φ, which lies in the ideal generated by the $k \times k$ minors of φ. Conversely, every $k \times k$ minor of φ is obtained by adjoining the last row and the last column. Hence adding one redundant generator does not change $\operatorname{Fitt}_i(M)$.

Finally, take two finite presentations of M. Let the first give generators $x_1, \ldots, x_n$, and let the second give generators $y_1, \ldots, y_{n'}$. Adjoin $y_1, \ldots, y_{n'}$ to the first presentation, and adjoin $x_1, \ldots, x_n$ to the second. The preceding paragraph shows that this does not change the corresponding Fitting ideals. After these enlargements, both presentations use the same generating set of M. For this fixed surjection from a free module onto M, we have already shown that $\operatorname{Fitt}_i(M)$ is independent of the chosen generators of the kernel. Therefore the two original presentations define the same ideal. □

Another important property of Fitting ideals is that they are compatible with base change and localization.

Proposition 11.17. Let B be an A-algebra and let M be a finitely presented A-module.
(i) (Base change): For each i, $\operatorname{Fitt}_i(M \otimes_A B) = \operatorname{Fitt}_i(M) \cdot B$.
(ii) (Localization): If $S \subset A$ is a multiplicatively closed set, then $\operatorname{Fitt}_i(S^{-1}M) = S^{-1} \operatorname{Fitt}_i(M)$.

Proof Tensoring a presentation $A^m \xrightarrow{\varphi} A^n \to M \to 0$ gives a presentation

$$B^m \xrightarrow{\varphi \otimes \mathrm{id}_B} B^n \to M \otimes_A B \to 0.$$

The matrix of $\varphi \otimes \mathrm{id}_B$ is obtained from the matrix of φ by applying the ring map $A \to B$ to its entries, so its minors generate the extended ideal $\operatorname{Fitt}_i(M) \cdot B$. The argument for localization is similar. □

In particular, if $I \subset A$ is an ideal and M is an A/I-module, then the Fitting ideals of M computed over A/I are the images of those computed over A under the quotient map.

The vanishing sets of the Fitting ideals define the closed subsets of $\operatorname{Spec} A$ where M has high fiber dimension. The proof of this is basically just linear algebra: after taking the fiber over a prime $\mathfrak{p}$, the only thing that determines the dimension of the cokernel is the rank of a matrix, and the rank of a matrix over a field is the largest size of a nonvanishing minor.

Theorem 11.18. Let A be a ring and let M be a finitely presented A-module. For each $i \geqslant 0$, the closed set $V(\operatorname{Fitt}_i(M))$ consists precisely of those primes $\mathfrak{p} \in \operatorname{Spec} A$ for which $\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})) > i$. Equivalently,

$$V(\operatorname{Fitt}_i(M)) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \text{ cannot be generated by } i \text{ elements} \,\}.$$

Proof Let $\mathfrak{p} \in \operatorname{Spec} A$ be a prime ideal. Tensoring a presentation $A^m \xrightarrow{\varphi} A^n \to M \to 0$ by $\kappa(\mathfrak{p})$ we obtain

$$\kappa(\mathfrak{p})^m \xrightarrow{\overline{\varphi}} \kappa(\mathfrak{p})^n \to M \otimes_A \kappa(\mathfrak{p}) \to 0,$$

and therefore $\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})) = n - \operatorname{rank}_{\kappa(\mathfrak{p})}(\overline{\varphi})$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Over the field $\kappa(\mathfrak{p})$, the rank of $\overline{\varphi}$ is the largest k for which some $k \times k$ minor of $\overline{\varphi}$ is nonzero. A minor of $\overline{\varphi}$ is the image in $\kappa(\mathfrak{p})$ of the corresponding minor of φ, so it is nonzero in $\kappa(\mathfrak{p})$ if and only if the corresponding minor of φ is not contained in $\mathfrak{p}$. Hence $\operatorname{rank}_{\kappa(\mathfrak{p})}(\overline{\varphi}) \geqslant k$ if and only if $\operatorname{Fitt}_{n-k}(M) \not\subset \mathfrak{p}$. In the case $k = n - i$, we obtain

$$\dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})) \leqslant i \Leftrightarrow \operatorname{rank}_{\kappa(\mathfrak{p})}(\overline{\varphi}) \geqslant n-i \Leftrightarrow \operatorname{Fitt}_i(M) \not\subset \mathfrak{p} \Leftrightarrow \mathfrak{p} \notin V(\operatorname{Fitt}_i(M)).$$

□

The next result characterizes freeness of a finitely presented module over a local ring entirely in terms of two Fitting ideals.

Proposition 11.19. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely presented A-module. Then M is free of rank r if and only if $\operatorname{Fitt}_r(M) = A$ and $\operatorname{Fitt}_{r-1}(M) = 0$. (For $r = 0$ the second condition is omitted.)

Proof If $M \simeq A^{\oplus r}$, the presentation $0 \to A^r \to M \to 0$ has no minors of size $\geqslant 1$, so $\operatorname{Fitt}_r(M) = A$ (by convention) and $\operatorname{Fitt}_{r-1}(M) = 0$.

Conversely, assume both conditions, and choose a finite presentation

$$A^m \xrightarrow{\varphi} A^n \to M \to 0.$$

The convention $\operatorname{Fitt}_i(M) = A$ for $i \geqslant n$, applied to $i = r - 1$, forces $r \leqslant n$ (else $\operatorname{Fitt}_{r-1}(M) = A \neq 0$, since the local ring is nonzero). If $r = n$, then $\operatorname{Fitt}_{r-1}(M) = \operatorname{Fitt}_{n-1}(M)$ is generated by the entries of φ, so $\operatorname{Fitt}_{r-1}(M) = 0$ gives $\varphi = 0$ and hence $M \simeq A^n = A^r$.

Assume therefore $r < n$. Since $\operatorname{Fitt}_r(M) = A$, some $(n - r) \times (n - r)$ minor of φ is a unit. After permuting basis vectors of A^n and A^m, we may assume this minor is the determinant of the upper-left $(n-r) \times (n-r)$ block U of φ. Since $\det U$ is a unit, U itself is invertible in $\mathrm{M}_{n-r}(A)$. Using U as a pivot block, elementary row operations on A^n and column operations on A^m reduce φ to block-diagonal form:

$$\varphi \sim \begin{pmatrix} I_{n-r} & 0 \\ 0 & \psi \end{pmatrix}, \qquad \psi \colon A^{m-(n-r)} \to A^r.$$

The $(n-r+1) \times (n-r+1)$ minors of this normal form are, by Laplace expansion through the identity block, exactly the entries of ψ. Hence $\operatorname{Fitt}_{r-1}(M)$ is the ideal generated by the entries of ψ, and the assumption $\operatorname{Fitt}_{r-1}(M) = 0$ forces $\psi = 0$. The cokernel is then $A^r / \operatorname{Im}(0) = A^r$, so $M \simeq A^r$. □

Let $\mathcal{E}$ be a coherent sheaf on the Noetherian scheme X. On any affine open $U = \operatorname{Spec} A$ with $\mathcal{E}|_U = \widetilde{M}$, the ideal $\operatorname{Fitt}_i(M) \subset A$ defines a quasi-coherent ideal sheaf on U. These ideal sheaves agree on overlaps by the base-change property of Proposition 11.17, and therefore glue to a global quasi-coherent ideal sheaf $\operatorname{Fitt}_i(\mathcal{E}) \subset \mathcal{O}_X$. By construction, $\operatorname{Fitt}_i(\mathcal{E})|_U$ is the sheaf associated to $\operatorname{Fitt}_i(M)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Corollary 11.20. Let $\mathcal{E}$ be a coherent sheaf on the Noetherian scheme X. Then

$$V(\mathrm{Fitt}_i(\mathcal{E})) = \{\, P \in X \mid \mathrm{rank}_P(\mathcal{E}) > i \,\}.$$

In particular, the set $\{\, P \in X \mid \mathrm{rank}_P(\mathcal{E}) \leqslant i \,\}$ is open, and the function $P \mapsto \mathrm{rank}_P(\mathcal{E})$ is upper semicontinuous: it can only jump 'up' upon specialization.

Proof We may check this by going to an affine open $U = \operatorname{Spec} A$ with $\mathcal{E}|_U = \widetilde{M}$, and here the equality reduces to Theorem 11.18. □

Similarly, Proposition 11.19 applied at each stalk gives an open description of the locally free locus.

Corollary 11.21. Let $\mathcal{E}$ be a coherent sheaf on the Noetherian scheme X and let $r \geqslant 0$. The set of points $P \in X$ at which $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module of rank r is open in X.

Proof The locus is in fact the intersection of the open sets

$$(X - V(\mathrm{Fitt}_r(\mathcal{E}))) \cap (X - \mathrm{Supp}(\mathrm{Fitt}_{r-1}(\mathcal{E}))).$$

To see this, let $P \in X$. Since $\mathcal{E}$ is coherent, $\mathcal{E}_P$ is a finitely presented $\mathcal{O}_{X,P}$-module. Fitting ideals commute with localization, so $\mathrm{Fitt}_i(\mathcal{E}_P) = \mathrm{Fitt}_i(\mathcal{E})_P$. Proposition 11.19 therefore shows that $\mathcal{E}_P$ is free of rank r if and only if $\mathrm{Fitt}_r(\mathcal{E})_P = \mathcal{O}_{X,P}$ and $\mathrm{Fitt}_{r-1}(\mathcal{E})_P = 0$. The first condition is equivalent to $P \notin V(\mathrm{Fitt}_r(\mathcal{E}))$, and the condition $\mathrm{Fitt}_{r-1}(\mathcal{E})_P = 0$ is equivalent to $P \notin \mathrm{Supp}(\mathrm{Fitt}_{r-1}(\mathcal{E}))$, which gives the above description. Finally, since $\mathcal{E}$ is coherent, freeness of $\mathcal{E}_P$ extends to a neighbourhood of P, so $\mathcal{E}$ is locally free of rank r on this open set. □

In particular, the locally free locus of a coherent sheaf on a Noetherian scheme is always open. When X is reduced, the criterion simplifies further: pointwise constancy of the rank is enough.

Theorem 11.22. Let X be a reduced Noetherian scheme and let $\mathcal{E}$ be a coherent sheaf on X. The following are equivalent:

(i) $\mathcal{E}$ is locally free of rank r.
(ii) $\mathcal{E}_P \simeq \mathcal{O}_{X,P}^r$ is a free $\mathcal{O}_{X,P}$-module for every $P \in X$.
(iii) $\mathrm{rank}_P(\mathcal{E}) = r$ for every $P \in X$.

Proof The implications (i)⇒(ii)⇒(iii) are immediate. For (iii)⇒(i), the hypothesis says that $\mathrm{rank}_P(\mathcal{E}) \leqslant r$ and $\mathrm{rank}_P(\mathcal{E}) \geqslant r$ at every P. By Corollary 11.20 the first inequality gives $V(\mathrm{Fitt}_r(\mathcal{E})) = \varnothing$, hence $\mathrm{Fitt}_r(\mathcal{E}) = \mathcal{O}_X$. The second inequality gives $V(\mathrm{Fitt}_{r-1}(\mathcal{E})) = X$, that is, $\mathrm{Fitt}_{r-1}(\mathcal{E})$ is contained in every prime ideal of every stalk. The intersection of all primes of a ring is the nilradical, so $\mathrm{Fitt}_{r-1}(\mathcal{E})$ is locally nilpotent, i.e., contained in the nilradical sheaf of $\mathcal{O}_X$. Since X is reduced, this nilradical is zero, so $\mathrm{Fitt}_{r-1}(\mathcal{E}) = 0$. By Corollary 11.21, $\mathcal{E}$ is locally free of rank r on all of X. □

At the generic point ξ of an integral scheme, the local ring $\mathcal{O}_{X,\xi} = K(X)$ is a field, so

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

every finitely generated $\mathcal{O}_{X,\xi}$-module is automatically free. Combined with the openness of the locally free locus, this immediately gives generic freeness.

Proposition 11.23 (Generic freeness). Let X be a Noetherian integral scheme and let $\mathcal{E}$ be a coherent sheaf on X. There is a dense open set $U \subset X$ on which $\mathcal{E}$ is locally free of rank $r = \operatorname{rank}(\mathcal{E})$.

Proof The stalk $\mathcal{E}_\xi$ is a finite-dimensional vector space over the field $\mathcal{O}_{X,\xi} = K(X)$, hence free of rank $r = \operatorname{rank}(\mathcal{E})$. By Proposition 11.19, $\operatorname{Fitt}_r(\mathcal{E})_\xi = \mathcal{O}_{X,\xi}$ and $\operatorname{Fitt}_{r-1}(\mathcal{E})_\xi = 0$.

The first equation says $\xi \notin V(\operatorname{Fitt}_r(\mathcal{E}))$, so $U_1 = X - V(\operatorname{Fitt}_r(\mathcal{E}))$ is a dense open containing ξ. The second says $\xi \notin \operatorname{Supp}(\operatorname{Fitt}_{r-1}(\mathcal{E}))$, so $U_2 = X - \operatorname{Supp}(\operatorname{Fitt}_{r-1}(\mathcal{E}))$ is open and contains ξ. On $U = U_1 \cap U_2$, Corollary 11.21 gives $\mathcal{E}|_U$ locally free of rank r. □

Proposition 11.24. Let X be a Noetherian scheme over a field k, let $k \to L$ be a field extension and let $q\colon X_L \to X$ be the base change morphism. Let $\mathcal{E}$ be a coherent sheaf on X. If $P \in X$ and $\overline{P} \in X_L$ lies above P, then $\mathcal{E}_P$ is free of rank r over $\mathcal{O}_{X,P}$ if and only if $(q^*\mathcal{E})_{\overline{P}}$ is free of rank r over $\mathcal{O}_{X_L,\overline{P}}$. In particular, $\mathcal{E}$ is locally free of rank r if and only if $q^*\mathcal{E}$ is locally free of rank r.

Proof Only the converse implication needs proof. Assume that $(q^*\mathcal{E})_{\overline{P}}$ is free of rank r.

Let $A = \mathcal{O}_{X,P}$, $B = \mathcal{O}_{X_L,\overline{P}}$, and $M = \mathcal{E}_P$. Then $(q^*\mathcal{E})_{\overline{P}} \simeq M \otimes_A B$. As Fitting ideals commute with base change, we obtain $\operatorname{Fitt}_i(M \otimes_A B) = \operatorname{Fitt}_i(M)B$. By Proposition 11.19, freeness of rank r is equivalent to the conditions $\operatorname{Fitt}_r = A$ and $\operatorname{Fitt}_{r-1} = 0$, with the second condition omitted when $r = 0$. Hence our assumption gives $\operatorname{Fitt}_r(M)B = B$ and $\operatorname{Fitt}_{r-1}(M)B = 0$.

Since $A \to B$ is a map of local rings, the equality $\operatorname{Fitt}_r(M)B = B$ implies $\operatorname{Fitt}_r(M) = A$. We claim that the second implies $\operatorname{Fitt}_{r-1}(M) = 0$. Indeed, B is the localization of $A \otimes_k L$ at a prime $\mathfrak{q}$ lying over the maximal ideal $\mathfrak{m}$ of A. Choose a k-basis (e_i) of L containing 1. If $a \in \operatorname{Fitt}_{r-1}(M)$, then $a \otimes 1$ vanishes in B, so there is an element $s = \sum_i a_i \otimes e_i \notin \mathfrak{q}$ such that $s(a \otimes 1) = 0$. Not all a_i lie in $\mathfrak{m}$, since otherwise $s \in \mathfrak{m}(A \otimes_k L) \subset \mathfrak{q}$. Hence some a_i is a unit in A. Comparing coefficients in $s(a \otimes 1) = 0$ shows that $a_i a = 0$, and hence $a = 0$. Therefore $\operatorname{Fitt}_{r-1}(M) = 0$, so Proposition 11.19 shows that M is free of rank r.

The final statement follows by applying the above to every $P \in X$, noting that every such P has a point of X_L lying above it. □

Example 11.25. Let $A = k[x, y]$, and let M be defined by the presentation

$$A \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} A^2 \to M \to 0.$$

Here $\operatorname{Fitt}_0(M) = 0$, $\operatorname{Fitt}_1(M) = (x, y)$, and $\operatorname{Fitt}_2(M) = A$. By Theorem 11.18, the locus $V(\operatorname{Fitt}_1(M)) = \{(0, 0)\}$ is exactly where the rank exceeds 1, and by Corollary 11.21, $\widetilde{M}$ is locally free of rank 1 on $\mathbb{A}^2_k - \{(0, 0)\}$. At the origin, the fiber has dimension 2, so $\widetilde{M}$ is not locally free there. △

The following example shows that on a non-reduced scheme, a coherent sheaf can have constant rank without being locally free.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 11.26. Let $A = k[\varepsilon]/(\varepsilon^2)$, and let $M = A/(\varepsilon) \simeq k$, viewed as an A-module. The presentation

$$A \xrightarrow{\varepsilon} A \to M \to 0$$

gives $\mathrm{Fitt}_0(M) = (\varepsilon)$ and $\mathrm{Fitt}_1(M) = A$. The ring A has a unique prime $\mathfrak{p} = (\varepsilon)$, and the fiber at this prime is $M(\mathfrak{p}) = M \otimes_A k \simeq k$, so $\mathrm{rank}_{\mathfrak{p}}(M) = 1$ everywhere on $\mathrm{Spec}\, A$. But $\mathrm{Fitt}_0(M) = (\varepsilon) \neq 0$, so by Proposition 11.19, M is not free of rank 1. This shows that the reducedness hypothesis in Theorem 11.22 is necessary: on the non-reduced scheme $\mathrm{Spec}\, A$, the sheaf $\widetilde{M}$ has constant pointwise rank 1 but is not locally free. △

Example 11.27. Let A be a DVR with fraction field K, and let s and ξ be respectively the closed and the open point of $X = \mathrm{Spec}\, A$. Let $\mathcal{E}$ be the $\mathcal{O}_X$-module with $\Gamma(X, \mathcal{E}) = A$ and $\Gamma(\{\xi\}, \mathcal{E}) = K$, and with the restriction map $A \to K$ equal to the zero map. Then $\mathcal{E}$ is an $\mathcal{O}_X$-module with exactly the same stalks as the structure sheaf $\mathcal{O}_X$, but it is not locally free (in fact, it is not even quasi-coherent), so Theorem 11.22 does not apply. △

Example 11.28. Let k be a field and let $R = k[x_0, \ldots, x_5]$. Consider the antisymmetric matrix

$$\phi = \begin{pmatrix} 0 & x_0 & x_1 & x_2 \\ -x_0 & 0 & x_3 & x_4 \\ -x_1 & -x_3 & 0 & x_5 \\ -x_2 & -x_4 & -x_5 & 0 \end{pmatrix}.$$

Computing, we find that $\det(\phi) = q^2$ where $q = x_0x_5 - x_1x_4 + x_2x_3$, that every 3×3-minor of ϕ is a multiple of q, and that the ideal generated by the 2×2-minors of ϕ has radical $(x_0, \ldots, x_5)$.

Let $A = R/(q)$ and $X = \mathrm{Spec}\, A$. The matrix ϕ gives an exact sequence

$$A^4 \xrightarrow{\phi} A^4 \longrightarrow M \longrightarrow 0$$

whose associated sheaf $\mathcal{E} = \widetilde{M}$ on X has Fitting ideals (computed over A) obtained as the images of the Fitting ideals over R. Since the source and target free modules have rank 4, the ideal $\mathrm{Fitt}_i(\mathcal{E})$ is generated by the $(4-i) \times (4-i)$ minors of ϕ. We therefore obtain $\mathrm{Fitt}_0(\mathcal{E}) = \mathrm{Fitt}_1(\mathcal{E}) = 0$, $\sqrt{\mathrm{Fitt}_2(\mathcal{E})} = (x_0, \ldots, x_5)$. By Corollary 11.21, on the open set $X - \{(0, \ldots, 0)\}$, where $\mathrm{Fitt}_2(\mathcal{E}) = A$ and $\mathrm{Fitt}_1(\mathcal{E}) = 0$, the sheaf $\mathcal{E}$ is locally free of rank 2. △

11.5 Vector bundles

For a locally free sheaf $\mathcal{E}$ of rank r, there is an associated scheme $V(\mathcal{E})$ together with a morphsim $\pi\colon V(\mathcal{E}) \to X$, whose fibers are affine r-spaces. The scheme $V(\mathcal{E})$ is called a *vector bundle of rank* r. To define it, consider the *symmetric algebra*

$$\mathrm{Sym}^*(\mathcal{E}) = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n(\mathcal{E}),$$

which is a quasi-coherent $\mathcal{O}_X$-algebra. Over an open set U where $\mathcal{E}|_U \simeq \mathcal{O}_U^r$, we have

$$\mathrm{Sym}^*(\mathcal{E})|_U \simeq \mathcal{O}_U[t_1, \ldots, t_r]. \tag{11.4}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

We define

$$V(\mathcal{E}) = \mathbf{Spec}(\mathrm{Sym}^*(\mathcal{E}))$$

which comes with an affine morphism $\pi\colon V(\mathcal{E}) \to X$. Over an affine open set $U \subset X$ where $\mathcal{E}|_U \simeq \mathcal{O}_U^r$, we have isomorphisms (of schemes over U)

$$\pi^{-1}(U) = V(\mathcal{E}) \times_X U \simeq \mathbf{Spec}(\mathrm{Sym}^*(\mathcal{E}|_U)) \simeq \operatorname{Spec} \mathcal{O}_X(U)[t_1, \ldots, t_r] \simeq \mathbb{A}^r_U.$$

This means that the morphism $\pi\colon V(\mathcal{E}) \to X$ is *locally trivial*; over $U \subset X$ it looks like the projection map $\mathbb{A}^r \times U \to U$. In particular, the scheme-theoretic fiber over a point $P \in X$ is isomorphic to $\mathbb{A}^r_{\kappa(P)}$.

In the literature, one often finds a direct definition of a vector bundle of rank r as a morphism $\pi\colon V \to X$ which is locally of the form $\mathbb{A}^r_U \to U$ such that the gluing maps $\mathbb{A}^r \times (U \cap V) \to \mathbb{A}^r \times (U \cap V)$ are 'linear' in the $\mathbb{A}^r$-coordinates. One can show that every such morphism arises from a locally free sheaf $\mathcal{E}$ as above. For this reason, it is common to swich between the terms *vector bundle* and *locally free sheaf*, as the two notions are essentially equivalent. A vector bundle corresponding to a locally free sheaf of rank 1 is called a *line bundle*.

Example 11.29. If $\mathcal{E} = \mathcal{O}_X^r$, then $V(\mathcal{E}) = \mathbb{A}^r_X$. △

Example 11.30. $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k}(-1)$, then $V(\mathcal{E})$ is the blow-up of $\mathbb{A}^2_k$ at a point. To see this, let $U_0 = \operatorname{Spec} k[x]$ and $U_1 = \operatorname{Spec} k[x^{-1}]$ be the standard covering of $\mathbb{P}^1_k$. Then $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}$ with transition function $g_{01}\colon \mathcal{O}_{U_{01}} \to \mathcal{O}_{U_{01}}$ given by multiplication by x^{-1}. Therefore, $V(\mathcal{E})$ is glued together by $V(\mathcal{E}|_{U_0}) = \operatorname{Spec} k[x,t]$ and $V(\mathcal{E}|_{U_1}) = \operatorname{Spec} k[x^{-1}, x^{-1}t]$. This is precisely the gluing data for the blow-up of $\mathbb{A}^2_k$ at the origin. More generally, $V(\mathcal{O}_{\mathbb{P}^{n-1}_k}(-1))$ is isomorphic to the blow-up of $\mathbb{A}^n_k$ at the origin. △

11.6 Exercises

Exercise 11.6.1. Let k be a field of characteristic $\neq 2$. Let $A = k[x,y]/(x^2+y^2-1)$ and $I = (x, y-1)$. Show that $I_\mathfrak{p} \simeq A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$. Show that $I \not\simeq A$ when $k = \mathbb{R}$ and $I \simeq A$ when $k = \mathbb{C}$.

Exercise 11.6.2. Let $A = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $X = \operatorname{Spec} A$. Consider the ideal $I = \mathbb{Z}/2 \times (0)$ and the sheaf $\mathcal{E} = \widetilde{I}$. Show that $\mathcal{E}$ is locally free, but not free.

Exercise 11.6.3. Let $\phi\colon L \to M$ be a map of invertible sheaves.

a) Show that if ϕ is surjective, then it is an isomorphism.
b) Give an example of an injective map ϕ which is not an isomorphism.

Exercise 11.6.4. Show that tensoring by a locally free sheaf of finite rank preserves exact sequences.

Exercise 11.6.5. Let X be an integral scheme. Show that if L is invertible and $\mathcal{E}$ is locally free, then any $\mathcal{O}_X$-linear map $\phi\colon L \to \mathcal{E}$ is either zero or injective.

Exercise 11.6.6. Let A be a ring, M be an A-module, and $S \subset A$ be a multiplicative set.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that there are natural isomorphisms $S^{-1}(\mathrm{Sym}^n_A M) \simeq \mathrm{Sym}^n_{S^{-1}A}(S^{-1}M)$ and $S^{-1}(\bigwedge^n M) \simeq \bigwedge^n (S^{-1}M)$.

b) Let $\mathcal{E}$ be a quasi-coherent sheaf on a scheme X. Define the *n-th symmetric power* $\mathrm{Sym}^n(\mathcal{E})$ and the *n-th exterior power*, $\bigwedge^n \mathcal{E}$, to be the sheaves associated to the presheaves

$$U \mapsto \mathrm{Sym}^n_{\mathcal{O}_X(U)}(\mathcal{E}(U)), \qquad U \mapsto \bigwedge^n (\mathcal{E}(U)).$$

Show that these sheaves are quasi-coherent and that for every affine open subset $U \subset X$, their sections are given by

$$\Gamma(U, \mathrm{Sym}^n \mathcal{E}) = \mathrm{Sym}^n_{\mathcal{O}_X(U)}(\mathcal{E}(U)), \qquad \Gamma(U, \bigwedge^n \mathcal{E}) = \bigwedge^n (\mathcal{E}(U)).$$

c) If $\mathcal{E}$ is locally free of rank r, show that $\mathrm{Sym}^n \mathcal{E}$ and $\bigwedge^n \mathcal{E}$ are locally free of ranks $\binom{n+r-1}{r-1}$ and $\binom{r}{n}$, respectively. Show also that $\bigwedge^n (\mathcal{E}^\vee) \simeq (\bigwedge^n \mathcal{E})^\vee$.

d) Compute $\mathrm{Sym}^n(\mathcal{O}_{\mathbb{P}^1_k}(a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(b))$ and $\bigwedge^2(\mathcal{O}_{\mathbb{P}^1_k}(a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(b) \oplus \mathcal{O}_{\mathbb{P}^1_k}(c))$.

Exercise 11.6.7. Let $0 \to \mathcal{E}' \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{E}'' \to 0$ be an exact sequence of locally free sheaves, and let $n \geqslant 1$.

a) Show that there is an exact sequence

$$\mathcal{E}' \otimes \mathrm{Sym}^{n-1} \mathcal{E} \longrightarrow \mathrm{Sym}^n \mathcal{E} \longrightarrow \mathrm{Sym}^n \mathcal{E}'' \to 0,$$

where the first map is induced by multiplication. Show moreover that if $\mathcal{E}'$ is invertible, then this sequence is short exact.

b) Show that there is an exact sequence

$$\mathcal{E}' \otimes \bigwedge^{n-1} \mathcal{E} \longrightarrow \bigwedge^n \mathcal{E} \longrightarrow \bigwedge^n \mathcal{E}'' \to 0,$$

where the first map is induced by wedge product.

c) If $\mathcal{E}'$ is invertible, show that the map in the previous item factors through $\mathcal{E}' \otimes \bigwedge^{n-1} \mathcal{E}''$ and gives a short exact sequence

$$0 \to \mathcal{E}' \otimes \bigwedge^{n-1} \mathcal{E}'' \longrightarrow \bigwedge^n \mathcal{E} \longrightarrow \bigwedge^n \mathcal{E}'' \to 0.$$

d) If $\mathcal{E}''$ is invertible, show that there is a short exact sequence

$$0 \to \bigwedge^n \mathcal{E}' \longrightarrow \bigwedge^n \mathcal{E} \longrightarrow \bigwedge^{n-1} \mathcal{E}' \otimes \mathcal{E}'' \to 0.$$

Exercise 11.6.8. Let $\mathcal{E}$ be locally free of rank r. The sheaf $\det(\mathcal{E}) = \bigwedge^r \mathcal{E}$ is called the determinant of $\mathcal{E}$.

a) Show that $\det(\mathcal{E})$ is invertible. Moreover, show that if $g_{ij} \in \mathrm{GL}_r(\mathcal{O}(U_i \cap U_j))$ are transition functions for $\mathcal{E}$, then the transition functions for $\det(\mathcal{E})$ are given by $\det(g_{ij}) \in \mathcal{O}^\times_X(U_i \cap U_j)$.

b) If $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r)$, show that $\det(\mathcal{E}) \simeq \mathcal{O}_{\mathbb{P}^1_k}(a_1 + \cdots + a_r)$.

c) Show that for a morphism $f\colon X \to Y$ and a locally free sheaf $\mathcal{E}$ on Y, there is a natural isomorphism $f^* \det(\mathcal{E}) \simeq \det(f^*\mathcal{E})$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

d) If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is exact sequence of locally free sheaves, show that $\det(\mathcal{E}) \simeq \det(\mathcal{E}') \otimes \det(\mathcal{E}'')$.

Exercise 11.6.9. Let $f\colon X \to Y$ and let $\mathcal{E}$ be a locally free sheaf of finite rank on Y. Find examples where the pullback map $f^*\colon \Gamma(Y, \mathcal{E}) \to \Gamma(X, f^*\mathcal{E})$ may fail to be injective/surjective.

Exercise 11.6.10 (The projection formula)**.** Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{O}_X$-modules. Use the adjunction between f_* and f^* to define a map

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \longrightarrow f_*(\mathcal{F}) \otimes \mathcal{E}. \tag{11.5}$$

Show that if $\mathcal{E}$ is locally free of finite rank then (11.5) is an isomorphism.

Exercise 11.6.11. Show that the subscheme $Z(s)$ for $s \in \Gamma(X, \mathcal{E})$, satisfies the following universal property: a morphism $f\colon T \to X$ satisfies $f^*s = 0$ if and only if it factors through $Z(s)$. HINT: Understand the case $X = \operatorname{Spec} A$ and $\mathcal{E} = \mathcal{O}_X$ first.

Exercise 11.6.12 (Projective modules)**.** Let A be a Noetherian integral domain. An A-module M is called *projective* if the functor $N \mapsto \operatorname{Hom}_A(M, N)$ is exact. Show that a finitely generated A-module M is projective if and only if it is a direct summand in a free module; that is, if there is another module N so that $M \oplus N \simeq A^I$.

Exercise 11.6.13. Let A be a Noetherian ring. Show that a finitely generated module is projective if and only if $M_{\mathfrak{p}}$ is projective for all $\mathfrak{p} \in \operatorname{Spec} A$. HINT: Show that $\operatorname{Hom}_A(M, N)_{\mathfrak{p}} = \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.

Exercise 11.6.14. Let $R = \prod_{i=1}^{\infty} \mathbb{F}_2$ be a direct product of countably many copies of the field $\mathbb{F}_2$ with two elements, and let $I = \bigoplus_{i=1}^{\infty} \mathbb{F}_2$.

a) Show that I is an ideal which is not finitely generated.
b) Show that R/I is not projective. HINT: I is generated by the elements $e_i \in R$ with a 1 in the i-th entry and 0 in all other entries. No nonzero element in R is killed by all the e_i's.
c) For a prime $\mathfrak{p} \in \operatorname{Spec} R$, show that $(R/I)_{\mathfrak{p}}$ is free. HINT: If $I \subset \mathfrak{p}$ and $\alpha \in I$, let $n \in \mathbb{N}$ be such that $\alpha_i = 0$ for every $i > n$ and define β by $\beta_i = 0$ for $i < n$ and $\beta_i = 1$ for $i > n$. Then $\beta \cdot \alpha = 0$ and $1 = \sum_{i=1}^{n} e_i + \beta$. Show that $\beta \notin \mathfrak{p}$ and deduce that α maps to 0 in $I_{\mathfrak{p}}$ and hence that $I_{\mathfrak{p}} = 0$.

Exercise 11.6.15. Find an example showing that if $\mathcal{E}$ is locally free, then $\mathcal{E}|_U$ needs not be free for *every* affine $U \subset X$.

Exercise 11.6.16. Consider the scheme $U = \mathbb{P}^2_k - \{(0:0:1)\}$, covered by the two affine open sets $U_0 = \operatorname{Spec} k[x, y]$ and $U_1 = \operatorname{Spec} k[x^{-1}, x^{-1}y]$ with $U_0 \cap U_1 = D(x) = D(x^{-1})$. Consider the matrix $\tau_{01} = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Show that τ_{01} defines a locally free sheaf $\mathcal{E}$ on U which fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

Compute $\Gamma(X, \mathcal{E})$. Is $\mathcal{E}$ globally generated?

Exercise 11.6.17. Let $f\colon X \to Y$ be a morphism.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that if $\mathcal{G}$ is a globally generated $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is globally generated on X.
b) Show that $f_*\mathcal{O}_X$ is globally generated if f is closed embedding.
c) Find an example of a finite morphism $f\colon X \to Y$ such that $f_*\mathcal{O}_X$ is not globally generated.

Exercise 11.6.18. Suppose $\mathcal{E}$ is locally free of rank r. Let U_i be a trivializing cover, and let τ_{ij} denote the transition functions for $\mathcal{E}$, interpretated as $r \times r$ matrices with entries in $\mathcal{O}_X(U_i \cap U_j)$. Show that $\mathcal{E}^\vee$ is obtained by the transition matrices $\nu_{ij} = (\tau_{ij}^t)^{-1}$.

Exercise 11.6.19 (Fitting ideal sheaves)**.** Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. For an affine open $U = \operatorname{Spec} A$ such that $\mathcal{F}|_U \simeq \widetilde{M}$ for a finitely generated A-module M, define the Fitting ideal (see Section **??**)

$$\operatorname{Fitt}_i(\mathcal{F})|_U = \widetilde{\operatorname{Fitt}_i(M)} \subset \mathcal{O}_U.$$

a) Show that these ideals are compatible on overlaps and glue to a coherent ideal sheaf $\operatorname{Fitt}_i(\mathcal{F}) \subset \mathcal{O}_X$.
b) Show that for $x \in X$, if $x \notin V(\operatorname{Fitt}_i(\mathcal{F}))$ then $\mathcal{F}_x$ can be generated by i elements over $\mathcal{O}_{X,x}$. Deduce that if $\mathcal{F}$ has constant rank r, then $\mathcal{F}$ is locally free of rank r on the open set $X - V(\operatorname{Fitt}_r(\mathcal{F}))$.
c) Let $f\colon Y \to X$ be a morphism of Noetherian schemes. Show that

$$\operatorname{Fitt}_i(f^*\mathcal{F}) = f^{-1}\operatorname{Fitt}_i(\mathcal{F}) \cdot \mathcal{O}_Y.$$

Exercise 11.6.20 (Koszul complex)**.** Let A be an integral domain and let $f, g \in A$. Assume that f, g form a *regular sequence*, that is, f and g are nonzerodivisors, and f is a nonzerodivisor modulo g. Consider the ideal $I = (f, g)$. Show that the sequences

$$0 \longrightarrow A \xrightarrow{\rho} A \oplus A \xrightarrow{\sigma} I \longrightarrow 0 \tag{11.6}$$

and

$$0 \longrightarrow A \xrightarrow{\rho} A \oplus A \xrightarrow{\sigma} A \longrightarrow A/I \longrightarrow 0. \tag{11.7}$$

are exact, where $\rho(a) = (-ag, af)$ and $\sigma(a, b) = af + bg$.

Exercise 11.6.21 (Koszul complex II)**.** Let X be an integral scheme and let L be an invertible sheaf on X. For two sections $s, t \in \Gamma(X, L)$, let $Y = V(s, t)$. Assume that the germs of s and t form a regular sequence in L_P for every $P \in X$.

a) Show that there are exact sequences

$$0 \longrightarrow L^{-2} \xrightarrow{\rho} L^{-1} \oplus L^{-1} \xrightarrow{\sigma} I \longrightarrow 0 \tag{11.8}$$

and

$$0 \longrightarrow L^{-2} \xrightarrow{\rho} L^{-1} \oplus L^{-1} \xrightarrow{\sigma} \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0. \tag{11.9}$$

where $\mathcal{I}$ is the ideal sheaf of the zero scheme $Y = V(s, t)$ of s and t.
b) Study this sequence when $X = \mathbb{P}^1_k$, when $L = \mathcal{O}_{\mathbb{P}^1_k}(1)$ and $s = x_0, t = x_1$. What happens on global sections?

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 11.6.22. Let k be a field, $R = k[x_0, \ldots, x_n]$ and $\mathbb{A}_k^{n+1} = \operatorname{Spec} R$. Consider the exact sequence of R-modules

$$0 \longrightarrow R \xrightarrow{\phi} R^{n+1} \longrightarrow M \longrightarrow 0 \tag{11.10}$$

where the map ϕ sends a polynomial p to $\sum_i px_ie_i$ where e_i is the i-th standard basis vector. Show that the restriction $\mathcal{E} = \widetilde{M}|_U$ is a locally free sheaf of rank n over the open set $U = \mathbb{A}_k^{n+1} - \{0\}$

Exercise 11.6.23. Let $f\colon Y \to X$ be a faithfully flat morphism of Noetherian schemes, and let $\mathcal{E}$ be a coherent sheaf on X. Show that $\mathcal{E}$ is locally free of rank r if and only if $f^*\mathcal{E}$ is locally free of rank r. HINT: Use Proposition 11.17 together with Corollary 11.21.

Exercise 11.6.24. Let M be a finitely presented A-module with presentation $A^m \xrightarrow{\varphi} A^n \to M \to 0$.

a) Show that $\operatorname{Fitt}_0(M) \subset \operatorname{Ann}(M)$. HINT: Let Δ be an $n \times n$ minor of φ, the determinant of some $n \times n$ submatrix B of φ. Use the identity $B \cdot \operatorname{adj}(B) = \Delta \cdot I_n$ together with the fact that the columns of B lie in $\operatorname{Ker}(A^n \to M)$.

b) Show that $\operatorname{Ann}(M)^n \subset \operatorname{Fitt}_0(M)$. HINT: Given $a_1, \ldots, a_n \in \operatorname{Ann}(M)$, choose $w_j \in A^m$ with $\varphi(w_j) = a_je_j$, form the matrix $C = (w_1 \mid \cdots \mid w_n)$, and apply the Cauchy–Binet formula to $\varphi C = \operatorname{diag}(a_1, \ldots, a_n)$.

c) Deduce that $\operatorname{Fitt}_0(M)$ and $\operatorname{Ann}(M)$ have the same radical, and hence the same vanishing locus in $\operatorname{Spec} A$.

Exercise 11.6.25. Let $A = k[x, y]$ and let $I = (x^2, xy) \subset A$.

a) Show that I has the presentation $A \xrightarrow{(y,\,-x)^T} A^2 \to I \to 0$. and compute $\operatorname{Fitt}_0(I)$, $\operatorname{Fitt}_1(I)$, and $\operatorname{Fitt}_2(I)$.

b) Determine the rank $\operatorname{rank}_P(\widetilde{I})$ at every point $P \in \mathbb{A}_k^2$, and show that $\widetilde{I}$ is locally free of rank 1 on $\mathbb{A}_k^2 - \{(0,0)\}$ but not at the origin.

Exercise 11.6.26. For a morphism $f\colon Y \to X$, and a locally free sheaf of rank r on X, show that $V(\mathcal{E}) \times_X Y \simeq V(f^*\mathcal{E})$, compatibly with the projections to Y.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

12

Sheaves on projective schemes

In this chapter, we study quasi-coherent sheaves on projective schemes. As in the affine case, there is a functor $M \mapsto \widetilde{M}$ that assigns to each graded R-module M a quasi-coherent sheaf on $\operatorname{Proj} R$. One of the main results of the chapter is that every quasi-coherent sheaf on a projective scheme arises in this way. In particular, this allows us to describe closed subschemes of a projective scheme in terms of homogeneous ideals.

There is, however, an important difference from the affine case: the functor $M \mapsto \widetilde{M}$ is no longer an equivalence of categories. Different graded modules can determine the same quasi-coherent sheaf, and distinct homogeneous ideals may define the same closed subscheme. This is one aspect in which projective schemes are more subtle than affine schemes (another is that morphisms of affine schemes are described in terms of ring maps, whereas morphisms of projective schemes are in general not).

A second main theme of the chapter is that sheaves on projective schemes are best studied using invertible sheaves, in particular, the twisting sheaves $\mathcal{O}_X(d)$. These enter in several ways: sections of invertible sheaves define closed subschemes, globally generating sections determine morphisms to projective space, and the graded module associated to a quasi-coherent sheaf $\mathcal{F}$ is recovered by taking global sections of all of its twists $\mathcal{F}(d)$. For this reason, invertible sheaves play an important role in algebraic geometry.

12.1 The sheaf associated to a graded module

Consider a projective scheme $X = V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$, defined by the ring

$$R = A[x_0, \dots, x_n]/\mathfrak{a} \tag{12.1}$$

with the standard grading. As in Chapter 6, the scheme X is covered by the standard affine opens $U_i = \operatorname{Spec}(R_{x_i})_0$, where $(R_{x_i})_0$ denotes the degree 0 part of the localization R_{x_i}.

Let M be a graded R-module. Then M determines a sheaf $\widetilde{M}$ on X as follows. Over each U_i, the localization $(M_{x_i})_0$ is a module over $(R_{x_i})_0$, and we take the sheaf

$$\widetilde{(M_{x_i})_0}$$

Over an overlap $U_i \cap U_j = \operatorname{Spec}(R_{x_i x_j})_0$, we identify

$$\widetilde{(M_{x_i})_0}|_{U_i \cap U_j} = \widetilde{(M_{x_i x_j})_0} = \widetilde{(M_{x_j})_0}|_{U_i \cap U_j}.$$

These identifications are compatible over the triple intersections $U_i \cap U_j \cap U_k = \operatorname{Spec}(R_{x_i x_j x_k})_0$, so the sheaves glue to a quasi-coherent $\mathcal{O}_X$-module, which we denote by $\widetilde{M}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This construction is functorial. If $\varphi\colon M \to N$ is a map of graded R-modules, then localizing and taking degree 0 parts gives $(R_{x_i})_0$-linear maps

$$(\varphi_{x_i})_0\colon (M_{x_i})_0 \longrightarrow (N_{x_i})_0.$$

These agree on the overlaps in the sense that $(\varphi_{x_i})_0$ and $(\varphi_{x_j})_0$ both localize to the same map $(M_{x_ix_j})_0 \longrightarrow (N_{x_ix_j})_0$ over $U_i \cap U_j$. After gluing, we obtain an $\mathcal{O}_X$-linear map $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$. Therefore, $M \mapsto \widetilde{M}$ defines a functor from graded R-modules to quasi-coherent sheaves on X.

This functor bears many similarities to the tilde-functor for Spec, but there are also important differences. The most striking is that non-isomorphic modules can give rise to the same sheaf.

Lemma 12.1. Let M be a graded R-module. The following are equivalent:
(i) $\widetilde{M} = 0$.
(ii) $M_{x_i} = 0$ for every $i = 0, \dots, n$.
(iii) For each $m \in M$, there exists $N \in \mathbb{N}$ such that $x_i^N \cdot m = 0$ for all i.
(iv) $\operatorname{Supp} M \subset V(x_0, \dots, x_n)$.

If M is finitely generated, these are further equivalent to
(v) $M_d = 0$ for all sufficiently large d.

The intuition behind the lemma is that $\widetilde{M}$ only sees M away from the irrelevant locus $V(x_0, \dots, x_n)$. A module supported there is invisible on X.

Proof (i) $\Leftrightarrow$ (ii): The sheaf $\widetilde{M}$ is zero if and only if its restriction $\widetilde{M}|_{U_i}$ to every open U_i is zero. Since U_i is affine, $\widetilde{M}|_{U_i}$ is zero if and only if the corresponding module $(M_{x_i})_0$ is zero. Since x_i acts invertibly on M_{x_i}, this is equivalent to $M_{x_i} = 0$. Therefore (i) and (ii) are equivalent.

(ii) $\Leftrightarrow$ (iii): This follows by the definition of localization.

(ii) $\Leftrightarrow$ (iv): Note that $D(x_0) \cup \cdots \cup D(x_n) = \operatorname{Spec} R - V(x_0, \dots, x_n)$. Now $M_{x_i} = 0$ if and only if $M_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in D(x_i)$, so the condition that all M_{x_i} vanish is equivalent to saying that $\operatorname{Supp}(M)$ is contained in $V(x_0, \dots, x_n)$.

(iii) $\Leftrightarrow$ (v) for M finitely generated: Choose homogeneous generators $m_1, \dots, m_r \in M$ of degrees $e_1, \dots, e_r$ and for each $j = 1, \dots, r$ choose an $N_j \in \mathbb{N}$ such that $x_i^{N_j} m_j = 0$ for all i. Let $e = \max_j e_j$ and $N = \max_j N_j$. Now assume $d > e + (n+1)N$. Every homogeneous element $m \in M_d$ can be written as $m = \sum a_j m_j \in M_d$ with a_j of degree $d - e_j$. Since each monomial in R_{d-e_j} has degree $> (n+1)N$, each monomial in a_j must contain some x_i^p with $p \geqslant N$, and so $a_j m_j = 0$. Hence $M_d = 0$ for all $d > e + nN$.

Conversely, if $M_d = 0$ for all $d \geqslant N$, then clearly $x_i^N m_j = 0$. □

Example 12.2. On $\mathbb{P}^1_k$, the graded module $M = k[x_0, x_1]/(x_0^2, x_1^2)$ satisfies $\widetilde{M} = 0$, yet it is nonzero. △

More generally, if $M_{\geqslant d}$ denotes the truncation $M_{\geqslant d} = \bigoplus_{e \geqslant d} M_e$, then Lemma 12.1 (applied to $M/M_{\geqslant d}$, which vanishes in large degrees) shows that M and $M_{\geqslant d}$ define the same sheaf. Consequently, if two graded R-modules M and N satisfy $M_{\geqslant d} \simeq N_{\geqslant d}$ for

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

some $d > 0$, then $\widetilde{M} \simeq \widetilde{N}$. The converse holds when M and N are finitely generated (see Exercise 12.8.3).

Proposition 12.3. Let R be a ring as above.

(i) The functor $M \mapsto \widetilde{M}$ is exact.

(ii) For graded R-modules $\{M_\alpha\}_{\alpha\in I}$, there is an $\mathcal{O}_X$-linear isomorphism

$$\widetilde{\bigoplus_{\alpha\in I} M_\alpha} \simeq \bigoplus_{\alpha\in I} \widetilde{M_\alpha}.$$

(iii) For two graded R-modules M and N, there is an $\mathcal{O}_X$-linear isomorphism

$$\widetilde{M \otimes_R N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

(iv) If M is a finitely generated graded R-module, then $\widetilde{M}$ is of finite type.

One word about (iii): if M and N are two graded R-modules, then the grading on the tensor product $M \otimes_R N$ is defined by letting $(M \otimes_R N)_d$ to be the additive subgroup of $M \otimes_R N$ generated by elements $x \otimes y$ where $x \in M_p$ and $y \in N_q$ where $p + q = d$. One checks that $M \otimes_R N$ is the direct sum of these graded parts as an R_0-module.

Proof (i): Let $M' \to M \to M''$ be an exact sequence of graded R-modules. It suffices to show that the corresponding sequence $\widetilde{M'} \to \widetilde{M} \to \widetilde{M''}$ of sheaves on X is exact locally. Restricting the sequence to $U_i = \operatorname{Spec}(R_{x_i})_0$, we obtain

$$\widetilde{(M'_{x_i})}_0 \longrightarrow \widetilde{(M_{x_i})}_0 \longrightarrow \widetilde{(M''_{x_i})}_0$$

The latter sequence is exact, because it is obtained from $M' \to M \to M''$ by localizing at x_i, taking degree 0 parts, and then applying the affine tilde-functor $\widetilde{(-)}$, all of which preserve exactness.

(ii): Since localization and taking degree 0 parts both commute with direct sums, there is a natural isomorphism of $(R_{x_i})_0$-modules

$$\big((\textstyle\bigoplus_\alpha M_\alpha)_{x_i}\big)_0 \;\simeq\; \bigoplus_\alpha \big((M_\alpha)_{x_i}\big)_0.$$

Taking the associated sheaves on U_i gives an isomorphism

$$\big(\widetilde{(\textstyle\bigoplus_\alpha M_\alpha)_{x_i}}\big)_0 \;\simeq\; \bigoplus_\alpha \big(\widetilde{(M_\alpha)_{x_i}}\big)_0.$$

These are compatible with the restriction maps, so they glue to the isomorphism in (ii).

(iii): Here $((M \otimes_R N)_{x_i})_0$ is an $(R_{x_i})_0$-module via the rule

$$a/x_i^s \cdot (m \otimes n)/x_i^t = (am \otimes n)/x_i^{s+t} = (m \otimes an)/x_i^{s+t}.$$

Consider the $(R_{x_i})_0$-linear map

$$\varphi_i \colon (M_{x_i})_0 \otimes_{(R_{x_i})_0} (N_{x_i})_0 \longrightarrow ((M \otimes_R N)_{x_i})_0, \tag{12.2}$$

induced by sending $m/x_i^d \otimes n/x_i^e$ to $(m \otimes n)/x_i^{d+e}$. One checks that φ_i is well-defined and $(R_{x_i})_0$-linear. It is surjective because $((M \otimes_R N)_{x_i})_0$ is generated by elements of the form $(m \otimes n)/x_i^{d+e}$ where $m \in M_d$, $n \in N_e$ are homogeneous. We define the inverse map ψ_i on

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

such elements by sending $(m \otimes n)/x_i^{d+e}$ to $m/x_i^d \otimes n/x_i^e$ in $(M_{x_i})_0 \otimes_{(R_{x_i})_0} (N_{x_i})_0$. One checks that the resulting map is well-defined and $(R_{x_i})_0$-linear. Moreover, $\varphi_i \circ \psi_i = \mathrm{id}$ and $\psi \circ \varphi_i = \mathrm{id}$, so φ_i is an isomorphism of $(R_{x_i})_0$-modules. These isomorphisms are moreover compatible, in the sense that φ_i and φ_j both localize to the same isomorphism

$$(M_{x_i x_j})_0 \otimes_{(R_{x_i x_j})_0} (N_{x_i x_j})_0 \longrightarrow ((M \otimes_R N)_{x_i x_j})_0, \tag{12.3}$$

sending $m/(x_i x_j)^d \otimes n/(x_i x_j)^e \mapsto (m \otimes n)/(x_i x_j)^{d+e}$. Therefore the map induces $\mathcal{O}_X$-linear isomorphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_R N}$.

If M is a finitely generated graded R-module, then each $(M_{x_i})_0$ is a finitely generated $(R_{x_i})_0$-module. Hence each restriction $\widetilde{M}|_{U_i}$ is of finite type, and therefore so is $\widetilde{M}$. □

Each element $m \in M_0$ of degree 0 defines a global section of the sheaf $\widetilde{M}$. Indeed, using the sheaf sequence

$$0 \longrightarrow \Gamma(X, \widetilde{M}) \longrightarrow \bigoplus_{i=0}^{n} (M_{x_i})_0 \longrightarrow \bigoplus_{i,j=0}^{n} (M_{x_i x_j})_0, \tag{12.4}$$

we may identify $\Gamma(X, \widetilde{M})$ with the R_0-module of $(n+1)$-tuples $(m_0, \ldots, m_n)$ where $m_i \in (M_{x_i})_0$ and $m_i = m_j$ in $M_{x_i x_j}$. In particular, the tuple $(m/1, \ldots, m/1)$ satisfies this condition. We will usually denote the corresponding section by $m \in \Gamma(X, \widetilde{M})$. This assignment defines an R_0-linear map

$$M_0 \longrightarrow \Gamma(X, \widetilde{M}) \tag{12.5}$$

To handle elements of nonzero degrees, we shift the grading. For $d \in \mathbb{Z}$, define $M(d)$ be the graded R-module with underlying R-module equal to M, but with the grading shifted by d steps:

$$M(d)_n = M_{n+d}.$$

So $M(d)$ has $(M(d))_0 = M_d$, $(M(d))_1 = M_{d+1}$ and so on, and in general an element from M_n will have degree $n - d$ when viewed in $M(d)$.

Over U_i we have

$$\big(M(d)_{x_i}\big)_0 = (M_{x_i})_d = x_i^d \cdot (M_{x_i})_0.$$

In particular, for $M = R$, we obtain the *twisting sheaves* $\mathcal{O}_X(d) = \widetilde{R(d)}$. These generalize the sheaves $\mathcal{O}_{\mathbb{P}^n_A}(d)$ on $\mathbb{P}^n_A$ which were introduced in Chapter 6.

Note that $\mathcal{O}_X(d)|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module generated by x_i^d. In particular, $\mathcal{O}_X(d)$ is an invertible sheaf. On the overlap $U_i \cap U_j$, the two generators are related by

$$x_i^d = \left(\frac{x_i}{x_j}\right)^d x_j^d,$$

so the transition functions of $\mathcal{O}_X(d)$ are exactly as in Example 11.8.

There is a canonical isomorphism $M(d) = M \otimes_R R(d)$. Indeed, both have M as underlying module, and the gradings coincide by the way we have defined the grading on the tensor product. By Proposition 12.3, we obtain a canonical isomorphism

$$\widetilde{M(d)} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

In particular, since $R(d+e) = R(d) \otimes_R R(e)$ for every $d, e \in \mathbb{Z}$, we obtain a canonical isomorphism

$$\mathcal{O}_X(d+e) \simeq \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(e).$$

To summarize, the two most important properties of $\mathcal{O}_X(d)$ are:

(i) Homogeneous elements of degree d in R give global sections of $\mathcal{O}_X(d)$

(ii) Tensoring a sheaf $\widetilde{M}$ by $\mathcal{O}_X(d)$ corresponds to shifting the grading of the module M.

The next goal is to show that any quasi-coherent sheaf on X is of the form $\widetilde{M}$ for some graded R-module M. To prove this, we need a construction that produces a graded module from a quasi-coherent sheaf $\mathcal{F}$. In the case $X = \operatorname{Spec} A$, this module was simply given to us by $M = \mathcal{F}(X)$. For $X = \operatorname{Proj} R$, taking global sections is not enough. Instead, one has to consider all the global sections of all twists $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$ at once.

For a quasi-coherent sheaf $\mathcal{F}$, we define its associated graded module of $\mathcal{F}$ by

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{F}(d)).$$

This is a graded R-module, because each element $a \in R_d$ can be regarded as a global section of $\mathcal{O}_X(d)$, and for $t \in \Gamma_*(\mathcal{F})_e = \Gamma(X, \mathcal{F}(e))$, the product $a \cdot t$ can be defined to be the image of $a \otimes t$ via the multiplication map

$$\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(e) \simeq \mathcal{F}(d+e)$$

Here is the main theorem of this chapter.

Theorem 12.4. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism

$$\widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}.$$

If $\mathcal{F}$ is of finite type, then there exists a finitely generated graded submodule $N \subset \Gamma_*(\mathcal{F})$ such that $\widetilde{N} \simeq \mathcal{F}$.

To prove this, we prove two general lemmas about sections over distinguished open sets.

Let L be an invertible sheaf on a scheme X, and let $s \in \Gamma(X, L)$. We define the nonvanishing locus of s by

$$D(s) = \{P \in X \mid s(P) \neq 0 \text{ in } L/\mathfrak{m}_P L\}.$$

Locally, after trivializing L, the section s becomes an ordinary regular function, so $D(s)$ is an open subset of X. The next lemma says that s generates L over $D(s)$.

Lemma 12.5. Let X be a scheme, let L be an invertible sheaf on X, and let $s \in \Gamma(X, L)$. Then the map $f \mapsto f \cdot s$ defines an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_{D(s)} \simeq L|_{D(s)}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof This is local on X, so we may assume that $L \simeq \mathcal{O}_X$ and $X = \operatorname{Spec} A$ is affine. Then s corresponds to a regular function $g \in A$ and $D(s) = D(g)$. Then the above map is simply the multiplication map $A_g \to A_g$, $f \mapsto fg$, which is an isomorphism because g is invertible. □

There is a useful fraction notation for sections over $D(s)$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, let $n \geqslant 0$, and consider a local section $t \in \Gamma(U, \mathcal{F} \otimes L^{\otimes n})$. By Lemma 12.5, we have $\mathcal{F} \otimes L^n|_{D(s)} \simeq \mathcal{F}|_{D(s)}$. Therefore, there exists a unique section

$$t/s^n \in \Gamma(U \cap D(s), \mathcal{F})$$

with the property that $t|_{U\cap D(s)} = t/s^n \otimes s^{\otimes n}$ in $\Gamma(U \cap D(s), \mathcal{F} \otimes L^{\otimes n})$. In particular, if $t \in \Gamma(U, L)$, then $t/s \in \Gamma(U \cap D(s), \mathcal{O}_X)$ is the unique regular function satisfying $t|_{U\cap D(s)} = t/s \cdot s$.

Locally, if $U = \operatorname{Spec} A$ is an affine open set over which $L|_U$ is free with generator e, we may write $s = ge$ for some $g \in A$, and any section of $\mathcal{F} \otimes L^{\otimes n}$ may be written uniquely as $t = m \otimes e^{\otimes n}$ with $m \in \Gamma(U, \mathcal{F})$. With this notation, $t/s^n = m/f^n$ in the usual sense, over $D(s) = D(g) = \operatorname{Spec} A_g$.

Example 12.6. Every homogeneous element $f \in R_d$ defines a global section of $\mathcal{O}_X(d)$. Under the trivialization of $\mathcal{O}_X(d)|_{U_i}$ by x_i^d, this section is represented by the regular function $f/x_i^d \in \mathcal{O}_X(U_i)$, which we recognize as the dehomogenization of f with respect to x_i. △

The next lemma says that every section of a quasi-coherent sheaf on $D(s)$ can be written uniquely as a homogeneous fraction with denominator a power of s.

Lemma 12.7. Let X be a scheme and let L be an invertible sheaf on X. Assume that X admits a finite affine cover $\{U_i\}_{i\in I}$ such that $L|_{U_i}$ is free and each overlap $U_i \cap U_j$ admits a finite affine cover. Let $s \in \Gamma(X, L)$, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Define

$$R = \bigoplus_{n\geqslant 0} \Gamma(X, L^{\otimes n}), \qquad M = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F} \otimes L^{\otimes n}).$$

Then there is a canonical isomorphism

$$\Gamma(D(s), \mathcal{F}) \simeq (M_s)_0.$$

Proof Consider the sheaf $\mathcal{M} = \bigoplus_{n\in\mathbb{N}} \mathcal{F} \otimes L^{\otimes n}$. The assumptions show that X is quasi-compact, so the direct sum sheaf coincides with the direct sum presheaf. We have $\mathcal{M}(X) = M$, and the restriction maps make each $\mathcal{M}(U)$ into a graded R-module.

For each pair $i, j \in I$, choose a finite affine cover $U_i \cap U_j = \bigcup_k U_{ijk}$. This gives an exact sequence of graded R-modules

$$0 \longrightarrow M \longrightarrow \bigoplus_i \mathcal{M}(U_i) \longrightarrow \bigoplus_{i,j,k} \mathcal{M}(U_{ijk}).$$

Because the covers are finite, we may use direct sums instead of products. Localizing at the homogeneous element $s \in R_1$ and taking the degree-0 part gives an exact sequence

$$0 \longrightarrow (M_s)_0 \longrightarrow \bigoplus_i (\mathcal{M}(U_i)_s)_0 \longrightarrow \bigoplus_{i,j,k} (\mathcal{M}(U_{ijk})_s)_0.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

We now compute $(\mathcal{M}(U)_s)_0$ for each affine open U among the U_i and U_{ijk}. The fraction notation gives a natural map

$$(\mathcal{M}(U)_s)_0 \longrightarrow \Gamma(U \cap D(s), \mathcal{F}), \qquad m/s^r \longmapsto m/s^r, \tag{12.6}$$

where $r \geqslant 0$ and $m \in \Gamma(U, \mathcal{F} \otimes L^{\otimes r})$.

To show that this map is an isomorphism, choose a generator e of the free $\mathcal{O}_U$-module $L|_U$, and write $s|_U = fe$ with $f \in \mathcal{O}_X(U)$. Then $U \cap D(s) = D(f)$. If $f = 0$, then $s = 0$, so $D(s) = \varnothing$ and there is nothing to show. Assume $f \neq 0$. For each $r \geqslant 0$, every section of $\mathcal{F} \otimes L^{\otimes r}$ over U can be written uniquely as $m \otimes e^{\otimes r}$ with $m \in \mathcal{F}(U)$. Under this identification, the map above becomes $m \otimes e^{\otimes r}/s^r \mapsto m/f^r$, so it identifies $(\mathcal{M}(U)_s)_0$ with $\Gamma(U, \mathcal{F})_f$. Since U is affine and $\mathcal{F}$ is quasi-coherent, there is a canonical isomorphism

$$\Gamma(U, \mathcal{F})_f \simeq \Gamma(D(f), \mathcal{F}) = \Gamma(U \cap D(s), \mathcal{F}).$$

The map (12.6) is exactly the composition of these isomorphisms, so it is an isomorphism.

On the other hand, the opens $U_{i,s} = U_i \cap D(s)$ cover $D(s)$, and the opens $U_{ijk} \cap D(s)$ cover the intersections $U_i \cap U_j \cap D(s)$. Therefore the sheaf condition for $\mathcal{F}$ gives an exact sequence

$$0 \longrightarrow \Gamma(D(s), \mathcal{F}) \longrightarrow \bigoplus_i \Gamma(U_i \cap D(s), \mathcal{F}) \longrightarrow \bigoplus_{i,j,k} \Gamma(U_{ijk} \cap D(s), \mathcal{F}).$$

Via the isomorphisms above, the first exact sequence becomes the second one. We obtain the desired isomorphism $\Gamma(D(s), \mathcal{F}) \simeq (M_s)_0$ by the 5-Lemma. □

Proof of Theorem 12.4 Set $M = \Gamma_*(\mathcal{F})$. We produce the isomorphism $\widetilde{M} \simeq \mathcal{F}$ chart by chart.

Apply Lemma 12.7 with $L = \mathcal{O}_X(1)$ and $s = x_i$. Since $D(x_i) = U_i$, we obtain canonical isomorphisms $\mathcal{F}(U_i) \simeq (M_{x_i})_0$. Concretely, every section of $\mathcal{F}$ on U_i can be written uniquely as a homogeneous fraction t/x_i^n with $t \in \Gamma(X, \mathcal{F}(n))$, and this is exactly an element of $(M_{x_i})_0$.

Because U_i is affine and $\mathcal{F}|_{U_i}$ is quasi-coherent, we conclude

$$\mathcal{F}|_{U_i} \simeq \widetilde{\mathcal{F}(U_i)} \simeq \widetilde{(M_{x_i})_0} = \widetilde{M}|_{U_i}.$$

These isomorphisms are compatible on overlaps, because the localization isomorphism of Lemma 12.7 is compatible with restriction. Hence they glue to an isomorphism $\mathcal{F} \simeq \widetilde{M}$.

Now assume that $\mathcal{F}$ is of finite type. For each i, choose finitely many sections $\sigma_{i1}, \ldots, \sigma_{ir_i} \in \mathcal{F}(U_i)$ that generate $\mathcal{F}|_{U_i}$. Via the isomorphism $\mathcal{F}(U_i) \simeq (M_{x_i})_0$, write $\sigma_{ij} = m_{ij}/x_i^{d_{ij}}$ with $m_{ij} \in M_{d_{ij}}$ and $d_{ij} \geqslant 0$. Let $N \subset M$ be the graded submodule generated by the finitely many homogeneous elements m_{ij}. For each i, the fractions $m_{ij}/x_i^{d_{ij}}$ lie in $(N_{x_i})_0$ and generate $(M_{x_i})_0 = \mathcal{F}(U_i)$. Hence $(N_{x_i})_0 \twoheadrightarrow (M_{x_i})_0$ for every i, so $((M/N)_{x_i})_0 = 0$ for every i. Lemma 12.1 then gives $\widetilde{M/N} = 0$. Applying tilde to the sequence $0 \to N \to M \to M/N \to 0$, then shows that $\widetilde{N} \simeq \widetilde{M} \simeq \mathcal{F}$. Since N is finitely generated, this proves the final statement. □

To summarize, we have constructed two functors, $\widetilde{(-)}$ and $\Gamma_*(-)$ between the category of graded modules to quasi-coherent sheaves on $\operatorname{Proj} R$. Since $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is an isomorphism,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

it follows that $\widetilde{(-)}$ is *essentially surjective*; that is, every quasi-coherent sheaf on $\operatorname{Proj} R$ is isomorphic to one of the form $\widetilde{M}$. It is however not faithful, as $\widetilde{M} = 0$ for any module M with support in $V_+(R_+)$. By Lemma 12.1 however, this is the only source of ambiguity.

Example 12.8. $\Gamma_*(\mathcal{O}_{\mathbb{P}^n_A}) = A[x_0, \dots, x_n]$, by Proposition 6.12. △

Example 12.9. It can happen that $\Gamma_*(\mathcal{O}_X)$ is different than R. A simple example is $X = \operatorname{Proj} k[t] = \operatorname{Spec} k$. Then $\mathcal{O}_X(1) \simeq \mathcal{O}_{\operatorname{Spec} k}$, and so $\Gamma(X, \mathcal{O}_X(m)) = k$ for all m, even $m < 0$. This example also shows that the module $\Gamma_*(\mathcal{F})$, which equals $\bigoplus_{n\in\mathbb{Z}} k$, can be non-finitely generated even if $\mathcal{F}$ is of finite type. △

A more interesting source of such examples is to take a projective scheme $X \subset \mathbb{P}^n_k$ and project it into $\mathbb{P}^{n-1}_k$. In some cases this will ramain a closed embedding, but in this new embedding X will be equal to $\operatorname{Proj} S$ for a different graded ring S.

Example 12.10 (A quartic rational curve)**.** Let $S = k[u^4, u^3v, uv^3, v^4]$, where each appearing monomial has degree 1 (note that the monomial u^2v^2 is missing). The projective scheme $X = \operatorname{Proj} S \subset \mathbb{P}^3_k$ is the image of the closed embedding

$$\mathbb{P}^1_k \longrightarrow \mathbb{P}^4_k, \qquad (u : v) \mapsto (u^4 : u^3v : uv^3 : v^4)$$

Geometrically, X is obtained by projecting the rational normal curve in $\mathbb{P}^4_k$ from the point $(0 : 0 : 1 : 0 : 0)$.

To compute $\Gamma(X, \mathcal{O}_X(1))$ we use the sheaf sequence for the affine cover consisting of $U_0 = D_+(u^4)$ and $U_1 = D_+(v^4)$. Here

$$\mathcal{O}_X(U_0) = (S_{u^4})_0 = k[v/u], \qquad \mathcal{O}_X(U_1) = k[u/v],$$

and $\mathcal{O}_X(1)$ is generated on U_0 by u^4 and on U_1 by v^4. The sheaf sequence reads

$$0 \to \Gamma(X, \mathcal{O}_X(1)) \to k[v/u]u^4 \oplus k[u/v]v^4 \to k[u/v, v/u]u^4.$$

Note that the monomial u^2v^2 satisfies $u^2v^2 = (v/u)^2u^4 = (u/v)^2v^4$, so it agrees on the two charts and defines a global section of $\mathcal{O}_X(1)$. In fact

$$\Gamma(X, \mathcal{O}_X(1)) = ku^4 \oplus ku^3v \oplus ku^2v^2 \oplus kuv^3 \oplus kv^4,$$

So $\Gamma(X, \mathcal{O}_X(1))$ contains all 5 monomials, while u^2v^2 is missing from S. In particular $\Gamma_*(\mathcal{O}_X)$ is strictly larger than S. △

Remark 12.11. Here is an alternative way to think about the module $\Gamma_*(\mathcal{F})$. Let

$$U = \operatorname{Spec} R - V(x_0, \dots, x_n),$$

and let $\iota\colon U \hookrightarrow \operatorname{Spec} R$ and $\pi\colon U \to X = \operatorname{Proj} R$ be the open embedding and the quotient morphism. If $\mathcal{F}$ is quasi-coherent on X, then $\pi^*\mathcal{F}$ is quasi-coherent on U. Since ι satisfies the hypotheses of Theorem 10.29, the sheaf $\iota_*\pi^*\mathcal{F}$ is quasi-coherent on $\operatorname{Spec} R$, hence is of the form $\widetilde{N}$ for some R-module N. To identify N, we compute

$$N = \Gamma(\operatorname{Spec} R, \iota_*\pi^*\mathcal{F}) = \Gamma(U, \pi^*\mathcal{F}).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

To simplify this, note that on $D(x_i) \subset U$ one has the decomposition $R_{x_i} = \bigoplus_{d\in\mathbb{Z}} (R_{x_i})_0\, x_i^d$, so over $U_i = D_+(x_i)$, there are natural local isomorphisms

$$\pi_*\pi^*\mathcal{F}|_{U_i} \simeq \bigoplus_{d\in\mathbb{Z}} \mathcal{F}(d)|_{U_i}.$$

These are compatible on overlaps, and glue to $\pi_*\pi^*\mathcal{F} \simeq \bigoplus_{d\in\mathbb{Z}} \mathcal{F}(d)$. Hence

$$N = \Gamma(U, \pi^*\mathcal{F}) = \Gamma(X, \pi_*\pi^*\mathcal{F}) = \Gamma_*(\mathcal{F}).$$

Similarly, localizing at x_i and taking 0 parts recovers $(N_{x_i})_0 \simeq \mathcal{F}(U_i)$, compatibly on overlaps. These isomorphisms glue to $\widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}$.

Corollary 12.12. Let $\mathcal{F}$ be a finite-type $\mathcal{O}_X$-module. Then there exist integers $d > 0$ and $N \geqslant 0$ together with a surjection

$$\mathcal{O}_X(-d)^{\oplus N} \twoheadrightarrow \mathcal{F}. \tag{12.7}$$

Proof By Theorem 12.4, we may choose a finitely generated graded R-module M with $\widetilde{M} \simeq \mathcal{F}$. For $d \gg 0$, the truncation $M_{\geqslant d}$ is generated by its degree-d part M_d, so there is a graded surjection $R(-d)^{\oplus N} \twoheadrightarrow M_{\geqslant d}$ for some N. The quotient $M/M_{\geqslant d}$ is concentrated in degrees $< d$, so Lemma 12.1 gives $\widetilde{M/M_{\geqslant d}} = 0$. Applying $M \mapsto \widetilde{M}$ to the exact sequence $0 \to M_{\geqslant d} \to M \to M/M_{\geqslant d} \to 0$, we obtain $\widetilde{M_{\geqslant d}} \simeq \widetilde{M} \simeq \mathcal{F}$. Since $M \mapsto \widetilde{M}$ is exact, we obtain a surjection as in (12.7). □

12.2 Closed subschemes of projective space

Every homogeneous ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$ determines a closed subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$. The main result of this section is that every closed subscheme arises in this way. However, as in the discussion on quasi-coherent sheaves, different ideals can give rise to the same closed subscheme.

Example 12.13. In $k[x_0, x_1]$, the two ideals $\mathfrak{a} = (x_0)$ and $\mathfrak{b} = (x_0^2, x_0x_1)$ both define the closed subscheme given by the point $(0 : 1)$ in $\mathbb{P}^1_k$. To see this, note that on the open set $x_1 \neq 0$, the ideal $\mathfrak{b}$ dehomogenizes to $(x_0^2/x_1^2, x_0x_1/x_1^2) = (x_0/x_1)$ in $k[x_0/x_1]$, while on the open set $x_0 \neq 0$ it dehomogenizes to the unit ideal in $k[x_1/x_0]$. Note also that $\mathfrak{b} = (x_0^2, x_1) \cap (x_0)$. So while both $\mathfrak{a}$ and $\mathfrak{b}$ define the same subscheme of $\mathbb{P}^1_k$, the ideal $\mathfrak{b}$ is in some sense inferior to $\mathfrak{a}$, as it has a component in the irrelevant ideal (x_0, x_1). △

To deal with this ambiguity, we introduce the *saturation* of an ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$ with respect to the irrelevant ideal $(x_0, \ldots, x_n)$, defined as

$$\mathfrak{a}^{\mathrm{sat}} = \{f \in A[x_0, \ldots, x_n] \;\mid\; \exists N \in \mathbb{N} \text{ such that } f \cdot x_i^N \in \mathfrak{a} \text{ for every } i\}.$$

We always have the inclusion $\mathfrak{a} \subset \mathfrak{a}^{\mathrm{sat}}$, and we say that $\mathfrak{a}$ is *saturated* if $\mathfrak{a} = \mathfrak{a}^{\mathrm{sat}}$. If $\mathfrak{a}$ is homogeneous, then so is $\mathfrak{a}^{\mathrm{sat}}$.

In the example above, the saturation of (x_0^2, x_0x_1) is the ideal (x_0). This example is typical: taking the saturation throws away components of $\mathfrak{a}$ supported on the irrelevant ideal. In particular, the saturation of the irrelevant ideal $(x_0, \ldots, x_n)$ itself is the unit ideal.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The saturation is connected to the operations $\widetilde{(-)}$ and Γ_* as follows. Consider a homogeneous ideal $\mathfrak{a} \subset A[x_0, \dots, x_n]$ and its associated closed subscheme $Y = V_+(\mathfrak{a})$, with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n_A}$. Applying the exact functor $M \mapsto \widetilde{M}$ to the exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A[x_0, \dots, x_n] \longrightarrow A[x_0, \dots, x_n]/\mathfrak{a} \longrightarrow 0$$

we see that $\widetilde{\mathfrak{a}} = \mathcal{I}_Y$. We also consider the associated graded module

$$\Gamma_*(\mathcal{I}_Y) = \bigoplus_{d \geqslant 0} \Gamma(\mathbb{P}^n_A, \mathcal{I}_Y(d)) \subset A[x_0, \dots, x_n].$$

Using $\widetilde{\mathfrak{a}} = \mathcal{I}_Y$, we obtain

$$\begin{aligned} \Gamma_*(\widetilde{\mathfrak{a}}) &= \bigoplus_{d \geqslant 0} \{\, f \in A[x_0, \dots, x_n]_d \mid f/x_i^d \in (\mathfrak{a}_{x_i})_0 \text{ for every } i \,\} \\ &= \bigoplus_{d \geqslant 0} \{\, f \in A[x_0, \dots, x_n]_d \mid \exists N \in \mathbb{N} \text{ such that } f \cdot x_i^N \in \mathfrak{a} \text{ for every } i \,\} \\ &= \mathfrak{a}^{\mathrm{sat}}. \end{aligned} \tag{12.8}$$

That is, applying $\mathfrak{a} \mapsto \widetilde{\mathfrak{a}}$ and then $\Gamma_*(-)$ corresponds to taking the saturation of $\mathfrak{a}$.

Theorem 12.14. For any ring A, there is a bijection

$$\left\{ \begin{array}{c} \text{closed subschemes} \\ Y \subset \mathbb{P}^n_A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{saturated homogeneous} \\ \text{ideals } \mathfrak{a} \subset A[x_0, \dots, x_n] \end{array} \right\}$$

given by $Y \mapsto \Gamma_*(\mathcal{I}_Y)$ and $\mathfrak{a} \mapsto V_+(\mathfrak{a})$.

Proof Let $Y \subset \mathbb{P}^n_A$ be a closed subscheme with ideal sheaf $\mathcal{I}_Y$, and set $\mathfrak{a} = \Gamma_*(\mathcal{I}_Y)$. We first show that $Y = V_+(\mathfrak{a})$ and that $\mathfrak{a}$ is saturated.

The ideal sheaf of $V_+(\mathfrak{a})$ is $\widetilde{\mathfrak{a}} = \widetilde{\Gamma_*(\mathcal{I}_Y)}$ and by Theorem 12.4, the natural map $\widetilde{\Gamma_*(\mathcal{I}_Y)} \to \mathcal{I}_Y$ is an isomorphism. Hence $V_+(\mathfrak{a})$ and Y have the same ideal sheaf, and therefore are the same closed subscheme of $\mathbb{P}^n_A$.

It remains to check that $\mathfrak{a}$ is saturated. Since $\mathfrak{a}$ is homogeneous, it suffices to treat homogeneous elements. Let $f \in \mathfrak{a}^{\mathrm{sat}}$ be homogeneous of degree d, so that $x_i^N f \in \mathfrak{a}$ for some N and all i. Then $x_i^N f$ defines a global section of $\mathcal{I}_Y(d+N)$. On the standard open set $U_i = \{x_i \neq 0\}$, multiplication by x_i^N is invertible, so the restriction of f to U_i lies in $\mathcal{I}_Y(d)|_{U_i}$. Since the opens U_i cover $\mathbb{P}^n_A$, it follows that f is a global section of $\mathcal{I}_Y(d)$. Hence $f \in \Gamma(\mathbb{P}^n_A, \mathcal{I}_Y(d)) \subset \mathfrak{a}$, and so $\mathfrak{a}$ is saturated.

Conversely, let $\mathfrak{a} \subset A[x_0, \dots, x_n]$ be a saturated homogeneous ideal, and put $Y = V_+(\mathfrak{a})$. Then $\mathcal{I}_Y = \widetilde{\mathfrak{a}}$, and by (12.8) we have $\Gamma_*(\mathcal{I}_Y) = \Gamma_*(\widetilde{\mathfrak{a}}) = \mathfrak{a}^{\mathrm{sat}} = \mathfrak{a}$. The two constructions are therefore mutually inverse. □

The sheaves $\mathcal{O}_{\mathbb{P}^n_k}(m)$ illustrate an important difference between $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$. On $\mathbb{A}^n_k$, closed subsets are cut out by regular functions, that is, by polynomials in $k[x_1, \dots, x_n]$. On $\mathbb{P}^n_k$, the only global regular functions are the constants. Nevertheless, every closed subset of $\mathbb{P}^n_k$ can be described as the zero set of sections of some $\mathcal{O}_{\mathbb{P}^n_k}(m)$, that is, of homogeneous polynomials.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

12.3 Coherent sheaves on projective space

We recall the following important theorem from commutative algebra, which describes graded modules over a polynomial ring in terms of free modules.

Let k be a field and let $R = k[x_0, \ldots, x_n]$. Hilbert's syzygy theorem says that any finitely generated graded R-module has a graded free resolution of the form

$$0 \longrightarrow E_n \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0, \tag{12.9}$$

where each $E_j = \bigoplus_{i=1}^{b_k} R(-d_{ij})$ is a free graded R-module of finite rank. E_i is called the i-th *syzygy module* of the resolution.

The maps between the free modules in (12.9) are defined by matrices with homogeneous entries in R. This sort of resolution is very useful when studying M, as it allows us to apply linear algebra techniques which can simplify computations and define invariants of M.

Applying the exact functor $\widetilde{(-)}$ to the sequence (12.9), we obtain an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \widetilde{M} \longrightarrow 0,$$

where $\mathcal{E}_j = \bigoplus_{i=1}^{b_k} \mathcal{O}_{\mathbb{P}^n_k}(-d_{ij})$ is a direct sum of twisting sheaves. Hence any finite type $\mathcal{O}_{\mathbb{P}^n_k}$-module admits a resolution where the terms are direct sums of sheaves of the form $\mathcal{O}_{\mathbb{P}^n_k}(d)$. In this sense, the invertible sheaves $\mathcal{O}_{\mathbb{P}^n_k}(d)$ are the building blocks of all coherent sheaves on $\mathbb{P}^n_k$.

Example 12.15 (Hypersurfaces)**.** Let $F \in R$ denote a nonzero homogeneous polynomial of degree $d > 0$ and let $X = \operatorname{Proj}(R/(F)) \subset \mathbb{P}^n_k$ be the corresponding projective hypersurface with closed embedding $\iota\colon X \hookrightarrow \mathbb{P}^n_k$.

Consider the map $R(-d) \to R$ given by multiplication by F. Note the shift in degrees here, in order to make this into a map of graded modules (the constant '1' gets sent to F, which should have degree d on both sides). We obtain an exact sequence

$$0 \longrightarrow R(-d) \longrightarrow R \longrightarrow R/(F) \longrightarrow 0.$$

Applying $\widetilde{(-)}$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \widetilde{R/(F)} \longrightarrow 0. \tag{12.10}$$

Note that $\widetilde{R/(F)}$ is isomorphic to the sheaf $\iota_*\mathcal{O}_X$; both sheaves take the value $(R_f/(F))_0$ over each distinguished open set $D_+(f)$. Therefore, comparing (12.10) to the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0,$$

we see that there is an isomorphism of $\mathcal{O}_{\mathbb{P}^n_k}$-modules $\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^n_k}(-d)$. △

Example 12.16 (The twisted cubic curve)**.** Let k be a field and write $\mathbb{P}^3_k = \operatorname{Proj} R$, where $R = k[x_0, x_1, x_2, x_3]$. Recall the *twisted cubic curve* $C \subset \mathbb{P}^3_k$, defined by the ideal $I \subset R$ generated by the 2×2-minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

that is, the 3 quadrics $(q_0, q_1, q_2) = (x_1x_3 - x_2^2,\ -x_0x_3 + x_1x_2,\ x_1^2 - x_0x_2)$.

Consider the graded surjection $\varphi\colon R(-2)^3 \to I$ sending $e_i \mapsto q_i$. Its kernel is the module of relations of the form $a_0q_0 + a_1q_1 + a_2q_2 = 0$ for $a_i \in R$. There are two obvious relations of this form,

$$x_0q_0 + x_1q_1 + x_2q_2 = 0, \qquad x_1q_0 + x_2q_1 + x_3q_2 = 0.$$

One way to see them is to append the row vectors $(x_0\, x_1\, x_2)$ and $(x_1\, x_2\, x_3)$ to the matrix A. In each case, the resulting 3×3-matrix has two equal rows, so that determinant is zero. Expanding the determinant gives the above relations.

It turns out that these are all the relations, so we obtain $\operatorname{Ker}(\varphi) \simeq R(-3)^2$ and an exact sequence

$$0 \longrightarrow R(-3)^2 \xrightarrow{A^t} R(-2)^3 \longrightarrow I \longrightarrow 0.$$

Taking the associated sheaves, we get a resolution of the ideal sheaf $\mathcal{I} = \widetilde{I}$ of C:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-3)^2 \xrightarrow{A^t} \mathcal{O}_{\mathbb{P}^3_k}(-2)^3 \longrightarrow \mathcal{I} \longrightarrow 0.$$

We will see later that such resolutions by direct sums of invertible sheaves will be very useful when computing cohomology groups of coherent sheaves. △

12.4 Remarks about non-standard gradings

So far in this chapter, we have assumed that the ring R is generated by finitely many elements $x_0, \dots, x_n$ of degree 1. For general graded rings R, much of the theory carries over, but there are a few subtleties that make the statements more complicated. The definition of the quasi-coherent sheaves $\widetilde{M}$ and $\mathcal{O}_X(d)$ still work without changes. However, it may happen that $\widetilde{M} \otimes \widetilde{N} \ncong \widetilde{M \otimes_R N}$ and the sheaf $\mathcal{O}_X(d)$ need not be locally free. Here is an example:

Example 12.17. Let $R = k[x_0, x_1, x_2]$, where x_0, x_1, x_2 have degrees 1,2,3 respectively and set $X = \operatorname{Proj} R$. We claim that the multiplication map

$$\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(2) \tag{12.11}$$

is not an isomorphism. To see this, work over the affine chart $U = D_+(x_2)$ corresponding to

$$A = (R_{x_2})_0 = k\left[\frac{x_0^3}{x_2},\ \frac{x_0x_1}{x_2},\ \frac{x_1^3}{x_2^2}\right].$$

Over U, the sections of $\mathcal{O}_X(1)$ and $\mathcal{O}_X(2)$ are given by

$$(R(1)_{x_2})_0 = A \cdot x_0 + A \cdot \frac{x_1^2}{x_2}, \qquad (R(2)_{x_2})_0 = A \cdot x_1 + A \cdot x_0^2,$$

and (12.11) is given by the multiplication map $(R(1)_{x_2})_0 \otimes_A (R(1)_{x_2})_0 \to (R(2)_{x_2})_0$.

The image of the multiplication map is generated as an A-module by the pairwise products of the generators of $(R(1)_{x_2})_0$, that is, x_0^2, $x_0x_1^2/x_2$, and x_1^4/x_2^2. This image does not contain x_1. Indeed, the module is a monomial A-submodule of $(R(2)_{x_2})_0$, and none of its monomial generators can be multiplied by a monomial of A to give x_1. But $x_1 \in (R(2)_{x_2})_0$, so the map is not surjective on U, and (12.11) is not an isomorphism. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

12.5 Pullbacks of invertible sheaves

Let $f\colon X \to Y$ be a morphism. For an invertible sheaf L on Y, the pullback f^*L is an invertible sheaf on X. Indeed, recall that pullbacks pull the structure sheaf back to the structure sheaf. Therefore, if L is trivial over an open cover $\{V_i\}_{i\in I}$, then f^*L is trivial over the open sets $f^{-1}(V_i)$. More concretely, if L is described the transition functions $g_{ij} \in \mathcal{O}_X(V_i \cap V_j)^\times$, then f^*L is described by pulled back functions $f^\sharp(g_{ij}) \in \mathcal{O}_X(f^{-1}(V_i \cap V_j))$.

If L and M are two invertible sheaves on Y, then

$$f^*(L \otimes_{\mathcal{O}_Y} M) \simeq f^*(L) \otimes_{\mathcal{O}_X} f^*(M).$$

It follows that f^* induces a map between the Picard groups

$$f^*\colon \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(X).$$

We can also pull back sections. For any open subset $W \subset Y$ and section $s \in L(W)$, we obtain a section $f^*(s) \in f^*L(f^{-1}(W))$ as follows: locally on $W \cap V_i$ where L is trivial, we can write $s = a_i \cdot s_i$ for some $a_i \in \mathcal{O}_Y(W \cap V_i)$ and local generator s_i of $L|_{V_i}$. The pullback section is then given by $f^*(s) = f^\sharp(a_i) \cdot f^*(e_i)$ on $f^{-1}(W \cap V_i)$, where $f^*(e_i)$ is the corresponding local generator of $f^*L|_{f^{-1}(V_i)}$. This construction is independent of the choice of trivialization and glues to give a well-defined global section.

Proposition 12.18. Let $f\colon X \to Y$ be a morphism of schemes, let L be an invertible sheaf on Y, and let $s \in \Gamma(Y, L)$. Then $D(f^*(s)) = f^{-1}(D(s))$. In particular, if $s_0, \ldots, s_n$ generate L, then $f^*(s_0), \ldots, f^*(s_n)$ generate f^*L.

Proof The statement is local on Y and X, so we may assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $L = \mathcal{O}_Y$. Then s is given by an element $a \in A$, and $f^*(s)$ is given by its image $f^\sharp(a) \in B$. Hence

$$D(f^*(s)) = D(f^\sharp(a)) = f^{-1}(D(a)) = f^{-1}(D(s)).$$

If $s_0, \ldots, s_n$ generate L, then the opens $D(s_i)$ cover Y. Therefore the opens $D(f^*(s_i)) = f^{-1}(D(s_i))$ cover X. Since f^*L is invertible, this means that $f^*(s_0), \ldots, f^*(s_n)$ generate f^*L. □

Example 12.19. Consider the morphism

$$f\colon \mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k, \qquad (x_0 : x_1) \longmapsto (x_0^2 : x_0x_1 : x_1^2).$$

We claim that $f^*\mathcal{O}_{\mathbb{P}^2_k}(1) \simeq \mathcal{O}_{\mathbb{P}^1_k}(2)$.

Let $U_i = D_+(x_i)$ be the standard open subsets of $\mathbb{P}^1_k$, and let $V_j = D_+(y_j)$ be the standard open subsets of $\mathbb{P}^2_k$. The image of f is covered by V_0 and V_2, with $f^{-1}(V_0) = U_0$ and $f^{-1}(V_2) = U_1$. The sheaf $\mathcal{O}_{\mathbb{P}^2_k}(1)$ has y_0 as a local generator on V_0 and y_2 on V_2. Since $y_2 = (y_2/y_0)y_0$ on $V_0 \cap V_2$, the corresponding transition function is y_2/y_0. Pulling back this regular function gives

$$f^\sharp(y_2/y_0) = x_1^2/x_0^2 = (x_1/x_0)^2.$$

This is exactly the transition function for $\mathcal{O}_{\mathbb{P}^1_k}(2)$ with respect to the trivializations by x_0^2 on U_0 and by x_1^2 on U_1. Hence $f^*\mathcal{O}_{\mathbb{P}^2_k}(1) \simeq \mathcal{O}_{\mathbb{P}^1_k}(2)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Under this isomorphism, a section $s = ay_0 + by_1 + cy_2 \in \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(1))$ pulls back to

$$f^*(s) = ax_0^2 + bx_0x_1 + cx_1^2$$

in $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(2))$. △

12.6 Morphisms to projective space

One reason why invertible sheaves are important is that they help us understand morphisms to projective space. Let A be a ring and let X be an A-scheme. We want to understand what data is needed to specify a morphism of A-schemes

$$f\colon X \longrightarrow \mathbb{P}^n_A. \tag{12.12}$$

The corresponding question for $\mathbb{A}^n_A$ has already been answered: morphisms of A-schemes $X \to \mathbb{A}^n_A$ are in one-to-one correspondence with n-tuples of regular functions $f_1, \dots, f_n \in \mathcal{O}_X(X)$. For projective space the right replacement for a tuple of regular functions is a tuple of sections of an invertible sheaf.

Given a morphism $f\colon X \to \mathbb{P}^n_A$, we obtain an invertible sheaf $L = f^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ on X, together with $n+1$ distinguished global sections $s_i = f^*x_i$ obtained by pulling back the homogeneous coordinates $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$. Since $x_0, \dots, x_n$ generate $\mathcal{O}_{\mathbb{P}^n_A}(1)$, the pullbacks $s_0, \dots, s_n$ generate L.

The main result in this section is that this process can be reversed: a morphism to projective space is the same thing as an invertible sheaf together with a generating set of global sections.

Theorem 12.20. Let X be a scheme over a ring A. Let L be an invertible sheaf on X and let $s_0, \dots, s_n \in \Gamma(X, L)$ be global sections that generate L. Then there is a unique morphism of A-schemes

$$f\colon X \longrightarrow \mathbb{P}^n_A$$

and an isomorphism $f^*\mathcal{O}(1) \simeq L$ that maps f^*x_i to s_i for $i = 0, \dots, n$.

Proof Write $R = A[x_0, \dots, x_n]$ and denote $\mathcal{O}_{\mathbb{P}^n_A}(1)$ by $\mathcal{O}(1)$, so that $\mathbb{P}^n_A = \operatorname{Proj} R$.

By Lemma 12.5, the open sets $D(s_i)$ cover X, and for each i the restriction $L|_{D(s_i)}$ is a free $\mathcal{O}_{D(s_i)}$-module with local generator s_i. Hence for each i and j, there is a unique section $s_j/s_i \in \mathcal{O}_X(D(s_i))$ such that $s_j = (s_j/s_i) \cdot s_i$ over $D(s_i)$.

These fractions define a map of A-algebras

$$(R_{x_i})_0 \longrightarrow \Gamma(D(s_i), \mathcal{O}_X), \qquad x_j/x_i \longmapsto s_j/s_i, \tag{12.13}$$

and the corresponding morphism of affine schemes is a map of A-schemes $f_i\colon D(s_i) \longrightarrow D_+(x_i) = \operatorname{Spec}(R_{x_i})_0$.

To check that the morphisms f_i glue, we check the overlaps $D(s_i) \cap D(s_j) = D(s_is_j)$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

and $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$. We have a commutative diagram

$$\begin{array}{ccc} (R_{x_i})_0 & \longrightarrow & \Gamma(D(s_i), \mathcal{O}_X) \\ \downarrow & & \downarrow \\ (R_{x_i x_j})_0 & \longrightarrow & \Gamma(D(s_i s_j), \mathcal{O}_X) \end{array} \tag{12.14}$$

As the bottom the map sends $\frac{x_k}{x_j} = \frac{x_k/x_i}{x_j/x_i}$ to $\frac{s_k}{s_j} = \frac{s_k/s_i}{s_j/s_i}$, we see that both f_i and f_j restrict to the same map $D(s_i s_j) \to \mathbb{P}^n_A$, so they glue to a morphism $f\colon X \to \mathbb{P}^n_A$.

Next, we show that $f^*\mathcal{O}(1) \simeq L$. Over $D_+(x_i)$, the sheaf $\mathcal{O}(1)$ is free with local generator x_i. Pulling back, $f^*\mathcal{O}(1)$ is free over $D(s_i) = f^{-1}(D_+(x_i))$ with local generator f^*x_i. Likewise, L is free over $D(s_i)$ with local generator s_i. Define an isomorphism $f^*\mathcal{O}(1)|_{D(s_i)} \to L|_{D(s_i)}$ by sending $f^*x_i \mapsto s_i$. To see that these local isomorphisms glue, recall that $\mathcal{O}(1)$ is obtained by gluing the trivial sheaves on the $D_+(x_i)$ with transition functions $g_{ij} = x_i/x_j$. Pulling back, $f^*\mathcal{O}(1)$ is glued from trivial sheaves on the $D(s_i)$ with transition functions $f^\sharp(x_i/x_j) = s_i/s_j$. But these are also the transition functions for L with respect to the local generators s_i, so the local maps $f^*x_i \mapsto s_i$ are compatible on overlaps and define a global isomorphism $f^*\mathcal{O}(1) \simeq L$ sending f^*x_i to s_i for every i.

Finally, the morphism f is unique, because the sections $s_0, \ldots, s_n$ determine the ratios s_j/s_i, which in term determine each $D_+(x_i) \to D_+(x_i)$ and f is uniquely determined by its restriction to each $D(s_i)$. □

Theorem 12.20 defines a bijection between the set of morphisms $f\colon X \to \mathbb{P}^n_A$ over A and the set of equivalence classes of tuples $(L, s_0, \ldots, s_n)$ where L is an invertible sheaf and $s_0, \ldots, s_n \in \Gamma(X, L)$ are global sections generating it. That is, a morphism f corresponds to the tuple $\bigl(f^*\mathcal{O}_{\mathbb{P}^n_A}(1),\ f^*x_0, \ldots, f^*x_n\bigr)$. Here two tuples $(L, s_0, \ldots, s_n)$ and $(M, t_0, \ldots, t_n)$ are equivalent if there is an isomorphism $\alpha\colon L \to M$ such that $\alpha(s_i) = t_i$ for all i. When the sheaf is the same, this just amounts to scaling: $(L, s_0, \ldots, s_n)$ and $(L, t_0, \ldots, t_n)$ determine the same morphism if and only if there is a global unit $\lambda \in \mathcal{O}_X^\times(X)$ with $t_i = \lambda\, s_i$ for every i.

It is common to write the morphism corresponding to $(L, s_0, \ldots, s_n)$ as

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n_A, \\ P &\longmapsto (s_0(P) : \cdots : s_n(P)). \end{aligned}$$

Here one should keep in mind that the s_i are sections of L, not regular functions. It is their ratios $s_j/s_i \in \mathcal{O}_X(D(s_i))$ that are regular and that define the morphism.

If the chosen sections $s_0, \ldots, s_n$ fail to generate L at every point of X, they still generate L on the open subset

$$U = \bigcup_{i=0}^{n} D(s_i) \subset X,$$

and define a morphism $U \to \mathbb{P}^n_A$, that is, a *rational map* $X \dashrightarrow \mathbb{P}^n_A$ defined on U.

Example 12.21. Let $X = \mathbb{A}^{n+1}_k = \operatorname{Spec} k[x_0, \ldots, x_n]$ and $L = \mathcal{O}_X$. Then $\Gamma(X, L) = k[x_0, \ldots, x_n]$. If we take the sections $x_0, \ldots, x_n$, then they generate L outside $V(x_0, \ldots, x_n)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Hence we get a morphism of schemes

$$\begin{aligned} \mathbb{A}_k^{n+1} - V(x_0, \dots, x_n) &\longrightarrow \mathbb{P}_k^n \\ (x_0, \dots, x_n) &\longmapsto (x_0 : \cdots : x_n) \end{aligned}$$

which is exactly the 'quotient space' description of $\mathbb{P}^n$ from (6.4). △

Example 12.22. On $\mathbb{P}_k^1$ the sections x_0^2, x_0x_1, x_1^2 generate $\mathcal{O}_{\mathbb{P}_k^1}(2)$. Theorem 12.20 therefore gives a morphism

$$\nu_2 \colon \mathbb{P}_k^1 \longrightarrow \mathbb{P}_k^2, \qquad (x_0 : x_1) \longmapsto (x_0^2 : x_0x_1 : x_1^2).$$

Its image is the smooth conic $V(x_0x_2 - x_1^2)$, and one has $\nu_2^*\mathcal{O}_{\mathbb{P}_k^2}(1) \simeq \mathcal{O}_{\mathbb{P}_k^1}(2)$. △

An important special case is $L = \mathcal{O}_X$ and $A = \mathbb{Z}$, in which case the sections are simply regular functions $g_0, \dots, g_n \in \mathcal{O}_X(X)$, and the requirement that they generate $L = \mathcal{O}_X$ means that the open sets $D(g_i)$ cover X. The associated morphism $f \colon X \to \mathbb{P}^n$ is then given on $D(g_i)$ by the regular functions $g_0/g_i, \dots, g_n/g_i$. For $X = \operatorname{Spec} k$ with k a field, this recovers the description of k-valued points of projective space: a tuple $(a_0, \dots, a_n) \in k^{n+1}$ not all zero determines a morphism $\operatorname{Spec} k \to \mathbb{P}^n$, and every such morphism is of this form. Hence $\mathbb{P}^n(k) = (k^{n+1} - \{0\})/k^\times$.

It is natural to ask whether the formula $\mathbb{P}^n(R) = (R^{n+1} - \{0\})/R^\times$ holds for all rings R. It does whenever $\operatorname{Pic}(\operatorname{Spec} R) = 0$ (in particular for local rings and for $R = \mathbb{Z}$), since then every invertible sheaf on $\operatorname{Spec} R$ is trivial. But it fails in general, as the following example shows.

Example 12.23. Let us consider the R-valued points of $\mathbb{P}^1$ for $R = \mathbb{Z}[\sqrt{-5}]$. The ring $\mathbb{Z}[\sqrt{-5}]$ is famously not a UFD, as

$$2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}). \tag{12.15}$$

We use this to define an R-point which is not of the form $(a : b)$ with $a, b \in R$.

Note that 2 and 3 generate the unit ideal of R, so $\operatorname{Spec} R$ is covered by $D(2) = \operatorname{Spec} R[\frac{1}{2}]$ and $D(3) = \operatorname{Spec} R[\frac{1}{3}]$, with intersection $D(6) = \operatorname{Spec} R[\frac{1}{6}]$.

Let $\alpha = (1 + \sqrt{-5})/2$ and $\beta = (1 - \sqrt{-5})/3$ and consider the ring maps $\phi_0 \colon \mathbb{Z}[x] \to R[\frac{1}{2}]$ and $\phi_1 \colon \mathbb{Z}[x^{-1}] \to R[\frac{1}{3}]$ sending $x \mapsto \alpha$ and $x^{-1} \mapsto \beta$. Because of (12.15), $\alpha\beta = 1$, and the maps ϕ_0 and ϕ_1 both extend to the *same* ring map

$$\mathbb{Z}[x, x^{-1}] \longrightarrow R[\tfrac{1}{6}].$$

Applying Spec, we see that the corresponding morphisms $D(2) \to \operatorname{Spec} \mathbb{Z}[x]$ and $D(3) \to \operatorname{Spec} \mathbb{Z}[x^{-1}]$ glue to a morphism $\operatorname{Spec} R \to \mathbb{P}^1$.

This morphism does not come from a global pair $(a : b)$. Indeed, on $D(2)$ it is given by $x \mapsto (1 + \sqrt{-5})/2$, so such a pair would have to be equivalent to $(1 + \sqrt{-5} : 2)$. That would force the ideal $(1 + \sqrt{-5}, 2)$ to be principal, but it is not. See Section A.7 for more details on this example.

What makes this example work is the fact that R is not a UFD and has many non-principal ideals. We will consider this example again in Chapter 19 when we discuss divisors. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

12.7 Application: Automorphisms of projective space

Let k be a field. An invertible $(n+1)\times(n+1)$ matrix $M \in \mathrm{GL}_{n+1}(k)$ acts linearly on $k[x_0,\dots,x_n]$ and so determines a linear automorphism $\mathbb{P}^n_k \to \mathbb{P}^n_k$. Two matrices M and M' define the same automorphism if and only if $M = \lambda M'$ for some $\lambda \in k^\times$, so the *projective linear group*

$$\mathrm{PGL}_{n+1}(k) = \mathrm{GL}_{n+1}(k)/k^\times$$

injects into the group of k-automorphisms of $\mathbb{P}^n_k$. We now show that this injection is surjective.

Theorem 12.24. $\mathrm{Aut}(\mathbb{P}^n_k) = \mathrm{PGL}_{n+1}(k)$.

Proof Let $\phi\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ be an automorphism. Then ϕ induces an isomorphism $\mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(\mathbb{P}^n_k)$. Since $\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, we must have either $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(1)$ or $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(-1)$. The latter case is however impossible, because $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ has nonzero global sections, whereas $\mathcal{O}_{\mathbb{P}^n_k}(-1)$ has none. Hence $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1) \simeq \mathcal{O}_{\mathbb{P}^1_k}(1)$. Fix such an isomorphism. The induced k-linear isomorphism $\phi^*\colon \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \to \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$ is given, in the basis $x_0,\dots,x_n$, by some invertible matrix $M \in \mathrm{GL}_{n+1}(k)$. By Theorem 12.20, the morphism ϕ is uniquely determined by the tuple $\big(\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1), \phi^*x_0, \dots, \phi^*x_n\big)$, and this is the same tuple defined by the linear automorphism associated to M. Therefore, by uniqueness, ϕ is the automorphism induced by M. □

Example 12.25 (Automorphisms of $\mathbb{A}^n_k$)**.** The automorphisms of $\mathbb{A}^1_k$ are all of the form $x \mapsto ax + b$ with $a \in k^\times$ and $b \in k$ (see Exercise 1.6.2).

For $n \geqslant 2$, there is no simple description of the automorphism group of $\mathbb{A}^n_k$. Any automorphism $\mathbb{A}^n_k \to \mathbb{A}^n_k$ is given by some tuple of polynomials $(f_1,\dots,f_n)$, but the degrees of the f_i can be arbitrarily large. For instance, on $\mathbb{A}^2_k$, every polynomial $p(x) \in k[x]$ gives an automorphism $(x,y) \mapsto (x, y + p(x))$, so $\mathrm{Aut}(\mathbb{A}^2_k)$ is infinite-dimensional. △

12.8 Exercises

Exercise 12.8.1. Let $R = k[x]$ with the standard grading and let $X = \operatorname{Proj} R = \operatorname{Spec} k$. Show that the R-modules $M = k[x]$ and $N = (x)$ give rise to isomorphic sheaves.

Exercise 12.8.2. Let $R = k[x]$ with $\deg(x) = 2$ and let $X = \operatorname{Proj} R = \operatorname{Spec} k$. Let $M = \bigoplus_{d\in\mathbb{Z}} k[x]y^{2i+1}$ with y in degree 1. Show that $\widetilde{M} = 0$ and that $\widetilde{M \otimes_R M} \not\simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{M}$.

Exercise 12.8.3. Let M, N be finitely generated graded R-modules. Show that $\widetilde{M} \simeq \widetilde{N}$ if and only if $M_{>d} \simeq N_{>d}$ for some $d > 0$. HINT: Consider the kernel and cokernel of $M \to \Gamma_*(X, \widetilde{M})$.

Exercise 12.8.4. Let $X = V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$ and set $R = A[x_0,\dots,x_n]/\mathfrak{a}$.

a) Show that if each x_i is a nonzerodivisor, then

$$\bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)) = \bigcap_{i=0}^{n} R_{x_i} \subset R_0[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}]. \tag{12.16}$$

b) Show that if each x_i is a prime element, then $\bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)) = R$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 12.8.5. Let k be an algebraically closed field and let $X \subset \mathbb{P}^n_k$ be a projective k-scheme. Show that $\Gamma(X, \mathcal{O}_X) = k$.

Exercise 12.8.6. Let $X = V_+(x^2 + y^2) \subset \mathbb{P}^1_{\mathbb{R}}$. Show that X is affine and that $\Gamma(X, \mathcal{O}_X) \simeq \mathbb{C}$.

Exercise 12.8.7. Let $\mathbb{P}^1_k = \operatorname{Proj} k[u_0, u_1]$ and $t = u_0/u_1$. Consider the closed subscheme $Z \subset \mathbb{P}^1_k$ which is supported at $(0 : 1)$ and which is locally given as $\operatorname{Spec} k[t]/t^n \subset D_+(u_0) = \operatorname{Spec} k[t]$. Describe the R module $\Gamma_*\mathcal{O}_Z$ and the canonical map $R = \Gamma_*\mathcal{O}_{\mathbb{P}^1_k} \to \Gamma_*\mathcal{O}_Z$.

Exercise 12.8.8. Let $X = \mathbb{P}^2_k$. Show that $L = \mathcal{O}_{\mathbb{P}^2}(2)$ is globally generated by the sections x_0^2, x_1^2, x_2^2.

Exercise 12.8.9. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ and $L = p^*\mathcal{O}_{\mathbb{P}^1_k}(1) \otimes q^*\mathcal{O}_{\mathbb{P}^1_k}(1)$ where $p, q \colon X \to \mathbb{P}^1$ are the two projections. If x_0, x_1 is a basis for $\Gamma(X, p^*\mathcal{O}_{\mathbb{P}^1_k}(1))$, and y_0, y_1 is a basis for $\Gamma(X, q^*\mathcal{O}_{\mathbb{P}^1_k}(1))$, show that $p^*\mathcal{O}_{\mathbb{P}^1_k}(1) \otimes q^*\mathcal{O}_{\mathbb{P}^1_k}(1)$ is globally generated by the four sections $s_0 = x_0y_0$, $s_1 = x_0y_1$, $s_2 = x_1y_0$, $s_3 = x_1y_1$. Describe the corresponding morphism $X \to \mathbb{P}^3_k$.

Exercise 12.8.10. Let $X = \mathbb{A}^1_k$ and $L = \mathcal{O}_X$. Show that the sections $1, t^2, t^3$ generate L and that the corresponding morphism $X \to \mathbb{P}^2_k$ has image equal to the cuspidal cubic $V(x_0x_2^2 - x_1^3)$ minus the point at infinity.

Exercise 12.8.11. Let $X = V_+(x_0) \subset \mathbb{P}^2_k$, and let $\iota \colon X \hookrightarrow \mathbb{P}^2_k$ be the inclusion. Under the natural identification $X \simeq \mathbb{P}^1_k$, show that $\iota^*\mathcal{O}_{\mathbb{P}^2_k}(1) \simeq \mathcal{O}_{\mathbb{P}^1_k}(1)$. Moreover, show that $\Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(1)) \simeq kx_0 \oplus kx_1 \oplus kx_2$, while $\Gamma(X, \iota^*\mathcal{O}_{\mathbb{P}^2_k}(1)) \simeq kx_1 \oplus kx_2$. So, unlike the pushforward, the pullback does not always give an isomorphism on global sections.

Exercise 12.8.12 (Closed subschemes of $\mathbb{P}^m \times \mathbb{P}^n$)**.** Let $R = A[x_0, \ldots, x_m, y_0, \ldots, y_n]$ with the *bigrading* defined by $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. An element $F \in R$ is called *bihomogeneous* of bidegree (d, e) if all the monomials in F have bidegree (d, e). Given a bihomogeneous polynomial F of bidegree (d, e), define its *dehomogenization* with respect to x_p and y_q as $F_{(p,q)} = x_p^{-d}y_q^{-e}F$ in the ring $A[x_i/x_p, y_j/y_q]$. For a bihomogeneous ideal $\mathfrak{a} \subset R$, define $\mathfrak{a}_{(p,q)} \subset A[x_i/x_p, y_j/y_q]$ to be the ideal generated by all dehomogenizations of elements of $\mathfrak{a}$.

a) Show that the closed subschemes
$$V_+(\mathfrak{a}_{(p,q)}) \subset D_+(x_p) \times_A D_+(y_q) = \mathbb{A}^m_A \times_A \mathbb{A}^n_A$$
glue to a closed subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^m_A \times_A \mathbb{P}^n_A$.

b) Show that there is a bijection between closed subschemes of $\mathbb{P}^m_A \times \mathbb{P}^n_A$ and bihomogeneous ideals in $A[x_0, \ldots, x_m, y_0, \ldots, y_n]$ which are saturated with respect to the 'irrelevant ideal' $(x_0, \ldots, x_m) \cap (y_0, \ldots, y_n)$

Exercise 12.8.13. The aim of this exercise is to investigate the functor of points of projective space $\mathbb{P}^n$. We will associate to a scheme T, its set of data $(L, s_0, \ldots, s_n)$ where L is an invertible sheaf L, with an $(n + 1)$-tuple of sections $s_0, \ldots, s_n$ that locally generate L everywhere. We declare $(L, s_0, \ldots, s_n) \sim (M, t_0, \ldots, t_n)$ if there is an isomorphism $f : L \to M$ so that $f^*(s_i) = \lambda \cdot t_i$ for some $\lambda \in \mathcal{O}_T^\times(T)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that $\sim$ is an equivalence relation.
b) Consider the assignment

$$F(T) = \big\{(L, s_0, \dots, s_n)\big| s_0, \dots, s_n \in \Gamma(T, L) \text{ generate } L \text{ everywhere}\big\} / \sim$$

Show that F is a functor.
c) Show that there is a natural transformation

$$\Phi(X) : \mathrm{Hom}(T, \mathbb{P}^n) \to F(T)$$

sending a morphism $f\colon T \to \mathbb{P}^n$ to the equivalence class of the data

$$(L, s_0, \dots, s_n) = (f^*\mathcal{O}(1), f^*x_0, \dots, f^*x_n)$$

d) Construct an inverse to Φ and deduce that F is represented by $\mathbb{P}^n$.
e) Show that elements of $F(\mathrm{Spec}\, k)$ are in correspondence with $(n+1)$-tuples $(a_0, \dots, a_n) \in k^{n+1}$, so that not all a_i are zero.
f) Show that the previous exercise also holds for a local ring.
g) Show that for a ring R, the set $F(\mathrm{Spec}\, R)$ is in bijection with the set of projective rank 1 summands of R^{n+1}, i.e., modules of rank 1 such that $M \oplus E \simeq R^{n+1}$ for some module E. This is the right generalization of a "line in k^n" for general rings.
h) Show that the more naive functor $F\colon \mathsf{Rings} \to \mathsf{Sets}$ defined by $F(R) = \{\text{free rank 1 submodules } L \subset R^{n+1}\}$ is *not* representable by a scheme. HINT: Consider $R = \mathbb{Z}[\sqrt{-5}]$.

Exercise 12.8.14. This is a continuation of Exercise 12.8.13. We will consider the product $X = \mathbb{P}^m \times \mathbb{P}^n$ and give a new interpretation of the Segre embedding $X \to \mathbb{P}^{(m+1)(n+1)-1}$ in terms of the functor of points.

a) Let T be a scheme and let $(L, s_0, \dots, s_m)$ and $(M, t_0, \dots, t_n)$ be elements of $h_{\mathbb{P}^m}(T)$ and $h_{\mathbb{P}^n}(T)$ respectively. Show that the $(m+1)(n+1)$ tensor products $u_{ij} = pr_1^* s_i \otimes pr_2^* t_j$ generate $pr_1^* L \otimes pr_2^* M$ on $T \times T$.
b) Show that

$$(pr_1^* L \otimes pr_2^* M, u_{00}, \dots, u_{mn}) \tag{12.17}$$

defines an element of $h_{\mathbb{P}^{(m+1)(n+1)-1}}(T)$, and that this defines a functor from $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$.
c) Deduce that there is a morphism $\phi\colon X \to \mathbb{P}^{(m+1)(n+1)-1}$.
d) Show that ϕ is an embedding. HINT: Show that the morphism ϕ has the property that $\phi^{-1}(D_+(u_{ij})) = D_+(x_0) \times D_+(y_0)$, and show that ϕ restricts to an embedding on distinguished subsets.

Exercise 12.8.15. This is a continuation of Exercise 12.8.13. We will consider the projective space $\mathbb{P}^n$ and give a new interpretation of the Veronese embedding $X \to \mathbb{P}^N$ in terms of the functor of points.

a) Let T be a scheme and let $(L, s_0, \dots, s_n)$ be an element of $h_{\mathbb{P}^m}(T)$. Show that for each $d \geqslant 1$, the $N = \binom{n+d}{d}$ monomials

$$s_0^{\otimes e_0} \otimes s_1^{\otimes e_1} \otimes \cdots \otimes s_n^{\otimes e_n} \tag{12.18}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

for $e_0 + \cdots + e_n = d$, generate $L^{\otimes d}$.

b) Show that $L^{\otimes}$ together with the N sections in (12.18) defines an element in $h_{\mathbb{P}^{N-1}}(T)$ and that this defines a functor from $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$.

c) Deduce that there is a morphism $\phi\colon X \to \mathbb{P}^{N-1}$.

d) Show that ϕ is an embedding. HINT: Consider distinguished open sets.

Exercise 12.8.16. Consider the projective line $\mathbb{P}^1_k$ and the squaring map $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$, which restricts to the squaring map $U_0 \to U_0$ on each $U_i \simeq \mathbb{A}^1_k$. Show that $f^*\mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(2)$.

Exercise 12.8.17 (A weighted projective plane)**.** Define the weighted projective plane $X = \mathbb{P}(1,1,2)$ over a field k.

Show that X is isomorphic to $V(xz - y^2) \subset \mathbb{P}^3_k$.

Use the standard exact sequences on $\mathbb{P}^3$ to compute $H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X)$.

Let $L = V(x, y)$ be the line on $X \subset \mathbb{P}^3_k$. Is the ideal sheaf $\mathcal{I}_L$ invertible? HINT: Consider the local ring at the origin in one of the affine opens in X.

Exercise 12.8.18. Let $R = k[x_0, x_1]$ with its standard grading, and let $X = \operatorname{Proj} R = \mathbf{P}^1_k$.

(i) Show that the product $\prod_{i=0}^{\infty} \mathcal{O}_X$ in the category of $\mathcal{O}_X$-modules is not quasi-coherent.

(ii) Show that the family $(\mathcal{O}_X)_{i \geqslant 0}$ nevertheless has a product $\mathcal{G}$ in $\mathrm{QCoh}(X)$. If $\mathcal{G} = \widetilde{N}$ for some graded R-module N, determine N. HINT: Write $N = \Gamma_*(\mathcal{G}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{G}(n))$. Compute each homogeneous piece $N_n = \Gamma(X, \mathcal{G}(n))$ separately.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

13

Separated morphisms

We have already seen that the topology on schemes behaves very differently from the usual Euclidean topology. In particular, even the most basic examples such as $\mathbb{A}^1$ are not Hausdorff, as the open sets are simply too large. Nevertheless, many of the useful consequences of the Hausdorff condition, such as 'uniqueness of limits', are important in algebraic geometry. The notion that replaces Hausdorffness in this setting is called *separatedness*.

The definition of separatedness is motivated the following basic result from point-set topology.

Proposition 13.1. A topological space X is Hausdorff if and only if the diagonal $\Delta = \{ (x, x) \mid x \in X \}$ is a closed subset of $X \times X$ in the product topology.

Proof The diagonal $\Delta \subset X \times X$ is closed if and only if its complement $X \times X - \Delta$ is open. With the product topology, this is equivalent to the following: any point $(x, y) \in X \times X$ with $x \neq y$ is contained in an open set of the form $U \times V$, where $U, V \subset X$ are open and $U \times V \subset X \times X - \Delta$. This in turn is equivalent to the Hausdorff condition; that there exists opens U, V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. □

The proposition suggests an alternative viewpoint based on the diagonal. Instead of asking whether the diagonal is closed in the product topology, we can consider the diagonal as a subset of the *fiber product* $X \times_{\mathbb{Z}} X$, equipped with its Zariski topology.

For example, for the affine line $X = \mathbb{A}^1$, we have $\mathbb{A}^1 \times_{\mathbb{Z}} \mathbb{A}^1 = \mathbb{A}^2$, where the Zariski topology is different from the product topology. While $\mathbb{A}^1$ is not Hausdorff, it makes perfect sense to talk about the diagonal locus where '$x = y$'; this is the Zariski closed subset $V(x - y)$.

This viewpoint via the diagonal gives a good substitute of Hausdorffness for schemes. Moreover, it extends to morphisms, so we may speak of what it means for a *morphism* $X \to S$ to be separated.

13.1 Separated morphisms

Let X be a scheme over a scheme S. The *diagonal morphism* is the morphism

$$\Delta_{X/S} \colon X \longrightarrow X \times_S X$$

defined by the two identity maps $X \to X$ as components. In other words, the defining property of $\Delta_{X/S}$ is that $p_i \circ \Delta_{X/S} = \mathrm{id}_X$ for $i = 1, 2$ where p_1 and p_2 denote the two projections.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The following little lemma gives intuition for the diagonal morphism.

Lemma 13.2. A morphism $f\colon Z \to X \times_S X$ factors via the diagonal if and only if $p_1 \circ f = p_2 \circ f$.

Proof If f can be factored as $f = \Delta_{X/S} \circ g$ for some map $g\colon Z \to X$, then $p_1 \circ f = p_1 \circ \Delta_{X/S} \circ g = \mathrm{id}_X \circ g = g$, and similarly for $p_2 \circ f$. Conversely, if $p_1 \circ f = p_2 \circ f$, put $g = p_1 \circ f\colon Z \to X$. Then the uniqueness part of the universal property of the fiber product shows that $\Delta_{X/S} \circ g = f$, so f factors via $\Delta_{X/S}$. □

In particular, if K is a field and $x_1, x_2 \in X(K)$ are two K-points, the induced K-point $x_1 \times x_2\colon \operatorname{Spec} K \to X \times_S X$ factors via the diagonal precisely whenever $x_1 = x_2$.

In the case that X and S are affine schemes, say $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$, the diagonal has a simple and natural interpretation in terms of algebras. It corresponds to the multiplication map

$$\mu\colon B \otimes_A B \longrightarrow B, \qquad b \otimes b' \longmapsto bb'.$$

The two projections correspond to the A-algebra maps $\beta_1(b) = b \otimes 1$ and $\beta_2(b) = 1 \otimes b$. By construction, $\mu \circ \beta_i = \mathrm{id}_B$ for $i = 1, 2$. Applying Spec, this translates into precisely the defining property of the diagonal morphism. Since the multiplication map μ is surjective, the diagonal morphism is a closed embedding in the affine case.

Proposition 13.3. If X is an affine scheme over the affine scheme S, then the diagonal morphism $\Delta_{X/S}\colon X \to X \times_S X$ is a closed embedding.

For general schemes, the diagonal need not be a closed embedding. However, the affine case implies that the diagonal is always *locally* closed.

Proposition 13.4. The diagonal $\Delta_{X/S}$ is a locally closed embedding.

Proof Cover S by open affine subsets S_i and cover each inverse image $f^{-1}(S_i)$ by affine open sets U_{ij}. The fiber products $U_{ij} \times_{S_i} U_{ij} = U_{ij} \times_S U_{ij}$ are affine open subsets of $X \times_S X$ whose union contain the image of the diagonal. By Proposition 13.3, the diagonal restricts to a closed embedding of U_i in $U_i \times_{S_i} U_i$, so it is a locally closed embedding. □

Definition 13.5. An S-scheme X is defined to be *separated* and that the structure morphism $X \to S$ is separated, if the diagonal map $\Delta_{X/S}\colon X \to X \times_S X$ is a closed embedding. A scheme X is said to be separated if it is separated over $\operatorname{Spec} \mathbb{Z}$.

Since the diagonal is always locally closed, it follows that to check separatedness it is enough to verify that the image $\Delta_{X/S}(X)$ is closed as a subset of $X \times_S X$. In particular, being separated is a condition that only involves the underlying topological part of the morphism $X \to S$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 13.6. Separated morphisms are stable under composition and base change and they are local on the target.

Proof *Composition.* Let $Y \to X$ and $X \to S$ be separated morphisms. Consider the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\Delta_{Y/X}} & Y \times_X Y & \longrightarrow & Y \times_S Y \\ & & \downarrow & & \downarrow \\ & & X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

The square is Cartesian. Since $X \to S$ is separated, $\Delta_{X/S}$ is a closed embedding, and therefore its base change $Y \times_X Y \to Y \times_S Y$ is a closed embedding. Since $Y \to X$ is separated, $\Delta_{Y/X}$ is also a closed embedding. Hence their composition, which is $\Delta_{Y/S}$, is a closed embedding. Hence $Y \to S$ is separated.

Base change. Let $X \to S$ be separated and let $T \to S$ be any morphism. Under the natural identification

$$X_T \times_T X_T \simeq (X \times_S X) \times_S T,$$

the diagonal $\Delta_{X_T/T}$ is the map $X \times_S T \to (X \times_S X) \times_S T$ given by the base change of $\Delta_{X/S}$ along $T \to S$. Since closed embeddings are stable under base change, $\Delta_{X_T/T}$ is a closed embedding. Hence $X_T \to T$ is separated.

Local on the target. As being a closed embedding is a local property on the target, a morphism $f\colon X \to S$ is separated if and only if there exists an open cover $\{S_i\}$ of S such that each restricted morphism $f^{-1}(S_i) \to S_i$ is separated. □

Example 13.7. Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is separated, by Proposition 13.3. This, together with the above paragraph, shows that any affine morphism $f\colon X \to Y$ is separated. In particular, closed embeddings are separated. △

Example 13.8. Open embeddings are separated. Indeed, if $U \subset X$ is an open set, then $U \times_X U \simeq U$ and under this identification, the diagonal map $\Delta_{U/X}\colon U \to U$ is simply the identity map, which is a closed embedding. △

Example 13.9. The affine line with two origins constructed in Section 5.4 is not separated over k. Recall that this scheme X is constructed by gluing two copies U_1, U_2 of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ along their common open subset $\operatorname{Spec} k[x, x^{-1}]$. Let $g_1\colon U_1 \to X$ and $g_2\colon U_2 \to X$ denote the two open embeddings. The scheme X has two distinct origins, namely the images 0_1 and 0_2 of the origin $0 \in \mathbb{A}^1_k$ under g_1 and g_2.

To see that X is not separated, we study the diagonal morphism. The product $X \times_k X$ is covered by four affine charts $U_i \times_k U_j$ for $i, j \in \{1, 2\}$, each isomorphic to $\mathbb{A}^2_k$. Each $\mathbb{A}^2_k$ contains an origin, giving four origins $0_i \times 0_j$ in total. These are the images of $0 \in \mathbb{A}^1_k$ under the four embeddings $g_{ij}\colon \mathbb{A}^1_k \to X \times_k X$ with components g_i and g_j. Away from the origin, all four morphisms g_{ij} coincide and agree with the diagonal morphism. By Lemma 13.2, only $0_1 \times 0_1$ and $0_2 \times 0_2$ lie on the diagonal. However, all four points lie in the closure of the diagonal. To see this, consider the point $0_1 \times 0_2$, which lies in the image of g_{12}. If V is any open subset containing $0_1 \times 0_2$, then its inverse image $g_{12}^{-1}V$ is a nonempty open subset of

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

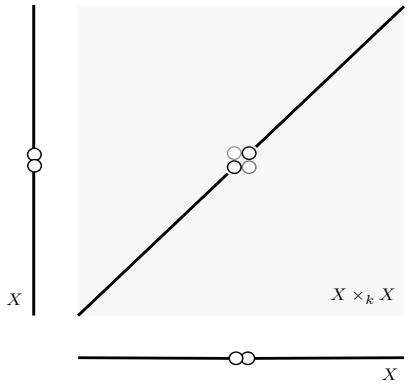


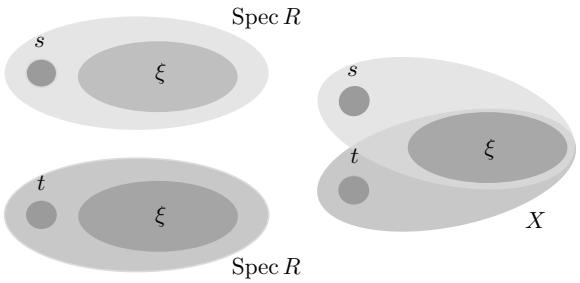
Figure 13.1 The diagonal of the affine line with two origins

$\mathbb{A}^1_k$. Such a subset must intersect $\mathbb{A}^1_k - \{0\}$, where g_{12} agrees with the diagonal. Consequently, V intersects the diagonal, and $0_1 \times 0_2$ lies in its closure. Therefore the diagonal is not closed in $X \times_k X$, and X is not separated over $\operatorname{Spec} k$. △

Example 13.10. An even more basic example of a scheme that is not separated is obtained by gluing the prime spectrum of a discrete valuation ring to itself along the generic point.

Let R be a DVR with fraction field K. Then $\operatorname{Spec} R = \{s, \xi\}$, where s is the closed point and ξ is the generic point (which is open). By Proposition 5.4, we may glue two copies of $\operatorname{Spec} R$ together by identifying the generic points, that is, the open subschemes $\operatorname{Spec} K$ in the two copies. In this way we obtain a scheme X together with two open embeddings $g_1, g_2 \colon \operatorname{Spec} R \to X$. Both maps send the generic point ξ to the same point $\gamma \in X$ and γ is open in X, but they differ on the closed point s. It follows that X is not separated.

The similar-looking examples of Examples **??** and **??** are separated, however, because they are affine.



△

13.2 Intersections of affine opens

Separatedness is introduced mainly because it guarantees good formal properties. Although the ability to freely glue schemes produces many non-separated schemes, most schemes one

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

finds in practice are separated. In particular, all affine schemes and all projective schemes are separated.

One particularly important property is that, on a separated scheme, the intersection of affine open subsets is again affine. This property will be important later on.

Proposition 13.11. Let X be an A-scheme. The following are equivalent:

(i) X is separated over $\operatorname{Spec} A$.

(ii) For any two affine open subschemes U and V, the intersection $U \cap V$ is also affine, and the natural multiplication map

$$\mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$$

is surjective.

(iii) There is an affine cover $\{U_i\}_{i\in I}$ such that all intersections $U_i \cap U_j$ are affine, and $\mathcal{O}_X(U_i) \otimes_A \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective for every $i, j \in I$.

Proof (i)⇒ (ii): Let U and V be affine open subsets of X. The fiber product $U \times_S V$ is an affine open subset of $X \times_S X$, and

$$U \cap V = \Delta_{X/S}(X) \cap (U \times_S V).$$

If X is separated over $S = \operatorname{Spec} A$, the diagonal $\Delta_{X/S}$ is a closed embedding. Hence $U \cap V$ is a closed subscheme of the affine scheme $U \times_S V$, hence affine (Proposition 4.22). Moreover, by the construction of the fiber product of affine schemes, we have

$$\Gamma(U \times_S V, \mathcal{O}_{U\times_S V}) = \Gamma(U, \mathcal{O}_U) \otimes_A \Gamma(V, \mathcal{O}_V),$$

and since $U \cap V$ is a closed subscheme of $U \times_S V$, the restriction map

$$\Gamma(U \times_S V, \mathcal{O}_{U\times_S V}) \longrightarrow \Gamma(U \cap V, \mathcal{O}_{U\cap V})$$

is surjective. This gives (ii).

The implication (ii)⇒(iii) is immediate.

For (iii)⇒(i), let $p_1, p_2 \colon X \times_A X \to X$ denote the two projections, and let $\Delta \colon X \to X \times_A X$ be the diagonal morphism. Let $U_i = \operatorname{Spec} B_i$ and $U_j = \operatorname{Spec} B_j$ be two affines in the cover $\{U_i\}$. Then we have

$$\Delta^{-1}(p_1^{-1}(U_i) \cap p_2^{-1}(U_j)) = \Delta^{-1}(p_1^{-1}(U_i)) \cap \Delta^{-1}(p_2^{-1}(U_j)) = U_i \cap U_j, \qquad (13.1)$$

Moreover, by the universal property of the fiber product,

$$p_1^{-1}(U_i) \cap p^{-1}(U_j) = U_i \times_A U_j \subset X \times_A X.$$

Since being a closed embedding is a local property on the target, it suffices to show that each restriction of Δ, $\Delta_{ij} \colon U_i \cap U_j \to U_i \times_A U_j$, is a closed embedding. By assumption, the intersection $U_i \cap U_j$ is affine, say $U_i \cap U_j = \operatorname{Spec} C_{ij}$, and the ring map $B_i \otimes_A B_j \to C_{ij}$ is surjective. This implies that Δ_{ij} is a closed embedding, as desired. □

Example 13.12. The above proposition gives a convenient criterion for checking that a scheme is separated. For instance, projective space $\mathbb{P}^n_A$ is separated over A. Indeed, it is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

covered by the standard affine covering, where all the pairwise intersections are affine, and the multiplication map

$$A\Big[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i}\Big]\otimes_A A\Big[\frac{x_0}{x_j},\dots,\frac{x_n}{x_j}\Big]\longrightarrow A\Big[\frac{x_0}{x_i},\dots,\frac{x_n}{x_i},\frac{x_i}{x_j}\Big]$$

is surjective (all monomials on the right-hand side are products of monomials on the left).

For affine space $\mathbb{A}^n_A$ and projective space $\mathbb{P}^n_A$ one can also see directly that the diagonal is closed. The diagonal in $\mathbb{A}^n_A \times_A \mathbb{A}^n_A \simeq \operatorname{Spec} A[x_1,\dots,x_n,y_1,\dots,y_n]$ is the closed subscheme

$$\Delta = V(x_1-y_1,\dots,x_n-y_n).$$

Similarly, for $\mathbb{P}^n_A$, the diagonal is the closed subscheme of $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ defined by the bihomogeneous ideal generated by the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{pmatrix}. \tag{13.2}$$

△

Example 13.13. Consider the affine plane with two origins X from Example 8.1. By construction, X is covered by two copies U_1, U_2, each isomorphic to $\mathbb{A}^2_k$. Their intersection is the open set $U_{12} = \mathbb{A}^2_k - V(x,y)$ which is not affine. Therefore X is not separated. Still, with $\mathcal{O}_X(U_1) = \mathcal{O}_X(U_2) = \mathcal{O}_X(U_{12}) = k[x,y]$, the multiplication map is

$$k[x,y]\otimes_k k[x,y] \to \Gamma(U_{12},\mathcal{O}_{U_1\cap U_2}) = k[x,y],$$

and this map *is* surjective. △

13.3 Graphs and Equalizers

In addition to the diagonal map, separatedness implies that other naturally occurring morphisms are closed embeddings.

Let $f\colon X\to Y$ be a morphism over S and assume that Y is separated over S. We define the *graph* of f to be the morphism $\Gamma_f\colon X\to X\times_S Y$ with components id_X and f. Equivalently, Γ_f is the base change of the diagonal morphism $\Delta_{Y/S}$ along the morphism $(f,\mathrm{id})\colon X\times_S Y\to Y\times_S Y$, as displayed in the Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow{\scriptstyle \Gamma_f} & & \big\downarrow{\scriptstyle \Delta_{Y/S}} \\ X\times_S Y & \xrightarrow{(f,\mathrm{id})} & Y\times_S Y \end{array}$$

As closed embeddings are stable under base change, Γ_f is a closed embedding and we may identify X with the closed subscheme $\Gamma_f\subset X\times_S Y$.

This subscheme is designed to parameterize pairs of the form $(x,f(x))$. More precisely, for a scheme T, the set of T-points of Γ_f is given by

$$\Gamma_f(T) = \{\,(x,y)\in (X\times_S Y)(T) \mid y = f(x) \text{ in } Y(T)\,\}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Note that Γ_f has two projections, $\Gamma_f \to X$ and $\Gamma_f \to Y$. The first projection $\Gamma_f \to X$ is an isomorphism.

A useful property of the graph morphism is that it allows us to write a morphism into a separated S-scheme as a composition

$$X \hookrightarrow X \times_S Y \to Y, \tag{13.3}$$

where the first arrow is a closed embedding and the second map is the projection. This is sometimes handy when proving statements about general morphisms, as we reduce to considering these two types of morphisms.

A similar construction works for studying the locus where two morphisms agree. Let $f, g\colon X \to Y$ be two morphisms over S. Together, they define a morphism

$$(f, g)\colon X \longrightarrow Y \times_S Y.$$

Consider the base change of $\Delta_{Y/S}$ via this morphism

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \Delta_{Y/S}} \\ X & \xrightarrow{(f,g)} & Y \times_S Y. \end{array}$$

If we assume that Y is separated over S, $\Delta_{Y/S}$ is a closed embedding. As closed embeddings are stable under base change, we may regard $E \subset X$ as a closed subscheme of X. This subscheme is sometimes called the *equalizer* of the morphisms f and g. By construction, E is the locus where the two morphisms f and g agree. For any scheme T,

$$E(T) = \{\, x \in X(T) \mid f_T(x) = g_T(x) \text{ in } Y(T) \,\}.$$

Moreover, $f = g$ if and only if the morphism $E \to X$ is an isomorphism.

Proposition 13.14. Let X, Y, and S be schemes. Assume that X is reduced and that Y is separated over S. Suppose $f, g\colon X \to Y$ are S-morphisms that agree on a dense open subset $U \subset X$. Then $f = g$ on all of X.

Proof As above, we regard $E \subset X$ as a closed subscheme. By assumption, f and g agree on the dense open set U, so we have $U \subset E$. As E is a closed subscheme of X and contains a dense open subset of X, we must have $X = E$ as topological spaces. As X is reduced, the only such closed subscheme is X itself. Therefore $E = X$, and hence $f = g$. □

Example 13.15. The proposition does no longer hold if the target is not separated. For instance, if Y is the affine line with two origins, then there are two morphisms $\iota_1, \iota_2\colon \mathbb{A}^1_k \to Y$ which agree on the dense open set $\mathbb{A}^1 - \{(0)\}$, but they are not equal. △

Example 13.16. Likewise, the conclusion may fail when the source is not reduced. Let $X = \operatorname{Spec} k[x, y]/(y^2, xy)$ and consider the morphisms $f, g\colon X \to \operatorname{Spec} k[u]$ defined by $u \mapsto x$ and $u \mapsto x + y$ respectively. These agree over the distinguished open set $D(x) \subset X$, but they are not equal. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

13.4 Rational maps

Let S be a scheme and let X and Y be S-schemes with X reduced. A *rational map* from X to Y is an equivalence class of pairs (U, f), where $U \subset X$ is a dense open subset and $f\colon U \to Y$ is an S-morphism. Two pairs (U, f) and (V, g) are equivalent if f and g agree on a dense open subset of $U \cap V$. A rational map is denoted with a dashed arrow $f\colon X \dashrightarrow Y$. We say that a rational map $X \dashrightarrow Y$ is *defined* at a point P if there is a representative $g\colon V \to Y$ with $P \in V$. When the target is separated, the representatives fit together into a single morphism, defined on a largest open set.

Proposition 13.17. Let X be a reduced S-scheme, let Y be separated over S, and let $f\colon X \dashrightarrow Y$ be a rational map. Then there is a largest open subset $U \subset X$ on which f is defined, and the representatives of f glue to an S-morphism $U \to Y$.

Proof Let $U \subset X$ be the set of points at which f is defined. For each $P \in U$ choose a representative $f_P\colon U_P \to Y$ with $P \in U_P$. Then $U_P \subset U$, so $U = \bigcup_{P\in U} U_P$ is open. For $P, Q \in U$ the morphisms f_P and f_Q agree on a dense open subset of $U_P \cap U_Q$. As $U_P \cap U_Q$ is reduced and Y is separated over S, Proposition 13.14 gives $f_P = f_Q$ on $U_P \cap U_Q$. The f_P therefore glue to an S-morphism $U \to Y$ (Proposition 5.1), which represents f. □

The complement $X - U$ is called the *indeterminacy locus* of the rational map.

13.5 Exercises

Exercise 13.5.1. Let X be the gluing of two copies of $\operatorname{Spec}\mathbb{Z}$ along the open set $U = \operatorname{Spec}\mathbb{Z} - \{(2)\}$. Show that X is not separated and describe the diagonal map.

Exercise 13.5.2. Let X/S be a scheme over S and let $U \subset X$ be an open set. Show that $\Delta_{X/S}|_U = \Delta_{U/S}$.

Exercise 13.5.3. Consider the scheme $X = V_+(x^3 + y^3 + tz^3)$ in $\mathbb{P}^2_{\mathbb{C}[t]}$.

a) Show that X is integral and compute $\dim X$.
b) Show that X is projective over $\mathbb{C}[t]$. Is it a projective scheme over $\mathbb{C}$?
c) Study the fibers of the morphsim $X \to \operatorname{Spec}\mathbb{C}[t]$. Which are reduced/irreducible?

Exercise 13.5.4. Let $X = \operatorname{Spec}\mathbb{C}$ and $S = \operatorname{Spec}\mathbb{R}$. Recall that the product $X \times_S X$ consists of two (closed) points. Which one corresponds to the diagonal? Can you find another $\mathbb{R}$-algebra A so that if $Y = \operatorname{Spec} A$ it holds that $Y \times_S Y \simeq X \times_S X$ and the diagonal is the other point?

Exercise 13.5.5. Let X and Y be schemes separated over a scheme S. Show that their product $X \times_S Y$ is separated over S.

Exercise 13.5.6. Let $f\colon \mathbb{A}^1_k - \{0\} \hookrightarrow \mathbb{A}^1_k$ be the open embedding, and let $g\colon \operatorname{Spec} k \to \mathbb{A}^1_k$ be the inclusion of the origin. Show that f is dominant. Compute the base change $(\mathbb{A}^1_k - \{0\}) \times_{\mathbb{A}^1_k} \operatorname{Spec} k$. Deduce that 'being dominant' is not stable under base change.

Exercise 13.5.7. Let X be a separated scheme. Show that for any affine opens $U_1, \ldots, U_r \subset X$, the intersection $U_1 \cap \cdots \cap U_r$ is affine.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 13.5.8. Show that if a scheme X is separated (over $\mathbb{Z}$), then for every scheme Y and every morphism $f\colon X \to Y$, the morphism f is separated.

Exercise 13.5.9. Let $T \to S$ be a morphism and let X and Y be two schemes over T. Show that there is a Cartesian diagram

$$\begin{array}{ccc} X \times_T X & \xrightarrow{\iota} & X \times_S X \\ \downarrow & & \downarrow{\scriptstyle f\times f} \\ T & \xrightarrow{\Delta_{T/S}} & T \times_S T, \end{array}$$

and conclude that the natural map $\iota\colon X \times_T Y \to X \times_S Y$ is a locally closed embedding. Hint: Use the functor of points to reduce to a statement of sets.

Exercise 13.5.10 (Diagonals and base change)**.** Let $X \to S$ and $T \to S$ be morphisms between schemes, and let $X_T = X \times_S T$. Show that the base change of the diagonal $\Delta_{X/S}$ is the diagonal $\Delta_{X_T/T}$, that is, the following diagram is Cartesian:

$$\begin{array}{ccc} X_T & \xrightarrow{\Delta_{X_T/T}} & X_T \times_T X_T \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

Exercise 13.5.11. Let X/S be a scheme and let $\iota\colon W \to X$ be an open subscheme or a closed subscheme (over S). Show that the diagram below is Cartesian

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \downarrow{\scriptstyle \Delta_{W/S}} & & \downarrow{\scriptstyle \Delta_{X/S}} \\ W \times_S W & \longrightarrow & X \times_S X \end{array}$$

Conclude that W/S is separated if X/S is.

Exercise 13.5.12. Let X and Y be schemes over S and $f, g\colon X \to Y$ two morphisms over S. Define the equalizer E of f, g as in the text.

a) Show that a morphism $h\colon Z \to X$ satisfies $f \circ h = g \circ h$ if and only if h factors via j.
b) Show that Y is separated if and only if for every pair of morphisms $f, g\colon Y \to X$, the equalizer $j\colon E \to X$ is a closed embedding.
c) Describe the equalizer of the two inlusions $\iota_1, \iota_2\colon \mathbb{A}^1 \to X$ into the affine line with two origins.

Exercise 13.5.13 (Monomorphisms)**.** Let $f\colon X \to Y$ be a morphism of schemes. We say that f is a *monomorphism* if the composition $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$ defined by $\alpha \mapsto f \circ \alpha$ is injective for any scheme T. That is, if $g_1\colon T \to X$ and $g_2\colon T \to X$ are morphisms such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

a) Prove that if f is a closed embedding or an open embedding then it is a monomorphism.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

b) Prove that f is a monomorphism if and only if the diagonal $\Delta\colon X \to X \times_Y X$ of the morphism f is an isomorphism.
c) Prove that if f is a monomorphism, then f is separated.
d) Show that $\operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathbb{R})$ is injective but not a monomorphism.

Exercise 13.5.14. Let X be scheme. We say that a scheme X has *affine diagonal* if for every affine opens U and V of X the intersection $U \cap V$ is affine.

a) Prove that X has affine diagonal if and only if the diagonal morphism $\Delta\colon X \to X \times_{\operatorname{Spec}\mathbb{Z}} X$ is an affine morphism. In particular, separated schemes (over $\operatorname{Spec}\mathbb{Z}$) have affine diagonal.
b) Find a scheme with affine diagonal which is not separated.

Exercise 13.5.15. Let $X \to S$ be a proper morphism and let $f\colon X \to Y$ be an open embedding. Show that if Y is irreducible, then f is an isomorphism.

Exercise 13.5.16 (The associated analytic space)**.** Let X be a scheme of finite type over $\mathbb{C}$.

a) If $X \subset \mathbb{A}^n_{\mathbb{C}}$ is a closed subscheme defined by complex polynomials $f_1, \dots, f_r$. Regarding the f_i's as holomorphic functions, show that we get an analytic subset of $\mathbb{C}^n$, denoted X^{an}. Show that as a set $X^{\mathrm{an}} = X(\mathbb{C})$, but the topologies differ in general.
b) More generally, for a scheme X of finite type over $\mathbb{C}$, show that this construction glues to produce an associated complex analytic space X^{an}.
c) Show that the assignment $X \mapsto X^{\mathrm{an}}$ is functorial.
d) Show that the analytic space of $\mathbb{P}^1_{\mathbb{C}}$ is homeomorphic to the 2-sphere S^2.

Exercise 13.5.17. Let X be a scheme of finite type over $\mathbb{C}$.

a) Show that an open subset $U \subset X$ is dense in X if and only if $U(\mathbb{C})$ is dense in $X(\mathbb{C})$ for the Euclidean topology.
b) Show more generally that a constructible subset $S \subset X$ is dense in X if and only if $S(\mathbb{C})$ is dense in $X(\mathbb{C})$. Deduce that a constructible subset $S \subset X$ is closed in X if and only if $S(\mathbb{C})$ is closed in $X(\mathbb{C})$.

Exercise 13.5.18 (Separatedness over $\mathbb{C}$ = Hausdorff)**.** Let X be a scheme of finite type over $\mathbb{C}$. Show that X is separated over $\mathbb{C}$ if and only if $X(\mathbb{C})$ is Hausdorff in the Euclidean topology. HINT: Use Exercise 13.5.17 to compare closedness of the algebraic diagonal with closedness of the Euclidean diagonal.

Exercise 13.5.19. Show that if a morphism is injective on points, then it is separated.

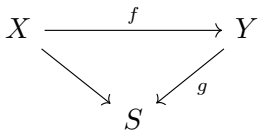
Exercise 13.5.20 (Stability under products)**.** Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be two morphisms. Show that $f \times f'$ factors as

$$X \times_{\mathbb{Z}} X' \to Y \times_{\mathbb{Z}} X' \to Y \times_{\mathbb{Z}} Y'.$$

The first map is a base change of f, and the second is a base change of f'. Deduce that if $\mathcal{P}$ is a property which is stable under base change and composition, then it is also stable under products.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 13.5.21 (Cancellation for properties of morphisms)**.** Let $\mathcal{P}$ be a property of morphisms which is stable under compositions and base change. Consider an S-morphism



Assume that $\Delta_{Y/S}\colon Y \to Y \times_S Y$ has $\mathcal{P}$ and $g \circ f$ has $\mathcal{P}$. Show that f has $\mathcal{P}$. HINT: Use the graph.

The most important special case of this is: Assume that closed embeddings satisfy $\mathcal{P}$. Then if $g \circ f$ has $\mathcal{P}$ and g is separated, then f has $\mathcal{P}$.

Show by an example that in general f is not necessarily a closed embedding even if $g \circ f$ is. HINT: Let X be the affine line with two origins.

Exercise 13.5.22. Let X and Y be two schemes of finite type over k. Show that any morphism $X \to Y$ is of finite type.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

14

Varieties and rational maps

14.1 Varieties

The term *variety* appears in the literature with slightly different meanings. What all definitions have in common is that a variety is a scheme of finite type over a field. Beyond this, some authors allow a variety to be reducible (though most require it to be reduced), and others assume that the base field is algebraically closed. In this book, we adopt the following convention:

Definition 14.1. A *variety* over a field k is an integral, separated scheme of finite type over k.

A *curve* is a variety of dimension 1. A *surface* is a variety of dimension 2. A *threefold* is a variety of dimension 3, and so on.

A *morphism* or simply *map* of varieties over k is a morphism $f\colon X \to Y$ of schemes over k. In this way, the varieties over k form a full subcategory Var/k of the category of k-schemes Sch/k.

Example 14.2 (Open subvarieties)**.** If X is a variety over a field k, then every open subscheme $U \subset X$ is a variety. Indeed, open subschemes of integral schemes are integral by Proposition 4.28, and open embeddings are separated (Example 13.8). Moreover, since X is of finite type over k, it is quasi-compact, so U can be covered by finitely many affine open subschemes, each of finite type over k by Proposition 8.8. Therefore U is also of finite type over k. △

Example 14.3 (Closed subvarieties)**.** Likewise, if X is a variety, then any integral closed subscheme $Y \subset X$ is also a variety. Indeed, Y is of finite type since it is a closed subscheme of a scheme of finite type (Exercise 8.11.30), and it is separated as a closed subscheme of a separated scheme (Proposition 13.7). △

The property of being a variety is not stable under base change. For example, $\operatorname{Spec}\mathbb{R}[x,y]/(x^2+y^2)$ is a variety over $\mathbb{R}$, but its base change to $\mathbb{C}$

$$\operatorname{Spec}\mathbb{C}[x,y]/(x^2+y^2) = \operatorname{Spec}\mathbb{C}[x,y]/((x+iy)(x-iy)) \tag{14.1}$$

is not a variety, because it is reducible (it is a union of two affine lines intersecting at a point). Similarly, the fiber product of two varieties need not be a variety.

When studying a variety X over a field k, it is common to study X by considering its base change $X_{\bar{k}}$ to the algebraic closure $\bar{k}$. If $\mathcal{P}$ is a property of schemes, we say that X is *geometrically* $\mathcal{P}$ if the base change $X_{\bar{k}}$ has $\mathcal{P}$. For example:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(i) X is *geometrically integral* if $X_{\bar{k}}$ is integral.
(ii) X is *geometrically irreducible* if $X_{\bar{k}}$ is irreducible.
(iii) X is *geometrically reduced* if $X_{\bar{k}}$ is reduced.

For instance, in the example above, the scheme X is geometrically reduced and geometrically connected, but not geometrically integral. The affine spaces $\mathbb{A}^n_k$ and the projective spaces $\mathbb{P}^n_k$ are geometrically integral over every field k.

14.2 Rational maps

Let S be a scheme and let X and Y be S-schemes with X reduced. A *rational map* from X to Y is an equivalence class of pairs (U, f), where $U \subset X$ is a dense open subset and $f\colon U \to Y$ is an S-morphism. Two pairs (U, f) and (V, g) are equivalent if f and g agree on a dense open subset of $U \cap V$. A rational map is denoted with a dashed arrow $f\colon X \dashrightarrow Y$.

We say that a rational map $X \dashrightarrow Y$ is *defined* at a point P if there is a representative $g\colon V \to Y$ with $P \in V$. When the target is separated, the representatives fit together into a single morphism, defined on a largest open set.

Proposition 14.4. Let X be a reduced S-scheme, let Y be separated over S, and let $f\colon X \dashrightarrow Y$ be a rational map. Then there is a largest open subset $U \subset X$ on which f is defined, and the representatives of f glue to an S-morphism $U \to Y$.

Proof Let $U \subset X$ be the set of points at which f is defined. For each $P \in U$ choose a representative $f_P\colon U_P \to Y$ with $P \in U_P$. Then $U_P \subset U$, so $U = \bigcup_{P\in U} U_P$ is open. For $P, Q \in U$ the morphisms f_P and f_Q agree on a dense open subset of $U_P \cap U_Q$. As $U_P \cap U_Q$ is reduced and Y is separated over S, Proposition 13.14 gives $f_P = f_Q$ on $U_P \cap U_Q$. The f_P therefore glue to an S-morphism $U \to Y$ (Proposition 5.1), which represents f. □

The complement $X - U$ is called the *indeterminacy locus* of the rational map.

In the rest of this section we work with varieties over a field k, taking $S = \operatorname{Spec} k$. A variety is integral, so its dense open subsets are exactly its nonempty ones, and it is separated over k. The proposition therefore applies to every rational map of varieties. In particular a rational function on X, regarded as a rational map to $\mathbb{A}^1_k$, has a largest open set on which it is regular.

A rational map $f\colon X \dashrightarrow Y$ is called *dominant* if $f(U)$ is dense in Y where U is some open set where f is defined (if this holds for one U, it holds for all).

Dominant rational maps can be composed: if $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ are rational maps which are defined over $U \subset X$ and $V \subset Y$, then there is a non-empty open set $W \subset V$ such that $f^{-1}(W) \subset U$ and we define $g \circ f\colon X \dashrightarrow Z$ to be the rational map corresponding to the morphism $f^{-1}(W) \to Z$. Hence the k-varieties together with the dominant rational maps form a category.

A dominant rational map $f\colon X \dashrightarrow Y$ is called *birational* if there exists a dominant rational map $g\colon Y \dashrightarrow X$ such that both $f \circ g$ and $g \circ f$ agree with the identity map on some nonempty open subsets. In this case, we say that X and Y are *birational*.

Birational equivalence is a much weaker relation than being isomorphic. For instance,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

any variety is birational to any nonempty open subset of itself. In particuar, $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are birational.

Example 14.5. The *Cremona transformation*

$$\begin{aligned} \phi\colon \mathbb{P}^2_k &\dashrightarrow \mathbb{P}^2_k \\ (x_0 : x_1 : x_2) &\longmapsto (x_1x_2 : x_0x_2 : x_0x_1) \end{aligned}$$

is a rational map defined away from the three coordinate points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$. It is a birational map, with itself as its inverse: $\phi \circ \phi = \mathrm{id}$. △

Example 14.6 (Cones)**.** Let $X \subset \mathbb{P}^n_k$ be a projective variety. Then the affine cone $C(X) \subset \mathbb{A}^{n+1}_k$ is birational to $X \times_k \mathbb{P}^1_k$. △

Rational maps and function fields

Let X be a variety over a field k. Recall that the function field of X is defined as the local ring $K(X) = \mathcal{O}_{X,\xi}$, where ξ is the generic point. We think of the elements of $K(X)$ as rational functions on X, that is, any nonzero $f \in K(X)$ defines a rational map $f\colon X \dashrightarrow \mathbb{A}^1_k$.

Proposition 14.7. Let X be a variety over a field k. There is a bijection

$$K(X) = \{\text{rational maps } f\colon X \dashrightarrow \mathbb{A}^1_k\}.$$

Proof By Theorem 4.6, for an open set $U \subset X$, the morphisms $U \to \mathbb{A}^1_k$ are in bijection with the elements of $\mathcal{O}_X(U)$. Taking the direct limit over all open sets U (which all contain the generic point ξ), then we get on one hand all rational maps to $\mathbb{A}^1$, whereas on the other, the stalk $\mathcal{O}_{X,\xi} = K(X)$. □

Let X and Y be varieties with generic points ξ and γ respectively and let $f\colon X \dashrightarrow Y$ be a dominant rational map. Then f is a morphism when restricted to some neighbourhood of ξ. By Proposition 4.39, we have $f(\xi) = \gamma$ and hence $f^\sharp$ induces a map between the local rings $\mathcal{O}_{Y,\gamma} \to \mathcal{O}_{X,\xi}$, that is,

$$f^\sharp\colon K(Y) \longrightarrow K(X). \tag{14.2}$$

If X and Y are affine varieties over a field k, and we interpret rational functions as morphisms to $\mathbb{A}^1_k$, then the pullback $f^\sharp(g)$ of a rational function $g \in K(Y)$ coincides with the pullback $f^\sharp(g) = g \circ f$, as discussed in Section 1.5.

Note that it is varieties together with *dominant* rational maps which form a category. If $f\colon X \dashrightarrow Y$ is not dominant, there is no hope of defining a pullback map as in (14.2). Consider for instance the closed embedding $\mathbb{A}^1_k \to \mathbb{A}^2_k$ of the 'x-axis' and the rational function $y^{-1} \in k(\mathbb{A}^2_k) = k(x,y)$.

Lemma 14.8. Let k be a field.
For a k-variety X, the function field $K(X)$ is a finitely generated field extension of k. Conversely, every finitely generated field extension K of k is of the form $K(X)$ for some variety X. We may choose X to be either affine or projective.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof Since $K(X) = K(U)$ for any open set $U \subset X$, we may assume that $X = \operatorname{Spec} A$ is affine, so that $K(X)$ is the fraction field of A. As X is of finite type over k, A is a finitely generated k-algebra. If $x_1, \ldots, x_n$ are generators for A as a k-algebra, then these elements generate $K(X)$ as a field over k.

Conversely, let $x_1, \ldots, x_n$ be a set of generators for K as a field extension of k and let $A \subset K$ be the k-subalgebra of K generated by them. Then A is an integral domain with fraction field equal to K. The scheme $V = \operatorname{Spec} A$ is an affine variety over k with $K(V) = K$. To get a projective variety, take a closed embedding $V \to \mathbb{A}^r_k$ and embed $\mathbb{A}^r_k$ as a distinguished open set in $\mathbb{P}^r_k$. Then if we define X to be the Zariski closure of V inside $\mathbb{P}^r_k$, then X is a projective variety with $K(X) = K(V) = K$. □

Theorem 14.9. Let X and Y be two varieties over k. Then there is a one-to-one correspondence between rational dominant maps $X \dashrightarrow Y$ and k-algebra homomorphisms $K(Y) \to K(X)$. In particular, two varieties are birationally equivalent if and only if their function fields are isomorphic as k-algebras.

Proof We have already showed that any rational map $f\colon X \dashrightarrow Y$ induces a map of k-algebras $K(Y) \to K(X)$. The inverse assignment is constructed as follows.

Let $U = \operatorname{Spec} B \subset X$ and $V = \operatorname{Spec} A \subset Y$ be open affine subsets. Then $K(X)$ is the fraction field of B and $K(Y)$ that of A, and we have $A \subset K(X)$ and $B \subset K(Y)$.

Let $\phi\colon K(Y) \to K(X)$ be a given map of k-algebras. If $a_1, \ldots, a_r$ generate A over k, then each $\phi(a_i)$ is of the form $\phi(a_i) = b_i/c_i$ with $b_i, c_i \in A$. Therefore, if we set $d = c_1 \cdots c_r$ we have $\phi(A) \subset B_d$. This means that ϕ induces a map of k-algebras $\bar{\phi}\colon A \to B_d$, and hence a morphism $\operatorname{Spec} B_d \to \operatorname{Spec} A = V \subset Y$. This means that we have a rational map $f\colon X \dashrightarrow Y$. Evidently, A maps injectively into B_d so the morphism is dominant. By construction, $\bar{\phi}$ localizes to the map ϕ, so f induces the map ϕ. □

In other words, $X \mapsto K(X)$ is a fully faithful contravariant functor from varieties over k and dominant rational maps to finitely generated field extensions of k, and by Lemma 14.8 it is essentially surjective. The same remains true if one restricts to affine varieties only, or to projective ones only. There is no functor in the other direction, since there is no systematic way of choosing one variety for each field.

Varieties are birational to hypersurfaces

Recall that a field k is said to be *perfect* if every algebraic extension $k \to L$ is separable (that is, for any $\alpha \in L$, the minimal polynomial $h(t) \in k[t]$ of α is coprime with $h'(t)$). All fields of characteristic 0 are perfect, as are algebraically closed fields, and all finite fields. The field $k = \mathbb{F}_p(t)$ is not perfect, as $L = \mathbb{F}_p(t^{1/p})$ is a non-separable extension. We use two standard facts about a perfect field k: a finitely generated extension of k has a separating transcendence basis (Theorem **??**), and a finite separable extension is generated by a single element (Theorem C.25). Together they give the following.

Proposition 14.10 (Varieties are birational to hypersurfaces). Let X be a variety of dimension n over a perfect field k. Then X is birational to a hypersurface in $\mathbb{A}^{n+1}_k$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof Only the function field $K = K(X)$ matters, since X is birational to each of its nonempty open subsets. It is a finitely generated extension of k of transcendence degree n, so by Theorem **??** there are $y_1, \dots, y_n \in K$, algebraically independent over k, such that K is a finite separable extension of $L = k(y_1, \dots, y_n)$. By Theorem C.25 there is a $u \in K$ with $K = L(u)$. Let

$$h(t) = t^r + a_{r-1}t^{r-1} + \cdots + a_0, \quad a_i \in L$$

be the minimal polynomial of u over L. Multiplying u with a suitable polynomial in the y_i, we may assume that $a_i \in k[y_1, \dots, y_n]$ for each i. Then h is irreducible in $k[y_1, \dots, y_n, t]$ by Gauss's Lemma, so $V(h) \subset \mathbb{A}_k^{n+1} = \operatorname{Spec} k[y_1, \dots, y_n, t]$ is an integral hypersurface with function field $L(u) = K$. By Theorem 14.9, X is birational to $V(h)$. □

14.3 Generically finite maps and degree

A dominant rational map $f\colon X \dashrightarrow Y$ of varieties induces an inclusion $K(Y) \hookrightarrow K(X)$ of function fields. In this section, we study when the extension is finite.

Proposition 14.11. Let $f\colon X \dashrightarrow Y$ be a dominant rational map of varieties over a field k. The following are equivalent:

(i) The field extension $K(X)/K(Y)$ is finite.
(ii) There are nonempty open subsets $U \subset X$ and $V \subset Y$ such that f restricts to a finite morphism $U \to V$.
(iii) We have $\dim X = \dim Y$.

Proof (i)⇒(ii): After restricting f and passing to affine open subsets, we obtain a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$, where $A \subset B$ are finitely generated integral k-algebras with $K(A) = K(Y)$ and $K(B) = K(X)$. Write $B = A[b_1, \dots, b_r]$. Each b_i is algebraic over $K(A)$, so after inverting a single nonzero element $s \in A$, every b_i is integral over A_s. Hence B_s is finite over A_s, and f restricts to a finite morphism between nonempty open subsets.

(ii)⇒(i): Replacing X and Y by U and V, we may assume that f is finite. Choose a nonempty affine open subset $\operatorname{Spec} A \subset Y$ with inverse image $\operatorname{Spec} B$. Then B is finite over A, so $K(X) = K(B)$ is finite over $K(Y) = K(A)$.

(i)⇔(iii): By (8.6), we have

$$\dim X = \operatorname{trdeg}_k K(X) = \operatorname{trdeg}_k K(Y) + \operatorname{trdeg}_{K(Y)} K(X),$$

we have $\dim X = \dim Y$ if and only if $K(X)$ is algebraic over $K(Y)$. Since $K(X)$ is finitely generated over $K(Y)$, this is equivalent to $K(X)/K(Y)$ being finite. □

A dominant rational map satisfying these conditions is called *generically finite*. If it is everywhere defined, it is called a *generically finite morphism*.

The *degree* of a dominant generically finite rational map $f\colon X \dashrightarrow Y$ is

$$\deg(f) = [K(X) : K(Y)].$$

If $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ are dominant generically finite rational maps, then $g \circ f$ is one as well, and $\deg(g \circ f) = \deg(f)\deg(g)$ by the degree formula for field extensions

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$K(Z) \subset K(Y) \subset K(X)$. A birational map is exactly a generically finite map of degree 1; in particular, birational varieties have the same dimension.

Heuristically, the degree of a morphism counts the number of points in a typical fiber (see Exercise 8.11.9).

Example 14.12. Consider the morphism

$$f\colon \operatorname{Spec} k[x,y]/(xy^2 - x^3 - 1) \longrightarrow \operatorname{Spec} k[x].$$

It is not finite, since its fiber over $x = 0$ is empty, whereas a finite dominant morphism of varieties is surjective. However, over the subset $D(x)$, the equation becomes $y^2 = (x^3+1)/x$, so the restricted morphism is finite there. The element $(x^3 + 1)/x$ is not a square in $k(x)$, so $K(X) = k(x)(y)$ is a degree 2 extension of $k(x)$, so f is generically finite of degree 2. △

14.4 Rationality

A variety X over a field k is called *rational* if it is birational to $\mathbb{A}^n_k$ for some n. Equivalently, X is rational if $K(X)$ is isomorphic to the function field of $\mathbb{A}^n_k$, that is, $K(X) \simeq k(t_1, \ldots, t_n)$.

Example 14.13. Let $X \subset \mathbb{A}^{n+1}_k$ be a hypersurface defined by an equation which is linear in one of the variables, say, $A + x_{n+1}B = 0$, where $A, B \in k[x_1, \ldots, x_n]$. Then X is rational, because the projection $\pi\colon X \to \mathbb{A}^n_k$ onto the first n coordinates, is birational, with inverse $\psi\colon \mathbb{A}^n_k \to X$ defined by

$$(x_1, \ldots, x_n) \longmapsto \Big(x_1, \ldots, x_n, -\frac{A(x_1, \ldots, x_n)}{B(x_1, \ldots, x_n)}\Big).$$

Algebraically, this corresponds to eliminating x_{n+1} using the relation $A + x_{n+1}B = 0$:

$$K(X) = k(x_1, \ldots, x_{n+1})/(A + x_{n+1}B) \simeq k(x_1, \ldots, x_n).$$

△

Example 14.14 (Quadrics)**.** Let k be an algebraically closed field, and let $X = V(q) \subset \mathbb{P}^{n+1}_k$ be a projective quadric hypersurface, defined by an irreducible homogeneous polynomial q of degree 2. After a linear change of coordinates, we may assume that X passes through the point $(1 : 0, \ldots, 0)$. This means that the polynomial q may be written as

$$q = x_0 L(x_1, \ldots, x_{n+1}) + Q(x_1, \ldots, x_{n+1}),$$

where L and Q are homogeneous of degree 1 and 2 respectively. If $L \neq 0$, then q is linear in x_0 and X is rational by Example 14.13. If $L = 0$, then X is the projective cone over a quadric of smaller dimension, hence rational by induction together with Example 14.6.

The same argument works over a general field k provided that X has a k-point $P \in X$, where at least one partial derivative $\partial q/\partial x_i(P)$ is nonzero (that is, q has a *smooth* k-point, see Section **??**).

On the other hand, if X is the curve defined by $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$, with no $\mathbb{R}$-points at all, then X cannot be rational, as rational varieties over $\mathbb{R}$ have a dense set of $\mathbb{R}$-points. Its base change $X_{\mathbb{C}}$ however, which is defined by the same equation in $\mathbb{P}^2_{\mathbb{C}}$, *is* rational. Varieties which become rational over the algebraic closure are called *geometrically rational*. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

A central question in algebraic geometry is to determine whether a given variety is rational. Here the main challenge is usually to *disprove* rationality, i.e., find a birational invariant that is non-trivial for the given example, but trivial for $\mathbb{P}^n_k$. Exercise 8.11.14 shows that the cubic curve $y^2 = x^3 - x$ is irrational. We will see more sophisticated examples in Section 20.17 after discussing differentials.

Exercise 14.4.1. Consider the rational map $f\colon \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^2_k$ defined by $(x_0 : x_1) \times (y_0 : y_1) \mapsto (x_0y_0 : x_1y_0 : x_1y_1)$. Show that f is birational and compute its inverse. Where are these morphisms defined?

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

15

Proper morphisms

Projective varieties have some good properties that affine varieties do not. Over an algebraically closed field the only regular functions on a projective variety are the constants, the image of a morphism between projective varieties is closed, and over $\mathbb{C}$ a projective variety is compact in the classical topology. These statements all fail in the affine case. For instance, the projection of the hyperbola $V(xy-1) \subset \mathbb{A}^2_k$ onto the first coordinate axis misses the origin, so its image is not closed. What is responsible for this good behavior is not projectivity itself, but a weaker condition called *properness*, which is the scheme-theoretic analogue of compactness in topology.

Every projective scheme over a ring is proper over that ring, and this is our main source of examples. The converse is not true, but the two notions stay close enough that results can usually be transferred from one to the other. What properness buys us are finiteness properties: pushforwards of coherent sheaves are coherent, a scheme proper over a field has only finitely many linearly independent global functions, and a proper morphism with finite fibers is finite.

15.1 Proper morphisms

A basic property of morphisms which does *not* behave well under base change is the property of being closed. This is illustrated in the next example.

Example 15.1. Consider the structure map $\pi\colon \mathbb{A}^1_k \to \operatorname{Spec} k$. This map is clearly closed, but if we pull back π along itself, we obtain $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$ and the projection onto the first factor $\mathbb{A}^2_k \to \mathbb{A}^1_k$. In $\mathbb{A}^2_k$ there are many closed sets which project to non-closed sets. For instance, the hyperbola $V(xy-1)$ projects onto the open set $\mathbb{A}^1_k - \{0\}$. △

This motivates the following strengthening of the notion of a closed morphism.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Definition 15.2. A morphism $f\colon X \to S$ is *universally closed* if it is closed and remains closed after any base change. That is, for any morphism $T \to S$, the induced morphism $f_T\colon X \times_S T \to T$ is closed.

$$\begin{array}{ccc} X \times_S T & \longrightarrow & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

A morphism f is said to be *proper* if it is separated, of finite type and universally closed. If X is a scheme over a ring A, we say that X is *proper over* A if $X \to \operatorname{Spec} A$ is proper.

Proper morphisms play a role in algebraic geometry analogous to that of proper maps in topology. In topology, a continuous map $f\colon X \to Y$ is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. For locally compact Hausdorff spaces, one can show that this is the same as asking that f be closed with compact fibers, and also the same as asking that for every map $Z \to Y$, the projection $X \times_Y Z \to Z$ is closed. In scheme theory, this last property, together with a finiteness condition and separatedness, is taken as the definition of properness.

For schemes of finite type over $\mathbb{C}$, separatedness and properness recover the familiar properties of the associated analytic space X^{an} with its classical topology:

- X is separated over $\mathbb{C}$ if and only if X^{an} is Hausdorff.
- X is proper over $\mathbb{C}$ if and only if X^{an} is compact.
- A morphism $f\colon X \to Y$ is proper if and only if $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper in the topological sense.

Proposition 15.3. Proper morphisms are stable under base change, composition and products, and properness is local on the target.

Proof Since separated morphisms and morphisms of finite type satisfy the same formal properties, it is enough to check these statements for 'universally closed'.

Base change. Let $f\colon X \to Y$ be universally closed and let $g\colon Z \to Y$ be any morphism. Consider the base change morphism $f_Z\colon X \times_Y Z \to Z$. To show that f_Z is universally closed, let $h\colon T \to Z$ be any morphism. We have the following commutative diagram:

$$\begin{array}{ccccc} X \times_Y Z \times_Z T & \longrightarrow & X \times_Y Z & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & {\scriptstyle f_Z}\downarrow & & {\scriptstyle f}\downarrow \\ T & \xrightarrow{h} & Z & \xrightarrow{g} & Y \end{array}$$

Both squares are Cartesian, so the outer rectangle is Cartesian as well. Since f is universally closed, p is closed. Hence f_Z is universally closed.

Composition. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be universally closed. For any morphism

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$h\colon T \to Z$, consider the following commutative diagram.

$$\begin{array}{ccccc} X \times_Y (Y \times_Z T) & \xrightarrow{p} & Y \times_Z T & \xrightarrow{q} & T \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

Again both squares are Cartesian, and hence the outer rectangle is Cartesian as well. As f and g are universally closed, p and q are closed. Therefore $q \circ p$ is closed, so $g \circ f$ is universally closed.

Products. Let $f\colon X \to Y$ and $g\colon X' \to Y'$ be two proper morphisms of schemes over a base scheme S; all products below are taken over S. The product morphism $f \times g$ factors as

$$X \times X' \xrightarrow{\mathrm{id}_X \times g} X \times Y' \xrightarrow{f \times \mathrm{id}_{Y'}} Y \times Y'.$$

Here $\mathrm{id}_X \times g$ is the base change of g along the projection $X \times Y' \to Y'$, and $f \times \mathrm{id}_{Y'}$ is the base change of f along the projection $Y \times Y' \to Y$. Hence both are proper by the base change case, and $f \times g$ is proper by the composition case.

Locality on target. Suppose $f\colon X \to Y$ is a morphism and assume that there is an open cover $\{U_i\}$ of Y such that each $f^{-1}(U_i) \to U_i$ is universally closed. Let $g\colon T \to Y$ be any morphism, and let $p\colon X \times_Y T \to T$ be the induced morphism. The open sets $g^{-1}(U_i)$ cover T, and $p^{-1}(g^{-1}(U_i)) = f^{-1}(U_i) \times_{U_i} g^{-1}(U_i)$. As $f^{-1}(U_i) \to U_i$ is universally closed, the restriction

$$f^{-1}(U_i) \times_{U_i} g^{-1}(U_i) \longrightarrow g^{-1}(U_i)$$

is closed for each i. This implies that p is closed as well, so f is universally closed. □

15.2 Projective space is proper

The standard example of a proper morphism is the structure morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$. In particular, any projective scheme over k is proper over k. This fits well with the intuition that projective varieties behave like compact spaces, whereas affine varieties do not.

Although projective schemes form the most important class of proper schemes, properness is more general than projectivity. There exist varieties over $\mathbb{C}$ which are proper but not projective; the first examples were constructed by Hironaka in 1960. Still, properness is a condition which is a bit more flexible and which is often easier to verify than projectivity.

Theorem 15.4. Projective n-space $\mathbb{P}^n$ is proper over $\mathbb{Z}$.

Proof We know that $\mathbb{P}^n$ is of finite type and separated over $\mathbb{Z}$, so it remains to show that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is universally closed. Let T be any scheme, and consider the base change $\pi\colon \mathbb{P}^n_T \to T$. Since the property of being closed is a local property on T, it suffices to treat the case where $T = \operatorname{Spec} A$ is affine.

Let $Z \subset \mathbb{P}^n_A$ be a closed subset, defined by a homogeneous ideal $I \subset A[x_0, \dots, x_n]$. We need to show that $\pi(Z)$ is closed in $\operatorname{Spec} A$.

Let $\mathfrak{p} \in \operatorname{Spec} A$ be a point which is not contained in $\pi(Z)$. We claim that there is some open neighborhood U of $\mathfrak{p}$ so that $\pi^{-1}(y) \cap Z = \varnothing$ for all $y \in U$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

If we identify $\pi^{-1}(\mathfrak{p})$ with $\mathbb{P}^n_{\kappa(\mathfrak{p})}$, the intersection $Z \cap \pi^{-1}(\mathfrak{p})$ is the closed subset defined by the ideal $I \otimes_A \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p})[x_0, \ldots, x_n]$. Since this fiber is empty by assumption, we must have for large $N > 0$,

$$I_N \otimes_A \kappa(\mathfrak{p}) = A[x_0, \ldots, x_n]_N \otimes_A \kappa(\mathfrak{p}). \tag{15.1}$$

Consider the A-module $M = A[x_0, \ldots, x_n]_N / I_N$. It is finitely generated, being a quotient of the free module $A[x_0, \ldots, x_n]_N$ of finite rank. Equation (15.1) says exactly that $M \otimes_A \kappa(\mathfrak{p}) = 0$, so Nakayama's lemma gives $M_{\mathfrak{p}} = 0$. As M is finitely generated, this means that $M_f = 0$ for some $f \in A - \mathfrak{p}$, that is,

$$(I_N)_f = (A[x_0, \ldots, x_n]_N)_f = A_f[x_0, \ldots, x_n]_N. \tag{15.2}$$

Now, $\pi^{-1}(D(f)) = \mathbb{P}^n_{A_f}$. Over this open subset, the closed set $Z \cap \pi^{-1}(D(f))$ is defined by the ideal $IA_f[x_0, \ldots, x_n]$. The equality above implies that $Z \cap \pi^{-1}(D(f)) = \varnothing$ as well. Therefore, $D(f)$ is an open neighborhood of $\mathfrak{p}$ disjoint from $\pi(Z)$, and hence $\pi(Z)$ is closed. □

As properness is stable under base change and composition, we deduce more generally:

Corollary 15.5.
(i) For any scheme S and $n \geqslant 0$, $\mathbb{P}^n_S = \mathbb{P}^n \times_{\operatorname{Spec} \mathbb{Z}} S$ is proper over S.
(ii) Every projective scheme X over a ring A is proper over A.

Another consequence of Theorem 15.4 is that the 'image of a projective variety is closed'. To prove this, we need the following lemma.

Lemma 15.6. If X is separated (resp. proper) over a ring A, and Y is separated over A, then any morphism $X \to Y$ over A is separated (resp. proper).

Proof Factor $f\colon X \to Y$ as the graph morphism $\Gamma_f\colon X \to X \times_A Y$ followed by the second projection $\pi_Y\colon X \times_A Y \to Y$. Since Y is separated over A, the graph morphism is a closed embedding, and in particular is proper. Also π_Y is separated (resp. proper), being a base change of $X \to \operatorname{Spec} A$. Therefore f is separated (resp. proper). □

One can compare this result to a fact from topology: if X is compact and Y is Hausdorff, then any continuous map $f\colon X \to Y$ is closed; in particular the image $f(X) \subset Y$ is closed.

Corollary 15.7. Let k be a field.
(i) Any morphism $f\colon X \to Y$ of projective k-schemes is proper. In particular, the image $f(X) \subset Y$ is closed in Y.
(ii) If $X \subset \mathbb{P}^n_k$ is quasi-projective, then X is proper over k if and only if X is closed in $\mathbb{P}^n_k$.

Proof (i): By assumption X and Y are closed subschemes of projective spaces over k, and so they are proper over k. By the lemma, the morphism $f\colon X \to Y$ is also proper.

(ii): The '⇐'-direction follows from Corollary 15.5. For the '⇒'-direction, apply Lemma

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

15.6 to the morphism $X \to \mathbb{P}^n_k$: since X is proper over k, and $\mathbb{P}^n_k$ is separated over k, X is closed in $\mathbb{P}^n_k$. □

The fact that the image $f(X)$ of a morphism of projective schemes is closed is a fundamental fact in projective geometry. As we noted in the introduction, the statement is not true with 'projective' replaced by 'affine'. It is yet another reason why one prefers projective varieties to non-projective ones.

Proposition 15.8. Let X be a proper variety over a field k. Then $\mathcal{O}_X(X)$ is a finite field extension of k. In particular, if k is algebraically closed, then $\mathcal{O}_X(X) = k$.

Proof A global function $f \in \mathcal{O}_X(X)$ is the same as a k-morphism $\varphi_f \colon X \to \mathbb{A}^1_k \subset \mathbb{P}^1_k$. By Lemma 15.6 the morphism $X \to \mathbb{P}^1_k$ is proper, so its image is closed in $\mathbb{P}^1_k$; being irreducible and contained in $\mathbb{A}^1_k$, it is a single closed point $\mathfrak{p} = V(g)$ with $g \in k[t]$ irreducible. Since X is reduced, φ_f factors through the reduced closed subscheme $\operatorname{Spec} \kappa(\mathfrak{p}) \subset \mathbb{A}^1_k$, and therefore f satisfies the algebraic equation $g(f) = 0$. Hence every element of $\mathcal{O}_X(X)$ is algebraic over k. As X is integral, $\mathcal{O}_X(X)$ is a domain contained in $K(X)$, so it is a field, algebraic over k. Since $K(X)/k$ is finitely generated, the algebraic closure of k in $k(X)$ is finite over k. □

The proof of Theorem 15.4 also describes the image explicitly when the base is affine.

Corollary 15.9. Let A be a Noetherian ring, and let $X = V_+(I) \subset \mathbb{P}^n_A$ for some homogeneous ideal $I \subset A[x_0, \ldots, x_n]$. If $\pi \colon \mathbb{P}^n_A \to \operatorname{Spec} A$ is the structure morphism and

$$J = (I : (x_0, \ldots, x_n)^\infty) \cap A,$$

then $\pi(X) = V(J)$.

Proof Let $\mathfrak{p} \in \operatorname{Spec} A$. Suppose first that $\mathfrak{p} \in \pi(X)$. Then there is a homogeneous prime ideal $\mathfrak{q} \subset A[x_0, \ldots, x_n]$ containing I and satisfying $\mathfrak{q} \cap A = \mathfrak{p}$. Since $\mathfrak{q} \in \operatorname{Proj} A[x_0, \ldots, x_n]$, not all of $x_0, \ldots, x_n$ belong to $\mathfrak{q}$, so we may choose an index i with $x_i \notin \mathfrak{q}$. If $G \in J$, then $(x_0, \ldots, x_n)^N G \subset I$ for some $N \geqslant 0$, hence $x_i^N G \in I \subset \mathfrak{q}$. Because $\mathfrak{q}$ is prime and $x_i \notin \mathfrak{q}$, it follows that $G \in \mathfrak{q}$. Since also $G \in A$, we get $G \in \mathfrak{q} \cap A = \mathfrak{p}$. Hence $J \subset \mathfrak{p}$, so $\mathfrak{p} \in V(J)$.

Now suppose that $\mathfrak{p} \notin \pi(X)$. As in (15.2), there exists $f \in A - \mathfrak{p}$ such that $V_+(I \cdot A_f[x_0, \ldots, x_n]) = \varnothing$. This implies that $(x_0, \ldots, x_n)^d \subset I \cdot A_f[x_0, \ldots, x_n]$ for some $d \geqslant 0$. After clearing denominators, we get $f^N \cdot (x_0, \ldots, x_n)^d \subset I$ for some $N \geqslant 0$ and hence $f^N \in (I : (x_0, \ldots, x_n)^\infty) \cap A = J$. Since $f^N \notin \mathfrak{p}$, we have $\mathfrak{p} \notin V(J)$. Hence $\pi(X) = V(J)$. □

Example 15.10. Let $A = \mathbb{C}[a, b, c]$ and let $F = ax^2 + bxy + cy^2 \in A[x, y]$ be a homogeneous polynomial of degree 2. Consider the ideal

$$\mathfrak{a} = \Big(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\Big) = (2ax + by,\ bx + 2cy) \subset A[x, y].$$

Note that $F \in \mathfrak{a}$, because $2F = x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}$. Consider the closed subscheme $X = V_+(\mathfrak{a}) \subset$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\mathbb{P}^1_A$. As X is projective over A, the image $\pi(X)$ under the projection $\pi\colon X \to \operatorname{Spec} A$ is closed. Geometrically, this subset parametrizes those (a, b, c) for which $ax^2 + bxy + cy^2$ has a double root. In fact,

$$\pi(X) = V(b^2 - 4ac). \tag{15.3}$$

To see this, we compute the elimination ideal $\mathfrak{b} = (\mathfrak{a} : (x, y)^\infty) \cap \mathbb{C}[a, b, c]$. The relations

$$b(bx + 2cy) - 2c(2ax + by) = (b^2 - 4ac)\, x,$$
$$b(2ax + by) - 2a(bx + 2cy) = (b^2 - 4ac)\, y,$$

show that $(b^2 - 4ac)\, x$ and $(b^2 - 4ac)\, y$ both lie in $\mathfrak{a}$, and hence that $b^2 - 4ac \in \mathfrak{b}$. By Corollary 15.9 this gives the inclusion $\pi(X) = V(\mathfrak{b}) \subset V(b^2 - 4ac)$.

For the opposite inclusion, it suffices to check closed points. If $b^2 = 4ac$ and $a \neq 0$, the point $(x : y) = (-b : 2a)$ satisfies $2ax + by = bx + 2cy = 0$. If $a = 0$ then also $b = 0$, and $(x : y) = (1 : 0)$ works instead, so in either case $(a, b, c) \in \pi(X)$. △

15.3 Finite morphisms and properness

Finite morphisms are important examples of proper morphisms. They are by definition affine and of finite type, hence separated, and since they are closed and stable under base change, they are universally closed. In this section we show the converse: a morphism which is both affine and proper is finite.

Lemma 15.11. Let $\phi\colon A \to B$ be a ring map such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. Then an element $a \in A$ is a unit if and only if $\phi(a)$ is a unit in B.

Proof If a is a unit, so is $\phi(a)$. Conversely, suppose that a is not a unit. Then a lies in some maximal ideal $\mathfrak{m} \subset A$, and by surjectivity there is a prime $\mathfrak{q} \subset B$ with $\phi^{-1}(\mathfrak{q}) = \mathfrak{m}$. Then $\phi(a) \in \mathfrak{q}$, so $\phi(a)$ is not a unit. □

Lemma 15.12. Let $A \to B$ be a ring map such that $\operatorname{Spec} B[t] \to \operatorname{Spec} A[t]$ is closed. Then B is integral over A.

Proof Fix $b \in B$, and consider the composition $\psi\colon A[t] \to B[t] \to B[t]/(bt - 1) = B_b$, which sends t to b^{-1}. Let $\mathfrak{a} = \operatorname{Ker} \psi$ and $C = A[t]/\mathfrak{a}$, so that $C \subset B_b$.

Since $B_b = B[t]/(bt - 1)$, the morphism $\operatorname{Spec} B_b \to \operatorname{Spec} A[t]$ is the restriction of $\operatorname{Spec} B[t] \to \operatorname{Spec} A[t]$ to the closed subset $V(bt - 1)$, and its image is therefore closed. On the other hand, since closure of the image of $\operatorname{Spec} R' \to \operatorname{Spec} R$ is $V(\operatorname{Ker}(R \to R'))$, we see that the image is exactly $V(\mathfrak{a}) = \operatorname{Spec} C$, and $\operatorname{Spec} B_b \to \operatorname{Spec} C$ is surjective.

As $\psi(t) = b^{-1}$ is a unit in B_b, Lemma 15.11 shows that the class of t is a unit in C. So $tP(t) - 1 \in \mathfrak{a}$ for some polynomial $P = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}$ in $A[t]$. Applying ψ gives $b^{-1}P(b^{-1}) = 1$ in B_b, and multiplying by b^n yields $b^n = c_0 b^{n-1} + \cdots + c_{n-1}$. This identity holds in B_b, so $b^m(b^n - c_0 b^{n-1} - \cdots - c_{n-1}) = 0$ in B for some $m \geqslant 0$, which shows that b is integral over A. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 15.13. Let $f\colon X \to Y$ be a morphism of Noetherian schemes. The following are equivalent:
(i) f is finite.
(ii) f is affine and proper.

Proof (i)⇒(ii): This was shown above.

(ii)⇒(i): Suppose that f is affine and proper, and take an affine open $V = \operatorname{Spec} A$ in Y, so that $f^{-1}V = \operatorname{Spec} B$ is affine as well. The restriction $\operatorname{Spec} B \to \operatorname{Spec} A$ is again proper. Hence B is of finite type and, by Lemma 15.12, integral over A; therefore it is finite as an A-module. □

A much deeper result about properness is the following:

Theorem 15.14. Any proper morphism with finite fibers is finite.

This fact follows from *Zariski's Main theorem*, which we will not cover in this book. For our purposes, the following version will be enough.

Proposition 15.15. Let k be a field, let X be a projective k-scheme, and let Y be a separated k-scheme of finite type. If $f\colon X \to Y$ has finite fibers over all closed points of Y, then f is finite.

Proof Recall that finiteness is local on the target. Let $U = \operatorname{Spec} A \subset Y$ be an affine open subset. Choose a closed embedding $X \subset \mathbb{P}^n_k$. The graph of f gives a closed embedding $X \to \mathbb{P}^n_k \times_k Y$ over Y. Restricting to U, we may regard $f^{-1}(U)$ as a closed subscheme of $\mathbb{P}^n_k \times_k U = \mathbb{P}^n_A$, with $f|_{f^{-1}(U)}$ induced by the projection to $\operatorname{Spec} A$. Replacing Y by U and X by $f^{-1}(U)$, we may therefore assume that $Y = \operatorname{Spec} A$ and that $X \subset \mathbb{P}^n_A$ is closed.

Let $y \in Y$ be a closed point. By homogeneous prime avoidance, there is a homogeneous polynomial $\overline{h} \in \kappa(y)[x_0, \dots, x_n]$ of positive degree which does not vanish at any point of the finite fiber X_y. Lift $\overline{h}$ to a homogeneous polynomial $h \in A[x_0, \dots, x_n]$. The closed subset $X \cap V_+(h)$ does not meet X_y. Since f is proper, its image $f(X \cap V_+(h))$ is closed in Y and does not contain y. We may therefore choose $g \in A$ such that $y \in D(g)$ and $D(g) \cap f(X \cap V_+(h)) = \varnothing$. It follows that $f^{-1}(D(g)) \subset X \cap D_+(h)$. The scheme $X \cap D_+(h)$ is affine, being closed in the affine scheme $D_+(h)$. Moreover, $f^{-1}(D(g))$ is a distinguished open subset of $X \cap D_+(h)$, and is therefore affine.

Doing this for every closed point of Y, we obtain a collection of distinguished open subsets whose inverse images are affine and which contain every closed point of Y. They cover Y, since otherwise their complement would be a nonempty closed subset of the Noetherian scheme Y, and hence would contain a closed point. Hence f is affine. Finally, a proper and affine morphism is finite by Proposition 15.13. □

15.4 Chow's Lemma

The following fundamental result explains the relation between projectivity and properness.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Theorem 15.16 (Chow's Lemma). Let A be a Noetherian ring and let $f\colon X \to \operatorname{Spec} A$ be proper. Then there exist a projective A-scheme X' and a proper birational A-morphism $\pi\colon X' \to X$ which is an isomorphism over a dense open subscheme of X.

Proof We sketch the proof, leading some details to the reader. Since X is Noetherian, it has only finitely many irreducible components. By looking at the morphism from their disjoint union onto X, we reduce to the case where X is irreducible.

Since f is of finite type, there is a finite affine open cover $X = U_1 \cup \cdots \cup U_r$ and closed embeddings $U_i \hookrightarrow \mathbb{A}^{m_i}_A$. We regard $\mathbb{A}^{m_i}$ as a standard open in $\mathbb{P}^{m_i}_A$ and define $\overline{U}_i$ to be the scheme-theoretic image of $U_i \to \mathbb{A}^{m_i}_A \to \mathbb{P}^{m_i}_A$ (Exercise 10.13.40). This gives an an open embedding $\nu_i\colon U_i \hookrightarrow \overline{U}_i$, where $\overline{U}_i$ is projective over A. The product $P = \overline{U}_1 \times_A \cdots \times_A \overline{U}_r$ is projective over A.

Since X is irreducible, $U = U_1 \cap \cdots \cap U_r$ is a dense open set. The maps $U \hookrightarrow U_i \xrightarrow{\nu_i} \overline{U}_i$ give a morphism $U \to P$, and hence its graph $\eta\colon U \longrightarrow X \times_A P$. We define X' to be the scheme-theoretic image of η, which fits into the diagram

$$\begin{array}{ccccc} U & \xhookrightarrow{\eta} & X' \subset X \times_A P & \xrightarrow{g} & P \\ & \searrow & \downarrow{\scriptstyle \pi} & & \\ & & X & & \end{array}$$

The morphism π is proper, since it is the restriction of the projection $X \times_A P \to X$. Moreover, over U the scheme-theoretic image of η is just the graph of $U \to P$, which is closed in $U \times_A P$, because P is separated over A. So π is an isomorphism over U.

Finally, one shows that g is a closed embedding. Let $q_i\colon P \to \overline{U}_i$ be the projection, and $W_i = q_i^{-1}\big(\nu_i(U_i)\big)$, $u_i = \nu_i^{-1} \circ q_i\colon W_i \to U_i$.

The key point is that $\pi^{-1}(U_i) = g^{-1}(W_i)$ and that on this open subscheme $\pi = u_i \circ g$. Indeed, $q_i \circ g$ and $\nu_i \circ \pi$ are two morphisms $\pi^{-1}(U_i) \to \overline{U}_i$ which agree on a schematically dense open U, hence everywhere, since $\overline{U}_i$ is separated. This implies $\pi^{-1}(U_i) \subset g^{-1}(W_i)$. Conversely, π and $u_i \circ g$ define morphisms $g^{-1}(W_i) \to X$ which agree on U, and hence everywhere because X is separated over A. Hence $\pi^{-1}(U_i) = g^{-1}(W_i)$.

Since $\pi = u_i \circ g$ on $\pi^{-1}(U_i)$, the restriction of the embedding $(\pi, g)\colon X' \hookrightarrow X \times_A P$ to $\pi^{-1}(U_i)$ factors through the graph of $u_i\colon W_i \to U_i$. Since this graph is isomorphic to W_i, the restriction $g\colon \pi^{-1}(U_i) \to W_i$ is a closed embedding. The opens $\pi^{-1}(U_i)$ cover X', so g is a locally closed embedding into P.

Properness of f enters only at the final step. Since $X' \to X$ and $X \to \operatorname{Spec} A$ are proper, X' is proper over A, and hence the A-morphism $g\colon X' \to P$ is proper. A proper locally closed embedding is a closed embedding, so X' is projective over A. If f is only assumed to be separated and of finite type, the same argument still gives a locally closed embedding $X' \hookrightarrow P$, and hence an X' which is quasi-projective over A. □

In particular, any proper variety admits a birational morphism from a projective variety.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

15.5 Exercises

Exercise 15.5.1. For each of the following morphisms, decide which of the three conditions 'separated', 'of finite type' and 'universally closed' hold, and in particular which of the morphisms are proper.

(i) $\operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\mathbb{Q}$.
(ii) $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$.
(iii) $\mathbb{A}^1_k \times_k \mathbb{P}^1_k \to \mathbb{A}^1_k$, the first projection.
(iv) The blow-up morphism $X \to \mathbb{P}^2_k$ of $\mathbb{P}^2_k$ at a k-point.
(v) $X \to \operatorname{Spec} k$, where X is $\mathbb{P}^1_k$ with a doubled origin.
(vi) $\mathbb{A}^2_k - \{(0,0)\} \to \mathbb{P}^1_k$, sending a point to the line through it and the origin.
(vii) $\mathbb{P}^2_{\mathbb{Z}} - \{(0:0:1)\} \to \operatorname{Spec}\mathbb{Z}$.

Exercise 15.5.2. Show that any proper locally closed embedding is a closed embedding. HINT: Check that the sheaf map is surjective.

Exercise 15.5.3. Let $f\colon X \to Y$ be a morphism of Noetherian schemes. Show that f is proper if and only if the induced morphism $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is proper. HINT: For universal closedness, use that $X_{\mathrm{red}} \to X$ is a surjective closed embedding, that this remains so after any base change, and that $X_{\mathrm{red}} \times_Y T = X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} (Y_{\mathrm{red}} \times_Y T)$.

Exercise 15.5.4. Let T be a topological space. Prove that T is compact if and only if, for every topological space S, the projection $p\colon T \times S \to S$ is closed, as follows:

a) (The Tube Lemma) Assume that T is compact. Let S be a topological space, and let $A \subset T \times S$ be closed. If $s_0 \notin p(A)$, prove that s_0 has an open neighborhood $V \subset S$ such that $V \cap p(A) = \varnothing$. Conclude that $p(A)$ is closed. HINT: For each $t \in T$, pick open sets $U_t \subset T$ and $V_t \subset S$ such that $(t, s_0) \in U_t \times V_t \subset (T \times S) - A$. Use compactness of T to reduce to finitely many U_t's.

b) Assume that $p\colon T \times S \to S$ is closed for every S, and suppose for contradiction that T is not compact. Let $\{U_i\}_{i\in I}$ be an open cover of T with no finite subcover. Let $\mathcal{J}$ be the set of finite subsets of I, and for $J \in \mathcal{J}$, set $C_J = T - \bigcup_{i\in J} U_i$. Show that each C_J is nonempty and closed, and that $C_K \subset C_J$ whenever $J \subset K$.

Let $S = \mathcal{J} \cup \{\infty\}$. Define a topology on S as follows: a subset $W \subset S$ is open if either $W \subset \mathcal{J}$ or else $\infty \in W$ and there exists $J_0 \in \mathcal{J}$ such that $\{K \in \mathcal{J} : J_0 \subset K\} \subset W$. Check that this defines a topology on S, and show that $\mathcal{J}$ is not closed in S.

c) Define $A = \{(t, J) \in T \times \mathcal{J} : t \in C_J\} \subset T \times S$. Prove that A is closed and that $p(A) = \mathcal{J}$, which is not closed in S.

HINT: If $(t, J) \notin A$ with $J \in \mathcal{J}$, use that C_J is closed. If the point is (t, ∞), choose $i \in I$ with $t \in U_i$. Then the product of U_i with the neighborhood $\{\infty\} \cup \{K \in \mathcal{J} : i \in K\}$ of ∞ is disjoint from A.

Exercise 15.5.5 (Properness over $\mathbb{C}$ = compact)**.** a) Show that complex projective space $\mathbb{CP}^n$ is compact in the Euclidean topology. Show that any complex projective variety is compact. HINT: Relate $\mathbb{CP}^n$ to the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

b) Let X be a separated scheme of finite type over $\mathbb{C}$. Show that X is proper over $\mathbb{C}$ if and only if $X(\mathbb{C})$ is compact in the Euclidean topology. HINT: For '$\Rightarrow$', use Chow's lemma to reduce to the projective case. For '$\Leftarrow$', use Chevalley's theorem and Exercises 15.5.4 and 13.5.17 to show that every projection $X \times_{\mathbb{C}} Y \to Y$, with Y of finite type over $\mathbb{C}$, is closed.

Exercise 15.5.6. Let k be a field. Show that the infinite disjoint union $\bigsqcup_{i\in\mathbb{N}} \operatorname{Spec} k$ is not universally closed over k.

Exercise 15.5.7. Let X be a variety over a field k, containing $\mathbb{A}^1_k$ as a closed subvariety. Show that X is not proper over k.

Exercise 15.5.8. Let $X \to S$ be a proper morphism with $X \neq \varnothing$, and let $f\colon X \to Y$ be an open embedding. Show that if Y is irreducible and separated over S, then f is an isomorphism.

Exercise 15.5.9. Show that $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{Q}$ has finite fibers, but is not finite. Why doesn't this contradict Proposition 15.15?

Exercise 15.5.10. Let k be a field, let $A = k[a,b,c,d]$, and let $X \subset \mathbb{P}^1_A$ be the closed subscheme defined by the two homogeneous equations $ax + by = 0$ and $cx + dy = 0$. Show that the image of $X \to \operatorname{Spec} A$ is the closed subset $V(ad - bc)$.

Exercise 15.5.11. Let k be a field, let $C = V_+(y^2 - xz, xw - yz, z^2 - yw) \subset \mathbb{P}^3_k$, and let $P = (0:0:1:0)$. Show that projection from P maps C onto the cuspidal cubic $V_+(y^3 - x^2w) \subset \mathbb{P}^2_k$, where the target coordinates are $(x:y:w)$.

Exercise 15.5.12 (Affine proper varieties)**.** Let X be an affine variety over a field k. Show that if X is proper over k, then X is finite over $\operatorname{Spec} k$. In particular, if k is algebraically closed, show that $X \simeq \operatorname{Spec} k$. HINT: Use Proposition 15.13.

Exercise 15.5.13. Let $X = V_+(x_2x_1^2 - x_0^3 - x_0^2x_2) \subset \mathbb{P}^2_{\mathbb{C}}$ be the nodal cubic, with node $P = (0:0:1)$, and set $U_1 = X - \{P\}$ and $U_2 = X - \{(0:1:0)\}$. Identify U_1 with $\mathbb{G}_m$ and embed it in $\mathbb{A}^2_{\mathbb{C}}$ by $t \mapsto (t, t^{-1})$; embed U_2 by its affine equation $y^2 = x^3 + x^2$. Show that their projective closures are $V_+(uv - w^2) \subset \mathbb{P}^2_{\mathbb{C}}$ and X, respectively, and that the scheme X' constructed in Chow's Lemma is $\mathbb{P}^1_{\mathbb{C}}$, with $\pi\colon X' \to X$ the normalization. Thus π need not be an isomorphism.

Exercise 15.5.14 (Chow's Lemma over non-affine base)**.** Let S be a Noetherian scheme and let $f\colon X \to S$ be a proper morphism. Show that there exist a scheme X' together with a closed embedding $X' \to \mathbb{P}^n_S$ over S and a proper S-morphism $\pi\colon X' \to X$ which is an isomorphism over a dense open set $U \subset X$. HINT: Cover S by affines S_i, and X by affine opens U_j such that each U_j maps into some S_i. Apply the proof of Chow's Lemma.

Exercise 15.5.15. Prove the following variant of Chow's Lemma. Let X be a separated scheme of finite type over a Noetherian ring A. Then there exists an A-scheme $\widetilde{X}$ and a surjective A-morphism $\pi\colon \widetilde{X} \to X$ such that (i) π factors as a closed embedding $\widetilde{X} \hookrightarrow \mathbb{P}^n_A \times_A X$ followed by the projection; (ii) $\widetilde{X}$ is quasi-projective over A; and (iii) there is a dense open $U \subset X$ such that π induces an isomorphism $\pi^{-1}(U) \simeq U$. Note that (ii) does

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

not follow from (i), since $\mathbb{P}^n_A \times_A X$ need not be quasi-projective; the immersion witnessing (ii) is the morphism $\widetilde{X} \to \overline{U}_1 \times_A \cdots \times_A \overline{U}_r$ of the proof of Chow's Lemma.

Exercise 15.5.16 (Valuative criterion)**.** Let $f\colon X \to S$ be a morphism of finite type between Noetherian schemes, and let R be a discrete valuation ring with fraction field K. Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{g} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \longrightarrow & S \end{array}$$

where j is the morphism induced by the inclusion $R \subset K$.

a) Show that if f is separated, then there exists *at most one* morphism $\operatorname{Spec} R \to X$ making the above diagram commute.

b) Show that if f is proper, then there exists a unique morphism $\operatorname{Spec} R \to X$ making the above diagram commute. HINT: Write $X_R = X \times_S \operatorname{Spec} R$. Let $P \in X_R$ be the image of $(j, g)\colon \operatorname{Spec} K \to X_R$ and let $W = \overline{\{P\}}_{\mathrm{red}} \subset X_R$, a closed subscheme, so that $f_R\colon W \to \operatorname{Spec} R$ is proper. Show that $f_R(W) = \operatorname{Spec} R$, and let $w \in W$ be a point mapping to the closed point of $\operatorname{Spec} R$. Then $R \subset \mathcal{O}_{W,w} \subset K$ and the maximal ideal of $\mathcal{O}_{W,w}$ meets R in the maximal ideal of R; conclude that $\mathcal{O}_{W,w} = R$, since a valuation ring has no such proper overring inside K.

In fact, the converses of a) and b) also hold. These statements are known as the *valuative criteria*. They are useful for checking that a morphism is separated or proper. In the next two exercises you may assume this fact.

c) Let k be a field, let $R = k[t]_{(t)}$ and $K = k(t)$, and consider the morphism $\operatorname{Spec} K \to \mathbb{A}^1_k$ corresponding to the element $t^{-1} \in K$. Show that this morphism does not extend to a morphism $\operatorname{Spec} R \to \mathbb{A}^1_k$. Deduce that $\mathbb{A}^1_k$ is not proper over k.

c) Let R be a discrete valuation ring with fraction field K. Show that every point of $\mathbb{P}^n(K)$ may be represented by homogeneous coordinates $(a_0 : \cdots : a_n)$ with all $a_i \in R$ and at least one $a_i \in R^\times$. HINT: Divide by a coordinate of minimal valuation. Deduce that every morphism $\operatorname{Spec} K \to \mathbb{P}^n_R$ over $\operatorname{Spec} R$ extends uniquely to a section $\operatorname{Spec} R \to \mathbb{P}^n_R$. Deduce that $\mathbb{P}^n$ is proper.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

16

Tangent spaces and regular schemes

In this chapter we study a scheme near a point by means of two vector spaces: the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ and its dual, the tangent space T_PX. For a scheme cut out by equations, both can be computed locally by the Jacobian criterion. A point is called *regular* if its tangent space has the expected dimension, namely the dimension of X at that point. Regular schemes, and likewise normal and smooth schemes, have good properties that make them easier to work with, and all three appear throughout the rest of the book.

16.1 Tangent spaces and cotangent spaces

Let X be a scheme and let $P \in X$. We want to define the tangent space T_PX of X at P. It turns out to be more convenient to define the cotangent space first and then define the tangent space as its dual.

Definition 16.1. Let X be a scheme and let $P \in X$.
(i) The *cotangent space* of X at P is the $\kappa(P)$-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$.
(ii) The *tangent space* of X at P is the dual vector space

$$T_PX = \operatorname{Hom}_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2, \kappa(P)).$$

The cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ measures functions vanishing at P modulo functions vanishing to second order. Since $\mathfrak{m}_P/\mathfrak{m}_P^2 = \mathfrak{m}_P \otimes_{\mathcal{O}_{X,P}} \kappa(P)$, this is naturally a $\kappa(P)$-vector space. For example, if k is a field and P is the origin in $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, then $\mathfrak{m}_P = (x_1, \dots, x_n)$, and $\mathfrak{m}_P/\mathfrak{m}_P^2$ is the k-vector space generated by the classes of $x_1, \dots, x_n$.

Given a morphism $f\colon X \to Y$ there is an induced map of cotangent spaces. Let $P \in X$ and $Q = f(P) \in Y$. The induced map on local rings $f_P^\sharp\colon \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$ sends $\mathfrak{m}_Q$ into $\mathfrak{m}_P$, and being a ring map it also sends $\mathfrak{m}_Q^2$ into $\mathfrak{m}_P^2$. Hence it induces a map of cotangent spaces

$$f_P^*\colon \mathfrak{m}_Q/\mathfrak{m}_Q^2 \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2. \tag{16.1}$$

This map is $\kappa(Q)$-linear, where $\mathfrak{m}_P/\mathfrak{m}_P^2$ is viewed as a $\kappa(Q)$-vector space via the map of residue fields $\kappa(Q) \to \kappa(P)$. If $g\colon Y \to Z$ is another morphism, then $(g\circ f)_P^* = f_P^* \circ g_{f(P)}^*$.

The tangent space, however, does not have the same functorial properties in general. The issue is that we are taking duals over different fields. More precisely, f_P^* is a map of $\kappa(Q)$-vector spaces, and there is no natural way to view $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ as a $\kappa(P)$-vector space.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

However, if f is a morphism of schemes of finite type over a field k and that $P \in X$ is a k-point. Then $Q = f(P)$ is a k-point as well, so $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ are both finite-dimensional k-vector spaces. Hence we can take duals to obtain a linear map

$$Df_P \colon T_P X \longrightarrow T_Q Y. \tag{16.2}$$

These tangent maps satisfy the chain rule $D(g \circ f)_P = Dg_{f(P)} \circ Df_P$ for any morphisms of k-schemes $f \colon X \to Y$ and $g \colon Y \to Z$.

To see how this definition compares with the classical notion of a tangent space, let k be a field and let $X = \operatorname{Spec} A$, where $A = R/I$ for $R = k[x_1, \ldots, x_n]$. Let $P = (a_1, \ldots, a_n) \in X(k)$ be a k-point and let $\mathfrak{m}_A$ and $\mathfrak{m}_R$ denote the corresponding maximal ideals $(x_1 - a_1, \ldots, x_n - a_n)$ in A and R respectively. Since $P \in X$, we have $I \subset \mathfrak{m}_R$, and the maximal ideal of $\mathcal{O}_{X,P}$ is induced by $\mathfrak{m}_A = \mathfrak{m}_R/I$.

We first note that one does not have to pass all the way to the local ring $\mathcal{O}_{X,P}$ in order to compute the cotangent space. If B is a ring and $\mathfrak{m} \subset B$ is a maximal ideal, then there is a natural isomorphism of $B/\mathfrak{m}$-vector spaces $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m}B_{\mathfrak{m}}/\mathfrak{m}^2 B_{\mathfrak{m}}$. Indeed, $B/\mathfrak{m} \simeq B_{\mathfrak{m}}/\mathfrak{m}B_{\mathfrak{m}}$, and therefore

$$\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m} \otimes_B B/\mathfrak{m} \simeq \mathfrak{m} \otimes_B B_{\mathfrak{m}}/\mathfrak{m}B_{\mathfrak{m}} \simeq \mathfrak{m}B_{\mathfrak{m}}/\mathfrak{m}^2 B_{\mathfrak{m}}.$$

Applying this to our setting, we find

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \simeq \frac{\mathfrak{m}_R}{I + \mathfrak{m}_R^2} \simeq \frac{\mathfrak{m}_R/\mathfrak{m}_R^2}{(I + \mathfrak{m}_R^2)/\mathfrak{m}_R^2}.$$

Equivalently, there is an exact sequence of k-vector spaces

$$0 \longrightarrow (I + \mathfrak{m}_R^2)/\mathfrak{m}_R^2 \longrightarrow \mathfrak{m}_R/\mathfrak{m}_R^2 \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2 \longrightarrow 0.$$

Every $f \in R$ has a Taylor expansion around P,

$$f = f(P) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P) \cdot (x_i - a_i) + \text{ higher degree terms in the } x_i - a_i. \tag{16.3}$$

Here the $\partial f/\partial x_i$ are the formal partial derivatives, defined on monomials by $\partial(x_1^{e_1} \cdots x_n^{e_n})/\partial x_i = e_i x_1^{e_1} \cdots x_i^{e_i - 1} \cdots x_n^{e_n}$ and extended by k-linearity. These obey the usual rules of differentiation, but can behave unexpectedly in characteristic $p > 0$. For instance, if $f = x^p$, then $\partial f/\partial x = p x^{p-1} = 0$ although f is not constant. For $f \in I$ we have $f(P) = 0$, so (16.3) gives

$$f \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)(x_i - a_i) \pmod{\mathfrak{m}_R^2}.$$

This means that $(I + \mathfrak{m}_R^2)/\mathfrak{m}_R^2$ is the subspace of $\mathfrak{m}_R/\mathfrak{m}_R^2$ generated by the linear parts $\sum_i \frac{\partial f}{\partial x_i}(P)(x_i - a_i)$ for $f \in I$. The cotangent space is the quotient of $\mathfrak{m}_R/\mathfrak{m}_R^2$ by this subspace.

Now identify $\mathfrak{m}_R/\mathfrak{m}_R^2$ with k^n using the basis $x_1 - a_1, \ldots, x_n - a_n$. A linear form on $\mathfrak{m}_R/\mathfrak{m}_R^2$ then corresponds to a vector $(v_1, \ldots, v_n) \in k^n$ by sending $x_i - a_i$ to v_i. Such a

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

linear form descends to $\mathfrak{m}_P/\mathfrak{m}_P^2$ exactly when it vanishes on the linear part of every $f \in I$. That is,

$$T_P X = \big\{ (v_1, \dots, v_n) \in k^n \mid \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P) v_i = 0 \text{ for every } f \in I \big\}.$$

If $I = (f_1, \dots, f_r)$, then it is enough to impose these conditions on the generators. Let

$$J(P) = \left(\frac{\partial f_i}{\partial x_j}(P) \right)$$

be the $r \times n$ Jacobian matrix evaluated at P, regarded as a linear map $k^n \to k^r$. Then

$$T_P X = \operatorname{Ker} J(P). \tag{16.4}$$

This is the classical tangent space of an affine algebraic set: it is the subspace of k^n cut out by the linear parts of the defining equations of X at P.

Dually, after identifying $\mathfrak{m}_R/\mathfrak{m}_R^2$ with k^n using the basis $x_1 - a_1, \dots, x_n - a_n$, the cotangent space is identified with $\operatorname{Coker} J(P)^t$.

Example 16.2. Let $X = \operatorname{Spec} \mathbb{C}[x, y]/(f)$ and let $P = (0, 0)$.

If $f = y^2 + x^3 + 3x + 4y$, then modulo $(x, y)^2$ the equation becomes $3x + 4y$. Hence

$$T_P X = \big\{ (v_1, v_2) \in \mathbb{C}^2 \mid 3v_1 + 4v_2 = 0 \big\}.$$

If $f = y^2 + x^3 + x^2$, then the equation has no linear part at the origin, and hence imposes no linear condition on tangent vectors. In this case $T_P X \simeq \mathbb{C}^2$. △

16.2 Regular schemes

Tangent and cotangent spaces lead naturally to the notion of regularity. Roughly speaking, a scheme is said to be regular if the tangent spaces have the expected dimension at every point.

Proposition 16.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then

$$\dim A \leqslant \dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2).$$

Equality holds if and only if $\mathfrak{m}$ can be generated by $\dim A$ elements.

Proof Choose elements $x_1, \dots, x_r \in \mathfrak{m}$ whose classes form a basis of $\mathfrak{m}/\mathfrak{m}^2$, where $r = \dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2)$. By Nakayama's lemma, these elements generate $\mathfrak{m}$, so $\mathfrak{m} = (x_1, \dots, x_r)$. So $\mathfrak{m}$ is a minimal prime over the ideal $(x_1, \dots, x_r)$. By Krull's principal ideal theorem, every minimal prime over an ideal generated by r elements has height at most r. Therefore $\dim A = \operatorname{ht}(\mathfrak{m}) \leqslant r$. The statement about equality also follows. □

A Noetherian local ring A with maximal ideal $\mathfrak{m}$ is called *regular* if

$$\dim A = \dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2).$$

A Noetherian ring A is called regular if $A_\mathfrak{p}$ is a regular local ring for every prime ideal $\mathfrak{p} \subset A$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

By Krull's principal ideal theorem (Theorem 8.52), $\dim A$ is the least number of elements $x_1, \dots, x_n \in \mathfrak{m}$ with $\sqrt{(x_1, \dots, x_n)} = \mathfrak{m}$, and such a tuple is called a system of parameters. It need not generate $\mathfrak{m}$, and by the proposition above, A is regular precisely when it can be chosen so that it does. The x_i then play the role of n 'coordinate functions' on $\operatorname{Spec} A$ near the closed point.

We say that a Noetherian scheme X is *regular* at a point P if $\mathcal{O}_{X,P}$ is a regular local ring. Otherwise, we say that P is a *singular point*. The scheme X is *regular* if it is regular at every point. The *regular locus* is $X_{\text{reg}} = \{ P \in X \mid \mathcal{O}_{X,P} \text{ is regular} \}$, and the set of singular points is $\operatorname{sing}(X) = X - X_{\text{reg}}$.

Example 16.4. The scheme $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ is regular. To see this, we consider the three types of points $P \in \mathbb{A}^2_{\mathbb{C}}$:

$P = (a, b)$ is a $\mathbb{C}$-point. The ring $\mathbb{C}[x, y]_{(x-a,y-b)}$ is regular: it has dimension 2, and the maximal ideal $(x - a, y - b)$ is generated by 2 elements.

$P = (f)$, where $f \in \mathbb{C}[x, y]$ is irreducible. Then $\mathbb{C}[x, y]_{(f)}$ has dimension 1 (e.g., by the Principal Ideal Theorem), and of course the maximal ideal (f) is generated by one element.

$P = (0)$ is the generic point. The local ring $\mathbb{C}[x, y]_{(0)} = \mathbb{C}(x, y)$ is a field, which is a regular local ring with maximal ideal (0). △

Example 16.5. For $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3)$ and $P = (0, 0)$, the local ring at P is

$$\mathcal{O}_{X,P} = \mathbb{C}[x, y]_{(x,y)}/(y^2 - x^3).$$

This ring has dimension 1, but $\mathfrak{m}_P = (x, y)$ cannot be generated by a single element. △

The condition of regularity is natural, but in order to work with it properly we need some nontrivial machinery from commutative algebra.

Theorem 16.6 (Basic facts about regular rings).

(i) Any localization of a regular ring is regular (Eisenbud, 2013, Corollary 19.14).

(ii) Every regular local ring is an integral domain (Eisenbud, 2013, Corollary 10.14); see also Exercise 16.5.21.

(iii) If A is Noetherian, then A is regular if and only if $A[x]$ is regular (Eisenbud, 2013, Exercise 19.3).

(iv) (Auslander–Buchsbaum): Every regular local ring is a UFD (Eisenbud, 2013, Theorem 19.11).

Item (ii) tells us that in a regular scheme, any point is contained in a unique irreducible component, and this component is an integral scheme. Hence the scheme is a disjoint union of integral schemes. In particular, any connected regular scheme is integral. This observation is sometimes useful when proving that a ring is an integral domain (see Exercise 16.5.19).

One consequence of (i) is that regularity can be checked on closed points.

Corollary 16.7. Let X be a scheme of finite type over a field. Then X is regular if and only if it is regular at every closed point.

Proof Let $P \in X$ and choose an affine open $U = \operatorname{Spec} A$ containing P so that P

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

corresponds to a prime ideal $\mathfrak{p}$. Choose a maximal ideal $\mathfrak{m} \supset \mathfrak{p}$. Then $\mathfrak{m}$ corresponds to a closed point, so if $A_{\mathfrak{m}}$ is regular, then $A_{\mathfrak{p}}$ is regular because it is a localization of $A_{\mathfrak{m}}$. □

Example 16.8. The schemes $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are regular. Indeed, their local rings are localizations of the polynomial ring $k[t_1, \ldots, t_n]$ which is regular by (iii). △

For completeness, we give a direct proof that polynomial rings are regular. The key step is the following lemma.

Lemma 16.9. Let $(S, \mathfrak{n})$ be a regular local ring of dimension d, set $R = S[x]$, and let $\mathfrak{p} \subset R$ be a prime ideal containing $\mathfrak{n}R$. Then $R_{\mathfrak{p}}$ is a regular local ring.

Proof Since S is regular of dimension d, its maximal ideal is generated by d elements $f_1, \ldots, f_d$ by Proposition 16.3. Fix also a chain of primes

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_d = \mathfrak{n}$$

in S, which exists because $\dim S = d$. Since $R/\mathfrak{q}_i R \simeq (S/\mathfrak{q}_i)[x]$ is an integral domain, each $\mathfrak{q}_i R$ is prime, and $\mathfrak{q}_i R \cap S = \mathfrak{q}_i$, so the inclusions remain strict in R. We write $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ for the maximal ideal of $R_{\mathfrak{p}}$.

Case 1: $\mathfrak{p} = \mathfrak{n}R$. Then $\mathfrak{m}$ is generated by the images of $f_1, \ldots, f_d$. Localizing the chain $\mathfrak{q}_0 R \subsetneq \cdots \subsetneq \mathfrak{q}_d R = \mathfrak{p}$ at $\mathfrak{p}$ gives a chain of length d in $R_{\mathfrak{p}}$, so Proposition 16.3 gives

$$d \leqslant \dim R_{\mathfrak{p}} \leqslant \dim_{\kappa(\mathfrak{p})} \mathfrak{m}/\mathfrak{m}^2 \leqslant d.$$

All of these are equalities, so $R_{\mathfrak{p}}$ is regular of dimension d.

Case 2: $\mathfrak{n}R \subsetneq \mathfrak{p}$. Let $K = S/\mathfrak{n}$. The image of $\mathfrak{p}$ in $R/\mathfrak{n}R \simeq K[x]$ is a nonzero prime ideal, hence is generated by an irreducible polynomial g. Choose $G \in R$ mapping to g. We claim that $\mathfrak{p} = (f_1, \ldots, f_d, G)$. The right hand side is contained in $\mathfrak{p}$, and if $F \in \mathfrak{p}$, then the image of F in $K[x]$ is a multiple of g, so $F \equiv HG \pmod{\mathfrak{n}R}$ for some $H \in R$, and therefore $F \in (\mathfrak{n}R, G)$.

Appending $\mathfrak{p}$ to the chain above gives $\mathfrak{q}_0 R \subsetneq \cdots \subsetneq \mathfrak{q}_d R = \mathfrak{n}R \subsetneq \mathfrak{p}$, a chain of length $d+1$, so $\dim R_{\mathfrak{p}} \geqslant d+1$. As $\mathfrak{m}$ is generated by the $d+1$ elements $f_1, \ldots, f_d, G$, the same argument as in Case 1 gives

$$d+1 \leqslant \dim R_{\mathfrak{p}} \leqslant \dim_{\kappa(\mathfrak{p})} \mathfrak{m}/\mathfrak{m}^2 \leqslant d+1,$$

so $R_{\mathfrak{p}}$ is regular of dimension $d+1$. □

Proposition 16.10. Let $R = k[x_1, \ldots, x_n]$ and let $\mathfrak{p} \subset R$ be a prime ideal. Then $R_{\mathfrak{p}}$ is a regular local ring.

Proof We argue by induction on n. If $n = 0$, then $R = k$ is a field, so there is nothing to prove. If $n > 0$, write $A = k[x_1, \ldots, x_{n-1}]$ and $R = A[x_n]$, and let $\mathfrak{p} \subset R$ be a prime ideal. Write $\mathfrak{q} = \mathfrak{p} \cap A$. By the induction hypothesis, $A_{\mathfrak{q}}$ is a regular local ring. Now $R_{\mathfrak{p}} \simeq (A_{\mathfrak{q}}[x_n])_{\mathfrak{p}A_{\mathfrak{q}}[x_n]}$, and the prime $\mathfrak{p}A_{\mathfrak{q}}[x_n]$ contains the extension of the maximal ideal of $A_{\mathfrak{q}}$. Therefore we may apply Lemma 16.9 to conclude that $R_{\mathfrak{p}}$ is regular. □

Example 16.11. The $\mathbb{C}$-scheme $X = \operatorname{Spec} \mathbb{C}[x]/(x^2+x)$ is regular, because it is isomorphic to the two-point scheme $\operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

On the other hand, the scheme $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ is singular, because $\mathbb{C}[x]/(x^2)$ is a local ring of dimension 0, and the maximal ideal $\mathfrak{m} = (x)$ is nonzero.

More generally, a regular local ring of dimension 0 must be a field (because it is both Artinian and an integral domain). Hence, if $X = \operatorname{Spec} A$ is a regular scheme of dimension 0, then A is a product of fields. △

The following basic result gives rise to many examples.

Proposition 16.12 (Complete intersections). Let X be a scheme that is regular at a point P, and let $Y \subset X$ be a closed subscheme containing P. Suppose that, in $\mathcal{O}_{X,P}$, the ideal of Y is generated by $f_1, \dots, f_r \in \mathfrak{m}_P$. Assume moreover that $\dim \mathcal{O}_{Y,P} = \dim \mathcal{O}_{X,P} - r$. Then Y is regular at P if and only if the images of $f_1, \dots, f_r$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$ are linearly independent over $\kappa(P)$.

Proof Write $A = \mathcal{O}_{X,P}$, $\mathfrak{m} = \mathfrak{m}_P$, and $I = (f_1, \dots, f_r)$. Then $\mathcal{O}_{Y,P} = A/I$. Let $\mathfrak{n} = \mathfrak{m}/I$ be the maximal ideal of A/I, and set $n = \dim A$. Since A is regular, $\dim_{\kappa(P)} \mathfrak{m}/\mathfrak{m}^2 = n$. Also

$$\mathfrak{n}/\mathfrak{n}^2 \simeq \mathfrak{m}/(\mathfrak{m}^2 + I).$$

If s is the dimension of the span of the images of $f_1, \dots, f_r$ in $\mathfrak{m}/\mathfrak{m}^2$, then $\dim_{\kappa(P)} \mathfrak{n}/\mathfrak{n}^2 = n - s$. Since $\dim A/I = n - r$, the quotient A/I is regular if and only if $s = r$. This means exactly that the images of $f_1, \dots, f_r$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. □

In particular, if $Y \subset X$ is a codimension 1 subscheme locally defined by a local equation $f \in \mathfrak{m}_P$ at P, then Y is regular at P if and only if $f \notin \mathfrak{m}_P^2$, that is, f vanishes at P with multiplicity 1.

Using the description of the tangent space in terms of the Jacobian matrix (16.4), we get the following useful criterion to check if a k-scheme is regular at a k-point.

Proposition 16.13 (Jacobian criterion). Let k be a field and let $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Let $J = \left(\partial f_i/\partial x_j\right)$ be the Jacobian matrix of the $f_1, \dots, f_r$. Then for each k-point P, we have

$$\operatorname{rank} J(P) \leqslant n - \dim \mathcal{O}_{X,P}, \tag{16.5}$$

and X is regular at P if and only if equality holds.

Proof By (16.4), we have $\dim_k T_P X = n - \operatorname{rank} J(P)$. Since P is a k-point, the tangent space and cotangent space have the same dimension, so $\dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2) = n - \operatorname{rank} J(P)$. Now (16.5) is just Proposition 16.3, and equality holds exactly when $\dim \mathcal{O}_{X,P} = \dim_k(\mathfrak{m}_P/\mathfrak{m}_P^2)$, that is, when $\mathcal{O}_{X,P}$ is regular. □

By Corollary 16.7, this means that if X is a scheme of finite type over an algebraically closed field k, then we can decide whether X is singular by computing the Jacobian matrices at k-points in affine charts.

Example 16.14. Consider the quadric surface $X = V(x^2 + yz - y)$ in $\mathbb{A}^3_{\mathbb{C}}$. Then the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Jacobian at a closed point $P = (a, b, c)$ is equal to

$$J(P) = \begin{pmatrix} 2a & c-1 & b \end{pmatrix}.$$

Therefore, X has a singular point at the point $(0, 0, 1)$ and it is regular at all other points. $\triangle$

Example 16.15 (A regular curve with vanishing Jacobian)**.** Let $p > 2$ be a prime, let $k = \mathbb{F}_p(t)$, and let $X = \operatorname{Spec} A$ with $A = k[x, y]/(f)$, where $f = y^2 + x^p + t$.

First of all, X is an integral curve over k, because f is irreducible (t is not a p-th power, so the polynomial $x^p + t$ is irreducible over k and hence so is f). The Jacobian of f is

$$J = \begin{pmatrix} p\,x^{p-1} & 2y \end{pmatrix} = \begin{pmatrix} 0 & 2y \end{pmatrix}.$$

This vanishes exactly at the closed point P corresponding to the maximal ideal $\mathfrak{m} = (x^p + t,\ y)$.

We claim that X is nevertheless regular. Let $Q \in X$ be a closed point. By Proposition 16.12 it suffices to show that $f \notin \mathfrak{m}_Q^2$.

Assume $f \in \mathfrak{m}_Q^2$. By the product rule both partial derivatives of f then lie in $\mathfrak{m}_Q$, so $2y \in \mathfrak{m}_Q$ and hence $y \in \mathfrak{m}_Q$. But since $f \in \mathfrak{m}_Q$, this implies $x^p + t \in \mathfrak{m}_Q$, hence $\mathfrak{m} \subset \mathfrak{m}_Q$ and so $Q = P$. It remains to rule out $Q = P$. At P, we have

$$\mathfrak{m}^2 = \big(y^2,\ y(x^p + t),\ (x^p + t)^2\big),$$

which is contained in $(y, (x^p + t)^2)$. Therefore, $f \in \mathfrak{m}^2$ would give $x^p + t = f - y^2 \in (y, (x^p + t)^2)$. However, reducing modulo y shows that this is impossible. Hence $f \notin \mathfrak{m}_Q^2$ for every closed point Q, and X is regular.

One explanation for this example is that the Jacobian matrix is the same if we consider the equations over k or the algebraic closure $\bar{k}$. If we pass to $\bar{k}$ and choose $s \in \bar{k}$ with $s^p = t$, then

$$\bar{k}[x, y]/(y^2 + x^p + t) \simeq \bar{k}[x, y]/(y^2 + (x + s)^p).$$

The latter curve is not regular at the point $(x, y) = (-s, 0)$, as $y^2 + (x + s)^p$ lies in the square of the maximal ideal $(x + s,\ y)$. $\triangle$

The example shows that regularity is not stable under base change. To remedy this, one introduces the stronger notion of *smoothness*, which is stable under base change. The two notions agree over a perfect field, in particular over an algebraically closed field, but differ in general. Roughly speaking, regularity asks that the tangent spaces have the expected dimension, whereas smoothness asks that the Jacobian matrices have the expected rank. Smoothness will be discussed in Chapter 17.

16.3 Regular embeddings

We now turn from a single scheme to a closed subscheme $Y \subset X$, and single out those subschemes that are cut out locally by a regular sequence. This is the best behaved situation: the number of equations is then as small as the codimension allows, and the ideal sheaf gives rise to two locally free sheaves on Y, the conormal and the normal sheaf. Regular schemes turn out to be a rich source of such subschemes, and these two sheaves will reappear when we study differentials on smooth schemes.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Let $i\colon Y \hookrightarrow X$ be a closed embedding of Noetherian schemes, defined by an ideal sheaf $\mathcal{I}$. We say that i is a *regular embedding* at a point $P \in Y$ if the ideal $\mathcal{I}_P \subset \mathcal{O}_{X,P}$ can be generated by a regular sequence in the sense of Definition 8.56. Equivalently, in a neighbourhood of P, the ideal sheaf $\mathcal{I}$ is generated by sections whose germs form a regular sequence at every point of Y in that neighborhood. If this holds at every point of Y, we call i a regular embedding. If the regular sequences can everywhere be chosen to have length r, then i is a regular embedding of codimension r. By Proposition 8.60, the subscheme Y then has pure codimension r in X, which is what justifies the name.

In this situation one also says that Y is a *locally complete intersection* in X. The name refers to the fact that near each point of Y the subscheme is cut out by exactly r equations. This is the smallest possible number, since by Krull's principal ideal theorem an ideal generated by fewer than r elements cannot cut out anything of codimension r.

Proposition 16.16. Let $(A, \mathfrak{m})$ be a regular local ring of dimension n, and let $f_1, \dots, f_r \in \mathfrak{m}$. If the classes of $f_1, \dots, f_r$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, then $A/(f_1, \dots, f_r)$ is a regular local ring of dimension $n - r$.

Proof We argue by induction on r. If $r = 1$, then $f_1 \notin \mathfrak{m}^2$, so in particular $f_1 \neq 0$. Since regular local rings are domains, Krull's principal ideal theorem gives $\dim A/(f_1) = n - 1$. Proposition 16.12 now shows that $A/(f_1)$ is regular.

Assume $r > 1$, and set $A' = A/(f_1, \dots, f_{r-1})$ with maximal ideal $\mathfrak{m}'$. By the induction hypothesis, A' is a regular local ring of dimension $n - r + 1$. Moreover, $\mathfrak{m}'/\mathfrak{m}'^2 \simeq \mathfrak{m}/(\mathfrak{m}^2 + (f_1, \dots, f_{r-1}))$, so the image of f_r in $\mathfrak{m}'/\mathfrak{m}'^2$ is nonzero. Applying the case $r = 1$ to A' and the image of f_r, we find that $A'/(f_r) = A/(f_1, \dots, f_r)$ is a regular local ring of dimension $n - r$. □

Theorem 16.17. Let X be a regular Noetherian scheme, and let $Y \subset X$ be a closed subscheme that is regular. Then $Y \hookrightarrow X$ is a regular embedding.

Proof Fix a point $P \in Y$. Write $A = \mathcal{O}_{X,P}$ and $I = \mathcal{I}_P$, and let $\mathfrak{m}$ be the maximal ideal of A. Let $B = A/I$, and let $\mathfrak{n} = \mathfrak{m}/I$ be the maximal ideal of B. Since A and B are regular local rings, the vector spaces $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{n}/\mathfrak{n}^2$ have dimensions $\dim A$ and $\dim B$.

Choose elements $y_1, \dots, y_d \in \mathfrak{m}$ whose images form a basis of $\mathfrak{n}/\mathfrak{n}^2$, where $d = \dim B$. Choose elements $x_1, \dots, x_c \in I$ whose classes form a basis of the kernel of $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$, where $c = \dim A - \dim B$. Then the classes of $x_1, \dots, x_c, y_1, \dots, y_d$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$.

By Proposition 16.16, each quotient $A/(x_1, \dots, x_j)$ is a regular local ring of dimension $n - j$, where $n = \dim A$; in particular it is an integral domain by Theorem 16.6(ii). The image of x_{j+1} in its cotangent space is nonzero, so x_{j+1} is a nonzero element of a domain, hence a nonzerodivisor. Therefore $x_1, \dots, x_c$ is a regular sequence, and $A/(x_1, \dots, x_c)$ is a regular local ring of dimension d.

Write $J = (x_1, \dots, x_c)$. Suppose that $J \neq I$. Then I/J is a nonzero ideal of the regular local ring A/J. By Theorem 16.6(ii), the ring A/J is an integral domain, so we may choose $0 \neq f \in I/J$. Then $\dim(A/J)/(f) \leqslant d - 1$. Since A/I is a quotient of $(A/J)/(f)$, it follows that $\dim A/I \leqslant d - 1$, a contradiction. Hence $I = J$ and is generated by the regular

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

sequence $x_1, \dots, x_c$. This proves that the embedding is regular at P. As P was arbitrary, $Y \hookrightarrow X$ is a regular embedding. □

The theorem asks that Y itself be regular. When the ambient scheme is regular there is a second criterion, which says nothing about the singularities of Y.

Proposition 16.18. Let $(A, \mathfrak{m})$ be a regular local ring, let $f_1, \dots, f_r \in \mathfrak{m}$, and set $I = (f_1, \dots, f_r)$. If $\dim A/I = \dim A - r$, then $f_1, \dots, f_r$ is a regular sequence in A.

Proof See (Eisenbud, 2013, Chapter 18). □

Geometrically, if X is regular and $Y \subset X$ is locally cut out by r equations and has codimension r, then $Y \hookrightarrow X$ is a regular embedding of codimension r, however singular Y may be. By Krull's principal ideal theorem, r equations can never cut out something of codimension more than r, so the hypothesis says that they are as independent as they can be. Some assumption on X is needed: in $A = k[x, y]/(x^2, xy)$ the ideal (y) has height 1 and one generator, but y is a zerodivisor, the culprit being the embedded point at the origin.

Proposition 16.19. Let A be a Noetherian ring, and let $I \subset A$ be an ideal generated by a regular sequence $f_1, \dots, f_r$. Then the map

$$\varphi\colon (A/I)^{\oplus r} \longrightarrow I/I^2, \qquad (\overline{a}_1, \dots, \overline{a}_r) \longmapsto \sum_{i=1}^{r} a_i f_i \bmod I^2,$$

is an isomorphism of A/I-modules. In particular, I/I^2 is a free A/I-module of rank r.

Proof The map is well-defined and surjective because $f_1, \dots, f_r$ generate I. For injectivity, we first observe that if $\sum_{i=1}^{r} c_i f_i = 0$, then every c_i belongs to I. Indeed, modulo $(f_1, \dots, f_{r-1})$ the relation becomes $c_r f_r = 0$. Since f_r is a nonzerodivisor on $A/(f_1, \dots, f_{r-1})$, we have $c_r \in (f_1, \dots, f_{r-1})$. Substituting an expression for c_r into the relation and arguing by induction on r shows that $c_i \in I$ for every i. If $(\overline{a}_1, \dots, \overline{a}_r)$ lies in the kernel, then $\sum_i a_i f_i \in I^2$, so $\sum_i a_i f_i = \sum_i b_i f_i$ for some $b_i \in I$. The observation above applied to $\sum_i (a_i - b_i) f_i$, shows that $a_i - b_i \in I$ for every i. This means that $\overline{a_i} = 0$ for all i so φ is injective. □

Corollary 16.20. Let $i\colon Y \hookrightarrow X$ be a regular embedding of locally Noetherian schemes of codimension r, and let $\mathcal{I}$ be its ideal sheaf. Then the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r on Y.

Proof The assertion is local on Y. Near any point of Y, choose an affine neighborhood $U = \operatorname{Spec} A \subset X$ on which $\mathcal{I}$ is defined by an ideal $I = (f_1, \dots, f_r)$ generated by a regular sequence. On $Y \cap U$, the conormal sheaf is the sheaf associated to I/I^2, which is free of rank r over A/I by Proposition 16.19. □

For a regular embedding $Y \hookrightarrow X$, the locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the *conormal sheaf* of Y in X. Its dual is called the *normal sheaf*:

$$N_Y = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Both have rank equal to the codimension of Y in X. Being locally free, they are also referred to as the conormal and the normal bundle.

16.4 Zero sets of sections

Let $\mathcal{E}$ be an locally free sheaf of rank r on a scheme X and let $s \in \Gamma(X, \mathcal{E})$ be a global section. We define the *zero set of* s by

$$V(s) = \{P \in X \mid s(P) = 0 \text{ in } \mathcal{E}(P)\}.$$

Equivalently, $V(s)$ consists of the points $P \in X$ such that $s_P \in \mathfrak{m}_P \mathcal{E}_P$.

The zero set $V(s)$ carries a canonical structure as a closed subscheme of X, which we will also denote by $V(s)$. The quickest way to define it is by specifying the ideal sheaf. Identifying $\Gamma(X, \mathcal{E}) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{E})$, we may regard s as a $\mathcal{O}_X$-linear map $s\colon \mathcal{O}_X \to \mathcal{E}$. Dualizing, we obtain a map of $\mathcal{O}_X$-modules

$$s^\vee : \mathcal{E}^\vee \longrightarrow \mathcal{O}_X, \tag{16.6}$$

whose image is a quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. The *zero scheme of* s of s is the closed subscheme defined by this ideal sheaf.

The zero scheme has a more explicit description locally. Let $U = \operatorname{Spec} A$ be an open affine subset such that $\mathcal{E}|_U \simeq \mathcal{O}_X^r|_U$ and let $e_1, \dots, e_r$ be a basis of $\mathcal{E}|_U$ with dual basis $e_1^\vee, \dots, e_r^\vee$ for $\mathcal{E}^\vee|_U$. Then we may uniquely write

$$s|_U = f_1 e_1 + \cdots + f_r e_r$$

for some $f_1, \dots, f_r \in A$. Moreover, $s^\vee(e_i^\vee) = f_i$, so over U, $s^\vee|_U$ is the tilde of the map $A^r \to A$ which sends $e_i^\vee$ to f_i. Therefore, $\mathcal{I}|_U = (f_1, \dots, f_r)^\sim$ and $V(s) \cap U$ is the closed subscheme cut out by the ideal $(f_1, \dots, f_r)$. Note that $(f_1, \dots, f_r)$ depends on the choice of basis, but the ideal they generate does not. From this, we also see that the underlying set of the zero scheme is the zero set.

Proposition 16.21. Let X be an integral scheme and let $s \in \Gamma(X, \mathcal{E})$ be a global section of locally free sheaf $\mathcal{E}$ of rank r. The zero scheme $V(s)$ has codimension at most r in X.

Proof It suffices to see that $V(s) \cap U$ has codimension at most r for an affine open set $U = \operatorname{Spec} A$ where $\mathcal{E}|_U \simeq \mathcal{O}_U^r$. In that case, s corresponds to an r-tuple $(f_1, \dots, f_r) \in A^r$ as above, and the statement follows from Krull's Principal Ideal theorem. □

In the special case when $\mathcal{E}$ has rank 1, then the map $s^\vee : \mathcal{E}^\vee \to \mathcal{O}_X$ is given by multiplication by s. In particular, if X is integral, this map is injective and identifies $\mathcal{E}^\vee$ with the ideal sheaf $\mathcal{I}$ of $V(s)$. The ideal sheaf sequence takes the form

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_{V(s)} \longrightarrow 0 \tag{16.7}$$

where $\iota\colon V(s) \to X$ is the inclusion.

Example 16.22. The monomial $s = x_0^2 x_1$ defines a global section of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k}(3)$. The zero scheme $V(s)$ is non-reduced, and the underlying zero set consists of two points, $(0 : 1)$ and $(1 : 0)$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 16.4.1. Let $f\colon Y \to X$ be a morphism of schemes and let $s \in \Gamma(X, \mathcal{E})$. Show that $V(f^*s) = f^{-1}(V(s))$ as closed subschemes of Y, where the right-hand side is the scheme-theoretic inverse image of Exercise 7.9.10.

By Proposition **??** the codimension of $V(s)$ is at most r, but it may be smaller. For instance, the zero scheme may be empty, and it may have components of smaller codimension. When the codimension is exactly r and the ambient scheme is regular, the r local equations are as independent as they can be, and the closed embedding $V(s) \hookrightarrow X$ is a regular embedding.

Proposition 16.23. Let X be a regular Noetherian scheme, let $\mathcal{E}$ be a locally free sheaf of rank r on X, and let $s \in \Gamma(X, \mathcal{E})$. If $Y = V(s)$ has pure codimension r in X, then $Y \hookrightarrow X$ is a regular embedding of codimension r, and

$$\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{E}^\vee|_Y, \qquad N_Y \simeq \mathcal{E}|_Y.$$

Proof To check that the embedding is regular, we may assume that $X = \operatorname{Spec} A$ is affine, $\mathcal{E}|_U = \mathcal{O}_U^r$ is trivial and the ideal sheaf $\mathcal{I}$ corresponds to the ideal $I = (f_1, \dots, f_r)$ in A. Let $P \in U$ be a point, corresponding to a prime $\mathfrak{p} \subset A$. Let $\mathfrak{q} \supset I$ be a minimal prime. By assumption, $\operatorname{ht} \mathfrak{q} = r$. A regular local ring is Cohen–Macaulay, hence catenary and equidimensional, so $\operatorname{ht} \mathfrak{q} + \dim A/\mathfrak{q} = \dim A$ for every prime $\mathfrak{q} \subset A$ (Eisenbud, 2013, Chapter 18). Hence $\dim A/\mathfrak{q} = \dim A - r$. Proposition 16.18 now shows that $f_1, \dots, f_r$ is a regular sequence. Therefore, $Y \hookrightarrow X$ is a regular embedding of codimension r.

For the conormal sheaf, compose $s^\vee$ with $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$ to obtain a map $\mathcal{E}^\vee \to \mathcal{I}/\mathcal{I}^2$. Viewing $\mathcal{I}/\mathcal{I}^2$ as a sheaf on Y, we get an $\mathcal{O}_Y$-linear map $\mathcal{E}^\vee|_Y \to \mathcal{I}/\mathcal{I}^2$. On an affine where $\mathcal{E}$ is trivial, this sends the dual basis $e_1^\vee, \dots, e_r^\vee$ to the r local equations of Y, which generate $\mathcal{I}/\mathcal{I}^2$, so the map is surjective. is surjective. Both $\mathcal{E}$ and $\mathcal{I}/\mathcal{I}^2$ are locally free of rank r on Y (the latter by Corollary 16.20), and a surjection between locally free sheaves of the same rank is an isomorphism. Dualizing, we also obtain $N_Y \simeq \mathcal{E}|_Y$. □

Example 16.24 (Complete intersections in projective space)**.** Let $Y \subset \mathbb{P}^n_k$ be a complete intersection of codimension r, defined by homogeneous polynomials $F_1, \dots, F_r$ of degrees $d_1, \dots, d_r$. Then Y is the zero scheme of the section $(F_1, \dots, F_r)$ of $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n_k}(d_i)$, so Proposition 16.23 gives

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \simeq \mathcal{O}_Y(-d_1) \oplus \cdots \oplus \mathcal{O}_Y(-d_r), \qquad N_Y \simeq \mathcal{O}_Y(d_1) \oplus \cdots \oplus \mathcal{O}_Y(d_r).$$

△

16.5 Exercises

Exercise 16.5.1. Let k be a field of characteristic $p > 0$ and let $f \in k[x_1, \dots, x_n]$.

a) Show that $\frac{\partial f}{\partial x_i} = 0$ for every i if and only if f lies in the subring $k[x_1^p, \dots, x_n^p]$.
b) Show that if k is perfect, then the condition in (a) holds if and only if $f = g^p$ for some $g \in k[x_1, \dots, x_n]$.
c) Show that (b) fails over $k = \mathbb{F}_p(t)$ by considering $f = x^p - t$.

Exercise 16.5.2 (The Jacobian criterion for plane curves)**.** Let k be an algebraically closed field of characteristic $\neq 2$, and for $f \in k[x, y]$ set $X = \operatorname{Spec} k[x, y]/(f)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) For the cuspidal cubic $f = y^2 - x^3$, show that X is singular exactly at the origin, and that $\dim_k T_P X = 2$ there.
b) For the nodal cubic $f = y^2 - x^2(x+1)$, show that X is again singular exactly at the origin.
c) For $f = y^2 - x^3 + x$, show that X is regular at every closed point.

Exercise 16.5.3. Let $S = \mathbb{N}a_1 + \cdots + \mathbb{N}a_r \subset \mathbb{Z}^n$ and $A = k[x^{a_1}, \ldots, x^{a_r}]$. Show that $\overline{A}$ is the k-algebra generated by monomials in $\bar{S} = \mathbb{Z}S \cap \mathbb{R}_{\geqslant 0}S$. Compute $\bar{R}$ for $R = k[x^4, x^3y, xy^3, y^4]$.

Exercise 16.5.4 (Multiplicity of a hypersurface)**.** Let k be an algebraically closed field, let $f \in R = k[x_1, \ldots, x_n]$ be a nonconstant polynomial vanishing at the origin P, and let $X = \operatorname{Spec} R/(f)$. Write $f = f_m + f_{m+1} + \ldots$ as a sum of homogeneous polynomials, where $f_m \neq 0$. The integer $m \geqslant 1$ is called the *multiplicity* of X at P, written $\operatorname{mult}_P X$. Let $\mathfrak{m} = (x_1, \ldots, x_n) \subset R$.

a) Show that $\mathfrak{m}^j$ consists of the polynomials all of whose terms have degree at least j, and deduce that m is the largest integer j with $f \in \mathfrak{m}^j$. Conclude that m depends only on the ideal $(f) \subset \mathcal{O}_{\mathbb{A}^n_k, P}$, and not on the chosen equation.
b) Show that $\dim_k T_P X = n - 1$ if $m = 1$, and that $\dim_k T_P X = n$ if $m \geqslant 2$.
c) Since $\dim \mathcal{O}_{X,P} = n - 1$, deduce that X is regular at P if and only if $m = 1$, and that in this case $T_P X$ is the hyperplane $f_1 = 0$.
d) Let $n = 2$, assume $\operatorname{char} k \neq 3$, and let $f = x^3 + y^3 - 3xy$. Show that X is singular only at the origin, and that the multiplicity there is 2.
e) Compute the multiplicity at the origin of the Whitney umbrella $f = y^2 + x^3 - x^2z^2$, and of the quadric $f = x_1^2 + \cdots + x_n^2$ when $\operatorname{char} k \neq 2$.
f) Let $g \in k[x_1, \ldots, x_n]$ be homogeneous of degree d. Show that the affine cone $\operatorname{Spec} R/(g)$ of Exercise 16.5.7 has multiplicity d at the origin.

Exercise 16.5.5. Let k be an algebraically closed field of characteristic $\neq 2$. Show that a hyperelliptic curve of the form $y^2 = f(x)$ is regular if and only if f has no repeated roots.

Exercise 16.5.6 (Singular points of a projective hypersurface)**.** Let k be an algebraically closed field, let $f \in k[x_0, \ldots, x_n]$ be homogeneous of degree $d \geqslant 1$, and let $X = V_+(f) \subset \mathbb{P}^n_k$.

a) Prove the Euler relation $\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = d \cdot f$.
b) Show that a closed point $P \in X$ is a singular point of X if and only if $\frac{\partial f}{\partial x_i}(P) = 0$ for every i. HINT: Work in a chart $D_+(x_j)$ with $x_j(P) \neq 0$ and use (a) to handle the j-th partial derivative.
c) Let $f = x_0^d + \cdots + x_n^d$ be the Fermat hypersurface. Show that X is regular if $\operatorname{char} k$ does not divide d, and that every point of X is singular otherwise.

Exercise 16.5.7 (Affine cones)**.** Let k be an algebraically closed field and let $f \in k[x_0, \ldots, x_n]$ be homogeneous of degree $d \geqslant 2$ with $\operatorname{char} k \nmid d$. Let $X = V_+(f) \subset \mathbb{P}^n_k$ and let $C(X) = \operatorname{Spec} k[x_0, \ldots, x_n]/(f) \subset \mathbb{A}^{n+1}_k$ be the affine cone over X.

a) Show that $C(X)$ is singular at the origin, with tangent space all of k^{n+1}.
b) Show that $C(X)$ is regular away from the origin if and only if X is regular.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 16.5.8 (The tangent space of projective space)**.** Let k be a field and let $\pi\colon \mathbb{A}_k^{n+1} - \{0\} \to \mathbb{P}_k^n$ be the projection. Let $P = (a_0, \ldots, a_n) \in \mathbb{A}_k^{n+1} - \{0\}$ be a k-point and let $p = \pi(P) = (a_0 : \cdots : a_n) \in \mathbb{P}_k^n$. Show that the differential $D\pi_P\colon T_P\mathbb{A}_k^{n+1} \to T_p\mathbb{P}_k^n$ is surjective, with kernel the line $k \cdot (a_0, \ldots, a_n) \subset k^{n+1}$. HINT: Work in a standard affine chart $D_+(x_i)$ with $a_i \neq 0$. Deduce that $T_p\mathbb{P}_k^n \simeq k^{n+1}/\big(k \cdot (a_0, \ldots, a_n)\big)$.

Exercise 16.5.9 (The Jacobian ideal cuts out the singular locus)**.** Let k be an algebraically closed field and let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$, and assume that every irreducible component of X has dimension d. Let $J \subset k[x_1, \ldots, x_n]$ be the ideal generated by $f_1, \ldots, f_r$ together with all the $(n-d) \times (n-d)$ minors of the Jacobian matrix $\big(\partial f_i/\partial x_j\big)$.

a) Show that a closed point $P \in X$ is a singular point of X if and only if $P \in V(J)$.
b) Deduce that X is regular if and only if $V(J) = \varnothing$. HINT: Use Corollary 16.7.

Exercise 16.5.10. Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of positive degree and let J be the ideal generated by f and all the partial derivatives $\partial f/\partial x_i$. Show that if $V_+(J) = \varnothing$ and $n \geqslant 2$, then f is irreducible.

Exercise 16.5.11. Let R be a Noetherian ring. Show that $\mathbb{A}_R^n$ and $\mathbb{P}_R^n$ are regular if $\operatorname{Spec} R$ is regular.

Exercise 16.5.12. Let k be a field with $\operatorname{char} k \neq 2$. Find all singular points of the curve $x^2y^2 + x^2z^2 + y^2z^2 = 0$ in $\mathbb{P}_k^2$. What happens if $\operatorname{char} k = 2$?

Exercise 16.5.13. Let k be an algebraically closed field. Compute the singular points of the curve $xy + x^3 + y^3 = 0$ in $\mathbb{A}_k^2$. Compare with Exercise 16.5.4(d).

Exercise 16.5.14. Let k be a field of characteristic $\neq 2, 3$. Compute the singular locus of the *Whitney umbrella* $A = k[x, y, z]/(y^2 + x^3 - x^2z^2)$.

Exercise 16.5.15. Let k be a field of characteristic $\neq 2, 3$ and let $X \subset \mathbb{A}_k^2$ be the curve defined by the equation $y^2 = x^3 + ax + b$, where $a, b \in k$. Show that X is regular if and only if $4a^3 + 27b^2 \neq 0$. What happens if k has characteristic 2 or 3? Compare with Exercise **??**.

Exercise 16.5.16. Assume $\operatorname{char} k \neq 2$ and let $X = V(I) \subset \mathbb{A}_k^3$ where $I = (xz + y^2, x^2 + y^2 + 2z)$.

a) Show that X is irreducible and of dimension 1.
b) Compute the singular points of X. What happens if $\operatorname{char} k = 2$?

Exercise 16.5.17. Determine the singular locus of the scheme $\operatorname{Spec} \mathbb{Z}[x]/(x^2 + 4)$ and compute the tangent space at each singular point.

Exercise 16.5.18. Let k be a field of characteristic $\neq 2, 3$ and let $C \subset \mathbb{A}_k^3$ be the image of the morphism $f\colon \mathbb{A}_k^1 \longrightarrow \mathbb{A}_k^3, t \mapsto (t^3 - 1, t^4 - t, t^5 - t^2)$. Explicitly, C is defined by the ideal $I = (y^2 - xz, x^2y + xy - z^2, x^3 + x^2 - yz)$.

a) Show that the singular locus of C consists of the origin $p = (0, 0, 0)$ and show that T_pC is 3-dimensional.
b) Show that the curve C does not lie on any regular surface in $\mathbb{A}_k^3$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 16.5.19. Let $X \subset \mathbb{P}^3_k$ be the twisted cubic curve, defined by the ideal $I = (y^2 - xz, xw - yz, z^2 - yw)$ in $k[x, y, z, w]$.

a) Use the Jacobian criterion to show that X is regular.
b) Show that X is connected.
c) Deduce that I is a prime ideal.

Exercise 16.5.20. Let k be a field and consider the space of 3×2 matrices over k as an affine space $\mathbb{A}^6_k = \operatorname{Spec} k[x_{11}, x_{12}, \ldots, x_{32}]$, where the variables represent the entries of a generic matrix

$$M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}.$$

Let $W \subset \mathbb{A}^6_k$ be the closed subscheme defined by the ideal generated by all 2×2 minors of M.

a) Show that the k-points of W correspond to the 3×2 matrices with entries in k of rank at most 1.
b) Show that W is an integral scheme.
c) Show that W has a unique singular point at the point corresponding to the zero matrix.
d) Consider the closed subscheme $\widetilde{W} \subset \mathbb{A}^6_k \times_k \mathbb{P}^1_k$ defined by the three equations $t_1 x_{i1} + t_2 x_{i2} = 0$ for $i = 1, 2, 3$, where $(t_1 : t_2)$ are homogeneous coordinates on $\mathbb{P}^1_k$. Let $p_1 \colon \widetilde{W} \to W$ and $p_2 \colon \widetilde{W} \to \mathbb{P}^1_k$ be the induced projections. Show that $\widetilde{W}$ is integral and its k-points correspond to pairs $\big(M, (t_1 : t_2)\big)$ such that $M(t_1, t_2)^{\mathsf{T}} = 0$.
e) Show that for every k-point $y \in \mathbb{P}^1_k$, the fiber $p_2^{-1}(y)$ is a linear affine 3-space in $\mathbb{A}^6_k$.
f) For $i = 1, 2$, define $U_i = p_2^{-1}(D_+(t_i)) \subset \widetilde{W}$. Construct explicit isomorphisms

$$\phi_i \colon U_i \simeq \mathbb{A}^3_k \times D_+(t_i)$$

that commute with the projection to $D_+(t_i)$. Conclude that $\widetilde{W}$ is regular and 4-dimensional.
g) Show that the projection $p_1 \colon \widetilde{W} \to W$ is birational and describe all its fibers.

Exercise 16.5.21. Let $(A, \mathfrak{m})$ be a regular local ring.

a) Assume $\dim A > 0$. Show that there exists an element $x \in \mathfrak{m} - \mathfrak{m}^2$ which is not a zerodivisor on A. HINT: In a Noetherian ring, the zerodivisors are the union of the associated primes.
b) Show that A/xA is a regular local ring of dimension $\dim A - 1$. Deduce by induction on $\dim A$, using Krull's intersection theorem, that A is an integral domain.

Exercise 16.5.22. Let X be a Noetherian scheme and let $f \in \mathcal{O}_X(X)$. Assume that f is a nonzerodivisor in every local ring $\mathcal{O}_{X,x}$. Show that every irreducible component of $V(f)$ has codimension 1 in every irreducible component of X containing it.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 16.5.23 (A scheme with a non-open regular locus)**.** Let k be a field. Consider the subring $A \subset k[X_1, X_2, \dots]$ generated by the squares and cubes of all variables:

$$A = k[X_1^2, X_1^3, X_2^2, X_2^3, \dots]$$

For each $i \geqslant 1$, let $\mathfrak{p}_i = (X_i^2, X_i^3) \subset A$. Define the multiplicative set $S = A - \bigcup_{i=1}^{\infty} \mathfrak{p}_i$ and let $R = S^{-1}A$.

a) Show that R is a 1-dimensional integral domain. Prove that $\operatorname{Spec} R$ consists of the generic point (0) and the infinite set of maximal ideals $\mathfrak{m}_i = \mathfrak{p}_i R$.

b) Show that R is Noetherian. HINT: Each nonzero polynomial in A involves only finitely many variables, so it belongs to only finitely many maximal ideals $\mathfrak{m}_i$.

c) For each i, let K_i be the fraction field of $k[X_j^2, X_j^3 \mid j \neq i]$. Show that

$$R_{\mathfrak{m}_i} \simeq K_i[X_i^2, X_i^3]_{(X_i^2, X_i^3)},$$

the local ring at the origin of a cuspidal curve over K_i. Deduce that the generic point (0) is regular but every closed point is singular. Conclude that the regular locus equals $\{(0)\}$, which is not open.

Exercise 16.5.24. Let k be a field of characteristic 2 containing an element $\alpha \in k$ which is not a square. Let $P \in \mathbb{A}_k^2$ be the closed point corresponding to the maximal ideal $\mathfrak{m} = (y, x^2 + \alpha)$ in $k[x, y]$, and let $X \subset \mathbb{A}_k^2$ be the plane affine curve defined by $f = y^2 - x(x^2 + \alpha)$.

a) Show that the Jacobian matrix of f is nonzero at every point of $X - \{P\}$, and that it vanishes at P.

b) Show that X is nevertheless regular at P, and conclude that X is regular.

c) Let $K = k(\beta)$ be the extension of k obtained by adjoining a square root β of α. Show that the curve $X_K = X \times_k K$ acquires a singular point. Compare with Example 16.15.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

17

Differentials

In Chapter 16 we studied a scheme X near a point P by means of the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ and its dual, the tangent space $T_P X$. We now want to globalize this picture by constructing two sheaves, the sheaf of differentials and the tangent sheaf, with these spaces as their fibers. Unlike their counterparts in differential geometry, these sheaves are defined for an arbitrary morphism, and they remain useful in the presence of singularities.

The construction of $\Omega_{X/k}$ is algebraic, and we construct it first on affine schemes. To a k-algebra A we associate an A-module $\Omega_{A/k}$, the module of Kähler differentials, built to satisfy the formal rules of differentiation: additivity, the product rule, and the vanishing of derivatives on constants. We also give a second description of $\Omega_{A/k}$ as I/I^2, where I is the ideal of the diagonal. This description is closest in spirit to $\mathfrak{m}_P/\mathfrak{m}_P^2$, and it is the one that globalizes well.

Once $\Omega_{X/k}$ has been constructed, it is natural to ask when it is locally free of the expected rank. For a scheme of finite type over a field, this is the condition of *smoothness*, which is closely related to regularity (Chapter **??**). The two conditions are close but not identical: smoothness is a property of X relative to k, whereas regularity is an intrinsic property of the local rings of X. Smooth schemes are always regular, and the converse holds over a perfect field.

Towards the end of the chapter, we discuss differential forms of higher degree $\Omega^p_{X/k}$. After $\mathcal{O}_X$, these are arguably the most important sheaves on a variety, as they let us define useful geometric invariants.

17.1 Differentials

We begin with the affine case, where the sheaf of differentials comes from a module of differentials attached to a ring. Over the next few sections, we let k be a ring and A be a k-algebra. Of course the main case we have in mind is when k is a field.

Definition 17.1. Let A be a k-algebra. Let F be the free A-module on symbols da, one for each $a \in A$. Let $N \subset F$ be the submodule generated by the elements

$$d(a+b) - da - db, \qquad d(ab) - a\,db - b\,da, \qquad d\lambda$$

for all $a, b \in A$ and $\lambda \in k$. The *module of differentials* of A over k is $\Omega_{A/k} = F/N$.

We continue to write da for the class of the generator da in $\Omega_{A/k}$. By construction, there is a map $d\colon A \to \Omega_{A/k}$ sending a to da, and $\Omega_{A/k}$ is generated by the symbols da subject to

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

the rules $d(a+b) = da + db$, $d(ab) = a\,db + b\,da$, and $d(\lambda) = 0$ for $a, b \in A$ and $\lambda \in k$. This means that d obeys exactly the same rules as differentiation in calculus: it is additive, it satisfies the product rule, and it vanishes on the 'constants' coming from k. For example, if $A = k[x]$, then repeated use of the product rule shows that $d(x^m) = mx^{m-1}\,dx$ for every $m \geqslant 1$. Therefore every polynomial $f \in k[x]$ satisfies $df = f'(x)\,dx$. More generally, if $A = k[x_1, \ldots, x_n]$, then the same argument shows that for $f \in A$,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i.$$

The map d is called a *k-linear derivation* (it is k-linear, but not A-linear in general). More generally, if M is an A-module, a k-derivation with values in M is a map $\partial\colon A \to M$ which is k-linear and satisfies the product rule. The A-module $\Omega_{A/k}$ is built so that every such derivation factors as $\partial = \ell \circ d$ for a unique A-linear map $\ell\colon \Omega_{A/k} \to M$. This perspective will not play a big role here, but it is occasionally useful for proving identities involving $\Omega_{A/k}$ (see Exercise 17.9.1).

If $\phi\colon A \to B$ is a map of k-algebras, then, viewing $\Omega_{B/k}$ as an A-module via ϕ, the assignment $da \mapsto d(\phi(a))$ respects the defining relations in the module of differentials and so induces an A-linear map

$$\phi^*\colon \Omega_{A/k} \longrightarrow \Omega_{B/k}. \tag{17.1}$$

Example 17.2. Let $A = k[y]$ and $B = k[x]$, and let $\phi\colon A \to B$ be the k-algebra map with $\phi(y) = x^2$, corresponding to the morphism $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$, $x \mapsto x^2$. By Proposition 17.4, $\Omega_{A/k} = A\,dy$ and $\Omega_{B/k} = B\,dx$. The pullback is computed as in the chain rule:

$$\phi^*(g(y)\,dy) = g(x^2) \cdot \phi^*(dy) = 2x\,g(x^2)\,dx$$

△

Differentials and the diagonal

There is a second construction of $\Omega_{A/k}$, which is a bit more geometric, and closer in spirit to the cotangent space $\mathfrak{m}/\mathfrak{m}^2$. This description will also be useful when we construct the *sheaf* of differentials, as it is better suited for gluing.

Let A be a k-algebra, let $\mu\colon A \otimes_k A \to A$ be the multiplication map $\mu(a \otimes b) = ab$ and let $I = \operatorname{Ker}(\mu) \subset A \otimes_k A$. As we saw in Section 13.1, I is the ideal describing the diagonal embedding

$$\Delta\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec}(A \otimes_k A).$$

The ideal I is generated by the elements of the form $1 \otimes a - a \otimes 1$ where $a \in A$. Indeed, if $\sum_i a_i \otimes b_i \in I$, then $\sum_i a_i b_i = 0$ in A, so

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1).$$

The next result shows that the module of differentials $\Omega_{A/k}$ can be identified with the quotient I/I^2. Just to clarify, there are a priori two A-module structures on $I \subset A \otimes_k A$, coming

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

from the two maps $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$. However, these coincide when passing to the quotient I/I^2, because for any $x \in I$,

$$(a \otimes 1)x - (1 \otimes a)x = (a \otimes 1 - 1 \otimes a) \cdot x \in I^2.$$

So I/I^2 has a canonical structure of an A-module.

Proposition 17.3. There is a canonical isomorphism of A-modules

$$\Omega_{A/k} \simeq I/I^2 \tag{17.2}$$

defined by $da \mapsto 1 \otimes a - a \otimes 1$.

Proof For $a \in A$, define $\delta(a) = 1 \otimes a - a \otimes 1 \in I$. Then it is clear that $\delta(a + b) = \delta(a) + \delta(b)$ and $\delta(\lambda) = 0$. Also, direct substitution shows that $\delta(ab) - a\,\delta(b) - b\,\delta(a) = \delta(a)\delta(b) \in I^2$. Therefore the assignment $da \mapsto \delta(a) \bmod I^2$ respects the defining relations and induces an A-linear map $\alpha\colon \Omega_{A/k} \to I/I^2$.

The inverse is defined as follows. Consider the A-linear map

$$\theta\colon A \otimes_k A \longrightarrow \Omega_{A/k}, \qquad a \otimes b \longmapsto a\,db,$$

where the A-module structure on $A \otimes_k A$ is through the left factor. We claim that θ induces an A-linear map $I/I^2 \to \Omega_{A/k}$. Since I is generated as a left A-module by the elements $\delta(a)$, it is enough to check that $\theta(\delta(a)\delta(b)) = 0$ for all $a, b \in A$. Here $\delta(a)\delta(b) = 1 \otimes ab - a \otimes b - b \otimes a + ab \otimes 1$, and θ sends this to

$$d(ab) - a\,db - b\,da + ab\,d(1) = 0.$$

Hence $I^2 \subset \operatorname{Ker}(\theta|_I)$, and so $\theta|_I$ induces an A-linear map $\beta\colon I/I^2 \to \Omega_{A/k}$.

Now $(\beta \circ \alpha)(da) = da$ for every generator da, so $\beta \circ \alpha = \mathrm{id}$. Conversely, I/I^2 is generated by the classes of the $\delta(a)$ and on these generators, we have $(\alpha \circ \beta)(\delta(a) \bmod I^2) = \delta(a) \bmod I^2$. So $\alpha \circ \beta = \mathrm{id}$ as well. □

So we have two descriptions of the same module $\Omega_{A/k}$. We can use either point of view, depending on which is more convenient.

Proposition 17.4 (Polynomial rings). For $A = k[x_1, \ldots, x_n]$, we have

$$\Omega_{A/k} \simeq A\,dx_1 \oplus \cdots \oplus A\,dx_n.$$

Proof By Proposition 17.3, we may identify $\Omega_{A/k}$ with I/I^2 where I is the diagonal ideal. We identify $A \otimes_k A \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by sending $x_i \otimes 1$ to x_i and $1 \otimes x_i$ to y_i. Under this identification, the multiplication map is evaluation $y_i = x_i$. Moreover, the diagonal ideal is given by $I = (y_1 - x_1, \ldots, y_n - x_n)$ and the classes of $y_i - x_i$ form a free basis of I/I^2. But under the isomorphism of Proposition 17.3, the element dx_i corresponds exactly to the class of $y_i - x_i$, so $\Omega_{A/k} \simeq A\,dx_1 \oplus \cdots \oplus A\,dx_n$. □

This polynomial ring example shows why the diagonal map is a natural thing to look at. If $A = k[x_1, \ldots, x_n]$, then $A \otimes_k A \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and the diagonal ideal is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

generated by $x_i - y_i$. Modulo the square of that ideal, one has

$$f(x_1, \dots, x_n) - f(y_1, \dots, y_n) \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(y_1, \dots, y_n)(x_i - y_i).$$

So I/I^2 remembers exactly the first-order change of f along the diagonal and forgets the terms lying in I^2.

Proposition 17.5 (Conormal sequence). Let R be a k-algebra, let $J \subset R$ be an ideal, and set $A = R/J$. Then there is an exact sequence of A-modules

$$J/J^2 \xrightarrow{\delta} \Omega_{R/k} \otimes_R A \longrightarrow \Omega_{A/k} \longrightarrow 0,$$

where $\delta(\overline{j}) = dj \otimes 1$.

Proof Define the A-linear map

$$\varphi\colon \Omega_{R/k} \otimes_R A \longrightarrow \Omega_{A/k}, \qquad dr \otimes 1 \longmapsto d\overline{r}.$$

This map is surjective, because $\Omega_{A/k}$ is generated by the differentials $d\overline{a}$ and every element of A has a lift to R.

If $j \in J$, then $\varphi(dj \otimes 1) = d\overline{j} = d0 = 0$. Also, if $j, j' \in J$, then in $\Omega_{R/k} \otimes_R A$ we have $d(jj') \otimes 1 = j\,dj' \otimes 1 + j'\,dj \otimes 1 = 0$, because j and j' act as 0 on A. Hence the map $J \to \Omega_{R/k} \otimes_R A$, $j \mapsto dj \otimes 1$, sends J^2 to 0. Moreover, for $r \in R$ and $j \in J$ one has $d(rj) \otimes 1 = r\,dj \otimes 1 + j\,dr \otimes 1 = r\,dj \otimes 1$, so this map is R-linear modulo J and therefore induces an A-linear map $\delta\colon J/J^2 \to \Omega_{R/k} \otimes_R A$. Hence φ factors through a surjective map

$$\overline{\varphi}\colon \big(\Omega_{R/k} \otimes_R A\big)/\operatorname{im}(\delta) \longrightarrow \Omega_{A/k}.$$

We construct the inverse as follows. Since $\Omega_{A/k}$ is defined by generators $d\overline{a}$ subject to the usual relations, it suffices to define the map on each generator and check that the relations are respected. For $\overline{a} \in A$, choose a lift $a \in R$ and let

$$\psi(d\overline{a}) = \text{class of } da \otimes 1 \text{ in } \big(\Omega_{R/k} \otimes_R A\big)/\operatorname{im}(\delta).$$

This is independent of the chosen lift, because if $a' = a + j$ with $j \in J$, then $da' \otimes 1 - da \otimes 1 = dj \otimes 1 \in \operatorname{im}(\delta)$. The defining relations also hold for these classes because they already hold in $\Omega_{R/k}$. Therefore ψ extends uniquely to an A-linear map

$$\psi\colon \Omega_{A/k} \longrightarrow \big(\Omega_{R/k} \otimes_R A\big)/\operatorname{im}(\delta).$$

By construction, the maps $\overline{\varphi}$ and ψ are inverse to each other, so $\overline{\varphi}$ is an isomorphism. This gives the desired exact sequence. □

The conormal sequence is the main tool for computing explicit presentations of modules of differentials. The main example is the following.

Let $A = k[x_1, \dots, x_n]/I$ be a finitely generated k-algebra, where $I = (f_1, \dots, f_r)$. Using the identification

$$\Omega_{k[x_1,\dots,x_n]/k} \otimes_{k[x_1,\dots,x_n]} A = \textstyle\bigoplus_{i=1}^{n} A\,dx_i.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

we may rewrite the conormal sequence as

$$I/I^2 \xrightarrow{\delta} \bigoplus_{i=1}^n A\,dx_i \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

As I/I^2 is generated as an A-module by the classes of the $f_1, \dots, f_r$ modulo I^2, there is a surjection $A^r \to I/I^2$ which gives the exact sequence

$$A^r \longrightarrow \bigoplus_{i=1}^n A\,dx_i \longrightarrow \Omega_{A/k} \longrightarrow 0. \tag{17.3}$$

Here the left-most map sends the i-th basis vector e_i to $df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\,dx_j$. This means that it can be identified with the A-linear map $J^t\colon A^r \to A^n$ defined by multiplication by the transpose of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \dots & \frac{\partial f_r}{\partial x_n}. \end{pmatrix}$$

Consequently, $\Omega_{A/k}$ is obtained from the free module on $dx_1, \dots, dx_n$ by modding out by the differentials of the defining equations $f_1, \dots, f_r$. This should be compared with the discussion of the classical tangent space in Section 16.1, where the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ was obtained by adding the linear parts of the defining equations.

In the following examples, k will be a field.

Example 17.6. Let $f(x) \in k[x]$ and set $A = k[x]/(f)$. Then

$$\Omega_{A/k} = \frac{A\,dx}{f'(x)\,dx}. \tag{17.4}$$

This module is supported on the points $P \in \operatorname{Spec} A$ where $f'(P) = 0$ in $\kappa(P)$. △

Example 17.7 (Plane curves)**.** Let $A = k[x,y]/(f)$ for $f \in k[x,y]$ and let $X = \operatorname{Spec} A$. Writing $\partial_x f$ and $\partial_y f$ for the partial derivatives of f, we have

$$\Omega_{A/k} = \frac{A\,dx \oplus A\,dy}{(\partial_x f\,dx + \partial_y f\,dy)}.$$

Assume that $V(\partial_x f, \partial_y f) = \varnothing$, so that $X = D(\partial_x f) \cup D(\partial_y f)$. Then $\Omega_{A/k}$ is locally free of rank 1. Indeed, on $D(\partial_x f)$ we may use the relation $\partial_x f\,dx + \partial_y f\,dy = 0$ to eliminate dx, so $(\Omega_{A/k})_{\partial_x f}$ is free with generator $dy/\partial_x f$, and likewise $(\Omega_{A/k})_{\partial_y f}$ is free with generator $-dx/\partial_y f$. On the overlap the same relation shows that the two generators differ by a unit, so they glue. △

Example 17.8 (The nodal cubic)**.** Let k be a field of characteristic $\neq 2, 3$ and let $A = k[x,y]/(y^2 - x^2(x+1))$. By the exact sequence (17.3),

$$\Omega_{A/k} = \frac{A\,dx \oplus A\,dy}{(2y\,dy - (3x^2 + 2x)\,dx)}.$$

Let $P = (0,0) \in \operatorname{Spec} A$ be the origin. Over the open set $\operatorname{Spec}(A) - \{P\}$ the module $\Omega_{A/k}$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is locally free of rank 1, generated by dx on $D(y)$, and by dy on $D(3x^2 + 2x)$. At P the terms $2y$ and $3x^2 + 2x$ both vanish, so

$$\Omega_{A/k} \otimes_A \kappa(P) \simeq k\,dx \oplus k\,dy,$$

which is 2-dimensional. In particular, $\Omega_{A/k}$ is not locally free at P. △

Proposition 17.9. Let k be a ring and let A be a k-algebra.
(i) (Localization): For a multiplicative set $S \subset A$, there is a canonical isomorphism $\Omega_{S^{-1}A/k} \simeq S^{-1}\Omega_{A/k}$
(ii) (Base change): Let B be a k-algebra and let $C = A \otimes_k B$. Then there is a canonical isomorphism $\Omega_{C/B} \simeq \Omega_{A/k} \otimes_A C$.

Proof See Exercises 17.9.6 and 17.9.7. □

17.2 The sheaf of differentials

Let X be a scheme over a ring k. By Proposition 17.9, the definition of the modules of differentials is compatible with localization. Using this fact, we define the *sheaf of Kähler differentials* to be the sheaf $\Omega_{X/k}$ obtained by gluing together $\widetilde{\Omega_{A/k}}$ for each open affine $U = \operatorname{Spec} A \subset X$.

Example 17.10. For the affine n-space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ over a ring k, we have

$$\Omega_{\mathbb{A}^n_k/k} = \widetilde{\Omega_{k[x_1,\ldots,x_n]/k}} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{A}^n_k}\,dx_i.$$

△

Using the diagonal definition of $\Omega_{A/k}$ there is also the following alternative definition, which works for an arbitrary separated morphism $X \to S$. Let $\mathcal{I}$ be the ideal sheaf defining the diagonal embedding $\Delta\colon X \to X \times_S X$. We define

$$\Omega_{X/S} = \Delta^*(\mathcal{I}/\mathcal{I}^2).$$

This agrees with the previous when $S = \operatorname{Spec} k$. More generally, if $U = \operatorname{Spec} B \subset X$ is an affine open subset that maps into an affine open $V = \operatorname{Spec} A \subset S$, then the restriction of the diagonal to U factors as $\Delta|_U\colon U \to U \times_S U \simeq U \times_V U = \operatorname{Spec}(B \otimes_A B)$. This is the closed embedding corresponding to the ideal $I = \operatorname{Ker}(B \otimes_A B \to B)$ (see Section 7.3). Moreover, $\mathcal{I}/\mathcal{I}^2$ restricts to $\widetilde{I/I^2}$ on $U \times_V U$. Therefore

$$\Omega_{X/S}|_U = \Delta^*(\mathcal{I}/\mathcal{I}^2)|_U \simeq \widetilde{I/I^2} \simeq \widetilde{\Omega_{B/A}}.$$

Hence the sheaf $\Omega_{X/S}$ is built out of the various $\Omega_{B/A}$ on affine opens. In particular, $\Omega_{X/S}$ is quasi-coherent. If $X \to S$ is of finite type, then $\Omega_{X/S}$ is of finite type, since each $\Omega_{B/A}$ is a finitely generated B-module.

Example 17.11. Consider the projective line $\mathbb{P}^1_k$ over a field k, with its standard affine cover $U_0 = \operatorname{Spec} k[x]$ and $U_1 = \operatorname{Spec} k[u]$, where $u = x^{-1}$ on the overlap. On the two charts we have $\Omega_{U_0/k} = \mathcal{O}_{U_0}\,dx$ and $\Omega_{U_1/k} = \mathcal{O}_{U_1}\,du$. Since $u = x^{-1}$, we get $du = -x^{-2}\,dx$ on

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$U_0 \cap U_1$. So the local generators are glued by the transition function $-x^{-2}$, which is the same as the transition function for $\mathcal{O}_{\mathbb{P}^1_k}(-2)$ up to a unit. It follows that $\Omega_{\mathbb{P}^1_k/k} \simeq \mathcal{O}_{\mathbb{P}^1_k}(-2)$. △

Example 17.12. Consider the projective plane $\mathbb{P}^2_k$ over a field k, with affine cover $U_0 = \operatorname{Spec} k[x,y]$, $U_1 = \operatorname{Spec} k[u,v]$, and $U_2 = \operatorname{Spec} k[s,t]$, where $u = x^{-1}$, $v = x^{-1}y$, $s = y^{-1}$, and $t = xy^{-1}$. On these charts we have

$$\Omega_{U_0/k} = \mathcal{O}_{U_0}\,dx \oplus \mathcal{O}_{U_0}\,dy, \quad \Omega_{U_1/k} = \mathcal{O}_{U_1}\,du \oplus \mathcal{O}_{U_1}\,dv, \quad \Omega_{U_2/k} = \mathcal{O}_{U_2}\,ds \oplus \mathcal{O}_{U_2}\,dt.$$

On the overlaps, the generators are related by the above relations. For example, on $U_0 \cap U_1$ we have $du = -x^{-2}dx$ and $dv = -x^{-2}y\,dx + x^{-1}dy$, so the transition matrix from (du, dv) to (dx, dy) is

$$\begin{pmatrix} -x^{-2} & 0 \\ -x^{-2}y & x^{-1} \end{pmatrix},$$

which is invertible there. Similar formulas hold on the other overlaps. It follows that $\Omega_{\mathbb{P}^2_k/k}$ is a locally free sheaf of rank 2.

In Section 17.7, we will see a more useful description of the sheaf of differentials on projective space. △

Let $f\colon X \to Y$ be a morphism of k-schemes. The maps ϕ^* of (17.1) glue in the same way, and give a pullback of differentials along f. To see this, let $V = \operatorname{Spec} A \subset Y$ and $U = \operatorname{Spec} B \subset f^{-1}(V)$ be affine opens, and let $\phi\colon A \to B$ be the ring map corresponding to $f|_U$. The construction of ϕ^* only used ϕ together with the generators and relations defining $\Omega_{A/k}$ and $\Omega_{B/k}$, both of which are compatible with localization (Proposition 17.9). Hence the maps $\Omega_{Y/k}(V) \to \Omega_{X/k}(U)$ obtained this way agree on overlaps and glue to a map $\Omega_{Y/k}(V) \to \Omega_{X/k}(f^{-1}(V))$ for every open $V \subset Y$. In particular, for $V = Y$ we obtain the *pullback map on differentials*

$$f^*\colon \Gamma(Y, \Omega_{Y/k}) \longrightarrow \Gamma(X, \Omega_{X/k}). \tag{17.5}$$

The construction can also be packaged as a map of $\mathcal{O}_Y$-modules $\Omega_{Y/k} \to f_*\Omega_{X/k}$ or, equivalently, as an $\mathcal{O}_X$-linear map $f^*\Omega_{Y/k} \to \Omega_{X/k}$, which on the affine opens above is the map $\Omega_{A/k} \otimes_A B \to \Omega_{B/k}$ sending $\omega \otimes b$ to $b\,\phi^*(\omega)$ (see Exercise 17.9.5).

Proposition 17.13 (Conormal sequence). Let k be a ring, let $i\colon Y \hookrightarrow X$ be a closed embedding of k-schemes, and let $\mathcal{I} \subset \mathcal{O}_X$ be its ideal sheaf. Then there is an exact sequence of $\mathcal{O}_Y$-modules

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Proof If X and Y are affine, this is just the sheaf version of the conormal sequence in Proposition 17.5. The general case follows because the sheaves and the maps involved are compatible with localization, so they glue to a map of sheaves. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

17.3 Differentials and cotangent spaces

Let X be a scheme of finite type over a field k and let $P \in X$ be a point. In this section, we relate the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ to the fiber of the sheaf of differentials

$$\Omega_{X/k}(P) = \Omega_{X/k} \otimes_{\mathcal{O}_{X,P}} \kappa(P).$$

Consider first the affine situation, $X = \operatorname{Spec} A$ where $A = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$, and let $J = \big(\partial f_i/\partial x_j\big)$ be the Jacobian matrix. Then the conormal sequence tensored by $\kappa(P)$ gives an exact sequence

$$\kappa(P)^{\oplus r} \xrightarrow{J(P)^t} \kappa(P)^{\oplus n} \longrightarrow \Omega_{X/k}(P) \longrightarrow 0. \tag{17.6}$$

In particular, when $P \in X(k)$ is a k-point, this shows that $\Omega_{X/k}(P)$ is identified with the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ as discussed in Section 16.1.

For a general point P, the relation between $\Omega_{X/k}(P)$ and the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a little bit more subtle. There is always an exact sequence relating the two. From Proposition 17.5 applied to the surjection k-algebras $\mathcal{O}_{X,P} \to \kappa(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$, we obtain

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \longrightarrow \Omega_{X/k}(P) \longrightarrow \Omega_{\kappa(P)/k} \longrightarrow 0.$$

The following result says that if P is closed and $\kappa(P)/k$ is separable, the left-most map is an isomorphism.

Proposition 17.14. Let X be a scheme of finite type over a field k, and let $P \in X$ be a closed point so that $\kappa(P)/k$ is separable. Then $\Omega_{\kappa(P)/k} = 0$ and the natural map $f \mapsto df \otimes 1$ defines an isomorphism of $\kappa(P)$-vector spaces.

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \longrightarrow \Omega_{X/k,P} \otimes_{\mathcal{O}_{X,P}} \kappa(P). \tag{17.7}$$

Proof This is a local statement, so we may assume $X = \operatorname{Spec} A$, where $A = R/I$ and $R = k[x_1, \dots, x_n]$. Write $\mathfrak{m}_A \subset A$ and $\mathfrak{m}_R$ for the maximal ideals corresponding to P with residue field $\kappa = A/\mathfrak{m}_A = R/\mathfrak{m}_R$.

We first prove the statement for R. The conormal sequence associated to the surjection $R \to \kappa$ with kernel $\mathfrak{m}_R$ takes the form

$$\mathfrak{m}_R/\mathfrak{m}_R^2 \xrightarrow{\beta} \Omega_{R/k} \otimes_R \kappa \longrightarrow \Omega_{\kappa/k} \longrightarrow 0. \tag{17.8}$$

Claim: When κ/k is separable, then $\Omega_{\kappa/k} = 0$.

Since the field extension is finite and separable, the Primitive Element Theorem implies that there exists $\alpha \in \kappa$ such that $\kappa = k(\alpha)$. Let $f(t) \in k[t]$ be the minimal polynomial of α, so that $\kappa \simeq k[t]/(f(t))$. As κ/k is separable, $f'(t)$ is invertible in κ, so by (17.4), we obtain

$$\Omega_{\kappa/k} = \frac{\kappa\, dt}{f'(t)\, dt} = 0.$$

Now, as $\Omega_{\kappa/k} = 0$, the sequence (17.8) shows that the map β is surjective. As R is regular, $\mathfrak{m}_R/\mathfrak{m}_R^2$ is a κ-vector space of dimension n. Also $\Omega_{R/k} \otimes_R \kappa$ has dimension n because $\Omega_{R/k}$ is free of rank n. Since β is a surjective map of κ-vector spaces of the same dimension, it is an isomorphism.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Next we prove the statement for $A = R/I$. The conormal sequence for $R \to A$ is

$$I/I^2 \longrightarrow \Omega_{R/k} \otimes_R A \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

Tensoring over A with κ gives a right exact sequence

$$(I/I^2) \otimes_A \kappa \longrightarrow \Omega_{R/k} \otimes_R \kappa \longrightarrow \Omega_{A/k} \otimes_A \kappa \longrightarrow 0, \tag{17.9}$$

where $f \bmod I^2$ maps to $df \otimes 1$. On the other hand, we have the sequence

$$I/I^2 \longrightarrow \mathfrak{m}_R/\mathfrak{m}_R^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow 0, \tag{17.10}$$

where $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is the quotient map to $\mathfrak{m}_A/\mathfrak{m}_A^2 = \mathfrak{m}_R/(\mathfrak{m}_R^2 + I)$.

These fit into a commutative diagram

$$\begin{array}{ccccccc} I/I^2 & \longrightarrow & \mathfrak{m}_R/\mathfrak{m}_R^2 & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ I/I^2 \otimes_A \kappa & \longrightarrow & \Omega_{R/k} \otimes_R \kappa & \longrightarrow & \Omega_{A/k} \otimes_A \kappa & \longrightarrow & 0 \end{array}$$

Here the left-most vertical map sends $\bar{f}$ to $\bar{f} \otimes 1$. Since this map is surjective and β is an isomorphism, the 5-lemma implies that the map γ is also an isomorphism, as required. □

17.4 Smoothness over a field

Regularity asks that the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ have the expected dimension. We now impose the corresponding condition on $\Omega_{X/k}$.

Throughout this section, k is a field and X is an integral scheme of finite type over k of dimension d. We will see in Proposition 17.15 that $\dim_{\kappa(P)} \Omega_{X/k}(P) \geqslant d$ at every point $P \in X$, so d is the smallest rank that $\Omega_{X/k}$ can have. We say that X is *smooth* over k at a point $P \in X$ if this smallest value is attained, that is, if $\Omega_{X/k,P}$ is a free $\mathcal{O}_{X,P}$-module of rank d. The *smooth locus* is the set $X_{\mathrm{sm}} = \{\, P \in X \mid X \text{ is smooth at } P \,\}$, and X is *smooth over* k if $X_{\mathrm{sm}} = X$.

Since $\Omega_{X/k}$ is coherent, freeness of a stalk spreads to a neighbourhood, so X is smooth at P if and only if $\Omega_{X/k}$ is locally free of rank d near P. In particular, the smooth locus X_{sm} is open, and X is smooth over k if and only if $\Omega_{X/k}$ is locally free of rank d. If X is a finite type k-scheme that is not integral, one calls $P \in X$ a smooth point over k if some open neighbourhood of P is integral and smooth in the above sense.

Proposition 17.15. Let $P \in X$. Then $\dim_{\kappa(P)} \Omega_{X/k}(P) \geqslant d$, and equality holds if and only if X is smooth at P.

Proof Let ξ be the generic point of X and let $K = K(X)$, so that $\Omega_{X/k,\xi} = \Omega_{K/k}$ and the rank of $\Omega_{X/k}$ equals $\dim_K \Omega_{K/k}$.

We first show that $\dim_K \Omega_{K/k} \geqslant d$. Choose a nonempty affine open $\operatorname{Spec} A \subset X$ and a presentation $A \simeq R/\mathfrak{p}$, where $R = k[x_1, \ldots, x_n]$ and $\mathfrak{p} \subset R$ is prime, and set $S = R_{\mathfrak{p}}$ and $\mathfrak{m} = \mathfrak{p}S$. By Proposition 16.10, the ring S is regular local, and its residue field is K. The

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

conormal sequence for $S \to S/\mathfrak{m} = K$ reads

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{S/k} \otimes_S K \longrightarrow \Omega_{K/k} \longrightarrow 0.$$

Here $\Omega_{S/k} \otimes_S K \simeq K^{\oplus n}$ by Proposition 17.4 and localization, while $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = \dim S = \operatorname{ht}(\mathfrak{p})$ because S is regular. Hence $\dim_K \Omega_{K/k} \geqslant n - \operatorname{ht}(\mathfrak{p})$, and this is at least d because $\operatorname{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) \leqslant \dim R = n$.

Every point of X is a specialization of ξ, so Theorem **??** gives $\operatorname{rank}_P(\Omega_{X/k}) \geqslant \dim_K \Omega_{K/k} \geqslant d$, which is the asserted inequality.

If X is smooth at P, then $\Omega_{X/k,P}$ is a free $\mathcal{O}_{X,P}$-module of rank d, so its fiber at P has dimension d. Conversely, assume that $\operatorname{rank}_P(\Omega_{X/k}) = d$. The rank r of $\Omega_{X/k}$ satisfies $r \leqslant d$ by Theorem **??** and $r \geqslant d$ by the bound above, so $r = d = \operatorname{rank}_P(\Omega_{X/k})$. The same theorem then shows that $\Omega_{X/k}$ is locally free of rank d near P, and hence that X is smooth at P. □

Example 17.16. The sheaf $\Omega_{X/k}$ can be locally free (of the wrong rank) even if X is not smooth. For instance, consider $k = \mathbb{F}_p(t)$ and $A = k[x]/(x^p - t)$. Then A is a field (isomorphic to $\mathbb{F}_p(t^{1/p})$), but since $\frac{\partial}{\partial x}(x^p - t) = p \cdot x^{p-1} = 0$, we have

$$\Omega_{A/k} \simeq A\,dx$$

which is free of rank 1. This also shows that the inequality in Proposition 17.15 may be strict. △

We now compare the two conditions. Smoothness refers to the morphism $X \to \operatorname{Spec} k$, whereas regularity is a property of the local rings of X alone.

Proposition 17.17. Let $P \in X$ and set $e = \dim \overline{\{P\}}$, so that $\dim \mathcal{O}_{X,P} = d - e$ by Proposition 8.50.

(i) If P is a closed point at which X is smooth, then $\mathcal{O}_{X,P}$ is a regular local ring.

(ii) If $\mathcal{O}_{X,P}$ is regular and $\dim_{\kappa(P)} \Omega_{\kappa(P)/k} = e$, then X is smooth at P.

Proof (i): Choose an affine neighbourhood $\operatorname{Spec} A$ of P and write $A \simeq R/(f_1, \dots, f_r)$ with $R = k[x_1, \dots, x_n]$. Let $\mathfrak{n} \subset R$ be the maximal ideal corresponding to P, and put $I = (f_1, \dots, f_r)R_{\mathfrak{n}}$, so that $\mathcal{O}_{X,P} \simeq R_{\mathfrak{n}}/I$. By Proposition 16.10 the ring $R_{\mathfrak{n}}$ is regular, and it has dimension n because $\mathfrak{n}$ is maximal, so $\dim_{\kappa(P)}(\mathfrak{n}/\mathfrak{n}^2) = n$. Since $\mathfrak{m}_P/\mathfrak{m}_P^2 = \mathfrak{n}/(I + \mathfrak{n}^2)$, this gives

$$\dim_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2) = n - \dim_{\kappa(P)}\big((I + \mathfrak{n}^2)/\mathfrak{n}^2\big). \tag{17.11}$$

Assume now that P is smooth, so that $\dim_{\kappa(P)} \Omega_{X/k}(P) = d$. By the exact sequence (17.6), the span of $df_1(P), \dots, df_r(P)$ in $\kappa(P)^{\oplus n}$ has dimension $n - d$, and after reordering we may assume that $df_1(P), \dots, df_{n-d}(P)$ are linearly independent. The assignment $f \mapsto df(P)$ induces a $\kappa(P)$-linear map

$$(I + \mathfrak{n}^2)/\mathfrak{n}^2 \longrightarrow \Omega_{R/k}(P) \simeq \kappa(P)^{\oplus n}.$$

Indeed, it kills $\mathfrak{n}^2$, because $f \in \mathfrak{n}^2$ gives $df \in \mathfrak{n}\,\Omega_{R_{\mathfrak{n}}/k}$ and hence $df(P) = 0$; and it is $\kappa(P)$-linear, because $d(af)(P) = a(P)\,df(P)$ for $a \in R_{\mathfrak{n}}$ and $f \in I$. This map

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

carries the classes of $f_1, \dots, f_{n-d}$ to $df_1(P), \dots, df_{n-d}(P)$, so those classes are linearly independent in $(I + \mathfrak{n}^2)/\mathfrak{n}^2$. Therefore $\dim_{\kappa(P)}\big((I + \mathfrak{n}^2)/\mathfrak{n}^2\big) \geqslant n - d$, and (17.11) gives $\dim_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2) \leqslant d$. The reverse inequality is Proposition 16.3, so $\dim_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2) = d = \dim \mathcal{O}_{X,P}$ and $\mathcal{O}_{X,P}$ is regular.

(ii): Applying Proposition 17.5 to the surjection $\mathcal{O}_{X,P} \to \kappa(P)$ gives an exact sequence

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \longrightarrow \Omega_{X/k}(P) \longrightarrow \Omega_{\kappa(P)/k} \longrightarrow 0.$$

Since $\mathcal{O}_{X,P}$ is regular we have $\dim_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2) = \dim \mathcal{O}_{X,P} = d-e$, and $\dim_{\kappa(P)} \Omega_{\kappa(P)/k} = e$ by assumption. The sequence therefore gives $\dim_{\kappa(P)} \Omega_{X/k}(P) \leqslant (d-e) + e = d$, and the reverse inequality is Proposition 17.15. Hence equality holds and X is smooth at P. □

In particular, every smooth scheme is regular. The hypothesis on $\kappa(P)$ in (ii) cannot be dropped: in Example 16.15 the curve X is regular at the closed point P corresponding to $\mathfrak{m} = (x^p + t, y)$ but not smooth there, and indeed $\kappa(P) = \mathbb{F}_p(t^{1/p})$ is inseparable over $k = \mathbb{F}_p(t)$, so $\Omega_{\kappa(P)/k}$ is nonzero while $e = 0$. We will see in Corollary 17.23 that the hypothesis holds at every point when k is perfect.

Proposition 17.18. Let $k \to L$ be a field extension. Then X is smooth over k if and only if X_L is smooth over L.

Proof Let $q\colon X_L \to X$ be the projection. On every affine open $U = \operatorname{Spec} A \subset X$, Proposition 17.9(ii) gives a canonical isomorphism $\Omega_{X_L/L} \simeq q^*\Omega_{X/k}$. By Proposition 11.24, the sheaf $\Omega_{X/k}$ is locally free of rank d if and only if $q^*\Omega_{X/k}$ is. □

Taking $L = \overline{k}$, the algebraic closure, this says that X is smooth over k exactly when $X_{\overline{k}}$ is smooth over $\overline{k}$. Since $\overline{k}$ is algebraically closed, smoothness and regularity coincide there. Putting these together, we see that X is smooth if and only if it is *geometrically regular*, i.e., $X_{\bar{k}}$ is regular.

Theorem 17.19. Let $P \in X$ be a closed point. The following are equivalent:
(i) P is smooth over k.
(ii) Every point of $X_{\overline{k}}$ lying over P is a regular point of $X_{\overline{k}}$.

Proof Let $q\colon X_{\overline{k}} \to X$ be the base-change morphism, so that $\Omega_{X_{\overline{k}}/\overline{k}} \simeq q^*\Omega_{X/k}$ by Proposition 17.9(ii).

Assume (i). After shrinking X we may assume $\Omega_{X/k}$ is locally free of rank d, so $X_{\overline{k}}$ is smooth over $\overline{k}$ by Proposition 17.18. Since $\overline{k}$ is algebraically closed, Proposition 17.17(i) shows that every closed point of $X_{\overline{k}}$ is regular, and hence $X_{\overline{k}}$ is regular by Corollary 16.7. In particular every point lying over P is regular.

Assume (ii). Since P is closed, the field $\kappa(P)$ is finite over k, so the fiber $q^{-1}(P) = \operatorname{Spec}(\kappa(P) \otimes_k \overline{k})$ is a finite $\overline{k}$-scheme. Every point $\overline{P}$ lying over P is therefore a closed point of $X_{\overline{k}}$ with $\kappa(\overline{P}) = \overline{k}$. Fix one; by hypothesis $\mathcal{O}_{X_{\overline{k}},\overline{P}}$ is regular, and its dimension is $\dim \mathcal{O}_{X,P} = d$, because P is closed and base change by a field extension does not change the dimension of the local ring. As $\kappa(\overline{P}) = \overline{k}$ we have $\Omega_{\kappa(\overline{P})/\overline{k}} = 0$, so Proposition 17.17(ii) gives that $\Omega_{X_{\overline{k}}/\overline{k},\overline{P}}$ is free of rank d. Through the isomorphism above, the pointwise form of Proposition 11.24 then shows that $\Omega_{X/k,P}$ is free of rank d, so P is smooth over k. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The smooth locus

Smoothness, like regularity, can be read off from a Jacobian matrix.

Proposition 17.20 (Jacobian criterion for smoothness). Let $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ be integral of dimension d, and let $J = \big(\partial f_i/\partial x_j\big)$ be the Jacobian matrix. Then

$$\operatorname{rank} J(Q) \leqslant n - d \qquad \text{for every } Q \in X,$$

and X is smooth at Q if and only if equality holds, that is, if and only if some $(n-d) \times (n-d)$ minor of J is nonzero in $\kappa(Q)$. Consequently $X - X_{\mathrm{sm}}$ is the closed subset of X cut out by the ideal generated by the $(n-d) \times (n-d)$ minors of J.

Proof By the exact sequence (17.6) we have $\dim_{\kappa(Q)} \Omega_{X/k}(Q) = n - \operatorname{rank} J(Q)$, so the first two assertions are Proposition 17.15 rewritten in terms of J. The last one follows because the minors are regular functions on X. □

Note that Proposition 16.13 applied only at k-points, and compared $\operatorname{rank} J(P)$ with the local dimension $\dim \mathcal{O}_{X,P}$, whereas the criterion above applies at every point of X, with the same bound $n - d$ at all of them. The reason is that the fiber of the sheaf $\Omega_{X/k}$ is computed by J at every point, while the identification of $\mathfrak{m}_P/\mathfrak{m}_P^2$ with $\operatorname{Coker} J(P)^t$ in Chapter 16 was available only at k-points.

Example 17.21. If $X = V(f) \subset \mathbb{A}^n_k$ is an integral hypersurface, then $\dim X = n - 1$ and the Jacobian matrix has just one row consisting of all the partial derivatives $\partial f/\partial x_i$. Hence a point $P \in X$ is smooth if and only if at least one partial derivative is nonzero at P. △

Corollary 17.22. Let X be a reduced scheme of finite type over a perfect field k. Then the smooth locus X_{sm} is a dense open subset of X.

Proof The smooth locus is open, so only density has to be proved. For each irreducible component, remove its intersections with the other components. It is enough to find a smooth point in the resulting nonempty open subset. We may therefore assume that X is integral.

Choose a hypersurface $Y = V(f) \subset \mathbb{A}^n_k$ birational to X, with f irreducible. If Y_{sm} were empty, every partial derivative of f would vanish on Y. Since $I(Y) = (f)$, this means that $\partial f/\partial x_i \in (f)$ for every i. But $\deg(\partial f/\partial x_i) < \deg(f)$, so each partial derivative must actually be the zero polynomial. If k has characteristic 0, then this implies that f is a constant, a contradiction. In positive characteristic this makes f a polynomial in $x_1^p, \dots, x_n^p$, and since k is perfect, f must be a p-th power, contradicting the assumption that it is irreducible. So in either case, Y_{sm} is a nonempty open subset of Y. As X and Y are birational, we also obtain that X_{sm} is nonempty, as required. □

Over a perfect field the smooth locus and the regular locus coincide.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Corollary 17.23. Assume that k is perfect and let $P \in X$. Then X is smooth at P if and only if $\mathcal{O}_{X,P}$ is a regular local ring. In particular $X_{\rm sm} = X_{\rm reg}$, so the regular locus is open and $\operatorname{sing}(X)$ is closed, and X is smooth over k if and only if it is regular.

Proof If X is smooth at P, choose an open neighbourhood $U \subset X_{\rm sm}$ of P. Every closed point of U is smooth, hence regular by Proposition 17.17(i), so U is regular by Corollary 16.7, and in particular $\mathcal{O}_{X,P}$ is regular.

Conversely, suppose that $\mathcal{O}_{X,P}$ is regular, and set $e = \dim \overline{\{P\}}$. The residue field $\kappa(P)$ is the function field of the integral scheme $\overline{\{P\}}$, which is of finite type over k of dimension e. Its smooth locus is dense by Corollary 17.22, hence contains its generic point, so Proposition 17.15 gives $\dim_{\kappa(P)} \Omega_{\kappa(P)/k} = e$. Now Proposition 17.17(ii) applies. □

In particular, for a variety over an algebraically closed field, regularity is equivalent to smoothness, and both are equivalent to $\Omega_{X/k}$ being locally free of rank $\dim X$.

Example 17.24. Over a non-perfect field, the smooth locus can even be empty. For instance, if $k = \mathbb{F}_p(t)$ for a prime p and X is the affine scheme defined by the polynomial $f = x^p - ty^p$ in $\mathbb{A}^2_k$, then the Jacobian matrix is the zero matrix, so $X_{\rm sm} = \varnothing$. This is explained by the fact that the base change $X_{\bar{k}}$ is non-reduced, hence singular at every point: if s is a p-th root of t, then $f = (x - sy)^p$. △

To summarize the discussion on smoothness and regularity: Smoothness is a relative notion and depends on the structure morphism $X \to \operatorname{Spec} k$. Regularity, by contrast, is intrinsic to the scheme. It is defined in terms of the local rings $\mathcal{O}_{X,P}$ and makes no reference to a ground field or to any other morphism.

For schemes of finite type over a field k, smoothness is the stronger condition. Every smooth scheme over k is regular, but over an imperfect field a regular scheme need not be smooth. Smoothness is also stable under arbitrary extensions of the ground field, whereas regularity need not be preserved by base change. In light of Proposition 17.19, smoothness over k can be rephrased by saying that $X_{\bar{k}}$ is regular, that is, it is *geometrically regular*.

17.5 The tangent sheaf

Let X be a scheme over a field k. The *tangent sheaf* of X is the dual of the sheaf of differentials,

$$\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

If X is of finite type over k, this $\mathcal{O}_X$-module is coherent by Proposition **??**.

An $\mathcal{O}_{X,P}$-linear map $\Omega_{X/k,P} \to \mathcal{O}_{X,P}$ induces a $\kappa(P)$-linear map on fibers, and this gives a natural map $\mathcal{T}_X(P) \to \operatorname{Hom}_{\kappa(P)}(\Omega_{X/k}(P), \kappa(P))$. At a k-point P we have $\Omega_{X/k}(P) \simeq \mathfrak{m}_P/\mathfrak{m}_P^2$, so this reads

$$\mathcal{T}_X(P) \longrightarrow \operatorname{Hom}_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2, \kappa(P)) = T_P X.$$

It is an isomorphism whenever $\Omega_{X/k,P}$ is a free $\mathcal{O}_{X,P}$-module, in particular at every smooth point of X, but not in general: taking duals does not commute with taking fibers (Exercises

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

17.9.13 and 17.9.14). So $\Omega_{X/k}$ has the cotangent spaces as its fibers at k-points, whereas $\mathcal{T}_X$ has the tangent spaces as its fibers only on the smooth locus.

Proposition 17.25. Let $X = \operatorname{Spec} A$, where $A = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Let $J = \left(\frac{\partial f_i}{\partial x_j}\right)$ be the Jacobian matrix. Then $\mathcal{T}_X$ is the quasi-coherent sheaf associated to the A-module

$$T = \operatorname{Ker}\bigl(J\colon A^n \to A^r\bigr).$$

Proof By the presentation (17.3), the module $\Omega_{A/k}$ is the cokernel of $J^t\colon A^r \to A^n$. Applying $\operatorname{Hom}_A(-, A)$ gives an exact sequence $0 \to \operatorname{Hom}_A(\Omega_{A/k}, A) \to A^n \xrightarrow{J} A^r$. Since X is affine, $\mathcal{T}_X$ is the sheaf associated to $\operatorname{Hom}_A(\Omega_{A/k}, A)$. □

Example 17.26. For $X = \mathbb{A}^n_k$, we have $\Omega_{\mathbb{A}^n_k/k} \simeq \mathcal{O}_{\mathbb{A}^n_k}^{\oplus n}$, so $\mathcal{T}_{\mathbb{A}^n_k} \simeq \mathcal{O}_{\mathbb{A}^n_k}^{\oplus n}$. △

One may wonder why one defines the sheaf $\Omega_{X/k}$ first and then defines the tangent sheaf as its dual. The reason is that $\Omega_{X/k}$ is the more fundamental object, especially when X has singularities. Dualizing always produces $\mathcal{T}_X$, but in general one cannot recover $\Omega_{X/k}$ from $\mathcal{T}_X$, as the next example shows.

Example 17.27 (The cuspidal cubic)**.** Let $A = k[x, y]/(y^2 - x^3)$ and assume that k has characteristic $\neq 2, 3$. By the exact sequence (17.3),

$$\Omega_{A/k} = \frac{A\,dx \oplus A\,dy}{(2y\,dy - 3x^2\,dx)},$$

and by the proposition above $T = \{\,(a, b) \in A^2 \mid -3x^2a + 2yb = 0\,\}$.

Unlike T, the module $\Omega_{A/k}$ contains torsion elements. Let $\eta = 3y\,dx - 2x\,dy$. Using the relation $2y\,dy = 3x^2\,dx$ and the identity $y^2 = x^3$, we obtain

$$\begin{aligned} y\,\eta &= 3y^2\,dx - 2xy\,dy = 3x^3\,dx - x \cdot 3x^2\,dx = 0, \\ x^2\,\eta &= 3x^2y\,dx - 2x^3\,dy = y \cdot 2y\,dy - 2x^3\,dy = 2(y^2 - x^3)\,dy = 0. \end{aligned}$$

It is also nonzero. If $\eta = 0$, then $(3y, -2x) = c(-3x^2, 2y)$ for some $c \in A$, and x is not a multiple of y in A.

The two elements used here, y and x^2, are up to units the partial derivatives of $y^2 - x^3$, and this is not a coincidence. The presentation above exhibits $\Omega_{A/k}$ as the cokernel of the 2×1 matrix with entries $-3x^2$ and $2y$, so its first Fitting ideal is

$$\operatorname{Fitt}_1(\Omega_{A/k}) = (3x^2, 2y) = (x^2, y),$$

and by Corollary 11.20 the vanishing locus of Fitt_1 is the set of points at which the rank of $\Omega_{A/k}$ exceeds its generic rank 1, which here is the origin. The torsion is therefore annihilated by the ideal cutting out the locus where $\Omega_{A/k}$ fails to be locally free, as it must be, since a locally free sheaf has no torsion.

Now write $A = k[t^2, t^3] \subset B = k[t]$, with $x = t^2$ and $y = t^3$, as in Example 1.28, and let $\iota\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding morphism. Inverting x gives $A[x^{-1}] = k[t, t^{-1}] = B[t^{-1}]$, so the two schemes agree away from the cusp. The induced map $\iota^*\colon \Omega_{A/k} \to \Omega_{B/k} = B\,dt$ sends $dx \mapsto 2t\,dt$ and $dy \mapsto 3t^2\,dt$, and is therefore an

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

isomorphism after inverting x. Hence $\operatorname{Ker} \iota^*$ is supported at the origin and consists of torsion elements. As $B\,dt$ is torsion free, the torsion of $\Omega_{A/k}$ is contained in $\operatorname{Ker} \iota^*$, so the two coincide. The image of ι^* is the A-module generated by $2t\,dt$ and $3t^2\,dt$, which is $tB\,dt$, and we obtain

$$0 \longrightarrow \operatorname{Tors}(\Omega_{A/k}) \longrightarrow \Omega_{A/k} \longrightarrow tB\,dt \longrightarrow 0.$$

The left-hand term contains η, and is in fact equal to $A\eta$ (Exercise 17.9.15).

For the duals, note that any A-linear map φ between nonzero A-submodules of $K = k(t)$ is multiplication by an element of K. Indeed, for nonzero m, m' in the source we have $m'\varphi(m) = \varphi(m'm) = m\varphi(m')$, so that $q = \varphi(m)/m$ does not depend on m. Since tB is generated over A by t and t^2, this gives

$$\operatorname{Hom}_A(tB, A) = \{\, q \in K \mid qt \in A \text{ and } qt^2 \in A \,\} = tB,$$

because $h = qt \in A = k \oplus t^2 B$ together with $th \in A$ forces the constant term of h to vanish, so that $h \in t^2 B$. Conversely $tB \cdot tB = t^2 B \subseteq A$. As homomorphisms into the domain A kill torsion, this one computation gives both duals:

$$T = \Omega^{\vee}_{A/k} \simeq \operatorname{Hom}_A(tB, A) = tB, \qquad \Omega^{\vee\vee}_{A/k} = T^{\vee} \simeq tB\,dt.$$

So $\Omega^{\vee\vee}_{A/k}$ is the torsion-free quotient of $\Omega_{A/k}$. Dualizing twice destroys the torsion and recovers nothing else.

Geometrically, the parametrization $t \mapsto (t^2, t^3)$ has vanishing derivative at $t = 0$, so ι^* lands in $tB\,dt$ and never reaches the generator dt of $\Omega_{B/k}$. Its kernel is the torsion, so η is a form that pulls back to zero on $\operatorname{Spec} B$, hence vanishes on the whole smooth locus of X. It lives only at the cusp, and dualizing erases it. △

17.6 Smooth closed subschemes

For an arbitrary closed embedding $i\colon Y \hookrightarrow X$ with ideal sheaf $\mathcal{I}$, Proposition 17.13 gives a right exact conormal sequence. When X and Y are smooth, it is exact on the left as well.

Proposition 17.28. Let X and Y be integral smooth schemes over k, and let $i\colon Y \hookrightarrow X$ be a closed embedding of codimension c with ideal sheaf $\mathcal{I}$. Then there are short exact sequences

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow i^*\mathcal{T}_X \longrightarrow N_Y \longrightarrow 0,$$

where $N_Y = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$.

Proof Smooth schemes over a field are regular, so $Y \hookrightarrow X$ is a regular embedding by Theorem 16.17, and $\mathcal{I}/\mathcal{I}^2$ is locally free of rank c by Corollary 16.20.

By Proposition 17.13 the sequence $\mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{X/k} \to \Omega_{Y/k} \to 0$ is exact, and its two right hand terms are locally free of ranks $\dim X$ and $\dim Y$. As $\Omega_{Y/k}$ is locally free, the surjection onto it splits locally, so its kernel is locally free of rank c. The induced map

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\mathcal{I}/\mathcal{I}^2 \to \mathrm{Ker}(i^*\Omega_{X/k} \to \Omega_{Y/k})$ is therefore a surjection between locally free sheaves of the same rank, hence an isomorphism, and the first sequence is exact. All three of its terms are locally free, so it is locally split, and dualizing gives the second sequence. □

In coordinates, left exactness says the following. If Y is cut out near a point $P \in Y$ by $f_1, \ldots, f_c$, then the differentials $df_1(P), \ldots, df_c(P)$ are linearly independent in $\Omega_{X/k}(P)$. In Chapter 16 the regularity of Y at P required only that the classes of $f_1, \ldots, f_c$ be linearly independent in $\mathfrak{m}_P/\mathfrak{m}_P^2$ (Proposition 16.12). Since the map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{X/k}(P)$ carries the class of f_i to $df_i(P)$, the condition on the differentials is the stronger of the two, in the same way that smoothness is stronger than regularity.

Taking fibers in the second sequence explains the name of N_Y. Let $P \in Y$ be a k-point. The sequence is locally split, so it remains exact after passing to fibers at P, and as X and Y are smooth the fibers of $i^*\mathcal{T}_X$ and $\mathcal{T}_Y$ at P are the tangent spaces T_PX and T_PY. We therefore obtain a short exact sequence of k-vector spaces

$$0 \longrightarrow T_PY \longrightarrow T_PX \longrightarrow N_Y(P) \longrightarrow 0.$$

So T_PY is a subspace of T_PX, and the fiber of the normal sheaf at P is the space of tangent directions of X at P modulo those along Y.

17.7 The Euler sequence and differentials of $\mathbb{P}^n_k$

In this section, we will give a concrete description of the sheaf of differentials on projective space. We continue to work over a general ring k.

Theorem 17.29 (The Euler Sequence). There is an exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n_k/k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0 \tag{17.12}$$

Proof Define the $\mathcal{O}_{\mathbb{P}^n_k}$-linear map

$$\varepsilon\colon \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}, \qquad (f_0, \ldots, f_n) \longmapsto \sum_{i=0}^{n} x_i f_i.$$

This map is surjective: on $U_i = D_+(x_i)$, the section with x_i^{-1} in the i-th component and 0 elsewhere maps to 1. Letting $K = \mathrm{Ker}(\varepsilon)$, we obtain a short exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)} \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0.$$

We will prove that $K \simeq \Omega_{\mathbb{P}^n_k/k}$.

Fix an index i. On U_i, the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(-1)|_{U_i}$ is free with x_i^{-1} as local generator, so a section of $\mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)}|_{U_i}$ may be uniquely expressed as $\sum_{j=0}^n a_j x_i^{-1} e_j$, and ε sends it to $a_i + \sum_{j\neq i} a_j x_j/x_i$. Such a section lies in $K|_{U_i}$ if and only if $a_i = -\sum_{j\neq i} a_j x_j/x_i$. Substituting, every section of $K|_{U_i}$ is uniquely of the form

$$\sum_{j\neq i} a_j \left(x_i^{-1} e_j - \frac{x_j}{x_i}\, x_i^{-1} e_i \right) = \sum_{j\neq i} a_j \frac{x_i e_j - x_j e_i}{x_i^2}.$$

Hence $K|_{U_i}$ is free with basis $\{(x_i e_j - x_j e_i)/x_i^2\}_{j\neq i}$. On the other hand, $\Omega_{\mathbb{P}^n_k/k}|_{U_i}$ is free

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

with basis $\{d(x_j/x_i)\}_{j\neq i}$, so we obtain an isomorphism $\phi_i\colon \Omega_{\mathbb{P}^n_k/k}|_{U_i} \xrightarrow{\sim} K|_{U_i}$ by sending $d(x_j/x_i)$ to $(x_ie_j - x_je_i)/x_i^2$.

It remains to verify that these isomorphisms glue. By symmetry, it suffices to check this on $U_0 \cap U_1$. Write $u_r = x_r/x_0$ for $r = 1, \dots, n$ for the coordinates on U_0, and $v_1 = x_0/x_1$, $v_r = x_r/x_1$ for $r = 2, \dots, n$ for coordinates on U_1. On the overlap, these are related by $v_1 = u_1^{-1}$ and $v_r = u_r/u_1$, so

$$dv_1 = -u_1^{-2}\,du_1 \qquad dv_r = u_1^{-1}\,du_r - u_ru_1^{-2}\,du_1 \text{ for } r \geqslant 2$$

Applying ϕ_0, we get

$$\phi_0(dv_1) = -u_1^{-2} \cdot \frac{x_0e_1 - x_1e_0}{x_0^2} = \frac{x_1e_0 - x_0e_1}{x_1^2} = \phi_1(dv_1),$$

and for $r \geqslant 2$,

$$\phi_0(dv_r) = u_1^{-1} \cdot \frac{x_0e_r - x_re_0}{x_0^2} - u_ru_1^{-2} \cdot \frac{x_0e_1 - x_1e_0}{x_0^2} = \frac{x_1e_r - x_re_1}{x_1^2} = \phi_1(dv_r).$$

Therefore ϕ_0 and ϕ_1 agree on the basis elements. By symmetry, the same holds on every overlap, so the local isomorphisms glue to a global isomorphism $\Omega_{\mathbb{P}^n_k/k} \simeq K$. Substituting this into the short exact sequence above gives the Euler sequence. □

In particular, since $\Omega_{\mathbb{P}^n_k/k}$ injects into $\mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)}$, we obtain $\Gamma(\mathbb{P}^n_k, \Omega_{\mathbb{P}^n_k/k}) = 0$.

Example 17.30. On $\mathbb{P}^1_k$, the Euler sequence takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0.$$

In particular, the exact sequence is not split exact. △

Example 17.31 (The tangent sheaf of $\mathbb{P}^n_k$)**.** Taking the dual of the Euler sequence, we obtain the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(1)^{\oplus(n+1)} \longrightarrow \mathcal{T}_{\mathbb{P}^n_k} \longrightarrow 0. \tag{17.13}$$

From the long exact sequence in cohomology, we find that $\Gamma(\mathbb{P}^n_k, \mathcal{T}_{\mathbb{P}^n_k})$ is a k-vector space of dimension $(n+1)^2 - 1$, and $H^i(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n_k}) = 0$ for all $i > 0$. △

17.8 The sheaf of p-forms

If X is a variety of dimension n over a field k, and p is a non-negative integer, we define the *sheaf of p-forms* as the $\mathcal{O}_X$-module

$$\Omega^p_{X/k} = \bigwedge^p \Omega_{X/k}.$$

When X is smooth over k, then $\Omega_{X/k}$ is locally free of rank n, so $\Omega^p_{X/k}$ is a locally free sheaf of rank $\binom{n}{p}$.

These sheaves are of significant interest because they allow us to define important invariants of the variety. The most important case is the sheaf $\Omega^n_{X/k}$, which is usually called the *canonical sheaf* or the *canonical bundle* of X. If X is smooth over k, then $\Omega^n_{X/k}$ is an invertible sheaf. The canonical sheaf plays a fundamental role in the classification of varieties. We will see this in the setting of curves in Chapter 22.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 17.32. $\Omega^n_{\mathbb{P}^n_k} = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$.

Proof Consider the Euler sequence for the cotangent bundle of $\mathbb{P}^n$

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0$$

Using Exercise **??**, we obtain

$$\det\left(\mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)}\right) = \mathcal{O}_{\mathbb{P}^n_k}(-n-1) = \det(\Omega_{\mathbb{P}^n_k}) \otimes \det(\mathcal{O}_{\mathbb{P}^n_k}) = \Omega^n_{\mathbb{P}^n_k}.$$

□

Using the conormal sequence, we can also study differentials on projective hypersurfaces.

Example 17.33 (Hypersurfaces)**.** Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree d, with ideal sheaf $\mathcal{I}$, and let $i\colon X \hookrightarrow \mathbb{P}^n_k$ be the inclusion. Since $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-d)$, Proposition 17.28 gives the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow i^*\Omega_{\mathbb{P}^n_k/k} \longrightarrow \Omega_{X/k} \longrightarrow 0. \tag{17.14}$$

Taking determinants, we get that $i^*\Omega^n_{\mathbb{P}^n_k} \simeq \Omega^{n-1}_X \otimes \mathcal{O}_X(-d)$, and hence

$$\Omega^{n-1}_X \simeq \mathcal{O}_X(d-n-1).$$

△

17.9 Exercises

Exercise 17.9.1. Let A be a k-algebra and let M be an A-module. A *k-derivation* $A \to M$ is a k-linear map $\partial\colon A \to M$ such that $\partial(ab) = a\,\partial(b) + b\,\partial(a)$ for all $a, b \in A$. Let $\mathrm{Der}_k(A, M)$ denote the set of such derivations. Show that the map $\alpha \mapsto \alpha \circ d$ defines a bijection $\mathrm{Hom}_A(\Omega_{A/k}, M) = \mathrm{Der}_k(A, M)$.

Exercise 17.9.2 (Tangent vectors and dual numbers)**.** Let X be a scheme over a field k, and let $P \in X$ be a k-point.

a) Show that there is a canonical bijection $\mathcal{T}_P X \simeq \mathrm{Der}_k(\mathcal{O}_{X,P}, k)$. HINT: If $\partial\colon \mathcal{O}_{X,P} \to k$ is a k-derivation, show that $\partial(\mathfrak{m}_P^2) = 0$.

b) Assume that $X = \operatorname{Spec} A$ is affine, and let $x\colon A \to k$ be the k-algebra map corresponding to P. Show that morphisms $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X$ of schemes over k whose closed point maps to P are in bijection with k-algebra maps $\varphi\colon A \to k[\epsilon]/(\epsilon^2)$ whose composition with $k[\epsilon]/(\epsilon^2) \to k$ is x. Show that every such φ can be written uniquely in the form $\varphi(a) = x(a) + \delta(a)\epsilon$ for some map $\delta\colon A \to k$, and that φ is a k-algebra map if and only if δ is a k-derivation.

c) Show that every k-derivation $A \to k$ extends uniquely to a k-derivation $\mathcal{O}_{X,P} \to k$. Conclude that tangent vectors at P are in canonical bijection with morphisms of k-schemes $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X$ whose closed point maps to P.

Exercise 17.9.3. Let k be a field. Compute $\Omega_{A/k}$ for the following k-algebras:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) $A = k[x,y]/(x^2+y^2)$. This will depend on the characteristic of k.
b) $A = k[x,y]/(xy)$.
c) $A = k[x,y]/(y^2-x^3)$.

Exercise 17.9.4. Let $A \subset B$ be a ring extension. Show that $\Omega_{B[t]/A} \simeq B[t]\,dt \oplus (\Omega_{B/A} \otimes_B B[t])$.

Exercise 17.9.5. Let k be a ring and let $\varphi\colon A \to B$ be a map of k-algebras.

a) Define $\varphi^*\colon \Omega_{A/k} \otimes_A B \to \Omega_{B/k}$ and $\beta\colon \Omega_{B/k} \to \Omega_{B/A}$ by $da \otimes b \mapsto b\,d(\varphi(a))$ and $db \mapsto db$, respectively. Show that these maps are well-defined and B-linear.
b) Show that the sequence $\Omega_{A/k} \otimes_A B \xrightarrow{\varphi^*} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \longrightarrow 0$ is right exact. HINT: The module $\Omega_{B/A}$ is obtained from $\Omega_{B/k}$ by imposing the additional relations $d(\varphi(a)) = 0$ for $a \in A$.
c) Let $f\colon X \to Y$ be a morphism of k-schemes. Show that these affine sequences glue to a right-exact sequence of sheaves $f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$.

Exercise 17.9.6 (Base change)**.** Show that for the k-algebra $C = A \otimes_k B$, there is an isomorphism $\Omega_{C/B} = \Omega_{A/k} \otimes_A C$. HINT: Compare the diagonal ideals $\operatorname{Ker}(A \otimes_k A \to A)$ and $\operatorname{Ker}(C \otimes_B C \to C)$ using $C \otimes_B C \simeq (A \otimes_k A) \otimes_k B$.

Exercise 17.9.7 (Localization)**.** For a multiplicative set $S \subset A$, show that there is an isomorphism of $S^{-1}A$-modules $\Omega_{S^{-1}A/k} \simeq S^{-1}(\Omega_{A/k})$. HINT: Compare the diagonal ideals before and after localizing $A \otimes_k A$ at the elements $s \otimes 1$ and $1 \otimes s$, for $s \in S$.

Exercise 17.9.8 (Products)**.** Let A and B be k-algebras.

a) Show that $\Omega_{(A \otimes_k B)/k} = (\Omega_{A/k} \otimes_k B) \oplus (\Omega_{B/k} \otimes_k A)$. HINT: $d(a \otimes b) = (d_A a) \otimes b + a \otimes (d_B b)$.
b) Show that if $X \to S$ and $Y \to S$ are two S-schemes, then

$$\Omega_{X \times_S Y/S} \simeq \pi_1^* \Omega_{X/S} \oplus \pi_2^* \Omega_{Y/S},$$

where π_1 and π_2 are the two projections.

Exercise 17.9.9. Let $f\colon X \to S$ be a morphism and let $P \in X$. Show that $(\Omega_{X/S})_P \simeq \Omega_{\mathcal{O}_{X,P}/\mathcal{O}_{S,f(P)}}$.

Exercise 17.9.10. In the text, $\Omega_{X/S}$ was defined for a separated morphism $X \to S$. Show that $\Omega_{X/S}$ can be defined for a general morphism by gluing together the various $\widetilde{\Omega_{B/A}}$ for each affine $\operatorname{Spec} A \subset S$ and $\operatorname{Spec} B \subset X$ with $\operatorname{Spec} B$ mapping into $\operatorname{Spec} A$.

Exercise 17.9.11 (Bott's formula for projective space)**.** Let k be a field, let $0 \leqslant p, q \leqslant n$ and let $d \in \mathbb{Z}$. Show that $H^q(\mathbb{P}^n_k, \Omega^p_{\mathbb{P}^n_k}(d)) = 0$ unless one of the following holds:

(i) $p = q$ and $d = 0$;
(ii) $q = 0$ and $d > p$;
(iii) $q = n$ and $d < p - n$.

HINT: Use the exterior powers of the Euler sequence, the cohomology of $\mathcal{O}_{\mathbb{P}^n_k}(d)$, and Serre duality.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 17.9.12. Let $X \subset \mathbb{A}^2_{\mathbb{Q}}$ be the subscheme defined by the ideal $I = (x^2, y^2)$. Show that the map $I/I^2 \to \Omega_{\mathbb{A}^2_{\mathbb{Q}}/\mathbb{Q}}|_X$ is not injective and compute its kernel.

Exercise 17.9.13. Let X be a scheme of finite type over a field k, and let $P \in X$ be a closed point such that $\kappa(P)/k$ is separable.

a) Show that there is a natural $\kappa(P)$-linear map

$$\mathcal{T}_{X,P} \otimes_{\mathcal{O}_{X,P}} \kappa(P) \longrightarrow \operatorname{Hom}_{\kappa(P)}\big(\Omega_{X/k,P} \otimes_{\mathcal{O}_{X,P}} \kappa(P), \kappa(P)\big).$$

b) Use Proposition **??** to identify the target with the classical tangent space $T_P X = \operatorname{Hom}_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2, \kappa(P))$.

c) Show that if $\Omega_{X/k,P}$ is free, then this map is an isomorphism.

Exercise 17.9.14. Let $A = k[x,y]/(x^2, xy, y^2)$ and let $X = \operatorname{Spec} A$. Compute $\Omega_{A/k}$. If P is the unique closed point, show that $\dim_k T_P X = 2$ and $\dim_k\big(\mathcal{T}_X \otimes_A k\big) = 4$. So taking the dual and taking the fiber do not commute in this example.

Exercise 17.9.15. Let k be a field of characteristic $\neq 2, 3$, let $A = k[x,y]/(y^2 - x^3)$, and let $\eta = 3y\,dx - 2x\,dy$ as in Example 17.27. Show that the torsion submodule of $\Omega_{A/k}$ equals $A\eta$. HINT: With $A \subset B = k[t]$ as in the example, a class $a\,dx + b\,dy$ lies in $\operatorname{Ker} \iota^*$ if and only if $2a + 3tb = 0$ in B. Deduce that $b \in (x, y)$, and write $b = xb_1 + yb_2$.

Exercise 17.9.16 (The conormal sequence of a locally complete intersection)**.** Let k be algebraically closed, let X be a smooth scheme over k, and let $i\colon Y \hookrightarrow X$ be an integral closed subscheme of codimension c which is a locally complete intersection in X. By Corollary 16.20, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free of rank c. Show that the conormal sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

is exact, even though Y need not be smooth. Explain what happens when $X = \mathbb{A}^2_k$ and Y is the nodal cubic of Example 17.8.

Exercise 17.9.17 (The normal sheaf of a graph)**.** Let X be a k-scheme, let Y be a smooth separated k-scheme, and let $f\colon X \to Y$ be a morphism. Let $\Gamma_f\colon X \hookrightarrow X \times_k Y$ be its graph. Show that Γ_f is a regular embedding and use the conormal sequence, together with $\Gamma_f^*\Omega_{X\times_k Y/k} \simeq \Omega_{X/k} \oplus f^*\Omega_{Y/k}$, to prove that $\mathcal{I}_{\Gamma_f}/\mathcal{I}_{\Gamma_f}^2 \simeq f^*\Omega_{Y/k}$ and $N_{\Gamma_f} \simeq f^*\mathcal{T}_Y$.

Exercise 17.9.18 (Adjunction for complete intersections)**.** Let k be a field and let $i\colon X \hookrightarrow \mathbb{P}^N_k$ be a smooth complete intersection of dimension $n = N - r > 0$, cut out by homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$, with ideal sheaf $\mathcal{I}$.

a) Using Theorem 16.17, show that $X \subset \mathbb{P}^N_k$ is a regular embedding and that $\mathcal{I}/\mathcal{I}^2 \simeq \bigoplus_{i=1}^r \mathcal{O}_X(-d_i)$.

b) Deduce from Proposition 17.28 that there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_X(-d_i) \longrightarrow i^*\Omega_{\mathbb{P}^N_k/k} \longrightarrow \Omega_{X/k} \longrightarrow 0.$$

c) Show that $\Omega^n_{X/k} \simeq \mathcal{O}_X(\sum_{i=1}^r d_i - N - 1)$.

d) Show that X is not rational when $\sum_{i=1}^r d_i \geqslant N + 1$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 17.9.19. Let k be an algebraically closed field, and let X and Y be two smooth hypersurfaces in $\mathbb{P}^n_k$ of different degrees d and e. Show that X and Y are not isomorphic unless $n = 2$ and either $(d, e) = (1, 2)$ or $(d, e) = (2, 1)$. HINT: Consider $\Omega_{X/k}$.

Exercise 17.9.20. Let k be an algebraically closed field and let $X \subset \mathbb{P}^{n+1}_k$ be a smooth quadric hypersurface. Show that X is not isomorphic to $\mathbb{P}^n_k$ unless $n = 1$.

Exercise 17.9.21. Let k be an algebraically closed field and let $X \subset \mathbb{P}^a_k \times \mathbb{P}^b_k$ be a smooth hypersurface of bidegree (d, e), where $a, b, d, e > 0$.

a) Show that $\Omega^{a+b-1}_{X/k} \simeq \mathcal{O}_X(d - a - 1, e - b - 1)$.
b) Deduce that X is not rational if $d \geqslant a + 1$ and $e \geqslant b + 1$.

Exercise 17.9.22. Let k be an algebraically closed field and let $X \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ be a smooth surface defined by a bihomogeneous equation of bidegree (d, e).

a) If $d = 1$, show that the projection $X \to \mathbb{P}^2_k$ is birational. In particular, X is rational.
b) If $e = 1$, show that the projection $X \to \mathbb{P}^1_k$ is a $\mathbb{P}^1$-bundle. In particular, X is rational.
c) Show that X is not rational if $d \geqslant 2$, $e \geqslant 3$.

Exercise 17.9.23. Let k be an algebraically closed field and let $X \subset \mathbb{P}^2_k \times \mathbb{P}^2_k$ be a smooth threefold defined by a bihomogeneous equation of bidegree (d, e). Show that if $d = 1$ or $e = 1$, then one of the projections makes X into a $\mathbb{P}^1$-bundle over $\mathbb{P}^2_k$. Deduce that X is rational. Show that X is not rational if $d, e \geqslant 3$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

18

Normal schemes

18.1 Normal schemes

Regularity is a strong condition, and many schemes that arise naturally are not regular. In this section we study a weaker condition called *normality*. An integral scheme is normal if all of its local rings are integrally closed in the function field. This is a purely algebraic requirement, and it is much less restrictive than regularity.

Every regular scheme is normal, and for curves the two conditions agree. In general the converse fails, but a normal scheme can only be singular in codimension at least 2.

Normality is useful because it controls the behaviour of rational functions. On a normal Noetherian scheme the local ring at a point of codimension 1 is a discrete valuation ring, so a rational function has a well-defined order of vanishing along each codimension 1 subscheme. Later in the chapter we use this to extend regular functions and rational maps across closed sets of codimension at least 2, and the same fact is what makes the theory of divisors work.

Normal rings

Let A be an integral domain with fraction field $K = K(A)$. An element $x \in K$ is said to be *integral over* A if it satisfies a monic equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \qquad a_i \in A.$$

The *integral closure* of A is defined as

$$\overline{A} = \big\{ x \in K \mid x \text{ is integral over } A \big\}. \tag{18.1}$$

It is straightforward to check that $\overline{A}$ is a subring of K containing A. We say that A is *normal* (or *integrally closed*) if $A = \overline{A}$.

Example 18.1. Any unique factorization domain A is normal. This includes rings such as $\mathbb{Z}$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[i]$ or $k[x_1, \ldots, x_n]$ where k is a field.

To see this, suppose $u/v \in K$ is integral over A. Since A is a UFD, we may assume that u and v have no common factors. By assumption, u/v satisfies a monic polynomial equation:

$$(u/v)^n + a_{n-1}(u/v)^{n-1} + \cdots + a_0 = 0,$$

where $a_i \in A$. Multiplying by v^n gives:

$$u^n + a_{n-1}u^{n-1}v + \cdots + a_0 v^n = 0.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This implies that v divides u^n, which by our assumption means that v must be a unit in A. Therefore, $u/v \in A$, and hence A is normal. △

Example 18.2. The ring $\mathbb{Z}[\sqrt{5}]$ is not normal. The fraction field equals $K = \mathbb{Q}(\sqrt{5})$, and this contains the golden ratio $\phi = \frac{1}{2}(1 + \sqrt{5})$. Note that ϕ satisfies the monic equation $\phi^2 - \phi - 1 = 0$, so ϕ is integral over $\mathbb{Z}$, hence over $\mathbb{Z}[\sqrt{5}]$, but of course $\phi \notin \mathbb{Z}[\sqrt{5}]$. △

Example 18.3 (The quadratic cone)**.** Let $X = \operatorname{Spec} A$ where $A = \mathbb{C}[x, y, z]/(xy - z^2)$. Note that A is not a UFD as $xy = z^2$ and one easily checks that x, y and z are all irreducible elements, so we cannot immediately conclude that A is normal. However, there is an isomorphism of rings $A \simeq \mathbb{C}[u^2, uv, v^2]$ sending $x \mapsto u^2$, $y \mapsto v^2$, $z \mapsto uv$. The ring $\mathbb{C}[u^2, uv, v^2]$ is normal in its fraction field $K = \mathbb{C}(u^2, uv, v^2)$. Indeed, suppose that $T = p/q \in \mathbb{C}(u^2, uv, v^2)$ satisfies a monic equation with coefficients in $\mathbb{C}[u^2, uv, v^2]$. Then T is integral over $\mathbb{C}[u, v]$, so $T \in \mathbb{C}[u, v]$ as $\mathbb{C}[u, v]$ is integrally closed. On the other hand, every element of $K = \mathbb{C}(u^2, uv, v^2)$ is invariant under the sign change $(u, v) \mapsto (-u, -v)$, so T is a polynomial in the even-degree monomials, and these are generated by u^2, uv, v^2. Hence $T \in \mathbb{C}[u^2, uv, v^2]$. (For another proof, see Exercise **??**.)

△

Example 18.4. If A is a normal ring, then so is $A[x]$ (see Exercise 18.6.2). △

Example 18.5. If A is a normal integral domain, then so is any localization $S^{-1}A$. Indeed, if $x \in K(A)$ is integral over $S^{-1}A$, then clearing the denominators in an integral equation shows that sx is integral over A for some $s \in S$. So $sx \in A$, and hence $x \in S^{-1}A$. △

There is a partial converse to this example:

Proposition 18.6. For an integral domain A, the following are equivalent:
(i) A is normal.
(ii) $A_{\mathfrak{p}}$ is normal for all prime ideals $\mathfrak{p}$.
(iii) $A_{\mathfrak{m}}$ is normal for all maximal ideals $\mathfrak{m}$.

Proof If A is normal, then so are all its localizations, by the previous example. Conversely, if each $A_{\mathfrak{m}}$ is normal, let $x \in K(A)$ be integral over A. Then x satisfies a monic equation with coefficients in A, hence in each $A_{\mathfrak{m}}$, so $x \in A_{\mathfrak{m}}$ for all $\mathfrak{m}$. For any integral domain A, one has $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ inside $K(A)$, so $x \in A$. □

The next result says that a Noetherian local domain of dimension 1 is normal exactly when it is a discrete valuation ring. Discrete valuation rings, in turn, have excellent properties. Their ideal structure is particularly simple, as the only nonzero ideals are powers of the maximal ideal.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 18.7. Let A be a Noetherian local domain of dimension 1 with maximal ideal $\mathfrak{m}$. The following statements are equivalent:
(i) A is a DVR.
(ii) The maximal ideal $\mathfrak{m}$ is principal.
(iii) Every nonzero ideal is a power of $\mathfrak{m}$.
(iv) A is a regular local ring, i.e., $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.
(v) A is normal.

Proof See Proposition C.37. □

The next theorem will be very important. It describes normality in terms of localizations at height 1 primes.

Theorem 18.8. Let A be a Noetherian integral domain. Then A is normal if and only if:
(i) $A_{\mathfrak{p}}$ is a discrete valuation ring for each height 1 prime $\mathfrak{p}$.
(ii) A equals the intersection of all of its height 1 localizations in $K(A)$:

$$A = \bigcap_{\operatorname{ht} \mathfrak{p}=1} A_{\mathfrak{p}}. \tag{18.2}$$

Proof If (i) and (ii) hold, then A is an intersection of normal rings, hence normal. Conversely, if A is normal, then each height 1 localization $A_{\mathfrak{p}}$ is a normal 1-dimensional ring, hence a discrete valuation ring by Proposition 18.7. The proof of the intersection formula (18.2) is outlined in Exercise 18.6.7. □

Normal schemes

Let X be an integral scheme and let $P \in X$. We say that X is *normal* at P if $\mathcal{O}_{X,P}$ is an integrally closed domain. We say that X is a *normal scheme* if it is normal at every point.

More generally, one may call a locally Noetherian, not necessarily integral scheme normal if all of its local rings are integrally closed domains. Such a scheme is a disjoint union of integral normal schemes. In this chapter we restrict to the integral case, so that a single function field is available throughout.

The primary example of a normal scheme is $X = \operatorname{Spec} A$, where A is a normal integral domain.

Lemma 18.9. Let X be an integral normal scheme. Then $\mathcal{O}_X(U)$ is normal for every nonempty open set $U \subset X$.

Proof Since X is integral, we regard all the section rings $\mathcal{O}_X(U)$ as subrings of the function field $K = K(X)$. Fix a nonempty open set U. If $U = \operatorname{Spec} A$ is affine, then since X is normal, all of the localizations $A_{\mathfrak{p}}$ are normal. Therefore, A is normal as well by Proposition 18.6.

For a general open $U \subset X$, let $z \in K$ satisfy a monic equation $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ with $a_i \in \mathcal{O}_X(U)$. For every affine open $V \subset U$, restricting this relation shows that z is integral over $\mathcal{O}_X(V)$, so $z \in \mathcal{O}_X(V)$ by the affine case. These sections all equal z

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

inside K, hence agree on overlaps and glue to a section of $\mathcal{O}_X(U)$ equal to z. Therefore $z \in \mathcal{O}_X(U)$. □

Example 18.10. $\mathbb{A}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_{\mathbb{Z}}$ are normal schemes, because the local rings are isomorphic to localizations of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, and these are normal. △

Although it is not obvious from the definition, normality is closely related to regularity. This is a consequence of the Auslander–Buchsbaum theorem: every regular local ring is a unique factorization domain, and every UFD is normal (Example 18.1).

Corollary 18.11. Any integral regular scheme is normal.

Although normal schemes form a strictly larger class than regular schemes, they still cannot be 'too singular'. The following result makes this precise.

Proposition 18.12. Let X be a normal integral Noetherian scheme, and let $P \in X$ be a point of codimension 1. Then $\mathcal{O}_{X,P}$ is a discrete valuation ring. In particular, X is regular at P.

Proof Since P has codimension 1, the local ring $\mathcal{O}_{X,P}$ is a Noetherian local domain of dimension 1, and it is integrally closed because X is normal. By Proposition 18.7, a normal Noetherian local domain of dimension 1 is a discrete valuation ring, and hence a regular local ring. □

Corollary 18.13. Let X be a normal integral Noetherian scheme. Then $\operatorname{sing}(X)$ contains no point of codimension $\leqslant 1$.

Corollary 18.14. Let X be a normal integral scheme of finite type over a perfect field. Then $\operatorname{sing}(X)$ is closed, and every irreducible component of $\operatorname{sing}(X)$ has codimension $\geqslant 2$.

Proof By Corollary 17.23, the regular locus of X coincides with the smooth locus, which is open. Hence $S = \operatorname{sing}(X)$ is closed, so it is a union of finitely many irreducible closed subsets. Let Z be one of these components, with generic point ζ; then the codimension of Z in X equals the codimension of ζ. The generic point of X is regular, since its local ring is the field $K(X)$, and by Proposition 18.12 every point of codimension 1 is regular as well. Hence ζ, which lies in S, has codimension at least 2, and therefore so does Z. □

Example 18.15 (Curves)**.** An integral curve X is normal if and only if it is regular. Indeed, the local ring at a closed point is a Noetherian local domain of dimension 1; for such rings normality and regularity are equivalent by Proposition 18.7. Also, the local ring at the generic point is the field $K(X)$, which is regular. △

To summarize:

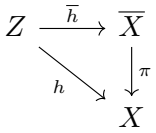
$$X \text{ regular} \implies \begin{array}{c}\text{All local rings } \mathcal{O}_{X,x} \\ \text{are UFDs}\end{array} \implies X \text{ normal} \implies \begin{array}{c}X \text{ regular at every} \\ \text{codimension 1 point}\end{array}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

18.2 Normalization

A useful fact is that any integral scheme admits a *normalization*. More precisely, if X is an integral scheme, its *normalization* is a normal scheme $\overline{X}$ together with a dominant morphism $\pi\colon \overline{X} \to X$ which is universal among dominant morphisms from normal schemes.

Theorem 18.16 (Normalization). Let X be an integral scheme. Then there exists a normal scheme $\overline{X}$ and a dominant morphism $\pi\colon \overline{X} \to X$ such that for any dominant morphism $h\colon Z \to X$ from a normal scheme Z, there is a unique morphism $\overline{h}\colon Z \to \overline{X}$ with $h = \pi \circ \overline{h}$.



Moreover, $\overline{X}$ is unique up to unique isomorphism over X.

Proof The uniqueness of the normalization $\overline{X}$ follows from the universal property.

For the existence, we start with the affine case. Let $X = \operatorname{Spec} A$ with fraction field $K = K(A)$ and let $\overline{A} \subset K$ be the integral closure of A in K. Let $\overline{X} = \operatorname{Spec} \overline{A}$ and $\pi\colon \overline{X} \to X$ be the morphism induced by the inclusion $A \subset \overline{A}$. Then $\overline{A}$ is a normal integral domain, hence $\overline{X}$ is normal. If $h\colon Z \to X$ is dominant with Z normal, then $h^\sharp\colon A \to \mathcal{O}_Z(Z)$ is injective (Example **??**) and $\mathcal{O}_Z(Z)$ is normal (Lemma 18.9). Therefore, $h^\sharp$ factors uniquely as $A \to \overline{A} \to \mathcal{O}_Z(Z)$. This gives a unique $\overline{h}\colon Z \to \overline{X}$ with $h = \pi \circ \overline{h}$.

For a general integral scheme X, choose an affine open cover $X = \bigcup_i U_i$, where $U_i = \operatorname{Spec} A_i$, and let $\pi_i\colon \overline{U_i} = \operatorname{Spec} \overline{A_i} \to U_i$ be the affine normalization constructed above. We first observe that if $V \subset U_i$ is a nonempty open subset, then $\pi_i^{-1}(V) \to V$ is a normalization of V. Indeed, a dominant morphism $Z \to V$ from a normal scheme remains dominant after composition with $V \hookrightarrow U_i$, so its unique factorization through $\overline{U_i}$ has image in $\pi_i^{-1}(V)$.

It follows that over every overlap $U_i \cap U_j$, the two schemes $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ have the same universal property. They are therefore canonically isomorphic over $U_i \cap U_j$. Uniqueness also gives the cocycle condition on triple overlaps, so the $\overline{U_i}$ glue to a normal scheme $\overline{X}$ and the maps π_i glue to a morphism $\pi\colon \overline{X} \to X$.

Finally, let $h\colon Z \to X$ be a dominant morphism from a normal scheme. For each i, the restricted morphism $h^{-1}(U_i) \to U_i$ factors uniquely through $\overline{U_i}$. These factorizations agree on overlaps by uniqueness and hence glue to a unique morphism $\overline{h}\colon Z \to \overline{X}$ such that $h = \pi \circ \overline{h}$. □

In the proof of the next theorem, we use the following finiteness result due to Emmy Noether.

Theorem 18.17 (Finite generation of integral closure). Let k be a field and let A be a finitely generated k-algebra which is an integral domain. Then the integral closure $\overline{A}$ is a finitely generated A-module. In particular, $\overline{A}$ is a finitely generated k-algebra.

This is a deep fact from commutative algebra (see (Eisenbud, 2013, Section 13.3) for

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a proof). Some assumptions are needed for the theorem to hold; there are examples of Noetherian integral domains A whose integral closure is not finite over A. For such rings the normalization morphism is not finite.

Theorem 18.18. Let X be a variety over a field k. Then
(i) $\overline{X}$ is a variety over k.
(ii) The normalization morphism $\pi\colon \overline{X} \to X$ is finite and birational.

Proof If $X = \operatorname{Spec} A$ with A a finitely generated k-domain and fraction field K, then $\overline{X} = \operatorname{Spec} \overline{A}$ where $\overline{A}$ is the integral closure of A in K, and π is induced by $A \hookrightarrow \overline{A}$. By Theorem 18.17, $\overline{A}$ is a finite A-module. Hence π is finite, $\overline{X}$ is affine of finite type over k, and since $K(\overline{A}) = K(A) = K$, the morphism π is birational.

For a general X, take an affine open cover $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$. Each normalization $\operatorname{Spec} \overline{A_i} \to \operatorname{Spec} A_i$ is finite and birational by the affine case, and these glue to give the normalization $\pi\colon \overline{X} \to X$, which is therefore finite and birational. Since $\overline{X} \to X$ is finite, and X is of finite type over k, $\overline{X}$ is also of finite type over k. Moreover, as finite morphisms are separated, $\overline{X}$ is separated over k. Finally, $\overline{X}$ is integral and normal by construction. Hence $\overline{X}$ is a normal variety over k. □

Examples

Example 18.19 (The cuspidal cubic)**.** Consider the cuspidal cubic curve $X = \operatorname{Spec} A$ where $A = k[x,y]/(y^2 - x^3)$. In Example 1.28, we showed that there is an isomorphism of k-algebras $A \to k[t^2, t^3]$ sending $x \mapsto t^2$ and $y \mapsto t^3$. The ring $k[t^2, t^3]$ is an integral domain with fraction field $K = k(t)$, but it is clearly not normal, because $t \notin k[t^2, t^3]$ and t satisfies the monic equation $T^2 - t^2 = 0$. As $k[t]$ is integrally closed in K (being a UFD), any element in K which is integral over A can be written as a polynomial in t. Therefore the integral closure of A is given by $\overline{A} = k[t]$. The normalization morphism $\operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is exactly the morphism $\pi\colon \mathbb{A}^1_k \to \operatorname{Spec} A$ from Example 1.28. △

Example 18.20 (The nodal cubic)**.** Consider the nodal cubic curve $X = \operatorname{Spec} A$ where $A = k[x,y]/(y^2 - x^3 - x^2)$, and assume that k has characteristic different from 2 (in characteristic 2 we are back in the cuspidal case).

Here it is less obvious what the normalization should be, and it helps to think geometrically. In the algebraic set $y^2 = x^3 + x^2$ in $\mathbb{A}^2(k)$, a line $y = tx$ through the origin meets the curve there and in one further point, namely $x = t^2 - 1$ and $y = tx = t^3 - t$. So $t \mapsto (t^2 - 1, t^3 - t)$ parameterizes the curve, one-to-one for $t \neq \pm 1$.

Algebraically, introduce the parameter $t = y/x$ in $K(X)$. Dividing $y^2 = x^3 + x^2$ by x^2 gives $t^2 = 1 + x$, so t satisfies the monic equation $T^2 - x - 1 = 0$ and is integral over A. As $x = t^2 - 1$ and $y = x \cdot y/x = t^3 - t$, we see that

$$A = k[t^2 - 1, t^3 - t] \subset k[t] \subset K = k(t),$$

and $k[t]$ is integrally closed in K, so $\overline{A} = k[t]$. The normalization map $\pi\colon \operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is the parameterization above. It is an isomorphism outside the origin, and identifies the two points $t = 1$ and $t = -1$ of $\mathbb{A}^1_k$ with the origin in X. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Passing from X to its normalization $\overline{X}$ is a common operation in algebraic geometry. This is because normal schemes have better properties than general schemes, and $\overline{X}$ is the 'best normal approximation' to X. One can then study X by performing computations on $\overline{X}$.

If X is a curve, its normalization even produces a regular scheme $\overline{X}$ and the normalization morphism $\pi\colon \overline{X} \to X$ is birational and an isomorphism outside the singular locus of X. In light of the relationship between the properties of normality and regularity, one can ask whether such a statement holds in higher dimensions, that is, whether there exists a *resolution of singularities* $\pi\colon Y \to X$, where Y is regular and π is birational. This holds if the ground field has characteristic zero, by the following famous theorem of Hironaka:

Theorem 18.21 (Hironaka). Let k be a field of characteristic zero and let X be an integral scheme of finite type over k. Then there exists a proper birational morphism $\pi\colon Y \to X$ with Y regular, which is an isomorphism over the regular locus of X.

Hironaka's result is in fact more precise, saying that one can construct the morphism π through a series of *blow-ups* along regular closed subschemes, which are systematically chosen to eliminate singularities. The analogous statement in positive characteristic is still open and is one of the most important unsolved problems in algebraic geometry.

18.3 Rational functions on normal schemes

In this section, we translate some of the algebraic properties of normal rings into geometric statements about normal schemes.

The first result is about extending a regular function defined on an open to a regular function on the whole scheme. This is not always possible: consider for instance x^{-1} on $\mathbb{A}^1_k - \{0\}$, which does not extend to a regular function on all of $\mathbb{A}^1_k$. However, if X is normal and the open set U is 'big' in the sense that $X - U$ has no codimension 1 components, then every regular function extends:

Theorem 18.22. Let X be a Noetherian normal scheme and let $U \subset X$ be an open subset with $\operatorname{codim}(X - U) \geqslant 2$. Then the restriction map induces an isomorphism

$$\mathcal{O}_X(X) \simeq \mathcal{O}_X(U). \tag{18.3}$$

Proof By our convention, X is integral. We may therefore view the rings of regular functions $\mathcal{O}_X(V)$ as subrings of the function field $K(X)$, and the restriction map (18.3) is simply the inclusion $\mathcal{O}_X(X) \subset \mathcal{O}_X(U)$. An element $f \in K(X)$ lies in $\mathcal{O}_X(U)$ if and only if $f \in \mathcal{O}_{X,P}$ for all $P \in U$. This condition can be checked on affine open subsets, so we reduce to the case $X = \operatorname{Spec} A$ for a normal integral domain A. The conclusion now follows from (18.2), because U contains every prime ideal of height 1. □

Normality is essential in Theorem 18.22. The following gives examples of rational functions on a surface which do not extend over a singular point.

Example 18.23. Consider the subring $A = k[x^2, xy, y^2, x^3, x^2y, xy^2, y^3]$ of $k[x, y]$. This ring is not normal, as x satisfies the monic equation $T^2 - x^2 = 0$, but $x \notin A$. Similarly, y

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

satisfies $T^3 - y^3 = 0$. In any case, these equations show that $A \subset k[x,y]$ is an integral extension and that $\overline{A} = k[x,y]$ is the integral closure of A. Consider the normalization map

$$\pi\colon \operatorname{Spec} k[x,y] \longrightarrow \operatorname{Spec} A.$$

Let $\mathfrak{m} = (x,y) \cap A = (x^2, xy, \ldots, y^3)$ and let $P \in \operatorname{Spec} A$ be the corresponding point. We claim that π is an isomorphism away from P. To prove this, we can work over the distinguished open sets $D(x^2)$ and $D(y^2)$, which together cover $\operatorname{Spec}(A) - \{P\}$. Localizing at x^2, we have $A_{x^2} = k[x,y]_{x^2} = k[x,y]_x$, so π is an isomorphism over $D(x^2)$. Similarly, π is an isomorphism over $D(y^2)$.

This means that every prime ideal of height 1 in A corresponds to a prime ideal of height 1 in $k[x,y]$ and vice versa. Therefore,

$$\bigcap_{\text{ht } \mathfrak{p}=1} A_{\mathfrak{p}} = \bigcap_{\operatorname{codim} \overline{\{p\}}=1} \mathcal{O}_{X,p} = \bigcap_{\operatorname{codim} \overline{\{p\}}=1} \mathcal{O}_{\mathbb{A}^2,p} = k[x,y] \supsetneq A,$$

so the formula (18.2) does not hold.

In this example, the singular locus is the point P, which is of codimension 2. △

Theorem 18.22 has the following consequence for morphisms from normal schemes.

Corollary 18.24. Let X and Y be affine varieties over a field k, let $Z \subset X$ be a closed subset whose components have codimension at least 2, and let $\phi\colon X - Z \to Y$ be a morphism. If X is normal, then ϕ extends uniquely to a morphism $\overline{\phi}\colon X \to Y$.

Proof Embed Y as a closed subvariety of $\mathbb{A}^n_k$. The composition $X - Z \to Y \to \mathbb{A}^n_k$ is defined by n regular functions on $X - Z$. By Theorem 18.22, these extend to regular functions on X, and hence define a morphism $\psi\colon X \to \mathbb{A}^n_k$. Every defining equation of Y vanishes after composition with ψ on the dense open subset $X - Z$. Since X is integral, it vanishes on all of X, so ψ factors through a morphism $\overline{\phi}\colon X \to Y$. Uniqueness follows because two morphisms to the separated scheme Y that agree on the dense open subset $X - Z$ agree on all of X. □

Example 18.25. The projection map $\pi\colon \mathbb{A}^2_k - \{(0,0)\} \to \mathbb{P}^1_k$ does not extend to a morphism $\pi\colon \mathbb{A}^2 \to \mathbb{P}^1_k$. In fact, it does not even extend as a continuous map. △

Proposition 18.26. Let $X \subset \mathbb{A}^{n+1}_k$ be an integral hypersurface over an infinite field k. If X is regular in codimension 1, then X is normal.

Proof After a general linear change of coordinates, projection onto $\mathbb{A}^n$ is finite. Fix a point $P \in \mathbb{A}^n$, and write $R = k[x_1, \ldots, x_n]_{\mathfrak{m}_P}$. Locally on $\mathbb{A}^n$, the equation of X can be taken to be monic in x_{n+1}, so $A = \mathcal{O}_X(X)_P$ is a finite free R-module.

Let $B = \overline{A}$ be the integral closure of A in $K(A)$. Since B is a domain, it is torsion-free as an R-module. Moreover, $A_Q = B_Q$ for every height-1 prime Q of R. Indeed, the primes of A lying over Q have height one, and X is regular, hence normal, at those points. It follows that the support of B/A has codimension at least 2 in $\operatorname{Spec} R$.

Write $M^* = \operatorname{Hom}_R(M, R)$ for the dual module. Since R is normal, a section of B^* over the complement of a subset of codimension at least 2 extends uniquely across that subset.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Hence the natural map $B^* \to A^*$ is an isomorphism, and consequently $A^{**} \to B^{**}$ is an isomorphism. On the other hand, $A \to A^{**}$ is an isomorphism because A is free over R, while $B \to B^{**}$ is injective because B is torsion-free. The commutative diagram of these natural maps now shows that $A \to B$ is surjective. Therefore $A = B$ and hence X is normal. □

18.4 Normal schemes and birational morphisms

A second important property of normal schemes is about birational morphisms. Recall that a morphism $f\colon X \to Y$ is birational if it induces an isomorphism between dense open subsets $U \subset X$ and $V \subset Y$. We have already seen several birational morphisms which are not isomorphisms. For instance the map $\mathbb{A}^2_k \to \mathbb{A}^2_k$ given by $(x, y) \mapsto (x, xy)$ is not an isomorphism since $V(x)$ is collapsed to a point. Even when f is a homeomorphism, it may not be an isomorphism, as in the example of the cuspidal cubic curve $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2, t^3]$ of Example 1.28. The next result shows that this phenomenon does not occur if the target is a normal scheme.

Proposition 18.27. Let X and Y be integral schemes and let $f\colon X \to Y$ be a finite and birational morphism. If Y is normal, then f is an isomorphism.

Proof Being an isomorphism is a property which is local on the target, and finite morphisms are affine, so we reduce to $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ where A and B are integral domains. Since f is dominant and finite, the corresponding ring map $\phi\colon A \to B$ is injective (Example **??**), so we may assume that it is given by an inclusion $A \subset B$ with B a finitely generated A-module. As f is birational, ϕ induces an isomorphism of the function fields $K(A) \simeq K(B)$, so we may view $A \subset B$ as subrings of $K(A)$. As B is finite over A and A is integrally closed in $K(A)$, the inclusion $A \subset B$ must be an equality, so f is an isomorphism. □

Finally, we mention the following result about rational maps of projective varieties.

Proposition 18.28. Let X be a normal variety over a field k and let Y be a projective variety. Let $f\colon X \dashrightarrow Y$ be a rational map. Then there is an open set $U \subset X$ such that $X - U$ has no components of codimension 1 and such that f extends to a unique morphism $\overline{f}\colon U \to Y$.

Proof Embed Y as a closed subvariety of $\mathbb{P}^n_k$, and let $K = K(X)$. The rational map f determines a K-point of Y and hence a point $(a_0 : \cdots : a_n) \in \mathbb{P}^n(K)$.

Let $P \in X$ be a point of codimension 1. Since X is normal, the local ring $R = \mathcal{O}_{X,P}$ is a DVR with fraction field K. Let v be its valuation and let t be a local parameter. After multiplying all the a_i by a common power of t, we may assume that every a_i lies in R and that at least one, say a_j, is a unit. The ratios $a_i/a_j \in R$ then define a morphism

$$\operatorname{Spec} R \longrightarrow D_+(x_j) \subset \mathbb{P}^n_k$$

whose restriction to the generic point is the original K-point. This morphism factors through

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Y: the inverse image of the closed subset $Y \subset \mathbb{P}^n_k$ contains the generic point of $\operatorname{Spec} R$, and hence is all of $\operatorname{Spec} R$.

We now extend this morphism to a neighborhood of P. Choose an affine open neighborhood W of P on which the finitely many elements $a_i/a_j \in R$ are represented by regular functions. These functions define a morphism from W to the affine chart $D_+(x_j)$. The finitely many equations defining $Y \cap D_+(x_j)$ vanish in the local ring at P, so after shrinking W they vanish on W. We obtain a morphism $W \to Y$ whose restriction to the generic point agrees with f. Hence f is defined at P. Let U be the largest open subset of X on which f is defined. The preceding argument shows that $X - U$ contains no point of codimension 1, so every component of $X - U$ has codimension at least 2. The extension $\overline{f}\colon U \to Y$ is unique because Y is separated and two morphisms that represent the same rational map agree on the dense generic point of the integral scheme U. □

Corollary 18.29. Let X and Y be normal projective varieties over k, and let $\phi\colon X \dashrightarrow Y$ be a birational map. Then ϕ is defined at every codimension-1 point of X, and ϕ^{-1} is defined at every codimension-1 point of Y.

Proof Apply Proposition 18.27 to ϕ and to its inverse rational map ϕ^{-1}. □

Example 18.30. Let $X \subset \mathbb{P}^2_k$ be the plane curve defined by $x_0x_2^2 = x_1^3 + x_0x_1^2$. Consider the projection from the point $(0:0:1)$,

$$f\colon X - \{(0:0:1)\} \longrightarrow \mathbb{P}^1_k, \qquad (x_0 : x_1 : x_2) \mapsto (x_0 : x_1).$$

To see that f extends over $(0:0:1)$, we work over the affine open set $D_+(x_2) = \mathbb{A}^2_k$ with affine coordinates $u = x_0/x_2$ and $v = x_1/x_2$. In this chart, the equation of X becomes $u = v^3 + uv^2$ and $(0:0:1)$ corresponds to the origin $P = (0,0)$. The restricted morphism f is then given by $(u,v) \mapsto (u:v)$. From the relation $u = v^3 + uv^2$, we can express u in terms of v as $u = v^3(1-v^2)^{-1}$ in $\mathcal{O}_{X,P}$. Since $1 - v^2$ is invertible in $\mathcal{O}_{X,P}$, this shows that $t = v$ is a local parameter at P. In terms of t, the map f at P is given by

$$(u,v) \longmapsto \left(\frac{t^3}{1-t^2} : t\right).$$

Dividing by t shows that the extension is given by $(u,v) \longmapsto (t^2 : 1 - t^2)$. Hence the extension of f sends $(0:0:1)$ to $(0:1)$.

Note that while f extends, the projection $\mathbb{P}^2 - \{(0:0:1)\} \to \mathbb{P}^1_k$ does not. △

18.5 Curves

In this final section, we apply the results of the previous sections to study curves. Recall that a *curve* is a variety of dimension 1 over a field k.

The normalization $\pi\colon \overline{X} \to X$ of a curve is finite and birational by Theorem 18.18, and that a normal curve is regular by Example 18.15. So every curve is birational to a regular curve, namely its normalization.

Because of this, much of the analysis reduces to studying regular curves. In this setting, the local ring $\mathcal{O}_{X,P}$ at each closed point P is a regular local ring of Krull dimension 1.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Equivalently, for each k-point P, the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ is 1-dimensional as a vector space over k. An element $t \in \mathfrak{m}_P$ whose class generates $\mathfrak{m}_P/\mathfrak{m}_P^2$ is called a *local parameter* at P. For actual computations, the following example will be useful.

Example 18.31 (Local parameters for plane curves)**.** Let $X \subset \mathbb{A}_k^2 = \operatorname{Spec} k[x,y]$ be the affine curve defined by an equation $f(x,y) = 0$ and let $P = (a,b)$ be a k-point of X. Then $\mathfrak{m}_P/\mathfrak{m}_P^2$ is generated by $x-a$ and $y-b$, and in $\mathfrak{m}_P/\mathfrak{m}_P^2$ these classes satisfy the relation

$$\frac{\partial f}{\partial x}(P)\cdot(x-a) + \frac{\partial f}{\partial y}(P)\cdot(y-b) = 0. \tag{18.4}$$

So if $\frac{\partial f}{\partial y}(P) \neq 0$, then we can express $y-b$ in terms of $x-a$, and $\mathfrak{m}_P/\mathfrak{m}_P^2$ is generated by $x-a$ alone. Consequently: if $\frac{\partial f}{\partial y}(P) \neq 0$, then $x-a$ is a local parameter at P. △

Another special fact is that when X is regular, any rational map $f\colon X \dashrightarrow Y$ to a projective curve Y extends to a morphism $f\colon X \to Y$.

Proposition 18.32. Let X be a regular curve and let Y be a projective k-scheme. Then every rational map $f\colon X \dashrightarrow Y$ extends uniquely to a morphism $\overline{f}\colon X \to Y$. In particular, two birational regular projective curves are isomorphic.

Proof Let $U \subset X$ be an open subset where f is a morphism. If there is a point $P \in X - U$, then P is a codimension 1 point since X is a curve, and f extends over P by Proposition 18.27. Repeating this for the finitely many points in $X - U$, we obtain the extension $\overline{f}\colon X \to Y$ extending f. This extension is unique because two morphisms to a separated scheme that agree on a dense open subset are equal (Proposition 13.14).

For the last statement, suppose X and Y are birational regular projective curves. Then the birational map and its inverse both extend to morphisms $g\colon X \to Y$ and $h\colon Y \to X$ by what we have just shown. Since $h \circ g$ and $g \circ h$ restrict to the identity on a dense open subset, they equal id_X and id_Y respectively, so g is an isomorphism. □

Morphisms between curves $f\colon X \to Y$ are particularly simple. There are two cases: either f is not dominant, in which case the image $f(X)$ is a point (its closure being a proper, closed and irreducible subset of Y); or it is dominant, in which case $f(X)$ is a dense open subset of Y.

Proposition 18.33. Let $f\colon X \to Y$ be a dominant morphism of curves over k. Then
(i) f is generically finite and $K(X)$ is a finite field extension of $K(Y)$.
(ii) If X is projective over k, then f is surjective and Y is also projective.
(iii) If both X and Y are projective, then f is finite.

Proof (i) follows from Proposition 14.11. (ii): If X is projective, then $f(X)$ is closed in Y, hence all of Y, as f is dominant. (iii): If both X and Y are projective, then f is finite, by Proposition 15.15. □

Proposition 18.34. Any regular curve is quasi-projective.

Proof Let X be a regular curve, and choose a finite affine open cover $\{U_1, \dots, U_r\}$. For each i, choose a closed embedding $U_i \hookrightarrow \mathbb{A}_k^{n_i}$, and let $\overline{U_i}$ be its closure in $\mathbb{P}_k^{n_i}$, with the reduced structure. Then $\overline{U_i}$ is an integral projective curve containing U_i as a dense open subscheme. By Proposition 18.31, the inclusion $U_i \hookrightarrow \overline{U_i}$ extends to a morphism $f_i \colon X \to \overline{U_i}$. Together these give a morphism

$$f = (f_1, \dots, f_r) \colon X \longrightarrow \overline{U_1} \times_k \cdots \times_k \overline{U_r}.$$

Let Y be the closure of $f(X)$, with the reduced structure. Since one of the f_i is nonconstant, Y is an integral projective curve. Let $p_i \colon Y \to \overline{U_i}$ be the restriction of the i-th projection, so that $f_i = p_i \circ f$.

We claim that $f_P^\sharp \colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is an isomorphism for every $P \in X$. Choose i such that $P \in U_i$. Since $p_i \circ f|_{U_i}$ is the inclusion $U_i \hookrightarrow \overline{U_i}$, the composition $\mathcal{O}_{\overline{U_i},P} \to \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is an isomorphism. It follows that $f_P^\sharp$ is surjective. Moreover, f is dominant by the definition of Y, so $f_P^\sharp$ is injective by Exercise 8.11.65. This proves the claim.

Exercise 8.11.65 now shows that $f \colon X \to Y$ is a local isomorphism. In particular, it is birational. We show that it is injective on points. Suppose that $f(P) = f(Q) = y$. Since f is a local isomorphism, there are open neighborhoods $P \in A$, $Q \in B$, and $y \in V_A, V_B \subset Y$ such that $f|_A \colon A \to V_A$ and $f|_B \colon B \to V_B$ are isomorphisms. After restricting to $V = V_A \cap V_B$, their inverses give two morphisms $s_P, s_Q \colon V \to X$. The open subset V is nonempty (it contains the generic point) and s_P and s_Q agree there, being the inverse of the birational map f. Since V is reduced and X is separated, Proposition 13.14 gives $s_P = s_Q$. Evaluating at y shows that $P = Q$, so f is injective.

By Exercise 8.11.65, an injective local isomorphism is an open embedding. Hence X is isomorphic to an open subscheme of the projective curve Y, and is therefore quasi-projective. □

Corollary 18.35. Let X be a projective curve over k. Then the normalization $\overline{X}$ is also projective.

Proof By the previous proposition, the regular curve $\overline{X}$ is quasi-projective, so we may view it as a locally closed subscheme of some projective space $\mathbb{P}_k^n$. On the other hand, the normalization map $\overline{X} \to X$ is finite, hence proper. Since X is proper over k, the scheme $\overline{X}$ is also proper over k. Therefore $\overline{X}$ is closed in $\mathbb{P}_k^n$ by Corollary **??**, and is therefore projective. □

Theorem 18.36. The assignment $X \mapsto K(X)$ defines an anti-equivalence between the following categories:

(i) The category of regular projective curves over k and dominant morphisms.
(ii) The category of finitely generated field extensions of k of transcendence degree 1 and maps of k-algebras.

Proof If X and Y are two regular projective curves, any rational map $X \dashrightarrow Y$ extends to a morphism. This, combined with Theorem 14.9 on page 259, shows that the functor $X \mapsto K(X)$ is fully faithful.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

By Lemma 14.8, any such field extension K arises as the function field of a projective curve X. Since X and its normalization $\overline{X}$ are birational, we have $K(X) = K(\overline{X})$. Since $\overline{X}$ is also projective, the contravariant functor $X \mapsto K(X)$ is essentially surjective. □

The theorem is not true if one of the conditions *regular*, *projective* or *dimension 1* is omitted. For instance, $\mathbb{P}^1_k$ and the cuspidal cubic $x_1^3 - x_0x_2^2 = 0$ are birational but not isomorphic (one is regular, the other is not). Likewise, the two affine schemes $\mathbb{A}^1_k$ and $\mathbb{A}^1_k - \{0\}$ are birational but not isomorphic (non-isomorphic coordinate rings). The same holds for $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ (non-isomorphic Picard groups).

Proposition 18.37. For a regular curve X over k, there is a one-to-one correspondence between nonconstant rational functions $g \in K(X) - k$ and nonconstant morphisms $f\colon X \to \mathbb{P}^1_k$.

Proof Write $K(\mathbb{P}^1) = k(t)$ for the function field of $\mathbb{P}^1_k$, where $t = t_1/t_0$. Given a nonconstant morphism $f\colon X \to \mathbb{P}^1_k$, the pullback $g = f^\sharp(t) \in K(X)$ is nonconstant, so $g \in K(X) - k$. Conversely, given $g \in K(X) - k$, let $U \subset X$ be the maximal open set where g is regular, and let $V \subset X$ be the maximal open set where $1/g$ is regular. Then $U \cap V$ is non-empty (it contains the generic point), and $U \cup V = X$ (for each $P \in X$, the local ring $\mathcal{O}_{X,P}$ is a DVR, hence either g or $1/g$ belongs to it). Now g defines a morphism $U \to D_+(t_0) \simeq \mathbb{A}^1_k$, and $1/g$ defines a morphism $V \to D_+(t_1) \simeq \mathbb{A}^1_k$. These glue to a morphism $f\colon X \to \mathbb{P}^1_k$, which is nonconstant because $g \notin k$. □

Proposition 18.38. Let X be a normal integral proper curve over a field k. Then X is projective.

Proof By Chow's Lemma there is a birational morphism $\pi\colon X' \to X$ with X' projective over k. Every fiber of π is finite: the fibers over the dense open U are single points, and a fiber of dimension 1 would contain an irreducible component of X', while every such component meets $U = \pi^{-1}(U)$. Hence π is finite by Proposition 15.15. Now let $V = \operatorname{Spec} A \subset X$ be affine, so that $\pi^{-1}(V) = \operatorname{Spec} B$ with B a finite A-module. As π is birational and X' is integral, $A \subset B \subset K(X)$, and B is integral over A. Since A is integrally closed in $K(X)$, we get $B = A$. Therefore π is an isomorphism and $X \simeq X'$ is projective. □

18.6 Exercises

Exercise 18.6.1. Let k be algebraically closed of characteristic different from 2 and 3, and let $X \subset \mathbb{A}^2_k$ be the curve given by $y^2 = x^3 - 1$. Show that all the local rings $\mathcal{O}_{X,p}$ are discrete valuation rings, and hence that X is a normal variety. More precisely, if (a, b) is a point of X, show that $x - a$ is a local parameter if $b \neq 0$ and that y is a local parameter when $b = 0$. HINT: Note that $y^2 - b^2 = x^3 - a^3$.

Exercise 18.6.2. Show that if A is a normal integral domain, then so is $A[x]$. HINT: Apply Gauss's Lemma.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 18.6.3. Let $K/\mathbb{Q}$ be a finite field extension of $\mathbb{Q}$ and let $\alpha \in K$. Show that α is integral over $\mathbb{Z}$ if and only if its minimal polynomial $f(x) \in \mathbb{Q}[x]$ has integer coefficients.

Exercise 18.6.4. Let $d \neq 1$ be a square-free integer and let $K = \mathbb{Q}(\sqrt{d})$. Let O_K denote the ring of integers in K, that is, the integral closure of $\mathbb{Z}$ in K.

a) Show that $\sqrt{d} \notin \mathbb{Q}$, so that $K/\mathbb{Q}$ has degree 2, and that the minimal polynomial of an element $w = a + b\sqrt{d}$, where $a, b \in \mathbb{Q}$ and $b \neq 0$, is
$$W^2 - 2aW + (a^2 - b^2 d).$$
b) Show that if w is integral over $\mathbb{Z}$, then $2a, a^2 - b^2 d \in \mathbb{Z}$. HINT: Use Exercise 18.6.3
c) Show that $O_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1 \mod 4$ and $O_K = \mathbb{Z}[\sqrt{d}]$ otherwise.

Exercise 18.6.5 (Involutions)**.** Let B be an integral domain and let $\sigma\colon B \to B$ be a ring automorphism satisfying $\sigma^2 = \mathrm{id}_B$. The map σ extends to an involution of the fraction field K of B by the assignment $\sigma(x/y) = \sigma(x)/\sigma(y)$. Let
$$A = B^\sigma = \{x \in B \mid \sigma(x) = x\}$$
be the ring of invariants and let $L = K(A)$ be its field of fractions.

a) Show that $L = K^\sigma = \{z \in K \mid \sigma(z) = z\}$.
b) Show that B is integral over A. Show that if B is normal, then A is normal. HINT: Any $x \in B$ satisfies a quadratic equation with coefficients in A.
c) Let k be a field with $\operatorname{char} k \neq 2$. Consider $B = k[x, y]$ and the involution $\sigma(x, y) = (-x, -y)$. Show that $B^\sigma = k[x^2, xy, y^2]$. Deduce that the ring $A = k[u, v, w]/(v^2 - uw)$ is normal.

Exercise 18.6.6. Let A be a unique factorization domain in which 2 is invertible. Recall that an element $f \in A$ is *squarefree* if it is not divisible by the square of any irreducible element. Let $f \in A$ be squarefree and such that $y^2 - f$ is irreducible in $A[y]$, and set $B = A[y]/(y^2 - f)$. Show that B is normal as follows.

a) Show that any $z \in K(B)$ can be expressed as $z = a + by$ for some $a, b \in K(A)$.
b) Show that z satisfies the monic equation $X^2 - 2aX + (a^2 - b^2 f) = 0$. Now assume that z is integral over B, and let $\overline{z} = a - by$ be its conjugate. Conjugation preserves integrality, and B is integral over A, so both z and $\overline{z}$ are integral over A. Hence their sum $2a$ and product $a^2 - b^2 f$ are integral over A; deduce from the normality of A that they belong to A.
c) Use that f is squarefree to show that $a, b \in A$. HINT: Write $b = u/v$ with $u, v \in A$ coprime, and consider $a^2 - b^2 f = (a^2 v^2 - u^2 f)/v^2$.
d) Taking $A = \mathbb{C}[x, y]$ and $f = xy$, deduce that the quadric cone $\mathbb{C}[x, y, z]/(z^2 - xy)$ is normal.

Exercise 18.6.7. This exercise will show that for a normal Noetherian integral domain A,
$$A = \bigcap_{\operatorname{ht} \mathfrak{p}=1} A_{\mathfrak{p}}. \tag{18.5}$$
Let $B = \bigcap_{\operatorname{ht} \mathfrak{p}=1} A_{\mathfrak{p}}$ and assume for a contradiction that $A \subset B$ is strict.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Let $x \in B - A$ and set $I = (A : x) = \{\, a \in A \mid ax \in A \,\}$.

a) Show that I is a proper nonzero ideal.
b) Show that $(A_{\mathfrak{p}} : x) = (A : x)_{\mathfrak{p}}$ for each $\mathfrak{p}$, using that A is Noetherian, and that if $\operatorname{ht} \mathfrak{p} = 1$, then $(A_{\mathfrak{p}} : x) = A_{\mathfrak{p}}$.
c) Deduce from *b*) that no height 1 prime contains I. Since $I \neq 0$, no prime of height 0 contains I either, so every prime containing I has height at least 2.
   Let $\mathfrak{q}$ be a minimal prime containing I and set $R = A_{\mathfrak{q}}$, $\mathfrak{m} = \mathfrak{q}R$.
d) Show that R is a Noetherian local domain and that $\sqrt{IR} = \mathfrak{m}$.
e) Prove that there exists an $n \geqslant 1$ with $\mathfrak{m}^n \subset IR$ and $\mathfrak{m}^{n-1} \not\subset IR$.
   Pick $y \in \mathfrak{m}^{n-1} - IR$ and set $z = yx$.
f) Show that $z \notin R$, but $\mathfrak{m}z \subset R$.
   Since $\mathfrak{m}z \subset R$, the set $\mathfrak{m}z$ is an ideal of R. There are two cases:
g) Case $\mathfrak{m}z \subset \mathfrak{m}$: Apply the Cayley-Hamilton theorem to the multiplication by z map $\mathfrak{m} \to \mathfrak{m}$ to show that z is integral over R, hence $z \in R$, contradicting *f*).
h) Case $\mathfrak{m}z \not\subset \mathfrak{m}$: Show that $\mathfrak{m}z = R$. Deduce that $\mathfrak{m} = (z^{-1})$ is principal. Apply the Principal Ideal Theorem to show that $\operatorname{ht} \mathfrak{m} \leqslant 1$, contradicting *c*).
i) Conclude that $A = B$.

Exercise 18.6.8. Let X be a Noetherian integral scheme. Show that X is normal if and only if $\mathcal{O}_{X,x}$ is normal for every *closed* point $x \in X$. HINT: If $y \in X$ is any point, there is a closed point in $\overline{\{y\}}$.

Exercise 18.6.9. Let X be a Noetherian normal scheme and let $\mathcal{E}$ be a locally free sheaf of finite rank on X. Show that if $U \subset X$ is an open set so that $X - U$ has codimension at least 2, then the restriction map $\Gamma(X, \mathcal{E}) \to \Gamma(U, \mathcal{E}|_U)$ is an isomorphism. HINT: This may be reduced to the usual statement for $\mathcal{E} = \mathcal{O}_X$.

Exercise 18.6.10. Show that the projective scheme $X = V_+(x_1^2 - 15x_0x_2) \subset \mathbb{P}^2_{\mathbb{Z}}$ is normal but not regular, and determine its singular points. HINT: Work in the standard affine charts. For normality, use that a hypersurface satisfies S_2 and check regularity at the codimension-one points.

Exercise 18.6.11. Use Theorem 18.22 to give another proof of the equality $\mathcal{O}_{\mathbb{A}^2_k}(U) = k[x, y]$, where $U = \mathbb{A}^2_k - V(x, y)$.

Exercise 18.6.12. Let X be the scheme obtained by identifying two distinct points $q_1, q_2 \in \mathbb{A}^2_k$, and let $p \in X$ be their common image. So $\mathcal{O}_X(U)$ consists of those regular functions on the preimage of U in $\mathbb{A}^2_k$ which take the same value at q_1 and q_2.

a) Show that X is an integral scheme of dimension 2.
b) Show that $\operatorname{sing}(X)$ consists of the single point p.
c) Show that $U = X - \{p\}$ is isomorphic to $\mathbb{A}^2_k - \{q_1, q_2\}$, and find a regular function $f \in \mathcal{O}_X(U)$ which does not extend to all of X.
d) Conclude that X is not normal, and that the conclusion of Theorem 18.22 fails for X.

Exercise 18.6.13 (The cone over a rational quartic curve)**.** Consider $X = \operatorname{Spec} A$, where A

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is the $\mathbb{C}$-algebra

$$A = \mathbb{C}[u^4, u^3v, uv^3, v^4] \simeq \mathbb{C}[t_0, t_1, t_3, t_4]/(t_0t_4 - t_1t_3, t_1^3 - t_0^2t_3, t_1^2t_4 - t_0t_3^2, t_3^3 - t_1t_4^2).$$

a) Show that X is a variety of dimension 2.
b) Show that X is regular outside the origin $P = V(t_0, t_1, t_3, t_4)$.
c) Show that $t_1^2/t_0 = u^2v^2 = t_3^2/t_4$ defines a regular function on $X - P$ which does not extend to all of X.
d) Conclude that X is regular in codimension 1 but not normal.
e) Show that the ideal (t_0, t_1^2) is not principal in A. HINT: Compare the monomials occurring in a hypothetical generator.

Exercise 18.6.14. Let $A = k[x, xy, y^2, y^3]$. Show that $\overline{A} = k[x, y]$ and that $\operatorname{Spec}(A)$ is regular at every codimension 1 point.

Exercise 18.6.15. Show that the normalization of the scheme $X = \operatorname{Spec} \mathbb{Z}[6i]$ is given by $\operatorname{Spec} \mathbb{Z}[i]$.

Exercise 18.6.16. Let $A = \mathbb{Z}[\sqrt{5}]$. By Exercise 18.6.4, $\overline{A} = \mathbb{Z}[(1 + \sqrt{5})/2]$. Describe the fibers of the normalization morphism $\operatorname{Spec} \overline{A} \to \operatorname{Spec} A$.

Exercise 18.6.17. Let k be a field and let $A = k[x, y]/(y^3 - x^4)$. Show that $\overline{A} = k[t]$, where $t = y/x$, and describe the fibers of the normalization map $\operatorname{Spec} \overline{A} \to \operatorname{Spec} A$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

19

Divisors

Much of the geometry of a variety is controlled by its closed subsets of codimension 1. On a curve these are the points, on a surface the curves lying on it, and in general they play the role of hypersurfaces. In studying them it is convenient to allow formal sums with integer coefficients. A formal sum $D = \sum_Z n_Z\, Z$ of codimension 1 subvarieties is called a *divisor*.

Divisors arise naturally from rational functions. The zeros and poles of a nonzero rational function f occur in codimension 1, each with a multiplicity, and together form its divisor $\operatorname{div}(f) = \sum_Z \operatorname{ord}_Z(f) \cdot Z$. On $\mathbb{P}^1_{\mathbb{C}}$ this is the sum of zeros minus poles, and a divisor arises like this precisely when the coefficients sum to zero. For general varieties the situation is much more subtle, and the failure of a divisor to be $\operatorname{div}(f)$ is measured by the *class group*.

Divisors are also closely tied to invertible sheaves. A codimension 1 subscheme is typically cut out by one equation locally, and in that case, its ideal sheaf is invertible. Conversely, a nonzero rational section of an invertible sheaf has orders of vanishing along codimension 1 subvarieties and hence a corresponding divisor. Making this correspondence precise is one of the main goals of the chapter.

With few exceptions we assume that X is integral, Noetherian and normal. Noetherianness lets us decompose closed subsets into finitely many irreducible components. Integrality lets us work in a fixed function field $K(X)$. Normality ensures that the local ring at the generic point of a prime divisor is a discrete valuation ring, so that the order of vanishing along it is well defined.

19.1 Weil divisors

Throughout this chapter, X denotes a separated Noetherian integral normal scheme.

A *prime divisor* is a closed integral subscheme $Z \subset X$ of codimension 1. A *Weil divisor*, or simply *divisor*, is a finite formal sum

$$D = \sum_Z m_Z Z$$

where $m_Z \in \mathbb{Z}$ and the Z's are prime divisors. We say that a divisor D is *effective* if each $m_Z \geqslant 0$, and we write $D \geqslant 0$. The *support* of D is defined to be the closed subset $\operatorname{Supp}(D) = \bigcup_{m_Z \neq 0} Z$.

The set of divisors forms a group, which will be denoted by $\operatorname{Div}(X)$.

Example 19.1. On $X = \mathbb{P}^1_{\mathbb{C}}$, the prime divisors are simply the closed points. Here are some

examples of Weil divisors:

$$D_1 = 3 \cdot (1:0) - 5 \cdot (0:1), \qquad D_2 = (1:1) + 5 \cdot (0:1)$$
$$D_1 + D_2 = 3 \cdot (1:0) + (1:1). \qquad \triangle$$

The reader may wonder why we include the assumption that X is normal. Certainly, the above definition can be made for any scheme, but normality gives us a well-behaved notion of 'order of vanishing', or 'multiplicity' for rational functions.

More precisely, as the scheme X is normal, all the local rings $\mathcal{O}_{X,P}$ are integrally closed in their fraction field, $K(X)$. This means that if $Z \subset X$ is a prime divisor, and $\zeta \in X$ is the generic point of Z, then the local ring $\mathcal{O}_{X,\zeta}$ is a 1-dimensional normal ring, and hence a discrete valuation ring (see Appendix **??**). We write

$$\operatorname{ord}_Z \colon K(X) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

for the corresponding valuation. More explicitly, the function ord_Z can be defined as follows. Given a nonzero $f \in \mathcal{O}_{X,\zeta}$, we can write it as $f = u \cdot t^m$, where t is a generator for the maximal ideal of $\mathcal{O}_{X,\zeta}$ and u is a unit, and define the valuation of f at Z to be $\operatorname{ord}_Z(f) = m$. Finally, we extend this to elements of $K(X)$ by setting $\operatorname{ord}_Z(f/g) = \operatorname{ord}_Z(f) - \operatorname{ord}_Z(g)$ and $\operatorname{ord}_Z(0) = \infty$. The number $\operatorname{ord}_Z(f)$ is called the *order of vanishing* of f at Z. If $\operatorname{ord}_Z(f)$ is positive, we say that f vanishes to that order along Z. If $\operatorname{ord}_Z(f)$ is negative, then we say that f has a *pole* of order $-\operatorname{ord}_Z(f)$ along Z.

If $X = \operatorname{Spec} A$ is affine, then every prime divisor is of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of height 1. Hence

$$\operatorname{Div}(\operatorname{Spec} A) = \bigoplus_{\operatorname{ht}\mathfrak{p}=1} \mathbb{Z} \cdot V(\mathfrak{p})$$

can be identified with the free abelian group on all height 1 prime ideals. For such a prime $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ is a discrete valuation ring, consisting of exactly the fractions $f = a/b \in K(A)$ where $b \notin \mathfrak{p}$, or equivalently, $\operatorname{ord}_{V(\mathfrak{p})}(f) \geqslant 0$. In light of Theorem **??**, which says that $A = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p}$, we conclude that $\operatorname{ord}_{V(\mathfrak{p})}(f) \geqslant 0$ for every prime divisor $V(\mathfrak{p})$ if and only if $f \in A$. Moreover, $f \in K(A)$ has zero order of vanishing along every prime divisor if and only if f is invertible in $A^\times$. In general, we have:

Proposition 19.2. For a rational function $f \in K(X)^\times$, we have
(i) $\operatorname{ord}_Z(f) \geqslant 0$ for all prime divisors $Z \subset X$ if and only if $f \in \mathcal{O}_X(X)$.
(ii) $\operatorname{ord}_Z(f) = 0$ for all prime divisors Z if and only if $f \in \mathcal{O}_X(X)^\times$.

Proof For an integral scheme X, a rational function $f \in K(X)$ is regular if and only if $f \in \mathcal{O}_{X,P}$ for every $P \in X$, which by the affine case above means that $\operatorname{ord}_Z(f) \geqslant 0$ for every prime divisor Z. Applying this to both f and f^{-1} proves the second item. □

This is the most basic example of the close links between rational functions and codimension 1 subschemes on normal schemes. If a rational function is regular outside a closed subset of codimension at least 2, then it is regular everywhere.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Definition 19.3 (Principal divisors)**.** For $f \in K(X)^\times$, we define its *principal divisor* by

$$\operatorname{div}(f) = \sum_Z \operatorname{ord}_Z(f) Z, \tag{19.1}$$

where the sum runs over all prime divisors.

The sum (19.1) is finite. Indeed, choose an affine open $U = \operatorname{Spec} A$ on which f is regular. A prime divisor Z either lies in $X - U$, and there are only finitely many such, since $X - U$ has finitely many codimension 1 components; or else Z meets U, in which case $\operatorname{ord}_Z(f) \geqslant 0$, and $\operatorname{ord}_Z(f) > 0$ only for the finitely many components of the zero set $V(f) \subset U$.

Note that if f is a regular function, i.e., $f \in \mathcal{O}_X(X)$, then $\operatorname{div}(f)$ is an effective divisor. It is supported along the zero set of f, with multiplicity $\operatorname{ord}_Z(f)$ along each component Z.

Example 19.4. On $\operatorname{Spec} \mathbb{Z}$, a Weil divisor is of the form

$$D = n_1 V(p_1) + \cdots + n_r V(p_r)$$

where the p_i are prime numbers. In the function field, $\mathbb{Q}$, the 'rational function' $f = p_1^{n_1} \cdots p_r^{n_r}$ satisfies $\operatorname{div}(f) = D$, so every divisor is a principal divisor. △

Example 19.5. Let k be an algebraically closed field and $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, with the standard chart $U_0 = D_+(x_0) = \operatorname{Spec} k[t]$, where $t = x_1/x_0$. Consider the rational function

$$f = t^2(t-1)^{-1} \in k(t) = K(\mathbb{P}^1_k).$$

For $a \in k$, both t and $t - 1$ are invertible in the local ring $\mathcal{O}_{U_0,(1:a)} = k[t]_{(t-a)}$ except when $a = 0$ or $a = 1$. At $(1 : 0)$ the element t is a local parameter and $f = t^2(\text{unit})$, so $\operatorname{ord}_{(1:0)}(f) = 2$. Similarly $\operatorname{ord}_{(1:1)}(f) = -1$. At the remaining point $(0 : 1)$ we use the coordinate $s = x_0/x_1 = t^{-1}$ and rewrite f as $(s^{-1})^2(s^{-1} - 1)^{-1} = s^{-1}(1 - s)^{-1}$. From this, we see that $\operatorname{ord}_{(0:1)}(f) = -1$ and hence

$$\operatorname{div}(f) = 2(1 : 0) - (1 : 1) - (0 : 1).$$

More generally, a nonzero polynomial $f(t) = (t-a_1)^{n_1} \cdots (t-a_r)^{n_r}$ has $\operatorname{ord}_{(1:a_i)}(f) = n_i$, and $\operatorname{ord}_{(0:1)}(f) = -\sum_i n_i$. △

Example 19.6. Let k be an algebraically closed field of characteristic different from 2, and let $X = V(x^3 - y^3 + y) \subset \mathbb{A}^2_k$. The function x vanishes at $(0, 0)$, $(0, 1)$, and $(0, -1)$. At $(0, 0)$ the relation $x^3 = y(y^2 - 1)$ shows that x is a local parameter and $y = x^3 \cdot (\text{unit})$. The same kind of local computation at the other two points gives

$$\operatorname{div}(x) = (0, 0) + (0, 1) + (0, -1), \qquad \operatorname{div}(y) = 3(0, 0).$$

△

The identity $\operatorname{ord}_Z(fg) = \operatorname{ord}_Z(f) + \operatorname{ord}_Z(g)$ shows that the principal divisors form a subgroup of $\operatorname{Div}(X)$. The *class group of* X is defined as the quotient group

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\langle \operatorname{div}(f) \mid f \in K(X)^\times \rangle.$$

It can be thought of as the group that captures the obstructions to a divisor being the divisor of a rational function.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Two Weil divisors D and D' are said to be *linearly equivalent* (written $D \sim D'$) if they have the same image in $\mathrm{Cl}(X)$; that is, $D - D' = \mathrm{div}(f)$ for some $f \in K(X)^\times$.

If $X = \operatorname{Spec} A$, the divisor of a rational function $f \in K(X)^\times$ is equal to

$$\mathrm{div}(f) = \sum_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{ord}_{\mathfrak{p}}(f)\, V(\mathfrak{p}).$$

Hence $\mathrm{div}(f) = 0$ if and only if $\mathrm{ord}_{\mathfrak{p}}(f) = 0$ for all height 1 primes, which by Proposition 19.2 happens if and only if $f \in A^\times$. Hence the kernel of the map $\mathrm{div}\colon K(X)^\times \to \mathrm{Div}(\operatorname{Spec} A)$ equals $A^\times$. The cokernel is by definition the class group $\mathrm{Cl}(\operatorname{Spec} A)$, so we have the exact sequence

$$0 \to A^\times \to K(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\operatorname{Spec} A) \to \mathrm{Cl}(\operatorname{Spec} A) \to 0. \tag{19.2}$$

Example 19.7. The class group of $\operatorname{Spec} \mathbb{Z}$ is trivial, as $\mathbb{Z}$ is a principal ideal domain, so every prime ideal is principal, not just the ones of height 1. (See also Example 19.4.) △

Example 19.8. The class group of $\operatorname{Spec} \mathbb{Z}[i]$ is also trivial. This follows by the analysis of Example 2.40, which showed that every prime ideal of $\mathbb{Z}[i]$ is principal. △

Example 19.9. Let A be a discrete valuation ring and let $X = \operatorname{Spec} A$. In A, the only nonzero prime ideal is the maximal ideal $\mathfrak{m}$. Therefore, if $P \in X$ denotes the closed point, we have $\mathrm{Div}(X) = \mathbb{Z} \cdot P$. Any Weil divisor on X is principal: if t is a generator for $\mathfrak{m}$, then $\mathrm{div}(t^n) = n \cdot P$ for each $n \in \mathbb{Z}$. Hence $\mathrm{Cl}(X) = 0$. △

Both the terms 'divisor' and 'class group' come from algebraic number theory and their origins can be traced back to Kummer's work on Fermat's Last Theorem. In this setting, the class groups are designed to measure how far the base ring is from being a unique factorization domain.

Proposition 19.10. Let A be a normal Noetherian integral domain. Then the following are equivalent:

(i) $\mathrm{Cl}(\operatorname{Spec} A) = 0$.
(ii) Every height 1 prime ideal in A is a principal ideal.
(iii) A is a unique factorization domain.

Proof The equivalence of (ii) and (iii) is a fact from commutative algebra (see Proposition C.42). It remains to show the equivalence (i) ⇔ (ii).

(ii) ⇒ (i): Write $X = \operatorname{Spec} A$. If Z is a prime divisor in X, then $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$, and as Z has codimension 1, $\mathfrak{p}$ is of height 1. By assumption (ii), $\mathfrak{p} = (f)$ for an element $f \in A$, and then $\mathrm{div}(f) = Z$, since f generates $\mathfrak{p}A_{\mathfrak{p}}$ and vanishes along no other prime divisor. This shows that $\mathrm{Cl}(\operatorname{Spec} A) = 0$.

(i) ⇒ (ii): Assume that $\mathrm{Cl}(\operatorname{Spec} A) = 0$. Let $\mathfrak{p}$ be a prime of height 1, and let $Z = V(\mathfrak{p}) \subset X$. By assumption, there is an $f \in K(X)^\times$ so that $\mathrm{div}(f) = Z$. We want to show that in fact $f \in A$ and that $\mathfrak{p} = (f)$. For the first statement: as $\mathrm{div}(f) = Z$, we have $\mathrm{ord}_{\mathfrak{q}}(f) = 0$ for $\mathfrak{q} \neq \mathfrak{p}$ and $\mathrm{ord}_{\mathfrak{p}}(f) = 1$, so f lies in $\mathcal{O}_X(X) = A$ by Proposition 19.2.

Secondly, to prove that f generates $\mathfrak{p}$, consider any nonzero element $g \in \mathfrak{p}$. Then $\mathrm{ord}_{\mathfrak{p}}(g) \geqslant 1$ and $\mathrm{ord}_{\mathfrak{q}}(g) \geqslant 0$ for all $\mathfrak{q} \neq \mathfrak{p}$. It follows that $\mathrm{ord}_{\mathfrak{q}}(g/f) = \mathrm{ord}_{\mathfrak{q}}(g) -$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\operatorname{ord}_{\mathfrak{q}}(f) \geqslant 0$ for *every* height 1 prime ideal $\mathfrak{q} \in \operatorname{Spec} A$, and hence $g/f \in A$ by Proposition 19.2. It follows that $g \in fA$, and so $\mathfrak{p} = (f)A$ is principal. □

Example 19.11. The class group of affine n-space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ is trivial, as the polynomial ring is a unique factorization domain. △

The class group of an open set

Let $U \subset X$ be a nonempty open subset. If $Z \subset X$ is a prime divisor, then $Z \cap U$ is either empty (if $Z \subset X - U$) or a prime divisor on U. This allows us to define the restriction of a divisor $D = \sum n_Z Z$ to U by the formula

$$D|_U = \sum_{Z \cap U \neq \varnothing} n_Z \cdot Z \cap U. \tag{19.3}$$

This defines a map of groups $\operatorname{Div}(X) \to \operatorname{Div}(U)$. Note that any divisor with support in $X - U$ is sent to 0 via this map.

If f is a rational function on X, the restriction $f|_U$ is a rational function on U, and $\operatorname{ord}_{Z\cap U}(f|_U) = \operatorname{ord}_Z(f)$ whenever $Z \cap U \neq \varnothing$ (the local rings at the generic points of Z and of $Z \cap U$ are the same). Therefore, the restriction map sends principal divisors to principal divisors.

In this setting, it is natural to ask how the class group of X and U are related. The answer is given by the theorem below.

Proposition 19.12 (Localization sequence). Let $W \subset X$ be a closed subset and let $U = X - W$ and let $Z_1, \ldots, Z_r$ be the prime divisors corresponding to the codimension 1 components of W. There is an exact sequence

$$\bigoplus_{i=1}^{r} \mathbb{Z}\, Z_i \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0, \tag{19.4}$$

where the map $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ is induced by $[Z] \mapsto [Z \cap U]$.

Proof If Z is a prime divisor on U, the closure in X is a prime divisor in X, so the rightmost map in (19.4) is surjective. The composition of the two maps is zero, since each Z_i is supported in W and hence restricts to 0 on U.

For exactness in the middle, suppose D is a divisor which is principal on U. Then $D|_U = \operatorname{div}(f)|_U$ for some $f \in K(U) = K(X)$, where $\operatorname{div}(f)$ is computed on X. The divisor $D - \operatorname{div}(f)$ therefore restricts to zero on U, so it is supported in $X - U$ and hence it must be a linear combination of the Z_i's. As $\operatorname{div}(f)$ is principal, the classes of D and of $D - \operatorname{div}(f)$ agree in $\operatorname{Cl}(X)$, so $[D]$ is in the image of the leftmost map. □

Corollary 19.13. If $Z \subset X$ is a closed subset of codimension at least 2, then $\operatorname{Cl}(X - Z) \simeq \operatorname{Cl}(X)$.

Example 19.14. If P denotes the origin in $\mathbb{A}^2_k$, then $\operatorname{Cl}(\mathbb{A}^n_k - P) = \operatorname{Cl}(\mathbb{A}^n_k) = 0$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This exact sequence, combined with Proposition 19.10, is very useful for computing class groups. Here is one example; we will see many more in Section 19.5.

Proposition 19.15. Let $X = \mathbb{P}_k^{n_1} \times_k \cdots \times_k \mathbb{P}_k^{n_r}$, and let $H_i = \mathrm{pr}_i^{-1}(h_i)$, where h_i is a hyperplane in the ith factor. Then

$$\mathrm{Cl}(X) \simeq \mathbb{Z}\, H_1 \oplus \cdots \oplus \mathbb{Z}\, H_r.$$

In particular, $\mathrm{Cl}(\mathbb{P}_k^n) \simeq \mathbb{Z}$.

Proof Let $N = \sum n_i$. We have

$$X - \bigcup_{i=1}^{r} H_i = \bigcap_{i=1}^{r} (X - H_i) \simeq \prod_{i=1}^{r} \mathbb{A}_k^{n_i} \simeq \mathbb{A}_k^N.$$

The localization sequence shows that $H_1, \ldots, H_r$ generate $\mathrm{Cl}(X)$. It remains to show that these classes are independent. Suppose $\sum_{i=1}^r a_i H_i = \mathrm{div}(f)$ for some $f \in K(X) = k(x_1, \ldots, x_N)$. Then $\mathrm{ord}_Z(f) = 0$ for every Z which intersects $X - \bigcup_{i=1}^r H_i$. Therefore f is a unit in $k[x_1, \ldots, x_N]$, hence a constant. So $\mathrm{div}(f) = 0$, and hence $a_1 = \cdots = a_r = 0$. □

The isomorphism $\mathrm{Cl}(\mathbb{P}_k^n) \simeq \mathbb{Z}$ is better described by the *degree map*. On projective space, a divisor is of the form $D = \sum n_i V_+(G_i)$, where $G_i \in k[x_0, \ldots, x_n]$ are irreducible homogeneous polynomials. We define

$$\deg D = \sum_i n_i \deg G_i.$$

As rational functions on $\mathbb{P}_k^n$ have the form F/G where $F, G \in k[x_0, \ldots, x_n]$ are of the same degree, this descends to a map $\mathrm{Cl}(\mathbb{P}_k^n) \to \mathbb{Z}$.

19.2 The sheaf associated to a Weil divisor

For a Weil divisor $D = \sum n_Z Z$, we define a sheaf $\mathcal{O}_X(D)$, whose sections are the rational functions with 'poles at worst along D'. There are several ways to express this condition. The simplest way is to require that a rational function $f \in K(X)$ satisfies $\mathrm{ord}_Z(f) \geqslant -n_Z$ for all Z, meaning that the order of the pole of f along Z is at most n_Z. Equivalently, we can formulate it by saying that $\mathrm{div}(f) + D \geqslant 0$. Informally, this means that the 'pole part' of $\mathrm{div}(f)$ is cancelled out by adding D.

We will let $\mathbf{K}(X)$ denote the constant sheaf on the function field $K(X)$.

Definition 19.16. For a divisor $D = \sum n_Z Z$, we define the subsheaf of $\mathbf{K}(X)$ by

$$\mathcal{O}_X(D)(U) = \{\, f \in K(X) \mid \mathrm{ord}_Z(f) \geqslant -n_Z \text{ for all } Z \text{ with } Z \cap U \neq \varnothing \,\}$$

for each nonempty open subset $U \subset X$.

By convention, $0 \in K(X)$ has infinite order of vanishing along every prime divisor, so $0 \in \mathcal{O}_X(D)(U)$ for every U.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The condition in the bracket becomes less restrictive on a smaller subset $U' \subset U$, so $\mathcal{O}_X(D)(U) \subset \mathcal{O}_X(D)(U')$. Hence $\mathcal{O}_X(D)$ becomes a presheaf with inclusion maps as the restriction maps. It is not hard to see that the sheaf axioms are satisfied, so $\mathcal{O}_X(D)$ is a sheaf. It is also an $\mathcal{O}_X$-module, because if $a \in \mathcal{O}_X(U)$ and $f \in \mathcal{O}_X(D)(U)$, then $\operatorname{ord}_Z(af) = \operatorname{ord}_Z(a) + \operatorname{ord}_Z(f) \geqslant -n_Z$ for all Z meeting U, so $\mathcal{O}_X(D)$ is stable under multiplication by elements of $\mathcal{O}_X$.

If $V \subset X$ is a nonempty open subset, then by identifying $K(V) = K(X)$, we have

$$\mathcal{O}_X(D)|_V = \mathcal{O}_V(D|_V), \tag{19.5}$$

where $D|_V$ is the restricted divisor (19.3).

Example 19.17. If $D = 0$, then $\mathcal{O}_X(D) = \mathcal{O}_X$. Indeed, $\Gamma(U, \mathcal{O}_X(0))$ consists of the rational functions $g \in K(X)$ such that $\operatorname{ord}_Z(g) \geqslant 0$ for every prime divisor Z intersecting U. Any such g must be regular on U, by Proposition 19.2, so $\Gamma(U, \mathcal{O}_X(0)) = \Gamma(U, \mathcal{O}_X)$. △

Example 19.18. If $Z \subset X$ is a prime divisor, then $\mathcal{O}_X(-Z) \simeq \mathcal{I}_Z$ is the ideal sheaf defining Z. Indeed, $\Gamma(U, \mathcal{O}_X(-Z))$ consists of the rational functions $g \in K(X)$ so that $\operatorname{ord}_Z(g) \geqslant 1$ and $\operatorname{ord}_{Z'}(g) \geqslant 0$ for every prime divisor $Z' \neq Z$. In other words, $g \in \mathcal{O}_X(U)$ and vanishes along $Z \cap U$. △

Example 19.19. Let k be a field and $X = \mathbb{P}^1_k$. We will compute $\Gamma(X, \mathcal{O}_X(D))$ for the divisor $D = V(x_1) = (1:0)$. We use the standard covering by $U_0 = \operatorname{Spec} k[x_1/x_0] = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[x_0/x_1] = \operatorname{Spec} k[t^{-1}]$.

The group $\Gamma(U_0, \mathcal{O}_X(D))$ consists of rational functions $f \in k(t)$ with a pole of order at most 1 at $t = 0$ and no other poles. This implies that

$$\Gamma(U_0, \mathcal{O}_X(D)) = \{\, \alpha t^{-1} + p(t) \mid \alpha \in k, p(t) \in k[t] \,\}.$$

As $(1:0)$ does not lie in U_1, we have $\Gamma(U_1, \mathcal{O}_X(D)) = \Gamma(U_1, \mathcal{O}_X) = k[t^{-1}]$.

The intersection of these subspaces in $k(t)$ is equal to

$$\Gamma(X, \mathcal{O}_X(D)) = k \oplus k\, t^{-1}.$$

We will see shortly that $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. △

Proposition 19.20. For every divisor D, the sheaf $\mathcal{O}_X(D)$ is a coherent sheaf.

Proof Write $D = \sum n_i Z_i$. We may assume that $X = \operatorname{Spec} A$ is affine. Define $M = \Gamma(X, \mathcal{O}_X(D))$. We first show that $\mathcal{O}_X(D) \simeq \widetilde{M}$. For every $g \in A$, we claim that

$$M_g = \Gamma(D(g), \mathcal{O}_X(D))$$

inside $K(X)$.

For the inclusion "$\subset$", let $h/g^m \in M_g$, and let Z be a prime divisor meeting $D(g)$. Then g is invertible at the generic point of Z, so $\operatorname{ord}_Z(g) = 0$. Therefore $\operatorname{ord}_Z(h/g^m) = \operatorname{ord}_Z(h) \geqslant -n_Z$ where n_Z is the coefficient of Z in D. Hence h/g^m belongs to $\Gamma(D(g), \mathcal{O}_X(D))$.

Conversely, let $f \in \Gamma(D(g), \mathcal{O}_X(D))$. The negative coefficients of $\operatorname{div}(f) + D$ can occur only along prime divisors contained in $V(g)$. There are only finitely many such divisors, and

$\operatorname{ord}_Z(g) > 0$ for each of them. Hence, for some large $m > 0$, we have $\operatorname{div}(g^m f) + D \geqslant 0$ on X. It follows that $g^m f \in M$ and hence $f \in M_g$. This proves that $\mathcal{O}_X(D) \simeq \widetilde{M}$.

It remains to show that M is a finitely generated A-module. Choose a nonzero $g \in A$ vanishing along every Z_i. After replacing g by a sufficiently large power, we may assume that $\operatorname{ord}_{Z_i}(g) \geqslant n_i$ whenever $n_i > 0$. If $f \in M$, then $\operatorname{ord}_Z(gf) \geqslant 0$ for every prime divisor Z, and hence $gf \in A$ by Proposition 19.2. Therefore gM is an ideal of A. Since A is Noetherian, the ideal gM is finitely generated. Multiplication by the nonzero element g gives an isomorphism $M \simeq gM$, so M is finitely generated. Hence $\mathcal{O}_X(D)$ is coherent. □

Lemma 19.21. Let D and E be two divisors on X. Then $\mathcal{O}_X(D) = \mathcal{O}_X(E)$ as subsheaves of $\mathbf{K}(X)$ if and only if $D = E$.

Proof One direction is clear. Assume $\mathcal{O}_X(D) = \mathcal{O}_X(E)$ and write $D = m_1 Z_1 + \cdots + m_r Z_r$ and $E = n_1 Z_1 + \cdots + n_r Z_r$ (where some of the m_i, n_i may be zero). Let ζ be the generic point of the prime divisor Z_1. Since X is normal, $\mathcal{O}_{X,\zeta}$ is a discrete valuation ring and we may choose a generator t for its maximal ideal. The open set $U = X - Z_2 \cup \cdots \cup Z_r$ contains ζ and $D|_U = m_1 Z_1$, so

$$\mathcal{O}_X(D)_\zeta = \{\, f \in K(X) \mid \operatorname{ord}_{Z_1}(f) \geqslant -m_1 \,\} = t^{-m_1}\mathcal{O}_{X,\zeta},$$

and likewise $\mathcal{O}_X(E)_\zeta = t^{-n_1}\mathcal{O}_{X,\zeta}$. Since $\mathcal{O}_X(D)_\zeta = \mathcal{O}_X(E)_\zeta$ inside $K(X)$, we see that $t^{n_1 - m_1}$ is invertible in $\mathcal{O}_{X,\zeta}$ and hence $m_1 = n_1$. Repeating the argument for the other components, we conclude that $D = E$. □

Lemma 19.22. For $h \in K(X)^\times$, we have

$$\mathcal{O}_X(\operatorname{div}(h)) = \frac{1}{h} \cdot \mathcal{O}_X \tag{19.6}$$

Moreover, a divisor D is principal if and only if $\mathcal{O}_X(D) \simeq \mathcal{O}_X$.

Proof Let $U \subset X$ be an open set and let $f \in K(X)$. For any $f \in K(X)^\times$,

$$\begin{aligned} f \in \Gamma(U, \mathcal{O}_X(\operatorname{div} h)) &\iff \operatorname{div}(f) + \operatorname{div}(h) \geqslant 0 \text{ on } U \\ &\iff \operatorname{div}(fh) \geqslant 0 \text{ on } U \\ &\iff fh \in \Gamma(U, \mathcal{O}_X), \end{aligned}$$

where the last equivalence follows from Proposition 19.2. This proves (19.6).

For the last statement: If $D = \operatorname{div}(h)$ is principal, multiplication by h gives an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{O}_X$. Conversely, suppose $\varphi\colon \mathcal{O}_X \to \mathcal{O}_X(D)$ is an isomorphism. Then $g = \varphi(1)$ is a nonzero rational function since φ is injective. As φ is $\mathcal{O}_X$-linear, $\varphi(s) = as$ for every local section s of $\mathcal{O}_X$, so $\mathcal{O}_X(D) = g\,\mathcal{O}_X$ as subsheaves of $\mathbf{K}(X)$. Applying (19.6) to $h = 1/g$, we obtain $\mathcal{O}_X(\operatorname{div}(1/g)) = g\,\mathcal{O}_X = \mathcal{O}_X(D)$, and hence $D = \operatorname{div}(1/g)$ by Lemma 19.21. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Definition 19.23. A divisor D is called *Cartier* if it is locally principal: there is an open covering $\{U_i\}_{i\in I}$ of X and rational functions $f_i \in K(X)^\times$ such that for each $i \in I$,

$$D|_{U_i} = \operatorname{div}(f_i). \tag{19.7}$$

Such a collection $\{(U_i, f_i)\}$ is called *Cartier data* for D.

The f_i can be thought of as the local equations that define the divisor. In general, not all codimension 1 subschemes can be locally defined by such an equation, so not all Weil divisors are Cartier.

The data (U_i, f_i) have to satisfy a certain condition to be able to arise from a divisor. On the overlap $U_i \cap U_j$, we must have $\operatorname{div}(f_i) = D|_{U_i\cap U_j} = \operatorname{div}(f_j)$. By Proposition 19.2, we see that $f_i f_j^{-1}$ and $f_j f_i^{-1}$ must be regular on $U_i \cap U_j$ for all i, j, hence

$$f_i f_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)^\times. \tag{19.8}$$

Conversely, given any (U_i, f_i) satisfying this requirement, we may construct a unique Cartier divisor D. Indeed, (19.8) implies that $\operatorname{ord}_Z(f_i) = \operatorname{ord}_Z(f_j)$ for any prime divisor Z intersecting $U_i \cap U_j$. This means that the coefficient of Z in D is well defined by $n_Z = \operatorname{ord}_Z(f_i)$ for any i such that $Z \cap U_i \neq \varnothing$. However, different data (U_i, f_i) and (V_j, g_j) may define the same divisor.

The Cartier divisors form a subgroup $\operatorname{CaDiv}(X) \subset \operatorname{Div}(X)$. Indeed, if D and E are Cartier, we may pass to a common refinement of the two coverings and represent both by data (U_i, f_i) and (U_i, g_i); then $(U_i, f_i g_i)$ is Cartier data for $D + E$, and (U_i, f_i^{-1}) is Cartier data for $-D$. Every principal divisor is Cartier, with the covering consisting of X alone.

The Cartier data (U_i, f_i) is often a more useful way to present a divisor than its components and multiplicities, especially in local computations. It also gives a natural way to describe of the sheaf $\mathcal{O}_X(D)$. Over each U_i, we have, by (19.5) and Lemma 19.22,

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i}. \tag{19.9}$$

In the notation of Section 11.1, $\mathcal{O}_X(D)$ is the invertible sheaf determined by the transition functions $g_{ij} = f_j f_i^{-1}$.

Let L be an invertible sheaf on X. A *rational section* of L is an element of the stalk L_ξ at the generic point ξ of X. Concretely, such an element is represented by a section $s \in \Gamma(V, L)$ on some nonempty open subset V.

Let $s \in \Gamma(V, L)$ be a nonzero rational section of L. Choose an open covering $\{U_i\}$ of X trivializing L, with local generators $e_i \in \Gamma(U_i, L)$ and transition functions $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$ determined by $e_i = g_{ij} e_j$. Over $U_i \cap V$ we may write

$$s|_{U_i\cap V} = f_i e_i, \qquad f_i \in \mathcal{O}_X(U_i \cap V). \tag{19.10}$$

Since $s \neq 0$ and X is integral, each f_i is a nonzero regular function on the dense open set $U_i \cap V$, hence a well-defined element of $K(X)^\times$. On $U_i \cap U_j \cap V$ we have $f_j e_j = s = f_i e_i = f_i g_{ij} e_j$, hence

$$g_{ij} = f_j / f_i, \tag{19.11}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

and so $f_i f_j^{-1} = g_{ij}^{-1} \in \mathcal{O}_X(U_i \cap U_j)^\times$. So (U_i, f_i) is Cartier data for a divisor which we denote by $\operatorname{div}(s)$. Concretely, the coefficient of a prime divisor Z in $\operatorname{div}(s)$ is equal to $\operatorname{ord}_Z(f_i)$ for any i with $Z \cap U_i \neq \varnothing$.

The divisor $\operatorname{div}(s)$ is independent of the choices made. Replacing s by a section agreeing with it on a smaller nonempty open set does not change the f_i as elements of $K(X)$. If $\{(W_\alpha, t_\alpha)\}$ is another trivializing cover with local generators, and $s = h_\alpha t_\alpha$ over $W_\alpha \cap V$, then over $U_i \cap W_\alpha$ the generators e_i and t_α differ by an invertible regular function, hence so do f_i and h_α; therefore they have the same order of vanishing along every prime divisor meeting $U_i \cap W_\alpha$.

Example 19.24. Consider the rational section of $L = \mathcal{O}_{\mathbb{P}^1}(-3)$ given by

$$s = \frac{x_0^2 - x_0x_1}{x_1^5},$$

Over $D_0(x_0) = \operatorname{Spec} k[t]$ with $t = x_1/x_0$, the sheaf L is generated by x_0^{-3}, and $s = f_0\, x_0^{-3}$ where

$$f_0 = \frac{(x_0^2 - x_0x_1)\, x_0^3}{x_1^5} = \frac{1-t}{t^5}.$$

This vanishes to order 1 at $t = 1$ and has a pole of order 5 at $t = 0$, so $\operatorname{div}(s)|_{U_0} = (1 : 1) - 5(1 : 0)$.

Over $U_1 = \operatorname{Spec} k[u]$ with $u = x_0/x_1$, L is generated by x_1^{-3}, and $s = f_1\, x_1^{-3}$ where $f_1 = (x_0^2 - x_0x_1)/x_1^2 = u^2 - u$. This has zeros of order 1 at $u = 0$ and $u = 1$, so $\operatorname{div}(s)|_{U_1} = (0 : 1) + (1 : 1)$. In all, we obtain

$$\operatorname{div}(s) = (0 : 1) + (1 : 1) - 5(1 : 0). \qquad \triangle$$

Theorem 19.25. Let D be a Weil divisor on X. The following are equivalent:

(i) D is a Cartier divisor;
(ii) $\mathcal{O}_X(D)$ is an invertible sheaf;
(iii) there is an invertible sheaf L on X and a nonzero rational section s of L such that $D = \operatorname{div}(s)$.

In this case, if (U_i, f_i) is Cartier data for D, then

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i}. \tag{19.12}$$

Proof (i) ⇒ (ii): If (U_i, f_i) is Cartier data for D, then (19.9) shows that $\mathcal{O}_X(D)|_{U_i} = f_i^{-1}\mathcal{O}_{U_i}$, which is (19.12). It is free of rank one, so $\mathcal{O}_X(D)$ is invertible.

(ii) ⇒ (i): Choose a covering $\{U_i\}$ of X with local generators $e_i \in \Gamma(U_i, \mathcal{O}_X(D))$. Since $\mathcal{O}_X(D)$ is a subsheaf of $\mathbf{K}(X)$, each e_i is a nonzero rational function and $\mathcal{O}_X(D)|_{U_i} = e_i\mathcal{O}_{U_i}$. Put $f_i = e_i^{-1}$. By (19.5) and Lemma 19.22 applied on U_i,

$$\mathcal{O}_{U_i}(D|_{U_i}) = \mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i} = \mathcal{O}_{U_i}(\operatorname{div}(f_i)),$$

so $D|_{U_i} = \operatorname{div}(f_i)$ by Lemma 19.21. Hence (U_i, f_i) is Cartier data for D.

(i) ⇒ (iii): Let (U_i, f_i) be Cartier data for D. By the first implication $L = \mathcal{O}_X(D)$ is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

invertible, with local generator $e_i = f_i^{-1}$ over U_i. Let $V = X - \operatorname{Supp}(D)$. Then $D|_V = 0$, so $\mathcal{O}_X(D)|_V = \mathcal{O}_V$ by (19.5) and the constant function 1 is a section of $\mathcal{O}_X(D)$ over V. Write s for this nonzero rational section. Over $U_i \cap V$ we have $s = 1 = f_i \cdot f_i^{-1} = f_i e_i$, so the local expression of s with respect to e_i is f_i, and (19.10) gives $\operatorname{div}(s) = D$.

(iii) $\Rightarrow$ (i): This is immediate from the construction of $\operatorname{div}(s)$, which produced Cartier data for it. □

By the above proof, there is a canonical rational section s_D of $\mathcal{O}_X(D)$ in $\operatorname{div}(s_D) = D$. It is given by the constant function 1 on $X - \operatorname{Supp}(D)$.

Proposition 19.26. Let L and L' be invertible sheaves on X, and let s and s' be nonzero rational sections of L and L'. Then $\operatorname{div}(s) = \operatorname{div}(s')$ if and only if there is an isomorphism $\varphi\colon L \to L'$ with $\varphi(s) = s'$.

Proof We keep the notation of (19.10) and (19.11): local generators e_i of L over a covering $\{U_i\}$, with $s = f_i e_i$ and $g_{ij} = f_j/f_i$.

If $\varphi(s) = s'$ for an isomorphism φ, then the $\varphi(e_i)$ are local generators of L' and $s' = f_i\varphi(e_i)$. The two sections have the same local expressions, so $\operatorname{div}(s') = \operatorname{div}(s)$.

For the converse it suffices to construct, for $D = \operatorname{div}(s)$, an isomorphism $\psi\colon L \to \mathcal{O}_X(D)$ with $\psi(s) = s_D$. Granting this, the same construction applied to s' gives $\psi'\colon L' \to \mathcal{O}_X(D)$ with $\psi'(s') = s_D$, and $\varphi = \psi'^{-1} \circ \psi$ is the required isomorphism. Since (U_i, f_i) is Cartier data for D, formula (19.12) says that $\mathcal{O}_X(D)|_{U_i}$ is generated by f_i^{-1}. So there is a unique isomorphism of $\mathcal{O}_{U_i}$-modules

$$\psi_i\colon L|_{U_i} \longrightarrow \mathcal{O}_X(D)|_{U_i}, \qquad \psi_i(e_i) = f_i^{-1}.$$

These agree on overlaps: over $U_i \cap U_j$,

$$\psi_j(e_i) = \psi_j(g_{ij}e_j) = g_{ij}\psi_j(e_j) = f_j/f_i \cdot 1/f_j = 1/f_i = \psi_i(e_i),$$

and e_i generates L over $U_i \cap U_j$, so $\psi_i = \psi_j$ there. Hence the ψ_i glue to an isomorphism $\psi\colon L \to \mathcal{O}_X(D)$, and over $U_i \cap V$ we get $\psi(s) = \psi(f_i e_i) = f_i \cdot f_i^{-1} = 1 = s_D$. □

Taking $(L', s') = (\mathcal{O}_X(D), s_D)$, we see that every pair (L, s) with $\operatorname{div}(s) = D$ is isomorphic to $(\mathcal{O}_X(D), s_D)$. In particular $L \simeq \mathcal{O}_X(D)$.

Example 19.27. Let $\mathbb{P}^n_k$. A nonzero homogeneous polynomial $F(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ of degree $d > 0$ defines a closed subscheme $V_+(F) \subset \mathbb{P}^n_k$ of codimension 1. The corresponding Weil divisor D is Cartier. Concretely, D is described by the standard covering $U_i = \operatorname{Spec} k[x_0/x_i, \dots, x_n/x_i]$ and dehomogenizations $f_i = x_i^{-d}F$. Over $U_i \cap U_j$, we have

$$f_i = (x_j/x_i)^d \cdot f_j \tag{19.13}$$

and x_j/x_i is an invertible regular function there. Two homogeneous polynomials F, G of the same degree d define linearly equivalent divisors, because their quotient is a rational function on $\mathbb{P}^n_k$.

Finally, the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(D)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$, because both sheaves are described by the same transition functions $(x_i/x_j)^d$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 19.28. Let D and E be Cartier divisors on X. Then:
(i) $\mathcal{O}_X(D+E) \simeq \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$;
(ii) $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ if and only if D and E are linearly equivalent.

Proof We may choose a common affine covering $\{U_i\}$ so that D and E are represented by Cartier data (U_i, f_i) and (U_i, g_i). Then $D+E$ is determined by $(U_i, f_i g_i)$, and by (19.12),

$$\mathcal{O}_X(D+E)|_{U_i} = (f_i g_i)^{-1}\mathcal{O}_{U_i} = f_i^{-1} g_i^{-1}\mathcal{O}_{U_i}.$$

Since U_i is affine, the tensor product is computed by

$$\big(f_i^{-1}\mathcal{O}_{U_i}\big) \otimes_{\mathcal{O}_{U_i}} \big(g_i^{-1}\mathcal{O}_{U_i}\big) \simeq f_i^{-1} g_i^{-1}\mathcal{O}_{U_i}, \quad (a f_i^{-1}) \otimes (b g_i^{-1}) \mapsto ab\, f_i^{-1} g_i^{-1}.$$

These isomorphisms are compatible with the transition functions on the two sides, so they glue to the isomorphism in (i).

For (ii), apply (i) to E and $-E$ to get $\mathcal{O}_X(E) \otimes \mathcal{O}_X(-E) \simeq \mathcal{O}_X$, so $\mathcal{O}_X(-E)$ is an inverse of $\mathcal{O}_X(E)$ in $\operatorname{Pic}(X)$. Tensoring by $\mathcal{O}_X(-E)$ and using (i) again, we find that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ if and only if $\mathcal{O}_X(D-E) \simeq \mathcal{O}_X$, which by Lemma 19.22 happens if and only if $D-E$ is principal. □

It is not true in general that every divisor is Cartier. We will see some basic counterexamples in Section 19.5. However, if X is *factorial*, meaning that all the local rings $\mathcal{O}_{X,P}$ are UFD's, then every divisor is Cartier:

Theorem 19.29. Let X be a Noetherian integral scheme such that every local ring $\mathcal{O}_{X,P}$ is a UFD. Then every Weil divisor is Cartier.

Proof Since the property of being Cartier is a local condition, we may assume that $X = \operatorname{Spec} A$ is affine. As Cartier divisors form a group, it suffices to show that every prime divisor is Cartier, that is, locally principal.

Let $Z = V(\mathfrak{p})$ be a prime divisor on X, where $\mathfrak{p} \subset A$ is a height 1 prime ideal. Over the open set $U = X - Z$, the divisor $Z|_U = 0$ is trivial, hence principal.

Let $Q \in Z$, corresponding to a prime ideal $\mathfrak{q} \supset \mathfrak{p}$. Then $\mathfrak{p}A_{\mathfrak{q}}$ has height 1, and $A_{\mathfrak{q}} = \mathcal{O}_{X,Q}$ is a UFD, so $\mathfrak{p}A_{\mathfrak{q}}$ is principal by Proposition C.42. Multiplying a generator by a unit, we may take it to be an element $f \in \mathfrak{p}$. Write $\mathfrak{p} = (f_1, \dots, f_r)$. Each f_i lies in $(f)A_{\mathfrak{q}}$, say $f_i = (a_i/b_i)f$ with $b_i \notin \mathfrak{q}$, so taking $b = b_1 \cdots b_r$ we get $\mathfrak{p}A_b \subseteq (f)A_b$. The reverse inclusion holds because $f \in \mathfrak{p}$, so $\mathfrak{p}A_b = (f)A_b$. The open set $W = D(b) \subset X$ now contains Q (since $b \notin \mathfrak{q}$), and the ideal of $Z \cap W$ is generated by f. In particular $Z \cap W$ is the only prime divisor of W along which f vanishes, and $\operatorname{ord}_Z(f) = 1$ because f generates $\mathfrak{p}A_{\mathfrak{p}}$. Therefore $Z|_W = \operatorname{div}(f)|_W$. As this holds for every $Q \in Z$, the sets W together with U cover X, so Z is Cartier. □

Corollary 19.30. Let X be a Noetherian regular scheme. Then every Weil divisor is Cartier.

Proof If X is regular, then each local ring $\mathcal{O}_{X,P}$ is a regular local ring, hence a UFD by the Auslander–Buchsbaum theorem (see Theorem 16.6). □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The classes of Cartier divisors form a subgroup of $\mathrm{Cl}(X)$. Indeed, Cartier divisors are closed under addition and negation, and the notions of principal Cartier divisor and principal Weil divisor agree.

Theorem 19.31. The map $D \mapsto \mathcal{O}_X(D)$ induces an isomorphism from the subgroup of $\mathrm{Cl}(X)$ generated by Cartier divisors onto $\mathrm{Pic}(X)$.

Proof By Proposition 19.28, the assignment $D \mapsto \mathcal{O}_X(D)$ is additive, and two Cartier divisors determine isomorphic invertible sheaves if and only if they are linearly equivalent. It therefore induces an injective homomorphism from the subgroup of $\mathrm{Cl}(X)$ generated by Cartier divisors to $\mathrm{Pic}(X)$.

To prove surjectivity, let L be an invertible sheaf on X. Choose a nonzero rational section s of L; such a section exists because L is trivial on some nonempty open subset of X. By Theorem 19.25, the divisor $\mathrm{div}(s)$ is Cartier, and by Proposition 19.26, $L \simeq \mathcal{O}_X(\mathrm{div}(s))$. Hence every element of $\mathrm{Pic}(X)$ comes from a Cartier divisor. □

In particular, if every Weil divisor on X is Cartier, then the preceding theorem gives a natural isomorphism

$$\mathrm{Cl}(X) \simeq \mathrm{Pic}(X).$$

For instance, this holds when X is regular.

Combining this with our computations of the class groups of affine and projective space, we obtain the following.

Theorem 19.32. Let k be a field and let $n \geqslant 1$. Then
(i) $\mathrm{Pic}(\mathbb{A}^n_k) = \mathrm{Cl}(\mathbb{A}^n_k) = 0$.
(ii) $\mathrm{Pic}(\mathbb{P}^n_k) = \mathrm{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$, generated by the class of a hyperplane.

In particular, every invertible sheaf on $\mathbb{P}^n_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(m)$ for some integer m.

It is quite common to move back and forth between divisors and invertible sheaves, and the two notions are often used interchangeably. Each has its advantages. A Weil divisor is both concrete and geometric object and clearly related to the codimension 1 subschemes. Invertible sheaves, on the other hand, have excellent algebraic properties: they can be tensored, dualized and pulled back, which makes them better suited for computations. Moreover, invertible sheaves make sense on an arbitrary scheme, without assuming that X is normal, integral or Noetherian.

19.3 Effective divisors and linear systems

We have seen that any Cartier divisor is the divisor of a rational section of an invertible sheaf. If we restrict to *global* sections, this divisor is effective, and it can be regarded as an honest closed subscheme.

Let L be an invertible sheaf and let $s \in \Gamma(X, L)$ be a nonzero global section. Writing $s|_{U_i} = f_i e_i$ as in (19.10), the $f_i \in \mathcal{O}_X(U_i)$ are now regular, so $\mathrm{ord}_Z(f_i) \geqslant 0$ for every

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

prime divisor Z and $\operatorname{div}(s)$ is effective. Conversely, if D is an effective Cartier divisor, then $1 \in \Gamma(X, \mathcal{O}_X(D))$, so the canonical section s_D of Theorem 19.25 is in fact a global section.

We may view such a D as the closed subscheme $V(s)$ cut out by the ideal sheaf $\mathcal{I}_D = \mathcal{O}_X(-D) \simeq L^\vee$, which is generated by f_i over U_i. The support of D is the zero set of s, with multiplicity $\operatorname{ord}_Z(s)$ along each component Z. Algebraically, this subscheme is defined by the ideal sheaf $\mathcal{I}_D = \mathcal{O}_X(-D) \simeq L^\vee$. The ideal sheaf sequence (16.7) takes the form

$$0 \longrightarrow L^\vee \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_D \longrightarrow 0. \tag{19.14}$$

Proposition 19.33. Let L be an invertible sheaf and let $s, s' \in \Gamma(X, L)$ be nonzero global sections. Then $\operatorname{div}(s) = \operatorname{div}(s')$ if and only if $s' = \lambda s$ for some $\lambda \in \mathcal{O}_X(X)^\times$.

Proof By Proposition 19.26 with $L' = L$, we have $\operatorname{div}(s) = \operatorname{div}(s')$ if and only if there is an isomorphism $\varphi\colon L \to L$ with $\varphi(s) = s'$. Now

$$\operatorname{Hom}_{\mathcal{O}_X}(L, L) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X(X),$$

so every such φ is multiplication by an element $\lambda \in \mathcal{O}_X(X)$, and φ is an isomorphism precisely when λ is invertible. □

Suppose now that X is a projective variety over an algebraically closed field k, so that $\mathcal{O}_X(X) = k$. Let D be a Cartier divisor and let $L = \mathcal{O}_X(D)$. If D' is an effective divisor with $D' \sim D$, then $\mathcal{O}_X(D') \simeq L$ and Proposition 19.28 gives a nonzero global section of L cutting out D'. By the above proposition, we obtain a bijection

$$\left\{ \begin{array}{c} \text{effective Cartier divisors } D' \\ \text{such that } D' \sim D \end{array} \right\} \longleftrightarrow \big(\Gamma(X, L) - 0\big)/k^\times. \tag{19.15}$$

The right-hand side is the set of k-points of a projective space of dimension $\dim_k \Gamma(X, L) - 1$. It is called the *complete linear system* of L, and is denoted $|L|$, or $|D|$ when $L \simeq \mathcal{O}_X(D)$. A *linear system* is a subset of some $|D|$ corresponding to a linear subspace of $\Gamma(X, L)$.

Example 19.34. Consider the curves $D_0 = V_+(x^3 + y^3 + z^3)$ and $D_1 = V_+(xyz)$ in $\mathbb{P}^2_{\mathbb{C}}$. The first is an irreducible cubic, the second a union of three lines. Although they look quite different, they are linearly equivalent: $g = (x^3 + y^3 + z^3)/xyz$ is a rational function with $\operatorname{div}(g) = D_0 - D_1$, so $[D_0] = [D_1]$ in $\operatorname{Cl}(\mathbb{P}^2_{\mathbb{C}})$. Geometrically, the family of cubics $s\,xyz + t(x^3 + y^3 + z^3) = 0$ connects them. This is one way of thinking about linear equivalence: linearly equivalent divisors move in a common family.

△

19.4 Pullbacks of divisors

For a morphism $f\colon X \to Y$ it is natural to ask whether one can pull back divisors from Y back to X. In general, this does not work, because the codimension may change under a morphism. For instance, if f is the inclusion of $X = V_+(x_0)$ into $Y = \mathbb{P}^2_k$, then $D = V_+(x_0)$ is a divisor on Y, but $f^{-1}D = X$ has codimension 0, so it is not a divisor there.

Invertible sheaves, however, always pull back. A pullback f^*L of an invertible sheaf is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

invertible on X and a section $s \in \Gamma(V, L)$ pulls back to a section $f^*(s)$ over $f^{-1}V$. The subtle point is that $f^*(s)$ may be zero, in which case it does not define a divisor on X.

A good situation is when the morphism $f\colon X \to Y$ is dominant. In that case $f^\sharp\colon K(Y) \to K(X)$ is injective, and the pullback of a nonzero rational section $f^*(s)$ is nonzero because it is locally the pullback of a nonzero rational function.

This allows us to pull back Cartier divisors. Suppose f is dominant and let D be a Cartier divisor on Y, given by the data (U_i, g_i) with $g_i \in K(Y)^\times$ and $g_i g_j^{-1} \in \mathcal{O}_Y(U_i \cap U_j)^\times$. Then each $f^\sharp(g_i)$ is a nonzero rational function on X, and

$$f^\sharp(g_i) f^\sharp(g_j)^{-1} = f^\sharp(g_i g_j^{-1})$$

is invertible on $f^{-1}(U_i) \cap f^{-1}(U_j) = f^{-1}(U_i \cap U_j)$. Hence the data $(f^{-1}(U_i),\ f^\sharp(g_i))$ defines a Cartier divisor on X, which we denote by f^*D.

Of course, this is just the pullback of sections in disguise. By (19.12), $\mathcal{O}_Y(D)$ is generated by g_i^{-1} over U_i, so $f^*\mathcal{O}_Y(D)$ is generated by $f^\sharp(g_i)^{-1}$ over $f^{-1}(U_i)$, and therefore

$$f^*\mathcal{O}_Y(D) \simeq \mathcal{O}_X(f^*D).$$

Under this isomorphism the canonical section s_D pulls back to a nonzero rational section of $f^*\mathcal{O}_Y(D)$ with $\operatorname{div}\bigl(f^*(s_D)\bigr) = f^*D$.

Example 19.35. Assume that $\operatorname{char} k \neq 2$ and consider the plane curve

$$X = V_+(x_0^3 - x_0x_1^2 + x_1x_2^2) \subset \mathbb{P}^2_k.$$

The sheaf $\mathcal{O}_{\mathbb{P}^2_k}(1)$ restricts to an invertible sheaf L on X and the sections x_0, x_1, x_2 restrict to nonzero sections of L. The corresponding divisors on X are:

$$\begin{aligned}
\operatorname{div}(x_0) &= (0:0:1) + 2(0:1:0)\\
\operatorname{div}(x_1) &= 3(0:0:1)\\
\operatorname{div}(x_2) &= (0:1:0) + (1:1:0) + (1:-1:0).
\end{aligned}$$

In particular, $2(0:0:1) \sim 2(0:1:0)$, as their difference is the principal divisor $\operatorname{div}(x_1/x_0)$. It turns out that $(0:0:1) \not\sim (0:1:0)$. This is in fact not so easy to prove (it will be a consequence of Proposition 23.18). More generally, any two lines define linearly equivalent divisors on X, since for two linear forms ℓ and ℓ', ℓ/ℓ' restricts to a rational function on X, so the divisors that the two lines cut out on X are linearly equivalent.

On the other hand, the section $s = x_0^3 - x_0x_1^2 + x_1x_2^2$ of $\mathcal{O}_{\mathbb{P}^2_k}(3)$ does not define a divisor on X, as it restricts to the zero section in $\mathcal{O}_X(3)$.

△

19.5 Extended examples

Proposition 19.36. Let $X = \operatorname{Spec} A$, where $A = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. Then

$$\operatorname{Cl}(X) \simeq \mathbb{Z}/2.$$

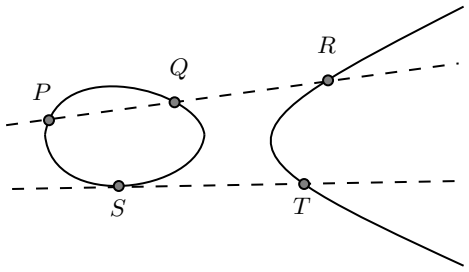


Figure 19.1 $P + Q + R \sim 2S + T$

Proof Consider the point $P = (0, 1)$ corresponding to the maximal ideal $\mathfrak{m} = (x, y - 1)$. The first step will be to show that the complement $X - P$ has trivial class group. Note that $X - P = D(y - 1)$ is the spectrum of the ring

$$A_{y-1} = \mathbb{R}[x, y]_{y-1}/(x^2 + y^2 - 1)$$

By the formulas for stereographic projection from P, there is an isomorphism of rings

$$A_{y-1} \simeq \mathbb{R}[t]_{1+t^2}, \qquad (x, y) \mapsto \left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2}\right)$$

with inverse sending t to $x/(1 - y)$. As $\mathbb{R}[t]_{1+t^2}$ is a localization of a UFD, we have $\mathrm{Cl}(X - P) = 0$.

The exact sequence (19.4) applied to the open set $X - P$ shows that P generates $\mathrm{Cl}(X)$. To determine the order of P, note that $\mathrm{div}(y - 1) = 2P$, because x is a local parameter at P and $y - 1 = -(y + 1)^{-1}x^2$ in $\mathcal{O}_{X,P}$. Hence $2[P] = 0$ in $\mathrm{Cl}(X)$. By Exercise 2.7.26, $\mathfrak{m}$ is not principal, so $\mathrm{Cl}(X) = \mathbb{Z}/2$. □

For comparison, the ring $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ is a UFD (Exercise 19.6.4), so its class group is trivial.

This example also shows that the class group can change under base change. For the complexified circle $X_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[x, y]/(x^2 + y^2 - 1)$, the class group is trivial, because $\mathbb{C}[x, y]/(x^2 + y^2 - 1) \simeq \mathbb{C}[u, v]/(uv - 1) \simeq \mathbb{C}[t, t^{-1}]$, which is a UFD.

The affine quadric cone

We next compute the class group of a singular quadric surface in $\mathbb{A}^3_k$.

Proposition 19.37. Let k be a field and let $X = \mathrm{Spec}\,A$ where $A = k[x, y, z]/(xy - z^2)$. Then

$$\mathrm{Cl}(X) = \mathbb{Z}/2$$

generated by the prime divisor $L = V(y, z)$.

Proof Note that L is a line in $\mathbb{A}^3_k$, which is indeed an integral subscheme of X of codimension 1. By $xy = z^2$, we have $V(y) = V(y, z) = L$, as sets, so $X - L = D(y)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

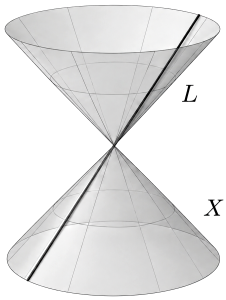


Figure 19.2 A singular quadric surface

Moreover,

$$A_y = k[x, y, y^{-1}, z]/(xy - z^2) \simeq k[y, y^{-1}, u],$$

using $x = y^{-1}z$. As this is a UFD, we obtain $\mathrm{Cl}(X - L) = 0$. Therefore $\mathrm{Cl}(X)$ is generated by L, by the Localization sequence.

We next claim that $\mathrm{div}(y) = 2L$, so $2[L] = 0$ in $\mathrm{Cl}(X)$. Since $V(y) = L$ as sets, the only prime divisor where y has nonzero order of vanishing is L. The local ring at the generic point of L is given by

$$A_{(y,z)} = k[x, y, z]_{(y,z)}/(xy - z^2).$$

Since x is invertible in this ring, we may write $y = x^{-1}z^2$ and z is a local parameter. This gives $\mathrm{ord}_L(y) = 2$ and hence $2L = \mathrm{div}(y)$.

To see that L is itself not a principal divisor, we work in the local ring at $P = (0, 0, 0)$:

$$\mathcal{O}_{X,P} = k[x, y, z]_{(x,y,z)}/(xy - z^2).$$

In this ring, $\mathfrak{p} = (y, z)$ is a height 1 prime ideal, but it cannot be principal. Indeed, the quotient $\mathfrak{m}/\mathfrak{m}^2$ (that is, the Zariski cotangent space at P) is 3-dimensional as a vector space over k, spanned by $\{\overline{x}, \overline{y}, \overline{z}\}$ and $\overline{y}, \overline{z}$ define a 2-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$. If $\mathfrak{p}$ were principal, the span of $\bar{y}$ and $\bar{z}$ would be contained in a 1-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$, which is not possible. This means that $[L] \neq 0$ in $\mathrm{Cl}(X)$ and hence $\mathrm{Cl}(X) = \mathbb{Z}/2$. □

Note that the open subscheme $U = X - \{(0, 0, 0)\}$ is regular, and the divisor L becomes Cartier on U. This example shows that removing a codimension 2 subset has no effect on Weil divisors, as $\mathrm{Cl}(U) = \mathrm{Cl}(X) = \mathbb{Z}/2$, but the subgroup of Cartier divisors might change.

Projective quadric surfaces

We next consider the projective version of the previous example.

Proposition 19.38. Let $X = V_+(xy - z^2) \subset \mathbb{P}^3_k = \mathrm{Proj}\, k[x, y, z, w]$ be the *projective quadric cone*, and let $D = V_+(x, z)$. Then

$$\mathrm{Cl}(X) = \mathbb{Z}\, D, \qquad \mathrm{Pic}(X) = 2\mathbb{Z}\, D.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof Let $H = V_+(w)$. This is a plane conic and hence a prime divisor on X. The complement $X - H$ is the affine quadric cone $\operatorname{Spec} k[x,y,z]/(xy - z^2)$, so the restriction of D generates $\operatorname{Cl}(X - H) \simeq \mathbb{Z}/2$ by the computation above. The localization sequence therefore shows that D and H generate $\operatorname{Cl}(X)$. Since the hyperplane $x = 0$ cuts out $V_+(x, z^2) = 2D$, we have $\operatorname{div}(x/w) = 2D - H$. Hence $H \sim 2D$, and consequently D alone generates $\operatorname{Cl}(X)$.

The class of H has infinite order: indeed $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^3_k}(1)|_X$, and for $d > 0$ we have $\dim_k \Gamma(X, \mathcal{O}_{\mathbb{P}^3_k}(d)|_X) > 1 = \dim_k \Gamma(X, \mathcal{O}_X)$, so $\mathcal{O}_{\mathbb{P}^3_k}(d)|_X \not\simeq \mathcal{O}_X$. As $2D \sim H$, the class of D has infinite order as well, and therefore $\operatorname{Cl}(X) \simeq \mathbb{Z}\, D$.

Finally, D is not Cartier: this is a local condition, and at the vertex it is the affine quadric cone computation above. On the other hand $2D \sim H$ is Cartier. In fact, the Cartier classes are precisely the even multiples of D, because if nD were Cartier for some odd n, then $D = nD - (n-1)D$ would be Cartier too. Hence $\operatorname{Pic}(X) = 2\mathbb{Z}\, D$. □

On the other hand, a regular quadric surface has a different class group. If k is algebraically closed and $Q \subset \mathbb{P}^3_k$ is a smooth quadric surface, then we may take $Q = V_+(xy - zw)$, which is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ by the Segre embedding. We computed the class group of this surface in Example 19.15:

$$\operatorname{Cl}(Q) = \operatorname{Pic}(Q) = \mathbb{Z}F_1 \oplus \mathbb{Z}F_2,$$

where F_1, F_2 are the fibres of the two projections.

We say that a divisor $D \sim aF_1 + bF_2$ has 'type' (a, b). For instance, a divisor of type $(1, 0)$ or $(0, 1)$ is a line on Q. We have $\mathcal{O}_{\mathbb{P}^3}(1)|_Q \simeq \mathcal{O}_Q(F_1 + F_2)$, so a hyperplane section of Q is a divisor of type $(1, 1)$, that is, a conic curve.

The 3-dimensional quadric cone

Proposition 19.39. Let $X = \operatorname{Spec} A$ where $A = k[x,y,z,w]/(xw - yz)$. Then

$$\operatorname{Cl}(X) \simeq \mathbb{Z},$$

generated by the class of the plane $V(x, y)$. Moreover, $\operatorname{Pic}(X) = 0$.

Proof Since $xw = yz$, the ring A is the prototype of a normal integral domain which is not a UFD (note that x, y, z, w are irreducible since they have degree 1 with respect to the standard grading on A.)

Consider the principal divisor $H = \operatorname{div}(y) = V(y)$, which decomposes as $Z + Z'$ where $Z = V(x, y)$ and $Z' = V(y, w)$. The complement $X - H = \operatorname{Spec}(A_y) = \operatorname{Spec}(k[x, y, y^{-1}, w])$ has trivial class group, as A_y is a UFD. From the exact sequence (19.4), we see that $\operatorname{Cl}(X)$ is generated by Z and Z'. As H is principal, we have $Z + Z' \sim 0$ in $\operatorname{Cl}(X)$, so $\operatorname{Cl}(X)$ is in fact generated by Z.

To prove that $\operatorname{Cl}(X) \simeq \mathbb{Z}$, we will show that no nonzero multiple of Z is principal. In fact, Z is a Weil divisor which has no multiple which is Cartier!

The key observation is that the open subset $X - Z$ is not affine (as shown in Exercise 19.6.9). Suppose that nZ were Cartier for some $n \neq 0$; replacing n by $-n$ if necessary, we may assume $n > 0$, so that nZ is effective. Then $L = \mathcal{O}_X(nZ)$ is invertible, and since

$nZ \geqslant 0$ the constant function 1 is a global section s of L, with $\operatorname{div}(s) = nZ$ by Theorem 19.25. As s is a global section, its zero scheme is the effective divisor nZ, whose support is Z; hence $X - V(s) = X - Z$. But then $X - Z$ would be affine, by Lemma 19.40. Hence no nonzero multiple of Z is Cartier, and in particular no nonzero multiple of Z is principal. Since $[Z]$ generates $\operatorname{Cl}(X)$, it follows that $\operatorname{Cl}(X) \simeq \mathbb{Z}[Z]$. □

Lemma 19.40. Let X be an affine Noetherian scheme, and let $V(s)$ be the zero scheme of a section s of an invertible sheaf L. Then $X - V(s)$ is also affine.

Proof Let $X = \operatorname{Spec} A$. Let $V_i = \operatorname{Spec} B_i$ be a finite affine cover of X that trivializes L with isomorphisms $\phi_i \colon L|_{V_i} \to \mathcal{O}_{V_i}$. Under these isomorphisms, the section $s|_{V_i}$ corresponds to some $f_i \in B_i$. Then $V(s) \cap V_i = \operatorname{Spec}(B_i/(f_i))$ and $V_i - V(s) = \operatorname{Spec}((B_i)_{f_i})$.

Consider the inclusion $\iota \colon U \to X$ of the open set $U = X - V(s)$. As $\iota^{-1}(V_i) = V_i - V(s)$ is affine, and the open sets $\iota^{-1}(V_i)$ form a cover of U, we see that ι is an affine morphism. But then $U = \iota^{-1}(X)$ is affine as well. □

Blow-ups of the plane at a point

Example 19.41. Let $\pi \colon X \to \mathbb{A}^2_k$ be the blow-up of the origin. With coordinates (x, y) on $\mathbb{A}^2_k$ and $(u : v)$ on $\mathbb{P}^1_k$, X embeds as a closed subscheme

$$X = V(xv - yu) \subset \mathbb{A}^2_k \times_k \mathbb{P}^1_k.$$

The exceptional divisor is $E = V(x, y)$. As a set, $E = \pi^{-1}(0,0) = \{(0,0)\} \times_k \mathbb{P}^1_k \simeq \mathbb{P}^1_k$.

We compute the class group. Since π is an isomorphism away from E, $X - E \simeq \mathbb{A}^2_k - \{(0,0)\}$, whose class group is zero. The localization sequence (19.4) therefore gives a shows that E generates $\operatorname{Cl}(X)$.

There is also a direct chart calculation of the restriction of $\mathcal{O}_X(E)$ to E. The ideal sheaf $\mathcal{I}_E = \mathcal{O}_X(-E)$ has local generator $e_U = x$ on U and $e_V = y$ on V. On the overlap they satisfy $e_V = te_U$. After restriction to $E \simeq \mathbb{P}^1_k$, this is the standard transition function for $\mathcal{O}_{\mathbb{P}^1_k}(1)$. Dualizing, we see that

$$\mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1_k}(-1).$$

Therefore E has infinite order in $\operatorname{Cl}(X)$, and hence $\operatorname{Cl}(X) = \operatorname{Pic}(X) = \mathbb{Z}E$. △

Example 19.42. Let $p = (0 : 0 : 1) \in \mathbb{P}^2_k$ and let X be the blow-up of $\mathbb{P}^2_k$ at p. Let $\pi \colon X \to \mathbb{P}^2_k$ be the blow-up morphism. The projection $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^1_k$ defined by $(x : y : z) \to (x : y)$ is defined everywhere, except p, and the closure of its graph gives the projective model

$$X = V(xv - yu) \subset \mathbb{P}^1_{k,(u:v)} \times_k \mathbb{P}^2_{k,(x:y:z)}. \tag{19.16}$$

Here π is the second projection. It is an isomorphism away from p, while $E = \pi^{-1}(p) = \mathbb{P}^1_k \times \{p\}$ is the exceptional divisor. Around p, the two charts are the affine blow-up charts computed in the previous example.

Let $L = V_+(z) \subset \mathbb{P}^2_k$. Since $p \notin L$, its inverse image $H = \pi^{-1}(L)$ is isomorphic to $L \simeq \mathbb{P}^1_k$; its class is the pullback of the hyperplane class. Removing H and E gives

$$X - (H \cup E) \simeq \mathbb{P}^2_k - (L \cup \{p\}) \simeq \mathbb{A}^2_k - \{(0,0)\}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The scheme on the right has trivial class group, so the localization sequence shows that H and E generate $\mathrm{Cl}(X)$. These generators are independent. Indeed, the two divisors are disjoint and

$$\begin{aligned} \mathcal{O}_X(H)|_H &\simeq \mathcal{O}_{\mathbb{P}^1_k}(1), & \mathcal{O}_X(E)|_H &\simeq \mathcal{O}_H. \\ \mathcal{O}_X(H)|_E &\simeq \mathcal{O}_E, & \mathcal{O}_X(E)|_E &\simeq \mathcal{O}_{\mathbb{P}^1_k}(-1), \end{aligned}$$

where the last isomorphism is the chart computation from the affine blow-up.

If $aH + bE$ were principal, restricting its associated invertible sheaf first to E and then to H would give $b = 0$ and $a = 0$. Hence

$$\mathrm{Cl}(X) = \mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E.$$

The two projections in (19.16) explain the geometry of the basic divisor classes. If $q\colon X \to \mathbb{P}^1_k$ is the first projection, then $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^2_k}(0,1)|_X$, and $\mathcal{O}_X(H-E) \simeq \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^2_k}(1,0)|_X$. So H gives the blow-down π, contracting E to p and $|H - E|$ gives q. The fiber of q over $(u : v)$ is the strict transform of the line $xv - yu = 0$ through p. Hence q makes X a $\mathbb{P}^1_k$-bundle over $\mathbb{P}^1_k$. The surface X is the Hirzebruch surface $\mathbb{F}_1$ (see Example ???). △

19.6 Exercises

Exercise 19.6.1. Let X be an integral normal Noetherian scheme. Verify that the set of principal divisors forms a subgroup of $\mathrm{Div}(X)$.

Exercise 19.6.2. Let $X = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3 - x)$. Compute the divisors of the rational functions x, y, and x^2/y.

Exercise 19.6.3. Let $X = V_+(x_2^3 + x_0^2 x_1 + x_0 x_1^2) \subset \mathbb{P}^2_{\mathbb{C}}$. Show that $\mathrm{div}(x_0) = 3(0:1:0)$ and $\mathrm{div}(x_1) = 3(1:0:0)$, and deduce that $3(0:1:0) \sim 3(1:0:0)$. Construct a morphism $f\colon X \to \mathbb{P}^1_k$ such that $f^*((0:1) - (1:0)) = D_1 - D_2$.

Exercise 19.6.4. Show that $\mathbb{R}[x,y]/(x^2 + y^2 + 1)$ is a UFD.

Exercise 19.6.5. For non-Noetherian schemes one quickly runs into difficulties when defining the divisor of a rational function. Imitating the construction of the affine line with two origins, construct the *affine line X with infinitely many origins*. Show that X is integral and normal, with fraction field $k(t)$, but that there are infinitely many closed points $p \in X$ for which $\mathrm{ord}_p(t) = 1$.

Exercise 19.6.6. Consider the smooth affine curve $X = V(y^2 - x^3 + 1) \subset \mathbb{A}^2_{\mathbb{C}}$. If $(a,b) \in X$, let $\sigma(a,b) = (a,-b)$, which also lies in X.

a) Show that for any $P \in X$, it holds that $P + \sigma(P) \sim 0$.
b) Show that if a non-vertical affine line meets X in the 0-cycle $P + Q + R$, counted with intersection multiplicity, then $P + Q + R \sim 0$.
c) Show that every Weil divisor on X is linearly equivalent either to 0 or to a prime divisor.

Exercise 19.6.7. The aim of this exercise is to prove the following statement, known as *Nagata's lemma*: Let A be a Noetherian integral domain, and let $x \in A - 0$. Suppose that (x) is prime and that A_x is a UFD. Then A is a UFD.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that A_x is normal.
b) Show that A is normal. HINT: If $t \in K(A)$ is integral over A, then $t \in A_x$.
c) Let $D = V(x) \subset \operatorname{Spec} A$. Show that there is an exact sequence

$$\mathbb{Z}D \to \operatorname{Cl}(\operatorname{Spec} A) \to \operatorname{Cl}(\operatorname{Spec} A_x) = 0.$$

d) Use the above sequence, together with the fact that $D = \operatorname{div}(x)$ is principal, to show that $\operatorname{Cl}(\operatorname{Spec} A) = 0$, and conclude that A is a UFD.

Exercise 19.6.8. Compute the class group of $\operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - z^3))$.

Exercise 19.6.9. Let $A = k[x,y,z,w]/(xw - yz)$, let $X = \operatorname{Spec} A$, and let $Z = V(x,y)$ be the prime divisor considered in Section 19.5. The aim of this exercise is to show that the open subscheme $X - Z$ is not affine.

a) Show that $X - Z = D(x) \cup D(y)$, and that

$$A_x = k[x, x^{-1}, y, z], \qquad A_y = k[x, y, y^{-1}, w],$$

where in the first ring $w = yz/x$ and in the second $z = xw/y$.
b) Let $u = z/x$. Show that $u = w/y$ in $K(X)$, and that the assignment $(x,y,u) \mapsto (x,y,xu,yu)$ defines an isomorphism

$$\big(\mathbb{A}^2_k - \{0\}\big) \times_k \mathbb{A}^1_k \simeq X - Z.$$

HINT: Check that it restricts to isomorphisms $D(x) \times \mathbb{A}^1_k \to \operatorname{Spec} A_x$ and $D(y) \times \mathbb{A}^1_k \to \operatorname{Spec} A_y$, and that these agree on the overlap.
c) Show that $\Gamma(X - Z, \mathcal{O}_X) = k[x,y,u]$. HINT: Compute the intersection $k[x,x^{-1},y,u] \cap k[x,y,y^{-1},u]$ inside $k(x,y,u)$.
d) Deduce that $X - Z$ is not affine. HINT: If it were, the inclusion of global sections would make it isomorphic to $\operatorname{Spec} k[x,y,u] = \mathbb{A}^3_k$; compare the images of the two schemes in $\mathbb{A}^2_k$ under (x,y).

Exercise 19.6.10. Let F be an irreducible homogeneous polynoial of degree d and let $U = \mathbb{P}^2_k - V_+(F)$. Show that $\operatorname{Cl}(U) \simeq \mathbb{Z}/d$.

Exercise 19.6.11. Let k be a field. Show that the Picard group of the general linear group $\mathrm{GL}_{n,k}$ is trivial. HINT: Use an exact sequence.

Exercise 19.6.12. Let X be a normal integral Noetherian scheme. Show that if two Weil divisors D and E are linearly equivalent, then $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ as $\mathcal{O}_X$-modules.

Exercise 19.6.13. Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$, and let $X = \operatorname{Spec} A$. Fix a local parameter $t \in A$, so that $\mathfrak{m} = (t)$, and let $P \in X$ be the closed point corresponding to $\mathfrak{m}$.

a) Show that $\mathcal{O}_X(nP) = t^{-n} \cdot \mathcal{O}_X$, as subsheaves of $\mathbf{K}(X)$.
b) Show that if $f \in K(X)^\times$, then $\operatorname{div}(f) = \operatorname{ord}_P(f) \cdot P$ and

$$\mathcal{O}_X(\operatorname{div}(f)) = t^{-\operatorname{ord}_P(f)} \mathcal{O}_X.$$

c) Show that $\operatorname{Div}(X) \simeq \mathbb{Z}$ and $\operatorname{Cl}(X) = 0$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 19.6.14. Let X be a scheme and let $D \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}$. Show that the following statements are equivalent:

(i) $\mathcal{I}$ is an invertible sheaf.
(ii) For every $P \in X$, the ideal $\mathcal{I}_P \subset \mathcal{O}_{X,P}$ is principal and generated by a nonzerodivisor.
(iii) There is an open covering U_i of X and nonzerodivisors $f_i \in \mathcal{O}_X(U_i)$ such that $\mathcal{I}|_{U_i} = f_i \mathcal{O}_{U_i}$.
(iv) For every $P \in X$, there is an open affine neighbourhood $U = \operatorname{Spec} A$ of P such that $U \cap D = \operatorname{Spec} A/(f)$, where $f \in A$ is a nonzerodivisor.
(v) D is the zero scheme of a regular global section s of an invertible sheaf L.

These conditions define D as an effective Cartier divisor. When X is normal, integral, and Noetherian, they are also equivalent to the associated effective Weil divisor being Cartier.

Exercise 19.6.15. Let X be a normal integral Noetherian scheme with generic point ξ, let L be an invertible sheaf, and let $t \in L_\xi$ be a rational section. For a prime divisor $Y \subset X$, define $\operatorname{ord}_Y(t)$ using a local trivialization of L near the generic point of Y. Show that the following are equivalent:

(i) t comes from a global section of L.
(ii) $t \in L_P$ for every codimension-1 point $P \in X$.
(iii) $\operatorname{ord}_Y(t) \geqslant 0$ for every prime divisor $Y \subset X$.

Deduce that

$$\Gamma(X, L) = \bigcap_{\operatorname{codim}(P)=1} L_P \subset L_\xi.$$

Exercise 19.6.16 (Norms)**.** Let $f\colon X \to Y$ be a finite dominant morphism of integral Noetherian normal schemes. Then f induces an inclusion of function fields $K(Y) \to K(X)$, making $K(X)$ into a finite extension of $K(Y)$.

a) Show that multiplication by $g \in K(X)$ induces a $K(Y)$-linear map of $K(Y)$-vector spaces $m_g\colon K(X) \to K(X)$.
b) Define the *norm* $N\colon K(X)^\times \to K(Y)^\times$ by $N(g) = \det(m_g)$. Show that N is multiplicative, i.e., $N(gh) = N(g)N(h)$ for each $g, h \in K(X)^\times$.
c) Define a morphism $f_*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ by setting

$$f_*[P] = [k(P) : k(f(P))]\,[f(P)]$$

for every prime divisor $P \subset X$. Show that this is well-defined.
d) Show that for every $g \in K(X)^\times$, one has $f_* \operatorname{div}(g) = \operatorname{div}(N(g))$.
e) Show that if $h \in K(Y)^\times$, then $f_* f^* \operatorname{div}(h) = (\deg f) \cdot \operatorname{div}(h)$.

Exercise 19.6.17. Let $X = \mathbb{A}^2_k - \{(0,0)\}$ and let $f\colon X \to \mathbb{P}^1_k$ be the quotient morphism. Show that $f^*\mathcal{O}_{\mathbb{P}^1_k}(1) \simeq \mathcal{O}_X$. Deduce that the induced map f^* on Picard groups may fail to be injective. Give an example of a dominant morphism $g\colon Y \to Z$ such that g^* is not surjective.

Exercise 19.6.18. Let X be a normal integral Noetherian k-scheme, and let $\pi\colon X \times_k \mathbb{A}^1 \to X$ be the projection and $\iota\colon X \to X \times_k \mathbb{A}^1$ the closed embedding defined by $x \mapsto (x, 0)$. We will consider the pullback map $\pi^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(X \times_k \mathbb{A}^1)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that the composition $\iota^* \circ \pi^*$ is the identity. Deduce that π^* is injective.
b) Show that for every $L \in \operatorname{Pic}(X \times_k \mathbb{A}^1)$, there is an affine open covering U_i of X such that L is trivial when restricted to each $\pi^{-1}(U_i)$. HINT: Use the affine case $\operatorname{Pic}(\operatorname{Spec} A) \simeq \operatorname{Pic}(\operatorname{Spec} A[t])$ for normal Noetherian domains A.
c) Let L be an invertible sheaf on $X \times_k \mathbb{A}^1$ and assume that there is an affine open covering U_i of X such that L is trivial when restricted to each $\pi^{-1}(U_i)$. Show that $L \simeq \pi^* A$ for some invertible sheaf A on X.
d) Using *c*), deduce that π^* is surjective.

Exercise 19.6.19. Let $X = \operatorname{Spec} k[x, y, z]/(y^2 - xz)$ and consider the morphism

$$\phi\colon \mathbb{A}^1_k \times_k X \to X, \quad (t, (x, y, z)) \mapsto (tx, ty, tz).$$

Let $\iota_0, \iota_1\colon X \to \mathbb{A}^1_k \times_k X$ denote the closed embeddings of the fibers over $t = 0$ and $t = 1$. Consider the maps ι_0^*, ι_1^*, and ϕ^* on Picard groups.

a) Show that both ι_0^* and ι_1^* are isomorphisms.
b) Show that $\iota_1^* \circ \phi^* = \mathrm{id}$ and $\iota_0^* \circ \phi^* = 0$ as maps $\operatorname{Pic}(X) \to \operatorname{Pic}(X)$. Deduce that ϕ^* is an isomorphism and conclude that $\operatorname{Pic}(X) = 0$.

Exercise 19.6.20. Let X be an integral scheme and let $\mathbf{K}(X)$ denote the constant sheaf on $K = K(X)$. Note that the sheaf $\mathcal{O}_X^\times$ of invertible sections of $\mathcal{O}_X$ embeds as a subsheaf of $\mathbf{K}(X)^\times$. Show that a Cartier divisor is the same thing as a global section of the quotient sheaf $\mathbf{K}(X)^\times/\mathcal{O}_X^\times$.

Exercise 19.6.21. For the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, d)$, let D denote the class of a coordinate divisor of weight 1. Show that $\operatorname{Cl}(\mathbb{P}) = \mathbb{Z}D$, and that the subgroup of Cartier divisor class group is generated by dD

Exercise 19.6.22. Compute the Picard group of the affine line with two origins. HINT: See the computation for $\mathbb{P}^1_k$.

Exercise 19.6.23 (Quadric hypersurfaces in higher dimension)**.** Assume that $\operatorname{char} k \neq 2$. Let

$$A = k[x_1, \ldots, x_n, y, z]/(x_1^2 + \cdots + x_n^2 - yz).$$

We will prove that A is a UFD for $n \geqslant 3$.

a) Show that A is an integral domain.
b) Show that A is normal. HINT: Show first that A_y is a UFD, hence normal. Then show that ideal (y) is prime for $n \geqslant 3$.
c) Show that $\operatorname{Cl}(\operatorname{Spec} A) = 0$ and that A is a UFD. HINT: Let $Z = \operatorname{div}(y)$.
d) Suppose now that k contains a square root of -1, and let $X = V(x_0^2 + \cdots + x_n^2) \subset \mathbb{A}^{n+1}_k = \operatorname{Spec} k[x_0, \ldots, x_n]$. Show that
   (i) if $n = 2$, then $\operatorname{Cl}(X) = \mathbb{Z}/2$;
   (ii) if $n = 3$, then $\operatorname{Cl}(X) = \mathbb{Z}$;
   (iii) if $n \geqslant 4$, then $\operatorname{Cl}(X) = 0$.

Exercise 19.6.24. Let k be an algebraically closed field of characteristic different from 2. Let $X \subset \mathbb{P}^n_k$ be a nonsingular quadric hypersurface. Show that
(i) if $n = 2$, then $\operatorname{Cl}(X) = \mathbb{Z}$;

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(ii) if $n = 3$, then $\mathrm{Cl}(X) = \mathbb{Z}^2$;
(iii) if $n \geqslant 4$, then $\mathrm{Cl}(X) = \mathbb{Z}$.

Exercise 19.6.25. Let $X = \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, and consider X as the quadric surface $X = V_+(x_0x_3 - x_1x_2) \subset \mathbb{P}^3_k$ under the Segre embedding. Show that X contains the twisted cubic curve C of Example 6.20, and that C defines a prime divisor of type $(1, 2)$ or $(2, 1)$. Conversely, show that any prime divisor of type $(1, 2)$ or $(2, 1)$ is isomorphic to the twisted cubic curve by a projective linear coordinate change.

Exercise 19.6.26. Let $\operatorname{char} k \neq 2$ and let $\phi\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the finite dominant morphism of degree 2 given by $(x_0 : x_1) \mapsto (x_0^2 : x_1^2)$.

a) Compute the pullback divisors $\phi^*(0 : 1)$, $\phi^*(1 : 0)$ and $\phi^*(1 : 1)$.
b) Check that $\phi^* \operatorname{div}(y_0/y_1) = \operatorname{div}\big(\phi^\sharp(y_0/y_1)\big)$, and that $\deg \phi^* D = 2 \deg D$ for each of the divisors D in part a).

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20

First steps in sheaf cohomology

One of the basic difficulties when working with sheaves is that surjective maps of sheaves do not always induce surjections on global sections. Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

one has an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}''), \tag{20.1}$$

where the right-most map may fail to be surjective. Equivalently, a global section of $\mathcal{F}''$ may fail to lift to a global section of $\mathcal{F}$. Understanding when such liftings exist is often essential in algebraic geometry. For instance, if $U \subset X$ is an open subscheme, one frequently wants to know when a regular function defined on U extends to a regular function on all of X.

Cohomology groups provide a systematic way to measure this failure of exactness. In good situations, they allow us to say something about the missing cokernel in (20.1). More precisely, the short exact sequence (20.1) extends to a *long exact sequence of cohomology groups*

$$\begin{aligned} 0 \longrightarrow \Gamma(X, \mathcal{F}') &\longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \\ \hookrightarrow H^1(X, \mathcal{F}') &\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}'') \\ \hookrightarrow H^2(X, \mathcal{F}') &\longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}'') \longrightarrow \cdots \end{aligned}$$

This means that the failure of surjectivity of the above is controlled by the group $H^1(X, \mathcal{F}')$ and the other groups in the sequence.

In addition to problems such as lifting sections, cohomology groups allow us to define geometric invariants of sheaves and schemes. These in turn allow us to distinguish schemes, that is, if two schemes have different cohomology groups they can not be isomorphic.

Cohomology groups can be defined for a sheaf of abelian groups on any topological space. There are several ways to define them. The modern approach uses the theory of derived functors. This is in most respects the 'right way' to define the groups in general, but going through the whole machinery of derived functors would take us too far astray. We therefore take a more concrete approach using *Čech cohomology* which is better suited for computations.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20.1 Some homological algebra

A *complex of abelian groups* $A^\bullet$ is a collection of groups A^i, indexed by the integers, together with maps between them

$$\cdots \xrightarrow{d^{p-2}} A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1} \xrightarrow{d^{p+1}} \cdots$$

called differentials, such that $d^p \circ d^{p-1} = 0$ for each p. A *map of complexes* $f\colon A^\bullet \to B^\bullet$ is a collection of group maps $f_p\colon A^p \to B^p$ that commute with the differentials, meaning that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{p-1} & \xrightarrow{d_A^{p-1}} & A^p & \xrightarrow{d_A^p} & A^{p+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{p-1}} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p+1}} & & \\
\cdots & \longrightarrow & B^{p-1} & \xrightarrow{d_B^{p-1}} & B^p & \xrightarrow{d_B^p} & B^{p+1} & \longrightarrow & \cdots
\end{array}$$

In this way, we may talk about kernels, images, cokernels, exact sequences of complexes, etc.

An element $\sigma \in A^p$ is called a *cocycle* if $d^p\sigma = 0$ and it is a *coboundary* if it lies in the image of d^{p-1}, that is, $\sigma = d^{p-1}\tau$ for some $\tau \in A^{p-1}$. The condition $d^p \circ d^{p-1} = 0$ implies that

$$\operatorname{Im} d^{p-1} \subset \operatorname{Ker} d^p,$$

so every coboundary is a cocycle. The *cohomology groups* of the complex $A^\bullet$ are designed to measure the difference between these two notions. The *p-th cohomology group* is defined as the quotient

$$H^p(A^\bullet) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

One may think of $H^p(A^\bullet)$ as a group that measures the failure of exactness of the complex $A^\bullet$ at stage p: the complex $A^\bullet$ is exact at p if and only if $H^p(A^\bullet) = 0$.

A map of complexes $f\colon A^\bullet \to B^\bullet$ takes cocycles to cocycles and coboundaries to coboundaries. Explicitly, if $a \in A^p$ is a cocycle (so that $d_A^p a = 0$), then $d_B^p(f_p a) = f_{p+1}(d_A^p a) = 0$, so $f_p a$ is also a cocycle. Similarly, if $a = d_A^{p-1} b$ is a coboundary, then $f_p a = d_B^{p-1}(f_{p-1} b)$ is also a coboundary. Consequently, f induces a map on cohomology groups

$$H^p(A^\bullet) \longrightarrow H^p(B^\bullet), \qquad [a] \mapsto [f_p a].$$

The following result is fundamental in the theory of cohomology:

Proposition 20.1. Let $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ be an exact sequence of complexes. Then there is a *long exact sequence* of cohomology groups

$$\cdots \longrightarrow H^p(A^\bullet) \longrightarrow H^p(B^\bullet) \longrightarrow H^p(C^\bullet) \longrightarrow H^{p+1}(A^\bullet) \longrightarrow H^{p+1}(B^\bullet) \longrightarrow H^{p+1}(C^\bullet) \longrightarrow \cdots$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof For each $p \in \mathbb{Z}$, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \xrightarrow{f_p} & B^p & \xrightarrow{g_p} & C^p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A^p} & & \downarrow{\scriptstyle d_B^p} & & \downarrow{\scriptstyle d_C^p} & & \\ 0 & \longrightarrow & A^{p+1} & \xrightarrow{f_{p+1}} & B^{p+1} & \xrightarrow{g_{p+1}} & C^{p+1} & \longrightarrow & 0 \end{array}$$

Applying the Snake Lemma, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Ker} d_A^p \xrightarrow{f_p} \operatorname{Ker} d_B^p \xrightarrow{g_p} \operatorname{Ker} d_C^p \xrightarrow{} A^{p+1}/\operatorname{Im} d_A^p \xrightarrow{f_{p+1}} B^{p+1}/\operatorname{Im} d_B^p \xrightarrow{g_{p+1}} C^p/\operatorname{Im} d_C^{p+1} \longrightarrow 0.$$

Now consider the diagram:

$$\begin{array}{ccccccccc} & & A^p/\operatorname{Im} d_A^{p-1} & \xrightarrow{f_p} & B^p/\operatorname{Im} d_B^{p-1} & \xrightarrow{g_p} & C^p/\operatorname{Im} d_C^{p-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{d}_A^p} & & \downarrow{\scriptstyle \overline{d}_B^p} & & \downarrow{\scriptstyle \overline{d}_C^p} & & \\ 0 & \longrightarrow & \operatorname{Ker} d_A^{p+1} & \xrightarrow{f_{p+1}} & \operatorname{Ker} d_B^{p+1} & \xrightarrow{g_{p+1}} & \operatorname{Ker} d_C^{p+1} & & \end{array}$$

where the rows are exact by the above. Observe that

$$H^p(A^\bullet) = \operatorname{Ker} d_A^p/\operatorname{Im} d_A^{p-1} = \operatorname{Ker} \overline{d}_A^p, \quad H^{p+1}(A^\bullet) = \operatorname{Coker} \overline{d}_A^p,$$

and similarly for $B^\bullet$ and $C^\bullet$. Applying the Snake Lemma again, we get the desired exact sequence. □

Two maps $f, g\colon A^\bullet \to B^\bullet$ are said to be *cochain homotopic* if there are maps $h_p\colon A^p \to B^{p-1}$ such that

$$f - g = d_B \circ h + h \circ d_A.$$

For any cocycle $a \in \operatorname{Ker} d_A^p$, this implies $f_p(a) - g_p(a) = d_B^{p-1}(h_p(a))$, and so $[f_p a] = [g_p a]$ in cohomology. Consequently, cochain-homotopic maps induce the *same* maps on cohomology.

Example 20.2. A common strategy for proving that $H^p(A^\bullet) = 0$ is to show that the identity map is cochain homotopic to the zero map. Explicitly, this means that one constructs maps $h\colon A^\bullet \to A^{\bullet-1}$ such that

$$d^{p-1}(h_p(a)) + h_{p+1}(d^p(a)) = a \quad \text{for all } a \in A^p \tag{20.2}$$

If $a \in A^p$ is a cocycle (so that $d^p a = 0$), then this equation implies $a = d^{p-1}(h_p a)$ so it is a coboundary, and hence $H^p(A^\bullet) = 0$. △

20.2 Čech cohomology

Let X be a topological space and let $\mathcal{U}$ be an open cover of X consisting of finitely many open sets $U_1, \dots, U_r$. For an increasing tuple $I = (i_0 < \cdots < i_p)$, write

$$U_I = U_{i_0} \cap \cdots \cap U_{i_p}.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For a sheaf $\mathcal{F}$ of abelian groups on X, the sheaf axiom gives an exact sequence of abelian groups

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i<j} \mathcal{F}(U_i \cap U_j). \tag{20.3}$$

The *Čech complex* $C^\bullet(\mathcal{U}, \mathcal{F})$ continues this sequence by including the sections over all finite intersections:

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

where the p-th term is given by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<i_1<\cdots<i_p} \mathcal{F}(U_{i_0 \dots i_p}),$$

and the differentials $d^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ are defined by

$$(d^p\sigma)_{i_0,\dots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\dots,\widehat{i_j},\dots,i_{p+1}}\big|_{U_{i_0 \cdots i_{p+1}}}.$$

Here $i_0, \dots \widehat{i_j}, \dots, i_{p+1}$ means $i_0, \dots, i_{p+1}$ with the index i_j omitted.

Since the open cover has r elements, $C^p(\mathcal{U}, \mathcal{F}) = 0$ for $p \geqslant r$. We also set $C^p(\mathcal{U}, \mathcal{F}) = 0$ for $p < 0$ unless an augmented complex is explicitly being used.

Example 20.3. The first two terms in the Čech complex are

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_i \mathcal{F}(U_i) \quad \text{and} \quad C^1(\mathcal{U}, \mathcal{F}) = \prod_{i<j} \mathcal{F}(U_i \cap U_j).$$

The group $C^0(\mathcal{U}, \mathcal{F})$ consists of r-tuples of sections $\sigma = (\sigma_1, \dots, \sigma_r)$, with $\sigma_i \in \mathcal{F}(U_i)$ for each i. Likewise, an element $\sigma = (\sigma_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$ is a collection of sections $\sigma_{ij} \in \mathcal{F}(U_i \cap U_j)$, one for each pair $i < j$.

The map $d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ sends an element $\sigma = (\sigma_1, \dots, \sigma_r)$ to the element $d^0\sigma \in C^1(\mathcal{U}, \mathcal{F})$ whose ij-component is

$$(d^0\sigma)_{ij} = \sigma_j - \sigma_i\,\big|_{U_{ij}}. \tag{20.4}$$

The map $d^1 \colon C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ sends $\sigma = (\sigma_{ij})$ to the element with ijk-th component equal to

$$(d^1\sigma)_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}\,\big|_{U_{ijk}} \tag{20.5}$$

△

Substituting (20.4) into (20.5), we see that all the terms cancel, so that $d^1 \circ d^0 = 0$. This happens also in higher degrees:

Lemma 20.4. $d^p \circ d^{p-1} = 0$ for every $p \geqslant 1$.

Proof For $\sigma \in C^{p-1}(\mathcal{U}, \mathcal{F})$ and $i_0 < \cdots < i_{p+1}$, we compute:

$$\begin{aligned} d^p(d^{p-1}(\sigma))_{i_0,\dots,i_{p+1}} &= \sum_{j=0}^{p+1} (-1)^j (d^{p-1}\sigma)_{i_0,\dots,\widehat{i_j},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}} \\ &= \sum_{j=0}^{p+1} (-1)^j \left(\sum_{l=0}^{j-1} (-1)^l \sigma_{i_0,\dots,\widehat{i_l},\dots,\widehat{i_j},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}} \right) \\ &\quad + \sum_{j=0}^{p+1} (-1)^j \left(\sum_{l=j}^{p} (-1)^l \sigma_{i_0,\dots,\widehat{i_j},\dots,\widehat{i_{l+1}},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}} \right). \end{aligned}$$

For integers $m < n$, the term $\sigma_{i_0,\dots,\widehat{i_m},\dots,\widehat{i_n},\dots,i_{p+1}}$ appears exactly twice in the sum: for $(l, j) = (m, n)$ in the first sum and $(l, j) = (n-1, m)$ in the second. The signs are opposite, so the two terms cancel. Hence $d^p \circ d^{p-1} = 0$. □

Therefore, the Čech complex is indeed a complex of abelian groups.

Definition 20.5. The p-th *Čech cohomology* of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as

$$H^p(\mathcal{U}, \mathcal{F}) = \frac{\operatorname{Ker} d^p}{\operatorname{Im} d^{p-1}}.$$

The Čech cohomology groups depend on the choice of open cover $\mathcal{U}$, but are independent of the ordering of the open sets U_i in the cover. Given two orderings, there is an isomorphism between the two associated Čech complexes which commutes with the differentials, and the cohomology groups are canonically isomorphic (see Exercise 20.18.14).

Example 20.6. The group $H^0(\mathcal{U}, \mathcal{F})$ is the kernel of the map $d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$, which is simply the usual map

$$\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i<j} \mathcal{F}(U_i \cap U_j).$$

This kernel is equal to $\mathcal{F}(X)$ by the sheaf axioms, so $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$. △

Example 20.7 (H^1 and lifting of sections)**.** The most interesting cohomology group is arguably $H^1(\mathcal{U}, \mathcal{F})$. It is the group of elements (σ_{ij}) such that $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$ modulo the elements of the form $\sigma_{ij} = \tau_j - \tau_i$ (restricted to $U_i \cap U_j$). As mentioned in the introduction, this group is closely related to the lifting of sections, as we now explain.

Suppose that we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

Suppose we want to lift a section $c \in \mathcal{C}(X)$ to a section of $\mathcal{B}(X)$. Since the sequence is exact, we can find lifts locally, i.e. there is an open cover $\mathcal{U} = \{\mathcal{U}_i\}$ and sections $b_i \in \mathcal{B}(U_i)$ that map to $c|_{U_i}$ over each U_i. Now we ask if we can assemble the b_i to a section $b \in \mathcal{B}(X)$. For this to be the case, we must have $b_j|_{U_{ij}} - b_i|_{U_{ij}} = 0$. In any case,

$$\sigma = \left(b_j|_{U_{ij}} - b_i|_{U_{ij}}\right)$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

defines an element of $C^1(\mathcal{U},\mathcal{A})$, because b_i and b_j map to the same element in $\mathcal{C}(U_{ij})$. Furthermore, $d\sigma = 0$, because

$$(d\sigma)_{ijk} = (b_k - b_j) - (b_k - b_i) + (b_j - b_i) = 0$$

(all terms restricted to U_{ijk}). When is σ zero in $H^1(\mathcal{U},\mathcal{A})$? This occurs if and only if there is an element $a = (a_i) \in C^0(\mathcal{U},\mathcal{A})$ such that

$$b_j|_{U_{ij}} - b_i|_{U_{ij}} = a_j|_{U_{ij}} - a_i|_{U_{ij}},$$

which is equivalent to saying that the elements $b_i - a_i \in \mathcal{B}(U_i)$ agree over the overlaps U_{ij}, or in other words, that they glue together to a section $b \in \mathcal{B}(X)$. Note that since $a_i \in \mathcal{A}(U_i)$, the image of $b_i - a_i$ is the same as that of b_i, i.e. b maps to c.

If the local lifts b_i are replaced by other lifts b_i', then $b_i' - b_i \in \mathcal{A}(U_i)$, and the resulting cocycle differs from σ by a coboundary. Thus the class $[\sigma] \in \check{H}^1(\mathcal{U},\mathcal{A})$ is independent of the chosen local lifts.

In summary, the section $c \in \mathcal{C}(X)$ can be lifted if and only if its class $[\sigma] \in H^1(\mathcal{U},\mathcal{A})$ is zero. In particular, if every section of $\mathcal{C}(X)$ lifts locally on the members of $\mathcal{U}$ and $\check{H}^1(\mathcal{U},\mathcal{A}) = 0$, then every global section of $\mathcal{C}$ lifts to $\mathcal{B}(X)$.

In Example 20.19 we will see a concrete example of a section which does not lift. △

A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a morphism of Čech cohomology groups. For each open set $U \subset X$, we have a map $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, and for each $p \geqslant 0$, we may define the map $\phi^p\colon C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{G})$ by applying ϕ componentwise. To see that this is a map of complexes, we check that $\phi^{p+1} \circ d^p_{\mathcal{F}} = d^p_{\mathcal{G}} \circ \phi^p$. For $\sigma \in C^p(\mathcal{U},\mathcal{F})$ and any $i_0 < \cdots < i_{p+1}$, we have

$$(\phi^{p+1}(d^p_{\mathcal{F}}\sigma))_{i_0,\dots,i_{p+1}} = \phi_{U_{i_0,\dots,i_{p+1}}}\left(\sum_{k=0}^{p+1}(-1)^k \sigma_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}}\right).$$

Since ϕ is a map of sheaves, it commutes with restrictions, so this equals

$$\sum_{k=0}^{p+1}(-1)^k \phi_{U_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}}(\sigma_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}})|_{U_{i_0,\dots,i_{p+1}}}.$$

On the other hand,

$$\begin{aligned}(d^p_{\mathcal{G}}(\phi^p\sigma))_{i_0,\dots,i_{p+1}} &= \sum_{k=0}^{p+1}(-1)^k(\phi^p\sigma)_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}} \\ &= \sum_{k=0}^{p+1}(-1)^k \phi_{U_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}}(\sigma_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}})|_{U_{i_0,\dots,i_{p+1}}}.\end{aligned}$$

Since these are equal, we see that the maps ϕ^p form a map of complexes. In particular, they induce a map between the corresponding cohomology groups

$$H^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(\mathcal{U},\mathcal{G}).$$

Hence $H^p(\mathcal{U},-)$ defines a functor from sheaves to abelian groups.

20.3 Examples

Example 20.8. For a cover $\mathcal{U} = \{U_0, U_1\}$ consisting of just two open sets, the Čech complex takes the following simple form

$$0 \longrightarrow \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_0 \cap U_1) \longrightarrow 0 \tag{20.6}$$

where $d(s_0, s_1) = s_1|_{U_0 \cap U_1} - s_0|_{U_0 \cap U_1}$. Then

$$H^0(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} d \quad \text{and} \quad H^1(\mathcal{U}, \mathcal{F}) = \operatorname{Coker} d,$$

and there is an exact sequence

$$0 \to H^0(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_0 \cap U_1) \to H^1(\mathcal{U}, \mathcal{F}) \to 0. \tag{20.7}$$

Concretely, $H^1(\mathcal{U}, \mathcal{F})$ is the group of sections in $\mathcal{F}(U_0 \cap U_1)$ modulo the sections of the form $s_1 - s_0$ where s_0 and s_1 are restrictions of sections in $\mathcal{F}(U_0)$ and $\mathcal{F}(U_1)$ respectively.

Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence and that the induced sequences of sections are exact over U_0, U_1 and $U_0 \cap U_1$. We can understand the connecting map δ in the long exact sequence by applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\mathcal{U}, \mathcal{F}') & \longrightarrow & C^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^0(\mathcal{U}, \mathcal{F}'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathcal{U}, \mathcal{F}') & \longrightarrow & C^1(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^1(\mathcal{U}, \mathcal{F}'') & \longrightarrow & 0 \end{array} \tag{20.8}$$

Concretely, for $s'' \in H^0(\mathcal{U}, \mathcal{F}'') \subset C^0(\mathcal{U}, \mathcal{F}'')$, lift to $(s_0, s_1) \in C^0(\mathcal{U}, \mathcal{F})$, then $\delta(s'')$ is represented by the class of $s_1|_{U_0 \cap U_1} - s_0|_{U_0 \cap U_1}$ in $H^1(\mathcal{U}, \mathcal{F}')$. △

Example 20.9 (The projective line)**.** Consider the projective line $\mathbb{P}^1_A$ over a ring A, with the standard affine cover given by $U_0 = \operatorname{Spec} A[x_1/x_0]$ and $U_1 = \operatorname{Spec} A[x_0/x_1]$ with intersection $U_0 \cap U_1 = \operatorname{Spec} A[x_0/x_1, x_1/x_0]$. Let us compute the cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(m)$ on $\mathbb{P}^1_A$. The Čech complex takes the form

$$0 \longrightarrow A\Big[\frac{x_1}{x_0}\Big]x_0^m \times A\Big[\frac{x_0}{x_1}\Big]x_1^m \xrightarrow{d} A\Big[\frac{x_1}{x_0}, \frac{x_0}{x_1}\Big]x_0^m = A\Big[\frac{x_1}{x_0}, \frac{x_0}{x_1}\Big]x_1^m \longrightarrow 0$$

The kernel of d consists of pairs (fx_0^m, gx_1^m) which agree on the overlap, which corresponds precisely to the space of homogeneous polynomials of degree m in $A[x_0, x_1]$ (which is zero if $m < 0$).

To analyze the cokernel of d, we examine the monomials in each term. The module $A[x_1/x_0]x_0^m$ is generated by $x_0^a x_1^b$ for $a + b = m$ and $b \geqslant 0$, while $A[x_0/x_1]x_1^m$ is generated by $x_0^a x_1^b$ for $a + b = m$ and $a \geqslant 0$. The target module $A[x_1/x_0, x_0/x_1]x_0^m$ is generated by all monomials $x_0^a x_1^b$ with $a + b = m$.

When $m \geqslant 0$, every monomial in $A[x_1/x_0, x_0/x_1]x_0^m$ is obtained from either $A[x_1/x_0]x_0^m$ or $A[x_0/x_1]x_1^m$, so

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_A}(m)) = 0 \qquad \text{for all } m \geqslant 0.$$

In fact, the same argument gives vanishing when $m = -1$. In particular, $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_A}) = 0$.

When $m \leqslant -2$, the cokernel is freely generated as an A-module by the monomials $x_0^a x_1^b$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

with $a, b < 0$ and $a + b = m$, namely, the monomials $x_0^{-1}x_1^{m+1}, x_0^{-2}x_1^{m+2}, \dots, x_0^{m+1}x_1^{-1}$. Hence

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_A}(m)) \simeq A^{-m+1} \qquad \text{for all } m \leqslant -2.$$

△

Example 20.10 (The cuspidal cubic)**.** Consider the cuspidal cubic curve $X = V_+(x_2^3 - x_0x_1^2)$ in $\mathbb{P}^2_k$. This admits an open cover $\mathcal{U}$ with two open sets, $U_0 = D_+(x_0)$ and $U_1 = D_+(x_1)$. Writing $u = x_1/x_0$, $v = x_2/x_0$ and $w = x_2/x_1$, we have

$$\begin{aligned} \mathcal{O}_X(U_0) &= k[x_1/x_0, x_2/x_0]/((x_2/x_0)^3 - (x_1/x_0)^2) = k[u, v]/(v^3 - u^2) \\ \mathcal{O}_X(U_1) &= k[x_0/x_1, x_2/x_1]/((x_2/x_1)^3 - (x_0/x_1)) = k[w] \\ \mathcal{O}_X(U_0 \cap U_1) &= k[w, w^{-1}]. \end{aligned}$$

where we used the relation $x_0/x_1 = (x_2/x_1)^3 = w^3$ to eliminate x_0/x_1. The images of u and v in $k[w, w^{-1}]$ are $u = x_1/x_0 = w^{-3}$, $v = x_2/x_0 = x_2/x_1 \cdot x_1/x_0 = w^{-2}$. Therefore, the differential is given by

$$d(p(u, v), q(w)) = q(w) - p(w^{-3}, w^{-2}).$$

The cokernel of d is generated by the monomial w^{-1}. Hence $\check{H}^1(\mathcal{U}, \mathcal{O}_X) \equiv k$. △

Example 20.11. Let $\mathcal{U} = \{U_0, \dots, U_r\}$ be a finite open cover such that one of the members is the whole space X. Then

$$H^p(\mathcal{U}, \mathcal{F}) = 0 \qquad \text{for all } p \geqslant 1.$$

To see this, we use the trick described in Example 20.2. Suppose for simplicity that $U_0 = X$ (otherwise, reorder the indexes) and define $h^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{U}, \mathcal{F})$ by

$$h^p(\sigma)_{j_0,\dots,j_{p-1}} = \begin{cases} \sigma_{0,j_0,\dots,j_{p-1}} & \text{if } j_0 \neq 0, \\ 0 & \text{if } j_0 = 0. \end{cases}$$

This is well-defined because $U_0 = X$, so $\sigma_{0,j_0,\dots,j_{p-1}} \in \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_{p-1}})$.

Let $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ with $p \geqslant 1$. We check that $dh + hd = \mathrm{id}$ in degree p. If $i_0 \neq 0$, then

$$\begin{aligned} (dh + hd)(\sigma)_{i_0,\dots,i_p} &= \sum_{j=0}^{p} (-1)^j \sigma_{0,i_0,\dots,\widehat{i_j},\dots,i_p} + (d\sigma)_{0,i_0,\dots,i_p} \\ &= \sum_{j=0}^{p} (-1)^j \sigma_{0,i_0,\dots,\widehat{i_j},\dots,i_p} + \sigma_{i_0,\dots,i_p} \\ &\quad + \sum_{j=0}^{p} (-1)^{j+1} \sigma_{0,i_0,\dots,\widehat{i_j},\dots,i_p} \\ &= \sigma_{i_0,\dots,i_p}. \end{aligned}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

If $i_0 = 0$, then

$$\begin{aligned}(dh + hd)(\sigma)_{0,i_1,\dots,i_p} &= \sum_{j=0}^{p} (-1)^j h(\sigma)_{0,i_1,\dots,\widehat{i_j},\dots,i_p} \\ &= \sigma_{0,i_1,\dots,i_p}.\end{aligned}$$

Hence h is a cochain homotopy between the identity and zero maps in every positive degree. Therefore $H^p(\mathcal{U}, \mathcal{F}) = 0$ for every $p \geqslant 1$ by Example 20.2. △

Example 20.12 (The unit circle)**.** Here is a basic example from topology. Let $X = S^1$ be the unit circle with its Euclidean topology, let $U = S^1 - \{(1,0)\}$ and $V = S^1 - \{(-1,0)\}$, and consider the open cover $\mathcal{U} = \{U, V\}$. We take coefficients in the constant sheaf $\mathbb{Z}_X$.

Both U and V are connected, so $\mathbb{Z}_X(U) = \mathbb{Z}$ and $\mathbb{Z}_X(V) = \mathbb{Z}$. On the other hand, $U \cap V = S^1 - \{(1,0), (-1,0)\}$ has two connected components, so $\mathbb{Z}_X(U \cap V) = \mathbb{Z} \times \mathbb{Z}$. Hence $C^0(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}\times\mathbb{Z}$ and $C^1(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}\times\mathbb{Z}$, and the differential $d^0 \colon C^0(\mathcal{U}, \mathbb{Z}_X) \to C^1(\mathcal{U}, \mathbb{Z}_X)$ is given by $d^0(a, b) = (b - a, b - a)$.

It follows that $H^0(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Ker}(d^0) = \mathbb{Z}$, while $H^1(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Coker}(d^0) = (\mathbb{Z} \times \mathbb{Z})/\mathbb{Z}(1,1) = \mathbb{Z}$.

Readers familiar with algebraic topology may recognize that this agrees with the singular cohomology of S^1. More generally, if X is homotopy equivalent to a CW complex and $\mathcal{U}$ is an open cover such that every open set in $\mathcal{U}$ and every finite intersection of such open sets is either empty or contractible, then the Čech cohomology groups $H^p(\mathcal{U}, \mathbb{Z}_X)$ agree with the singular cohomology groups $H^p_{\text{sing}}(X, \mathbb{Z})$ (see for example (Griffiths and Harris, 1979, p. 42)). △

Example 20.13 (Constant sheaves on irreducible spaces)**.** Let X be an *irreducible* topological space and let A_X denote the constant sheaf associated to an abelian group A. We claim that for any finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X,

$$H^p(\mathcal{U}, A_X) = 0 \qquad \text{for all } p \geqslant 1.$$

As X is irreducible, all the intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ are nonempty and connected. Therefore, the Čech complex takes the following form:

$$\prod_i A \longrightarrow \prod_{i<j} A \longrightarrow \prod_{i<j<k} A \longrightarrow \cdots \tag{20.9}$$

Note that this complex does not depend on X nor on the open cover $\mathcal{U}$; only the index set I plays a role. We can therefore replace $\mathcal{U}$ by the trivial cover where all $U_i = X$. In this case, the higher cohomology groups vanish by Example 20.11.

For this reason, constant sheaves are not particularly useful for studying geometric properties of schemes with the Zariski topology. For schemes, it is the quasi-coherent sheaves which give us richer and more interesting geometric invariants. △

20.4 Čech cohomology on schemes

As the previous examples illustrate, the cohomology groups $H^p(\mathcal{U}, \mathcal{F})$ can be computed if we have a good understaning of the sections of $\mathcal{F}$ over the open sets in the finite cover $\mathcal{U}$. In

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

such cases, the maps in the Čech complex are entirely explicit, and computing their kernels and images can often be done quite algorithmically.

On the other hand, the current definition of the cohomology groups is unsatisfactory for a few reasons. First, we have restricted to working with finite open covers. Second, the groups $H^p(\mathcal{U}, \mathcal{F})$ depend on the open cover $\mathcal{U}$, whereas we want intrinsic invariants that depend only on $\mathcal{F}$. This dependency is important: for instance, $\mathcal{U}$ could consist of the single open set X, and so $H^i(\mathcal{U}, \mathcal{F}) = 0$ for all $i \geqslant 1$. Moreover, Example 20.11 shows that adding X to any cover $\mathcal{U}$ makes all the higher cohomology groups zero.

The solution to these issues is typically to use more refined covers, where the open sets are 'simple' in some sense. In the context of schemes, this means considering open covers $\mathcal{U}$ consisting of affine open sets. We will show that when X is Noetherian and separated and the sheaf $\mathcal{F}$ is quasi-coherent, the group $H^i(\mathcal{U}, \mathcal{F})$ is independent of the cover $\mathcal{U}$.

Theorem 20.14 (Main properties of Čech cohomology). Let X be a Noetherian separated scheme and let $\mathcal{U} = \{U_i\}$ be a finite affine cover of X. Then

(i) The Čech cohomology groups define functors $H^i(\mathcal{U}, -)\colon \mathsf{Ab}(X) \to \mathsf{Ab}$.

(ii) $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

(iii) Short exact sequences of quasi-coherent sheaves induce long exact sequences

$$\cdots \to H^p(\mathcal{U}, \mathcal{F}') \to H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F}'') \to H^{p+1}(\mathcal{U}, \mathcal{F}') \to \cdots$$

(iv) For any other finite affine cover $\mathcal{V} = \{V_j\}$ of X and quasi-coherent sheaf $\mathcal{F}$, there are canonical isomorphisms

$$H^p(\mathcal{U}, \mathcal{F}) = H^p(\mathcal{V}, \mathcal{F}) \qquad \text{for all } p \geqslant 0$$

(v) For every quasi-coherent sheaf $\mathcal{F}$, one has $H^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > \dim X$.

We have already proved items (i) and (ii). These statements hold for any finite cover; we do not need to assume that X is separated, nor that the cover is affine. For the remaining properties, we work with a finite affine cover $\{U_i\}$ such that all the intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ is affine. A Noetherian scheme is quasi-compact and hence has a finite affine cover, while separatedness ensures that all finite intersections of affine opens are affine. In light of item (iv), we make the following definition.

Definition 20.15. For a Noetherian separated scheme X, and a quasi-coherent sheaf $\mathcal{F}$, we write $H^p(X, \mathcal{F})$ for the group $H^p(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ is any finite affine cover of X.

Let us prove item (iii), namely that short exact sequences of sheaves induce long exact sequences in cohomology. Consider a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

By Corollary 10.11, for any affine open subset $U = \operatorname{Spec} A$, the sequence of sections is exact:

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0. \tag{20.10}$$

For our cover, $\mathcal{U} = \{U_i\}_{i \in I}$, all finite intersections are affine. Taking products of the exact

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

sequences (20.10) over all relevant intersections, we obtain exact sequences

$$0 \longrightarrow C^p(\mathcal{U}, \mathcal{F}') \longrightarrow C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{F}'') \longrightarrow 0,$$

and consequently a short exact sequence of Čech complexes

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}') \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}'') \longrightarrow 0.$$

Finally, applying Proposition 20.1, we obtain the desired long exact sequence

$$\cdots \longrightarrow H^i(\mathcal{U}, \mathcal{F}') \longrightarrow H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}, \mathcal{F}'') \longrightarrow H^{i+1}(\mathcal{U}, \mathcal{F}') \longrightarrow \cdots .$$

20.5 Cohomology of sheaves on affine schemes

The following result is fundamental in the study of sheaf cohomology groups on schemes.

Theorem 20.16 (Cohomology on affine schemes). Let $X = \operatorname{Spec} A$ and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for any finite affine cover $\mathcal{U}$ of X,

$$H^p(\mathcal{U}, \mathcal{F}) = 0 \text{ for all } p > 0.$$

This is the first example of a *vanishing theorem* for cohomology. The fact that higher cohomology groups vanish is good thing – after all, these groups were introduced to measure the 'failure' of certain exactness and lifting statements.

Proof Let $\mathcal{U} = \{U_0, \ldots, U_r\}$ be the affine cover. As X is affine, it is separated, so all intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ are also affine. We want to show that the complex of A-modules

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \tag{20.11}$$

is exact. We know that the theorem holds in the trivial case when one of the U_i, say, U_0 is equal to X by Example 20.11. For the general case, we reduce to the that case as follows.

As (20.11) is a complex of A-modules, exactness can be checked after localizing at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Fix $\mathfrak{p}$ and, after relabeling, assume that $\mathfrak{p} \in U_0$.

Consider the restriction $\mathcal{F}|_{U_0}$ and the induced cover $\mathcal{U} \cap U_0$ of U_0. The corresponding Čech complex

$$0 \longrightarrow \mathcal{F}(U_0) \longrightarrow C^0(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0}) \longrightarrow C^1(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0}) \longrightarrow \cdots \tag{20.12}$$

is exact by the trivial case, because the induced cover contains U_0 itself. Localization commutes with finite products and preserves exactness, so the following sequence is also exact:

$$0 \longrightarrow \mathcal{F}(U_0)_\mathfrak{p} \longrightarrow C^0(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0})_\mathfrak{p} \longrightarrow C^1(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0})_\mathfrak{p} \longrightarrow \cdots \tag{20.13}$$

As $\mathcal{F}$ is quasi-coherent, the localization of (20.11) at $\mathfrak{p}$ coincides with the complex (20.13). Since the latter is exact, so is (20.11), completing the proof. □

The fact that higher cohomology groups vanish for every quasi-coherent sheaf is a distinctive property of affine schemes. In fact, this property characterizes affine schemes among all schemes (see (Stacks Project, 2018, Tag 01XE)). Example 20.19 shows that even the most basic non-affine scheme, $\mathbb{P}^1_k$, admits sheaves with nonvanishing higher cohomology. Here is another example.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 20.17. Consider the affine line with two origins X, constructed by gluing two copies of $\operatorname{Spec} k[x]$ along the open subset $D(x) = \operatorname{Spec} k[x, x^{-1}]$ via the identity map. The Čech complex for this cover takes the form

$$0 \longrightarrow k[x] \times k[x] \xrightarrow{d} k[x, x^{-1}] \longrightarrow 0.$$

As we verified previously, $H^0(\mathcal{U}, \mathcal{O}_X) = \operatorname{Ker} d = k[x]$. For the H^1, note that

$$H^1(\mathcal{U}, \mathcal{O}_X) = \operatorname{Coker} d = k[x, x^{-1}]/k[x] = \bigoplus_{i>0} k \cdot x^{-i}.$$

In particular, $H^1(\mathcal{U}, \mathcal{O}_X)$ is infinite-dimensional as a k-vector space. △

20.6 Independence of the cover

Our first application of the vanishing theorem for affine schemes is the proof of item (iv) in Theorem 20.14: the Čech cohomology groups are independent of the chosen finite affine cover.

We first consider what happens when we add a single affine open set U_0 to an affine cover $\mathcal{V} = \{U_1, \ldots, U_r\}$. Write $\mathcal{U} = \{U_0, U_1, \ldots, U_r\}$ for the new cover.

Let $\mathcal{F}$ be a quasi-coherent sheaf. The Čech complex with respect to $\mathcal{U}$ is given by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{0 \leqslant i_0 < \cdots < i_p \leqslant r} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

Every increasing $(p+1)$-tuple either contains no 0 or begins with 0. This gives a decomposition

$$C^p(\mathcal{U}, \mathcal{F}) = C^p(\mathcal{V}, \mathcal{F}) \oplus C^{p-1}(\mathcal{W}, \mathcal{F}), \tag{20.14}$$

where

$$C^p(\mathcal{V}, \mathcal{F}) = \prod_{1 \leqslant i_0 < \cdots < i_p \leqslant r} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

and

$$C^q(\mathcal{W}, \mathcal{F}) = \prod_{1 \leqslant j_0 < \cdots < j_q \leqslant r} \mathcal{F}(U_0 \cap U_{j_0} \cap \cdots \cap U_{j_q}) \qquad q \geqslant 0$$

and we set $C^{-1}(\mathcal{W}, \mathcal{F}) = \mathcal{F}(U_0)$. In this notation, $C^\bullet(\mathcal{W}, \mathcal{F})$ is identified with the augmented Čech complex of the cover $\mathcal{W}$ of U_0 consisting of the affine open sets $U_0 \cap U_1, \ldots, U_0 \cap U_r$.

The decomposition (20.14) is generally not a decomposition of complexes. If $d_{\mathcal{U}}$, $d_{\mathcal{V}}$ and $d_{\mathcal{W}}$ denote the Čech differentials of the three complexes involved, then

$$d_{\mathcal{U}}(a, b) \;=\; \big(d_{\mathcal{V}} a\,,\;\; a|_{\mathcal{W}} - d_{\mathcal{W}} b\big)$$

where $a|_{\mathcal{W}}$ denotes the image of a under the restriction map $C^p(\mathcal{V}, \mathcal{F}) \to C^p(\mathcal{W}, \mathcal{F})$ to U_0. This formula follows by evaluating $d_{\mathcal{U}}$ on the two kinds of index tuples: those containing no 0 and those beginning with 0. It follows that the first projection $\pi\colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{V}, \mathcal{F})$ satisfies $\pi \circ d_{\mathcal{U}} = d_{\mathcal{V}} \circ \pi$, so it is a map of complexes.

The kernel $K = \operatorname{Ker} \pi$ is the subcomplex $K^p = \{0\} \times C^{p-1}(\mathcal{W}, \mathcal{F})$. As a complex, it is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

identified with $\big(C^{p-1}(\mathcal{W},\mathcal{F}),-d_{\mathcal{W}}\big)$, namely the shifted augmented Čech complex of $\mathcal{F}|_{U_0}$ with differential multiplied by -1. Since U_0 is affine and $\mathcal{F}|_{U_0}$ is quasi-coherent, the complex is exact, so $H^p(K^\bullet)=0$ for all p. The long exact sequence in cohomology associated to the short exact sequence

$$0\longrightarrow K^\bullet\longrightarrow C^\bullet(\mathcal{U},\mathcal{F})\xrightarrow{\pi}(C^\bullet(\mathcal{V},\mathcal{F}),d_{\mathcal{V}})\longrightarrow 0,$$

then gives natural isomorphisms $H^p(\mathcal{U},\mathcal{F})=H^p(\mathcal{V},\mathcal{F})$ for every p.

Finally, let $\mathcal{U}=\{U_1,\dots,U_r\}$ and $\mathcal{V}=\{V_1,\dots,V_s\}$ be two finite affine covers and consider their union $\mathcal{U}\cup\mathcal{V}=\{U_1,\dots,U_r,V_1,\dots,V_s\}$. By what we just showed, adding the sets of $\mathcal{V}$ to $\mathcal{U}$ one by one induces isomorphisms on cohomology, so $H^p(\mathcal{U},\mathcal{F})=H^p(\mathcal{U}\cup\mathcal{V},\mathcal{F})$. Doing the same for the cover $\mathcal{V}$, we obtain the desired isomorphisms

$$H^p(\mathcal{U},\mathcal{F})=H^p(\mathcal{U}\cup\mathcal{V},\mathcal{F})=H^p(\mathcal{V},\mathcal{F}).$$

Hence the Čech cohomology groups are independent of the choice of finite affine cover.

See Exercise 20.18.22 for an example where two different affine covers give nonisomorphic Čech cohomology groups when X is nonseparated.

20.7 Pushforwards and cohomology

The following lemma is very useful, and it will be applied several times later on. For an affine morphism $f\colon X\to Y$, it allows us to transport computations of cohomology groups on X to computations on Y, at the cost of replacing $\mathcal{F}$ with $f_*\mathcal{F}$.

Lemma 20.18. Let $f\colon X\to Y$ be an affine morphism of Noetherian, separated schemes. Then for each quasi-coherent sheaf $\mathcal{F}$ on X, and $p\geqslant 0$, we have a canonical isomorphism

$$H^p(X,\mathcal{F})=H^p(Y,f_*\mathcal{F}). \tag{20.15}$$

Proof Let $\mathcal{V}=\{V_i\}$ be a finite affine covering of Y such that $H^p(X,f_*\mathcal{F})$ is computed by the Čech complex $C^\bullet(V_i,f_*\mathcal{F})$. Note that the latter complex equals $C^\bullet(f^{-1}V_i,\mathcal{F})$. As f is affine, the affine subsets $U_i=f^{-1}(V_i)$ form a finite affine covering $\mathcal{U}$ of X, and the lemma follows simply because the Čech complexes computing the two sides of (20.15) are identical. □

As a special case, if $\iota\colon Y\hookrightarrow X$ is a closed embedding, then ι is affine, and hence $H^i(Y,\mathcal{F})\simeq H^i(X,\iota_*\mathcal{F})$. In particular, $H^i(Y,\mathcal{O}_Y)\simeq H^i(X,\iota_*\mathcal{O}_Y)$. Using the ideal sheaf sequence $0\to\mathcal{I}\to\mathcal{O}_X\to\iota_*\mathcal{O}_Y\to 0$, we may then relate the cohomology groups of $\mathcal{O}_Y$ to those of $\mathcal{O}_X$ and $\mathcal{I}$ on X.

Example 20.19. Let $Z\subset\mathbb{P}^1_k$ be the subscheme given by two closed distinct k-points p and q and let $\iota\colon Z\to\mathbb{P}^1_k$ be the inclusion. The ideal sheaf sequence 12.15 is

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^1_k}(-2)\longrightarrow\mathcal{O}_{\mathbb{P}^1_k}\longrightarrow\iota_*\mathcal{O}_Z\longrightarrow 0.$$

Under the identification $\Gamma(\mathbb{P}^1_k,\iota_*\mathcal{O}_Z)=\Gamma(Z,\mathcal{O}_Z)\simeq k\oplus k$, the element $(0,1)\in k\oplus k$ defines the function which vanishes at p and has value 1 at q. This section cannot lift to a global section of $\mathcal{O}_{\mathbb{P}^1_k}$, since any regular function on $\mathbb{P}^1_k$ is constant.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This failure of lifting can be explained by the cohomology group $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2))$ which is 1-dimensional. Since $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0$, one can think of the elements of this group as the group of elements of $k \oplus k$ modulo those that lift to $\mathcal{O}_{\mathbb{P}^1_k}$. Here it is clear that an element $(a, b) \in k \oplus k$ lifts if and only if $a = b$. In fact, in this example, the connecting map

$$\delta\colon H^0(\mathbb{P}^1_k, i_*\mathcal{O}_Z) \longrightarrow H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2))$$

can be identified with the map $k \oplus k \to k$ sending (a, b) to $a - b$. △

20.8 Cohomology and dimension

The next result is another 'vanishing theorem' for cohomology groups. It states that the cohomology groups vanish in degrees above the dimension of the scheme.

Theorem 20.20. Let X be a quasi-projective scheme of dimension n over a Noetherian ring A. Then:

(i) X admits an open cover $\mathcal{U}$ by at most $\dim X + 1$ affine open subsets.

(ii) For any quasi-coherent sheaf $\mathcal{F}$ on X, we have

$$H^p(X, \mathcal{F}) = 0 \quad \text{for all } p > \dim X.$$

Proof (i): Since X is quasi-projective, we may write $X = \overline{X} - W$ where $\overline{X} \subset \mathbb{P}^r_A$ where $\overline{X}$ and $\overline{W}$ are closed subschemes of $\mathbb{P}^r_A$. We may assume that no irreducible component of $\widetilde{X}$ is contained in W (simply by disregarding such components).

We prove (i) by induction on $n = \dim X$, showing that X can be covered by at most $n + 1$ distuinguished open sets from $\mathbb{P}^n_A$. When $n = 0$, the scheme X consists of finitely many closed points, which correspond to homogeneous maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ in $A[x_0, \ldots, x_n]$. By prime avoidance, we may choose a homogeneous element of positive degree $f \in A[x_0, \ldots, x_n]$ not contained in any $\mathfrak{p}_i$. This means that $V_+(f)$ is disjoint from X, and X is contained in $D_+(f)$, which is affine. Therefore, the statement holds for $n = 0$.

For $n > 0$, let $I_{\overline{X}}, I_W \subset A[x_0, \ldots, x_r]$ be the homogeneous ideals defining $\overline{X}$ and W respectively. Again by prime avoidance, we may choose a homogeneous polynomial $f \in I_W$ which does not vanish on any component of $\overline{X}$. By construction, the hypersurface $H = V_+(f)$ contains W and $\mathbb{P}^r_A - H = D_+(f)$ is affine. The intersection $\overline{X} \cap D_+(f)$ is therefore an affine open subscheme of X. Since f does not vanish identically on any component of $\overline{X}$, the intersection $\overline{X} \cap H$ has dimension $< n$. By the induction hypothesis, $\overline{X} \cap H$ can be covered by at most n distinguished open sets in $\mathbb{P}^r_A$. Combining $\overline{X} \cap D_+(f)$ with these affine opens gives the required cover of X by at most $n + 1$ affine opens.

(ii): The vanishing follows directly from the existence of the affine cover in (i). For any quasi-coherent sheaf $\mathcal{F}$, the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ associated to this cover has length $n + 1$, and consequently $H^p(X, \mathcal{F}) = 0$ for all $p > n$. □

The vanishing $H^p(X, \mathcal{F}) = 0$ for $p > \dim X$ holds more generally for arbitrary sheaves on Noetherian topological spaces, though the proof requires different techniques. The above special case for quasi-coherent sheaves will be sufficient for the applications in this book.

20.9 Cohomology of the sheaves $\mathcal{O}(m)$

In Example 20.9, we computed the cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(m)$. For $m \geqslant 0$, we found that $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m))$ could be identified with the space of homogeneous polynomials of degree m, and $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m)) = 0$. On the other hand, for $m \leqslant -2$, $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m)) = 0$, while $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m))$ was nonzero.

We will now carry out a more general computation of the cohomology groups of $\mathcal{O}_{\mathbb{P}^n_A}(m)$ for any projective space $\mathbb{P}^n_A$ over a ring A. The strategy is the same: we use the Čech complex associated to the standard cover $D_+(x_0), \dots, D_+(x_n)$ to to compute the cohomology groups.

Write $R = A[x_0, \dots, x_n]$. Then the nonzero groups in the Čech complex are

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \prod_{0 \leqslant i \leqslant n} (R_{x_i})_m \\ C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \prod_{0 \leqslant i < j \leqslant n} \big(R_{x_i x_j}\big)_m \\ &\vdots \\ C^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= (R_{x_0 x_1 \cdots x_n})_m \end{aligned}$$

and the Cech complex takes the form

$$\prod_{0 \leqslant i \leqslant n} (R_{x_i})_m \xrightarrow{d^0} \prod_{0 \leqslant i < j \leqslant n} \big(R_{x_i x_j}\big)_m \xrightarrow{d^1} \prod_{0 \leqslant i < j < k \leqslant n} \big(R_{x_i x_j x_k}\big)_m \xrightarrow{d^2} \cdots \tag{20.16}$$

where the maps are as usual composed of alternating sums of localization maps.

Theorem 20.21 (Cohomology of sheaves on $\mathbb{P}^n$). Let A be a ring and let $\mathbb{P}^n_A$ be the projective n-space over A. Then:

(i) For each $m \in \mathbb{Z}$,

$$H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = A[x_0, \dots, x_n]_m.$$

(ii) For each $m \in \mathbb{Z}$,

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \big(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \dots, x_n^{-1}]\big)_m \tag{20.17}$$

In particular, there is a canonical isomorphism

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)) \simeq A.$$

(iii) For each $m \in \mathbb{Z}$ and $p \neq 0, n$,

$$H^p(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = 0.$$

Proof (i): In degree 0, we recover the following isomorphism:

$$\begin{aligned} H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \operatorname{Ker} d^0 \\ &= A[x_0, \dots, x_n]_m. \end{aligned}$$

(ii): Observe that

$$C^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) = (A[x_0, \dots, x_n]_{x_0 \cdots x_n})_m$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is a free graded A-module spanned by monomials of the form $x_0^{e_0}\cdots x_n^{e_n}$ of degree $\sum e_i = m$. The image of d^{n-1} is generated by such monomials where at least one e_i is non-negative. Therefore, $H^n(X, \mathcal{O}_{\mathbb{P}^n_A}(m))$, which is the cokernel of d^{n-1} is given by

$$\begin{aligned} H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= A\big\{x_0^{e_0}\cdots x_n^{e_n} \;\big|\; e_i < 0 \text{ for every } i \text{ and } \sum e_i = m\big\} \\ &= \big(x_0^{-1}\cdots x_n^{-1}A[x_0^{-1},\dots,x_n^{-1}]\big)_m\,. \end{aligned}$$

In degree $m = -n-1$, there is only one such monomial, namely $x_0^{-1}\cdots x_n^{-1}$, so

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)) = A\cdot x_0^{-1}\cdots x_n^{-1}.$$

(ii): We argue by induction on n. For $n = 0$ and $n = 1$, there is nothing to prove. Let $n \geqslant 2$, and write $R = S[x_n]$ where $S = A[x_0,\dots,x_{n-1}]$. Rewriting to single out the variable x_n, we express the Cech complex as the degree m part of the following complex:

$$\mathcal{C}^\bullet: \qquad 0 \to \prod_{i=0}^{n-1} R_{x_i} \times R_{x_n} \to \cdots \to \prod_{\substack{|I|=p\\ I\not\ni n}} R_{x_I} \times \prod_{\substack{|I|=p-1\\ I\not\ni n}} R_{x_I x_n} \to \cdots \tag{20.18}$$

To finish the proof, we need to prove that $H^p(\mathcal{C}^\bullet) = 0$ for $p = 1,\dots,n-1$.

Localizing the complex with respect to x_n, we see that $\mathcal{C}^\bullet$ embeds as a subcomplex of the following complex

$$\mathcal{D}^\bullet: \qquad 0 \to \prod_{i=0}^{n-1} R_{x_i x_n} \times R_{x_n} \to \cdots \to \prod_{\substack{|I|=p\\ I\not\ni n}} R_{x_I x_n} \times \prod_{\substack{|I|=p-1\\ I\not\ni n}} R_{x_I x_n} \to \cdots \tag{20.19}$$

This is the same complex as the Cech complex of $\mathcal{O}_{D(x_n)}$ on $D(x_n) = \operatorname{Spec}(R_{x_n})$ for the cover with $n+1$ open affines $D(x_0x_n),\ \dots,\ D(x_{n-1}x_n),\ D(x_n)$. In particular, since $D(x_n)$ itself is part of the cover, the complex (20.19) is exact in all positive degrees, i.e., $H^p(\mathcal{D}^\bullet) = 0$ for all $p > 0$.

Let $\mathcal{E}^\bullet = \mathcal{D}^\bullet/\mathcal{C}^\bullet$. Decomposing the elements in $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ according to their x_n-degree, we see that $\mathcal{E}^\bullet$ decomposes as $\mathcal{E}^\bullet = \bigoplus_{l>0} \mathcal{E}_l^\bullet \cdot x_n^{-l}$ where $\mathcal{E}_l^\bullet$ is the complex

$$0 \to \prod_{i=0}^{n-1} S_{x_i} \times 0 \to \cdots \to \prod_{\substack{|I|=p\\ I\not\ni n}} S_{x_I} \times \prod_{\substack{|I|=p-1\\ I\not\ni n}} 0 \to \cdots. \tag{20.20}$$

Ignoring the zero summands in (20.20), we see that $\mathcal{E}_l^\bullet$ is just the Čech complex associated to the standard cover of $\mathbb{P}^{n-1}_A$. More precisely, for each $l > 0$, the degree-m part of the complex $\mathcal{E}_l^\bullet \cdot x_n^{-l}$ is the degree-$(m+l)$ part of that Čech complex. Therefore, by induction, we have $H^p((\mathcal{E}_l^\bullet \cdot x_n^{-l})_m) = 0$ for $p = 1,\dots,n-2$. Since this holds for every $l > 0$, it follows that $H^p(\mathcal{E}_m^\bullet) = 0$ for $p = 1,\dots,n-2$.

Now, taking the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathcal{D}^\bullet \longrightarrow \mathcal{E}^\bullet \longrightarrow 0, \tag{20.21}$$

we obtain the exact sequence

$$0 \to H^0(\mathcal{C}^\bullet) \to H^0(\mathcal{D}^\bullet) \to H^0(\mathcal{E}^\bullet) \to H^1(\mathcal{C}^\bullet) \to 0 \tag{20.22}$$

and isomorphisms $H^p(\mathcal{C}^\bullet) \simeq H^{p-1}(\mathcal{E}^\bullet)$ for $p > 1$. Hence $H^p(\mathcal{C}^\bullet) = 0$ for $2 \leqslant p \leqslant n-1$.

For the remaining group $H^1(\mathcal{C}^\bullet)$, note that (20.22) takes the form

$$0 \to R \to R_{x_n} \to \bigoplus_{l>0} S \cdot x_n^{-l} \to H^1(\mathcal{C}^\bullet) \to 0$$

Here the map $R_{x_n} \to \bigoplus_{l>0} S \cdot x_n^{-l}$ is clearly surjective, so we deduce that $H^1(\mathcal{C}^\bullet) = 0$. This completes the proof of (iii). □

The proof of Theorem 20.21 also gives an important duality between H^0 and H^n on $\mathbb{P}^n_A$. Recall that, if M and N are A-modules, then an A-bilinear map $M \times N \to A$ is called a *perfect pairing* if the induced map $M \to \mathrm{Hom}_A(N, A)$ is an isomorphism.

Consider the A-bilinear pairing

$$(\,,\,)\colon A[x_0, \dots, x_n] \times \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \dots, x_n^{-1}]\right) \to A \tag{20.23}$$

sending a pair (p, q) of Laurent polynomials p and q to the coefficient of $x_0^{-1} \cdots x_n^{-1}$ in the product pq. In terms of the standard monomial bases, this pairing is given by

$$(x_0^{d_0} \cdots x_n^{d_n}, x_0^{e_0} \cdots x_n^{e_n}) = \begin{cases} 1 & \text{if } d_i + e_i = -1 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is perfect and allows us to canonically identify

$$\left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \dots, x_n^{-1}]\right)_d = \mathrm{Hom}_A(A[x_0, \dots, x_n]_{-n-1-m}, A).$$

This allows us to regard the n-th cohomology group $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ as the *dual* of a corresponding H^0:

Corollary 20.22 (Serre duality for $\mathbb{P}^n$). For each $m \in \mathbb{Z}$, there is an isomorphism

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \mathrm{Hom}_A(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_A}(-m-n-1)), A). \tag{20.24}$$

When $A = k$ is a field, the dimensions of the cohomology groups are easily computed:

Corollary 20.23. Let k be a field. Then for $m \geqslant 0$

$$\dim_k H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)) = \binom{m+n}{n},$$
$$\dim_k H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-m)) = \binom{m-1}{n},$$

and all other cohomology groups are 0.

20.10 Cohomology of sheaves on projective schemes

By the results of the previous section, the cohomology groups of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ over a field k are always finite-dimensional k-vector spaces. This is part of a more general result, saying that on projective schemes of over a ring, the cohomology groups of coherent sheaves are

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

always finitely generated. Note that this is definitely not the case for affine schemes: even the H^0 of the structure sheaf on $\mathbb{A}^1_k$ is infinite dimensional, as it equals $k[t]$.

Theorem 20.24 (Serre). Let $X \subset \mathbb{P}^n_A$ be a projective scheme over a Noetherian ring A and let $\mathcal{F}$ be a coherent sheaf. Then:

(i) For each i, $H^i(X, \mathcal{F})$ is a finitely generated A-module.

(ii) There exists an $m_0 > 0$ such that

$$H^i(X, \mathcal{F}(m)) = 0.$$

for all $m \geqslant m_0$ and $i > 0$.

Proof Let $\iota\colon X \to \mathbb{P}^n_A$ denote the closed embedding and consider the sheaf $\iota_*\mathcal{F}$. Since ι is finite, the sheaf $\iota_*\mathcal{F}$ is again coherent (Exercise 10.13.27) and

$$\begin{aligned} H^i(X, \mathcal{F}(d)) &= H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \iota^*\mathcal{O}_{\mathbb{P}^n}(d)) \\ &= H^i(\mathbb{P}^n, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \iota^*\mathcal{O}_{\mathbb{P}^n}(d))) \\ &= H^i(\mathbb{P}^n, \iota_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)) \\ &= H^i(\mathbb{P}^n, (\iota_*\mathcal{F})(d)). \end{aligned} \tag{20.25}$$

Therefore we reduce to the case $X = \mathbb{P}^n_A$.

Note that both parts of the theorem are trivially satisfied if $i > n$ because the cohomology groups are zero in this case ($\mathbb{P}^n_A$ can be covered by $n+1$ affines). The proof will take this as the base case and proceed by downwards induction on i.

By Theorem 12.4, any coherent $\mathcal{O}_{\mathbb{P}^n_A}$-module is of the form $\widetilde{M}$ for some finitely graded module M over $R = A[x_0, \ldots, x_n]$.

(i): As M is finitely genenerated, we may pick a graded surjection $\bigoplus_j R(-a_j) \to M$ for M. Letting K be the kernel, we have an exact sequence of finitely generated graded R-modules

$$0 \longrightarrow K \longrightarrow \bigoplus_i R(-a_j) \longrightarrow M \longrightarrow 0$$

Applying tilde, we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^n_A}(-a_j) \longrightarrow \mathcal{F} \longrightarrow 0.$$

If we take the long exact sequence of cohomology, we get

$$\cdots \to H^i(\mathbb{P}^n_A, \mathcal{K}) \to \bigoplus_j H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-a_j)) \to H^i(\mathbb{P}^n_A, \mathcal{F}) \to H^{i+1}(\mathbb{P}^n_A, \mathcal{K}) \to \ldots$$

By induction on i, the cohomology group $H^{i+1}(\mathbb{P}^n_A, \mathcal{K})$ is a finitely generated A-module, and the same holds for each $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-a_j))$ by Theorem 20.21. $H^i(\mathbb{P}^n_A, \mathcal{F})$ is therefore squeezed between two finitely generated A-modules, so by exactness, it must itself be finitely generated.

(ii): Twist the above sequence by $\mathcal{O}_{\mathbb{P}^n_A}(m)$ and take the long exact sequence in cohomology

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

to get

$$H^i(\mathbb{P}^n_A, \bigoplus_j \mathcal{O}_{\mathbb{P}^n_A}(m-a_j)) \longrightarrow H^i(\mathbb{P}^n_A, \mathcal{F}(m)) \longrightarrow H^{i+1}(\mathbb{P}^n_A, \mathcal{K}(m))$$

Again, by downward induction on i, and the fact that $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m-a_j)) = 0$ for all $i > 0$ and $m > a_j$, we find that $H^i(\mathbb{P}^n_A, \mathcal{F}(m)) = 0$. □

Corollary 20.25. Let $X \subset \mathbb{P}^n_A$ be a projective scheme over an Noetherian ring A and let $f\colon X \to \operatorname{Spec} A$ be the projection. If $\mathcal{F}$ is a coherent sheaf on X, then $f_*\mathcal{F}$ is a coherent sheaf on $\operatorname{Spec} A$.

Proof The sheaf $f_*\mathcal{F}$ is in any case quasi-coherent by Theorem 10.29. In fact, $f_*\mathcal{F} = \widetilde{\Gamma(X,\mathcal{F})}$, so it suffices to show that $\Gamma(X,\mathcal{F}) = H^0(X,\mathcal{F})$ is finitely generated as an A-module. This is clear if f is a closed embedding $X \to \mathbb{P}^n_A$, because f is finite and $f_*\mathcal{F}$ is coherent on $\mathbb{P}^n_A$ (Exericise 10.13.27). It is also clear if f is the projection $\mathbb{P}^n_A \to \operatorname{Spec} A$, by item (i) in the above theorem. Now the general case follows by factoring f as a closed embedding followed by the projection. □

The assumption that X is projective over A is essential: for the structure morphism $f\colon \operatorname{Spec} k[x] \to \operatorname{Spec} k$, we have $f_*\mathcal{O}_{\operatorname{Spec} k[x]} = \widetilde{k[x]}$, but $k[x]$ is not finite as a k-vector space.

The Euler characteristic

If X is a projective scheme over a field k and $\mathcal{F}$ is a coherent sheaf, Serre's theorem tells us that the cohomology groups $H^i(X,\mathcal{F})$ are finite-dimensional k-vector spaces. In particular, we can ask about their dimensions. We define

$$h^i(X,\mathcal{F}) = \dim_k H^i(X,\mathcal{F}).$$

It turns out that the alternating sum of these dimensions has very good formal properties, so we make the following definition:

Definition 20.26. Let X be a projective scheme over a field k. We define the *Euler characteristic* of $\mathcal{F}$ as

$$\chi(\mathcal{F}) = \sum_{i \geqslant 0} (-1)^i h^i(X,\mathcal{F}).$$

Note that the sum is well-defined, as there are only finitely many nonzero cohomology groups appearing on the right-hand side.

Proposition 20.27. The Euler characteristic χ is additive on exact sequences, i.e., if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof This follows from the fact that if $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of k-vector spaces, then $\sum_i (-1)^i \dim_k V_i = 0$. Applying this to the long exact sequence in cohomology gives the claim. □

Example 20.28. Let $X = \mathbb{P}^n_k$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_k}(d)$. If $d \geqslant 0$, then the only possible nonzero $h^i(X, \mathcal{F})$ is

$$h^0(\mathbb{P}^n_k, \mathcal{O}(d)) = \binom{n+d}{n},$$

and when $d < 0$ only the h^n is nonzero:

$$h^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{-d-1}{n}.$$

If we use the extended binomial coefficient $\binom{x}{n} = x(x-1)\cdots(x-n+1)/n!$ we may write

$$\binom{-d-1}{n} = (-1)^n \binom{n+d}{n}.$$

This means that for any $d \in \mathbb{Z}$, we have

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{n}.$$

This is a polynomial in d of degree n which agrees with $h^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ for all $d \geqslant 0$. △

The example shows that for a direct sum $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_k}(a_r)$, the Euler characteristic $\chi(\mathcal{E}(m))$ is a polynomial in m. Even more generally, we can take any coherent sheaf $\mathcal{F}$ and a free resolution of it:

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where the $\mathcal{E}_i$ are direct sums of invertible sheaves of the form $\mathcal{O}(d)$. If we tensor this sequence by $\mathcal{O}_{\mathbb{P}^n_A}(m)$, we get[1]

$$0 \to \mathcal{E}_n(m) \to \cdots \to \mathcal{E}_1(m) \to \mathcal{E}_0(m) \to \mathcal{F}(m) \to 0$$

Note that each of the terms $\chi(\mathcal{E}_i(m))$ is a polynomial in m. Then since the Euler characteristic is additive on exact sequences, also $\chi(\mathcal{F}(m))$ is a polynomial in m. Moreover, again by Serre's theorem, we have $H^i(X, \mathcal{F}(m)) = 0$ for $m \gg 0$ and $i > 0$, so $\chi(\mathcal{F}(m)) = H^0(\mathcal{F}(m))$ for m large.

If we start with a coherent sheaf $\mathcal{F}$ on $X \subset \mathbb{P}^n_k$, and apply the previous discussion to $i_*\mathcal{F}$ on $\mathbb{P}^n_k$, we have proved the following:

Corollary 20.29. Let $X \subset \mathbb{P}^n_k$ be a projective scheme over k. Then the function

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m))$$

is a polynomial in m, and for large m, $P_{\mathcal{F}}(m) = H^0(X, \mathcal{F}(m))$.

[1] The stalk of $\mathcal{O}(m)$ at a point P is equal to $\mathcal{O}_{\mathbb{P}^n_k, P}$, so tensoring by $\mathcal{O}(m)$ preserves exactness.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This polynomial is called the *Hilbert polynomial* of $\mathcal{F}$. While $\chi(\mathcal{F})$ is an intrinsic invariant of $\mathcal{F}$, the Hilbert polynomial is not, as it depends on the choice of embedding $X \subset \mathbb{P}^n_k$.

When $\mathcal{F} = \widetilde{M}$ for a graded module M, $P_{\mathcal{F}}(m)$ coincides with the usual Hilbert polynomial of M as defined in commutative algebra.

20.11 Example: Plane curves

Let $X = V_+(f) \subset \mathbb{P}^2_k$ be a plane curve, defined by an homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. Let us compute the groups of the structure sheaf $H^i(X, \mathcal{O}_X)$. We have the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^2_k} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

where the ideal sheaf $\mathcal{I}_X$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X$. By Example 12.15, $\mathcal{I}_X \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$, and the sequence can be rewritten as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2_k} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0 \tag{20.26}$$

From the short exact sequence, we get the long exact sequence as follows:

$$\begin{aligned} 0 \to H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) &\to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2_k}) \to H^0(X, \mathcal{O}_X) \\ &\to H^1(\mathbb{P}^2_k, \mathcal{O}(-d)) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2_k}) \to H^1(X, \mathcal{O}_X) \\ &\to H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) \to H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}) \longrightarrow 0. \end{aligned}$$

From the formulas for the cohomology groups on $\mathbb{P}^2_k$, we find $H^0(X, \mathcal{O}_X) \simeq k$ and

$$\dim_k H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}.$$

The dimension of the cohomology group on the left is the *genus* of the curve X (it will be introduced properly in Chapter 22). So the above can be rephrased as saying *the genus of a plane curve of degree d is* $(d-1)(d-2)/2$.

Tensoring the sequence (20.26) by $\mathcal{O}_{\mathbb{P}^2_k}(m)$, we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2_k}(m-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2_k}(m) \longrightarrow i_*\mathcal{O}_X(m) \longrightarrow 0$$

and the long exact sequence gives that the Hilbert polynomial of $\mathcal{O}_X$ equals

$$P(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = dm - \frac{d^2-3d}{2}.$$

Example 20.30. A plane curve of degree 1, i.e., a projective line, is isomorphic to $\mathbb{P}^1_k$, and the genus is 0 in accordance with the above result. △

Example 20.31. Suppose k has characteristic different from 2 and contains an element $i \in k$ with $i^2 = -1$. Then the curve

$$X = V_+(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_k$$

is isomorphic to $\mathbb{P}^1_k$. Indeed, the variable change $u_0 = x_0 + ix_2$, $u_1 = ix_1$, $u_2 = x_0 - ix_2$ brings the equation into the form $u_1^2 - u_0u_2 = 0$. This is the image of the standard embedding $(s : t) \mapsto (s^2 : st : t^2)$. △

Example 20.32. Every plane cubic has arithmetic genus 1. In particular,

$$X = V_+(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}^2_k$$

is not isomorphic to $\mathbb{P}^1_k$ over any field, since its first cohomology group has dimension 1. △

20.12 Example: The twisted cubic

Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will continue Example 12.16 and consider the twisted cubic curve $X = V_+(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Let us compute the group $H^1(X, \mathcal{O}_X)$. Of course we know what the answer should be, because $X \simeq \mathbb{P}^1$, and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.

Now, to compute $H^1(X, \mathcal{O}_X)$ on X, it is convenient to relate it to a cohomology group on $\mathbb{P}^3$. We have $H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$ where $i\colon X \to \mathbb{P}^3$ is the inclusion. The sheaf $i_*\mathcal{O}_X$ fits into the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow i_*\mathcal{O}_X \to 0$$

where $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^3$. Applying the long exact sequence in cohomology, we get

$$\cdots \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$$
$$\hookrightarrow H^2(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow \cdots$$

By our description of sheaf cohomology on $\mathbb{P}^3$, $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$, which implies that $H^1(X, \mathcal{O}_X) = H^2(\mathbb{P}^3, \mathcal{I})$. We can compute the latter cohomology group using the exact sequence of Example 12.16:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{I} \longrightarrow 0.$$

Now, taking the long exact sequence we get

$$\cdots \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}(-3)^2) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \longrightarrow H^2(\mathbb{P}^3, \mathcal{I})$$
$$\hookrightarrow H^3(\mathbb{P}^3, \mathcal{O}(-3)^2) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \longrightarrow H^3(\mathbb{P}^3, \mathcal{I}).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Here $H^2(\mathbb{P}^3, \mathcal{O}(-2)) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(-3)) = 0$ by our previsous computations. Hence by exactness, we find $H^2(\mathbb{P}^3, \mathcal{I}) = 0$. It follows that $H^1(X, \mathcal{O}_X) = 0$ also, as expected.

$\mathcal{O}_X(U) \times \mathcal{O}_X(V) = A \times B$ and $\mathcal{O}_X(U \cap V) = A_x$.

20.13 Example: Hyperelliptic curves

Let us recall the hyperelliptic curves defined in Section 5.4. Assume that $\operatorname{char} k \neq 2$, let $g \geqslant 1$, and let $f(x) \in k[x]$ be squarefree of degree $2g + 1$ or $2g + 2$. Let X be the scheme obtained by gluing the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x, y]}{(y^2 - f(x))}, \qquad B = \frac{k[u, v]}{(v^2 - u^{2g+2} f(u^{-1}))},$$

As before, we glue $D(x) \subset U$ to $D(u) \subset V$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

We compute the Čech cohomology groups of $\mathcal{O}_X$ with respect to the affine cover $\mathcal{U} = \{U, V\}$. Since the cover has only two open sets, the Čech complex is

$$\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \xrightarrow{d} \mathcal{O}_X(U \cap V).$$

To simplify the computation, we use the relation $y^2 = f(x)$ to write

$$A \simeq k[x] \oplus k[x]\, y,$$

as a $k[x]$-module. Similarly, using $v = x^{-g-1}y$, we write $B \simeq k[x^{-1}] \oplus k[x^{-1}]\, x^{-g-1}y$. Also,

$$\mathcal{O}_X(U \cap V) = A_x \simeq k[x^{\pm 1}] \oplus k[x^{\pm 1}]\, y.$$

With these identifications, the Čech complex becomes

$$\big(k[x] \oplus k[x]\, y\big) \oplus \big(k[x^{-1}] \oplus k[x^{-1}]\, x^{-g-1}y\big) \xrightarrow{d} k[x^{\pm 1}] \oplus k[x^{\pm 1}]\, y,$$

$$\big(p(x)+q(x)y,\ r(x^{-1})+s(x^{-1})x^{-g-1}y\big) \mapsto -p(x)+r(x^{-1})+\big(-q(x)+s(x^{-1})x^{-g-1}\big)y.$$

To find $H^0(X, \mathcal{O}_X) = \operatorname{Ker} d^0$ and $H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^0$, we compare monomials on each side. First of all, any element of the kernel must have $p(x) = r(x^{-1})$ in $k[x^{\pm 1}]$, so p and r are constants. Similarly, $q(x) = s(x^{-1})x^{-g-1}$, which implies that $q = s = 0$. Therefore,

$$H^0(X, \mathcal{O}_X) \simeq k.$$

For $H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^0$, we note that we may obtain all the monomials $x^a y^b$ on the right-hand side from one of the summands on the left, except for $x^{-1}y, \ldots, x^{-g}y$. Therefore,

$$H^1(X, \mathcal{O}_X) = k\, x^{-1}y \oplus \cdots \oplus k\, x^{-g}y \simeq k^g.$$

An interesting special case is when $g = 2$. Then X is a projective curve which cannot be embedded in $\mathbb{P}^2_k$. Indeed, we showed that for any curve in $\mathbb{P}^2_k$ of degree d, the corresponding cohomology group $H^1(X, \mathcal{O}_X)$ has dimension equal to $\frac{1}{2}(d-1)(d-2)$, but this does not take the value 2 if d is an integer.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20.14 Example: Complete intersections

Let k be a field, let $R = k[x_0, \dots, x_n]$ with $n \geqslant 2$, and let $F, G \in R$ be nonzero homogeneous polynomials of degrees $d, e > 0$. Assume that G is a nonzerodivisor in $S = R/(F)$. Then the projective scheme $X = V_+(F, G) \subset \mathbb{P}^n_k$ is a complete intersection of dimension $n - 2$. We will compute the cohomology groups of X and Y using two exact sequences.

Let $Y = V_+(F) \subset \mathbb{P}^n_k$, and write $\iota\colon Y \to \mathbb{P}^n_k$, $j\colon X \to Y$ for the two inclusions. Since G is a nonzerodivisor in S, we have two exact sequences of graded rings

$$0 \to R(-d) \xrightarrow{\cdot F} R \to S \to 0, \qquad 0 \to S(-e) \xrightarrow{\cdot G} S \to S/(G) \to 0. \tag{20.27}$$

Taking the associated sheaves gives

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0, \tag{20.28}$$

and

$$0 \longrightarrow \mathcal{O}_Y(-e) \longrightarrow \mathcal{O}_Y \longrightarrow j_*\mathcal{O}_X \longrightarrow 0. \tag{20.29}$$

Taking the long exact sequence in cohomology of (20.28), we obtain

$$H^0(Y, \mathcal{O}_Y) = k, \quad H^p(Y, \mathcal{O}_Y) = 0 \ \text{ for } 1 \leqslant p \leqslant n-2, \quad h^{n-1}(Y, \mathcal{O}_Y) = \binom{d-1}{n}. \tag{20.30}$$

Twisting (20.28) by $\mathcal{O}_{\mathbb{P}^n_k}(-e)$ gives

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-e) \longrightarrow \iota_*\mathcal{O}_Y(-e) \longrightarrow 0, \tag{20.31}$$

and the long exact sequence shows that

$$\begin{aligned} H^p(Y, \mathcal{O}_Y(-e)) &= 0, && p \leqslant n-2 \\ h^{n-1}(Y, \mathcal{O}_Y(-e)) &= \binom{d+e-1}{n} - \binom{e-1}{n}. \end{aligned} \tag{20.32}$$

From these sequences, we may compute all the cohomology groups $H^p(X, \mathcal{O}_X)$.

Case $n = 2$: Here X is a zero-dimensional subscheme of $\mathbb{P}^2_k$. From the exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y(-e)) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow 0$$

we obtain

$$\begin{aligned} h^0(X, \mathcal{O}_X) &= h^0(Y, \mathcal{O}_Y) + h^1(Y, \mathcal{O}_Y(-e)) - h^1(Y, \mathcal{O}_Y) \\ &= 1 + \left(\binom{d+e-1}{2} - \binom{e-1}{2}\right) - \binom{d-1}{2} = de. \end{aligned} \tag{20.33}$$

Case $n \geqslant 3$. By (20.29) and (20.32), we obtain

$$H^0(X, \mathcal{O}_X) = k, \qquad H^p(X, \mathcal{O}_X) = 0 \ \text{ for } p = 1, \dots, n-3,$$

and the exact sequence

$$0 \longrightarrow H^{n-2}(X, \mathcal{O}_X) \longrightarrow H^{n-1}(Y, \mathcal{O}_Y(-e)) \longrightarrow H^{n-1}(Y, \mathcal{O}_Y) \longrightarrow 0,$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

since $H^{n-1}(X, \mathcal{O}_X) = 0$ since X has dimension $n-2$. Therefore

$$h^{n-2}(X, \mathcal{O}_X) = \binom{d+e-1}{n} - \binom{d-1}{n} - \binom{e-1}{n}.$$

Example 20.33. When $n = 3$, $X \subset \mathbb{P}^3_k$ is a curve, and

$$h^1(X, \mathcal{O}_X) = \binom{d+e-1}{3} - \binom{d-1}{3} - \binom{e-1}{3} = \frac{1}{2}\, de\,(d+e-4) + 1.$$

For instance, for $(d, e) = (1, 2)$ we recover $h^1(X, \mathcal{O}_X) = 0$ for a conic. △

Example 20.34 (Connectedness)**.** For $n \geqslant 3$, we have

$$H^0(X, \mathcal{O}_X) = k.$$

This implies that every codimension 2 complete intersection X is connected: if $X = Y \sqcup Z$ is a disjoint union, then $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) \oplus H^0(Z, \mathcal{O}_Z)$ is at least 2-dimensional. Note that we allow X to be both reducible and nonreduced. △

Example 20.35 (Bézout's theorem)**.** Let us consider the case $n = 2$ in more detail. In this case, $C = V_+(F)$ and $D = V_+(G)$ are two curves in $\mathbb{P}^2_k$ of degrees d and e respectively, and $X = V_+(F, G)$ is the scheme-theoretic intersection. The nonzerodivisor hypothesis means that C and D have no common component.

Suppose that k is algebraically closed. As a set, $X = \{P_1, \dots, P_r\}$ consists of finitely many k-points. After a linear coordinate change, we may assume that $X \subset D_+(x_0) \equiv \operatorname{Spec} k[x, y]$. Then the classical form of *Bézout's theorem* says that

$$\sum_{i=1}^{r} \operatorname{mult}_{P_i}(C, D) = de, \tag{20.34}$$

where $\operatorname{mult}_{P_i}(C, D)$ is the *intersection multiplicity* of C and D at P_i, defined as

$$\operatorname{mult}_{P_i}(C, D) = \dim_k \mathcal{O}_{X, P_i} = \dim_k \left(\frac{k[x, y]}{(f, g)} \right)_{\mathfrak{m}_{P_i}}$$

and f and g are the dehomogenizations of F and G.

This formula follows by computing $\dim_k H^0(X, \mathcal{O}_X)$ in two ways. On the one hand, (20.33) implies that $\dim_k H^0(X, \mathcal{O}_X) = de$. On the other hand, since $\mathcal{O}_X(X)$ is an Artinian ring, we have

$$H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X) = \prod_{i=1}^{r} \mathcal{O}_{X, P_i} = \prod_{i=1}^{r} \left(\frac{k[x, y]}{(f, g)} \right)_{\mathfrak{m}_{P_i}} \tag{20.35}$$

Taking dimensions as k-vector spaces, we obtain (20.34).

△

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20.15 Coherence of pushforwards

Projective schemes have much stronger finiteness properties than affine schemes. The most important instance of this is Serre's theorem (20.24), that the cohomology groups of coherent sheaves are finitely generated. Using Chow's lemma, we may upgrade this to the following finiteness result for proper morphisms.

Theorem 20.36. Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.

Proof Coherence is local on the target, so we may replace Y by an affine open subscheme and assume that $Y = \operatorname{Spec} A$, where A is Noetherian. Since f is quasi-compact and separated, the pushforward of a quasi-coherent sheaf along f is quasi-coherent. Thus $f_*\mathcal{F} \simeq \widetilde{\Gamma(X,\mathcal{F})}$, and it is enough to prove that $\Gamma(X,\mathcal{F})$ is a finitely generated A-module.

For a closed subset $T \subset X$, let $\mathcal{P}(T)$ be the assertion that $\Gamma(X,\mathcal{E})$ is a finitely generated A-module for every coherent sheaf $\mathcal{E}$ with $\operatorname{Supp}(\mathcal{E}) \subset T$. We prove $\mathcal{P}(T)$ for all closed subsets T by Noetherian induction. The assertion is clear for $T = \varnothing$. Suppose that T is nonempty and that $\mathcal{P}(T')$ holds for every proper closed subset $T' \subsetneq T$.

Let $\mathcal{E}$ be a coherent sheaf with support contained in T. If $\operatorname{Supp}(\mathcal{E}) \subsetneq T$, the result follows immediately from the induction hypothesis, so we may assume that $\operatorname{Supp}(\mathcal{E}) = T$. Let $\mathcal{I} = \operatorname{Ann}_{\mathcal{O}_X}(\mathcal{E})$, let $i\colon Z \hookrightarrow X$ be the closed subscheme defined by $\mathcal{I}$, and write $\mathcal{E} = i_*\mathcal{G}$. Then $\mathcal{G}$ is coherent on Z, the underlying space of Z is T, and $p = f \circ i\colon Z \to \operatorname{Spec} A$ is proper.

Apply Chow's lemma, Theorem 15.16, to p. We obtain a proper birational morphism $g\colon Z' \to Z$ which is an isomorphism over a dense open subscheme $U \subset Z$, while $h = p \circ g\colon Z' \to \operatorname{Spec} A$ is projective. Consider the adjunction morphism $\alpha\colon \mathcal{G} \to g_*g^*\mathcal{G}$ and put $\mathcal{K} = \operatorname{Ker}(\alpha)$. The sheaf $g_*g^*\mathcal{G}$ is quasi-coherent because g is quasi-compact and separated, and consequently $\mathcal{K}$ is a quasi-coherent subsheaf of the coherent sheaf $\mathcal{G}$. Since Z is Noetherian, $\mathcal{K}$ is coherent. Over U the morphism g is an isomorphism, so $\alpha|_U$ is an isomorphism. Hence $\operatorname{Supp}(\mathcal{K}) \subset Z - U$, which is a proper closed subset of T. The induction hypothesis therefore shows that $\Gamma(Z,\mathcal{K}) = \Gamma(X, i_*\mathcal{K})$ is a finitely generated A-module.

Taking global sections in $0 \to \mathcal{K} \to \mathcal{G} \to g_*g^*\mathcal{G}$ gives an exact sequence

$$0 \longrightarrow \Gamma(Z,\mathcal{K}) \longrightarrow \Gamma(Z,\mathcal{G}) \longrightarrow \Gamma(Z', g^*\mathcal{G}).$$

The last module is finitely generated over A by Proposition **??**, since h is projective and $g^*\mathcal{G}$ is coherent. Thus $\Gamma(Z,\mathcal{G})/\Gamma(Z,\mathcal{K})$ is isomorphic to a submodule of a finitely generated A-module. It is finitely generated because A is Noetherian. As $\Gamma(Z,\mathcal{K})$ is also finitely generated, the same is true of $\Gamma(X,\mathcal{E}) = \Gamma(Z,\mathcal{G})$. This proves $\mathcal{P}(T)$ and completes the Noetherian induction. □

Applying this to $\mathcal{F} = \mathcal{O}_X$ and $Y = \operatorname{Spec} k$, we obtain the following generalization of Proposition 15.8.

Corollary 20.37. Let X be a scheme which is proper over a field k. Then $\mathcal{O}_X(X)$ is a finite-dimensional k-algebra.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20.16 Hodge numbers

We can use the Euler sequence to compute the cohomology groups of the sheaves of p-forms on $\mathbb{P}^n_k$ as follows. By Exercise **??**, there are exact sequences

$$\begin{aligned}
0 \longrightarrow \Omega^1_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\binom{n+1}{1}} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0.\\
0 \longrightarrow \Omega^2_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-2)^{\binom{n+1}{2}} \longrightarrow \Omega^1_{\mathbb{P}^n_k} \longrightarrow 0.\\
\vdots\\
0 \longrightarrow \Omega^n_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-n)^{\binom{n+1}{n}} \longrightarrow \Omega^{n-1}_{\mathbb{P}^n_k} \longrightarrow 0.
\end{aligned} \tag{20.36}$$

The long exact sequence applied to (20.36) shows that for $p = 1, \dots n$

$$H^q(\mathbb{P}^n_k, \Omega^p_{\mathbb{P}^n_k}) = H^{q-1}(\mathbb{P}^n_k, \Omega^{p-1}_{\mathbb{P}^n_k}).$$

By induction on p, starting with $p = 0$, we obtain the following:

Proposition 20.38. Let k be a field and let $n \in \mathbb{N}$. Then

$$H^q(\mathbb{P}^n_k, \Omega^p_{\mathbb{P}^n_k}) = \begin{cases} k & \text{if } p = q \text{ and } 0 \leqslant p \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{20.37}$$

In general, the numbers $h^{p,q}(X) = \dim_k H^q(X, \Omega^p_X)$ are called the *Hodge numbers* of X. They are important geometric invariants.

Example 20.39 (Quartic surfaces)**.** Let $X \subset \mathbb{P}^3_k$ be a smooth surface defined by a homogeneous polynomial of degree 4. We can compute the cohomology groups of Ω^1_X and $\Omega^2_X \simeq \mathcal{O}_X$ using the following three exact sequences

$$\begin{aligned}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0\\
0 \longrightarrow \mathcal{O}_X(-4) \longrightarrow \Omega^1_{\mathbb{P}^3_k/k}|_X \longrightarrow \Omega^1_{X/k} \longrightarrow 0\\
0 \longrightarrow \Omega_{\mathbb{P}^3_k/k}(-4) \longrightarrow \Omega^1_{\mathbb{P}^3_k/k} \longrightarrow \Omega^1_{\mathbb{P}^3_k/k}|_X \longrightarrow 0
\end{aligned}$$

Here the first sequence is the ideal sheaf sequence of X, the second is the conormal sequence, and the third is obtained by tensoring the first by $\Omega_{\mathbb{P}^3_k/k}$. From the corresponding long exact sequences in cohomology, we compute the various $h^{p,q} = \dim_k H^q(X, \Omega^p_X)$, shown in the following table (the *Hodge diamond* of X):

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

△

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

20.17 Application: irrationality of hypersurfaces

One of the main reasons for introducing the sheaves $\Omega_{X/k}$, $\mathcal{T}_X$ and $\Omega^p_{X/k}$ is to define geometric invariants. In fact, some of these invariants are strong enough to even distinguish varieties up to birational equivalence:

Theorem 20.40. Let $f\colon X \dashrightarrow Y$ be a birational map between smooth projective varieties over an algebraically closed field. Then, for any integer $p \geqslant 0$, there is an isomorphism of global sections of the sheaves of differential forms:

$$\Gamma(X, \Omega^p_X) \simeq \Gamma(Y, \Omega^p_Y).$$

To prove this, we first extend the pullback of differentials from Chapter 17 to p-forms. Recall that a map of k-algebras $\varphi\colon A \to B$ induces an A-linear map $\varphi^*\colon \Omega_{A/k} \to \Omega_{B/k}$. Applying $\bigwedge^p$, we obtain a map $\Omega^p_{A/k} \to \Omega^p_{B/k}$. Gluing these over affine opens, exactly as in the case $p = 1$, gives for every morphism of k-schemes $f\colon X \to Y$ and every $p \geqslant 0$ a pullback map

$$f^*\colon \Gamma(Y, \Omega^p_{Y/k}) \longrightarrow \Gamma(X, \Omega^p_{X/k}). \tag{20.38}$$

Example 20.41. Let $f\colon \mathbb{A}^2_k \to \mathbb{A}^2_k$ be the morphism $(x, y) \mapsto (x, xy)$. Write $A = k[u, v]$, $B = k[x, y]$ so that f is induced by the ring map $\varphi\colon A \to B$ sending $u \mapsto x$, $v \mapsto xy$. Then $\Omega^2_{A/k}$ is generated by $du \wedge dv$. Here $f^*(du) = dx$ and $f^*(dv) = d(xy) = y\,dx + x\,dy$, and the pullback $f^*(du \wedge dv)$ is given by

$$f^*(du \wedge dv) = dx \wedge (y\,dx + x\,dy) = y(dx \wedge dx) + x(dx \wedge dy) = x\,dx \wedge dy.$$

△

We will need the following compatibility property.

Lemma 20.42. Let $f\colon X \to Y$ be a dominant morphism of integral k-schemes and let $p \geqslant 0$. Then the following diagram commutes:

$$\begin{array}{ccc} \Gamma(Y, \Omega^p_{Y/k}) & \xrightarrow{f^*} & \Gamma(X, \Omega^p_{X/k}) \\ \downarrow & & \downarrow \\ \Omega^p_{K(Y)/k} & \longrightarrow & \Omega^p_{K(X)/k} \end{array} \tag{20.39}$$

If f is birational then the bottom map is an isomorphism.

Proof Let $V = \operatorname{Spec} A \subset Y$ and $W = \operatorname{Spec} B \subset f^{-1}(V)$ be affine opens, with $\varphi\colon A \to B$ the corresponding ring map. Since f is dominant, φ is injective and so extends to $K(Y) = K(A) \hookrightarrow K(B) = K(X)$. By Proposition 17.9, $\Omega^p_{K(Y)/k}$ and $\Omega^p_{K(X)/k}$ are the localizations of $\Omega^p_{A/k}$ and $\Omega^p_{B/k}$, and under this identification the germ of f^* is the localization of φ^*. Hence the square commutes.

If f is birational, then $K(Y) \to K(X)$ is an isomorphism of k-algebras, and hence induces an isomorphism $\Omega^p_{K(Y)/k} \to \Omega^p_{K(X)/k}$ by functoriality. This is the bottom map of the square. □

Proof of Theorem 20.40 Let $U \subset X$ be the largest open subset on which f is defined. Since X is smooth, hence normal, $X - U$ has codimension at least 2.

The morphism $f|_U \colon U \to Y$ is dominant and birational, so Lemma 20.42 applies. As Y and U are smooth and integral, the sheaves Ω^p_Y and Ω^p_U are locally free, so the vertical maps in (20.39) are injective. The bottom map is an isomorphism, and hence $f^* \colon \Gamma(Y, \Omega^p_Y) \to \Gamma(U, \Omega^p_U)$ is injective.

On the other hand, Ω^p_X is locally free, and $X - U$ has codimension at least 2 so Exercise 18.6.9 shows that restriction induces an isomorphism $\Gamma(X, \Omega^p_X) \simeq \Gamma(U, \Omega^p_U)$. Composing gives an injection $\Gamma(Y, \Omega^p_Y) \hookrightarrow \Gamma(X, \Omega^p_X)$, and the same argument applied to the inverse birational map gives an injection the other way. As both spaces are finite dimensional over k, they are isomorphic. □

By the theorem, the numbers $h^0(X, \Omega^p_X)$ are birational invariants. We will use them as an obstruction to rationality: since $h^0(\mathbb{P}^n_k, \Omega^p_{\mathbb{P}^n_k}) = 0$ for all $p > 0$, any smooth projective variety where some Ω^p_X has a nonzero global section cannot be rational.

Theorem 20.43. Let k be an algebraically closed field and let $n, d \geqslant 2$. Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree d. Then

$$\dim_k \Gamma(X, \Omega^{n-1}_X) = \binom{d-1}{n}.$$

In particular, if $d \geqslant n+1$, then X is not rational.

Proof Recall that $\Omega^{n-1}_X \simeq \mathcal{O}_X(d-n-1)$. We compute the global sections using

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(d-n-1) \longrightarrow \mathcal{O}_X(d-n-1) \longrightarrow 0.$$

Since $n \geqslant 2$, we have $H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)) = 0$ for all $m \in \mathbb{Z}$, so $H^0(\mathcal{O}_X(d-n-1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^n_k}(d-n-1))$ which has the indicated dimension. When $d \geqslant n+1$, this number is positive, so X is not rational. □

Example 20.44. In the theorem, it is essential that the hypersurface is smooth. For instance, for any $d \geqslant 1$, the plane curve defined by $x_0 x_1^{d-1} - x_2^d = 0$ is rational (see Example 14.13). △

20.18 Exercises

Exercise 20.18.1. Let X be a scheme and let $\mathcal{F}$ be a sheaf on X. For two coverings $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ of a scheme X, we say that $\mathcal{V}$ is a *refinement* if there is a *refinement map* $\tau \colon J \to I$ so that V_j is contained in $U_{\tau(j)}$ for all $j \in J$.

a) Show that τ induces maps $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^p(\mathcal{V}, \mathcal{F})$ by $(\tau^*\sigma)_{j_0,\dots,j_p} = \sigma_{\tau(j_0),\dots,\tau(j_p)}$ restricted to $V_{j_0} \cap \cdots \cap V_{j_p}$.

b) Show that the maps τ^* commute with the differentials, i.e., $d \circ \tau^* = \tau^* \circ d$. Conclude that there are induced maps $\tau^* \colon H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{V})$.

c) Show that if $\sigma, \tau \colon \mathcal{V} \to \mathcal{U}$ are two refinement maps, then $\sigma^* = \tau^*$. HINT: Construct a homotopy between σ^* and τ^*.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

d) Show that the collection of all $H^p(\mathcal{U}, \mathcal{F})$ with the τ^* as maps between them form a directed system of abelian groups.
e) Define $\check{H}^p(X, \mathcal{F})$ as the direct limit of the system in *d*), show that the $\check{H}^p(X, \mathcal{F})$ are functorial in $\mathcal{F}$ and have long exact sequences.
f) Show that for a Noetherian separated scheme and each quasi-coherent sheaf $\mathcal{F}$, there is a natural isomorphism $\check{H}^p(X, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F})$

Exercise 20.18.2. Generalize Example 20.10 to show that the curve $V(x_2^d - x_0 x_1^{d-1}) \subset \mathbb{P}^2_k$ has an $H^1(X, \mathcal{O}_X)$ of dimension $\frac{1}{2}(d-1)(d-2)$ as a k-vector space.

Exercise 20.18.3. Let $X = S^1$ and let $\mathcal{U}$ be the covering of X with three pairwise intersecting open intervals with empty intersection. Show that the Čech complex takes the form $\mathbb{Z}^3 \xrightarrow{d^0} \mathbb{Z}^3 \to 0$. Compute the map d^0 and use it to verify that $H^0(\mathcal{U}, \mathbb{Z}_X) = H^1(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}$.

Exercise 20.18.4. Let $X = \mathbb{A}^n_k - \{0\}$ be the complement of the origin.
a) Compute $H^i(X, \mathcal{O}_X)$ for all i.
b) Give a new proof that X is not an affine scheme for $n \geqslant 2$.

Exercise 20.18.5. With reference to the twisted cubic example in Section 20.12, show that X is not a complete intersection. Furthermore, show that

- $H^0(\mathbb{P}^3, \mathcal{I}(2)) = k^3$
- $H^1(\mathbb{P}^3, \mathcal{I}(m)) = 0$ for all $m \in \mathbb{Z}$.
- $H^2(\mathbb{P}^3, \mathcal{I}(-1)) = k$.

Exercise 20.18.6 (Degree)**.** Let $R = k[x_0, \ldots, x_n]$ and let M be a graded R-module. Consider the Hilbert polynomial of $\mathcal{F} = \widetilde{M}$

$$\chi(\mathcal{F}(t)) = \frac{d}{m!} t^m + \text{lower order terms in } t \tag{20.40}$$

a) Show that $m = \dim(\mathrm{Supp}(\mathcal{F}))$. HINT: Use induction on the dimension of the support and the short exact sequence associated with a hyperplane section.
b) Let $X \subset \mathbb{P}^n_k$ be a projective variety. We define the *degree of* X, denoted $\deg(X)$, as the integer d appearing in the leading coefficient of the Hilbert polynomial for $\mathcal{F} = \mathcal{O}_X$.
   Show that $\deg(X)$ can be computed as follows: if $\dim X = 0$ then $\deg(X) = h^0(\mathcal{O}_X)$. If $\dim X > 0$, and H is a hyperplane such that $\dim(X \cap H) < \dim X$, then $\deg(X) = \deg(X \cap H)$.
c) Let $X = V_+(f_1, \ldots, f_r) \subset \mathbb{P}^n_k$ be a complete intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$. Show that $\deg(X) = d_1 \cdots d_r$.
d) Show that the twisted cubic curve $C \subset \mathbb{P}^3_k$ has degree 3. Deduce that C is not scheme-theoretically the intersection of two surfaces. On the other hand, find two polynomials $f, g \in k[x_0, \ldots, x_3]$ such that $C = V_+(f, g)$ as sets.

Exercise 20.18.7. Let $n > 0$ be an integer and consider the integral projective scheme $X = \mathrm{Proj}(R)$, where R is the ring

$$R = k[x, y, z, w]/(x^2, xy, y^2, u^n x - v^n y).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that X is irreducible, non-reduced, and of dimension 1.
b) Compute $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$.

Exercise 20.18.8. Let $Y_r \subset \mathbb{P}^n_k$ be a complete intersection of r hypersurfaces $H_1, \ldots, H_r$ of degrees $d_1, \ldots, d_r \geqslant 1$.

a) Using $0 \to \mathcal{O}_{Y_{r-1}}(-H_r) \to \mathcal{O}_{Y_{r-1}} \to \mathcal{O}_{Y_r} \to 0$, show that the maps

$$H^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) \longrightarrow H^i(Y_r, \mathcal{O}_{Y_r}(d))$$

are isomorphisms for all $i < n - r$ and all $d \in \mathbb{Z}$.
b) Show that Y_r is connected if $\dim Y_r \geqslant 1$.

Exercise 20.18.9. Let X be a projective scheme and let Y, Z be closed subschemes of X with $Z \subset Y$. Write P_Y and P_Z for the Hilbert polynomials of $\mathcal{O}_Y$ and $\mathcal{O}_Z$ respectively. Show that $P_Z(m) \leqslant P_Y(m)$ for $m \gg 0$.

Exercise 20.18.10 (Split locally free sheaves)**.** A locally free sheaf is said to be *split* if it is isomorphic to a direct sum of locally free sheaves of smaller rank.

a) Show that any locally free sheaf on $\mathbb{A}^1_k$ and $\mathbb{P}^1_k$ is split.
b) Show that a sheaf $\mathcal{E}$ on $\mathbb{P}^n_k$ splits as a sum of invertible sheaves, then $H^i(X, \mathcal{E}(m)) = 0$ for all $i = 1, \ldots, n-1$ and $m \in \mathbb{Z}$.
c) Consider the rank 2 locally free sheaf $\mathcal{E}$ on $\mathbb{P}^2_k$ which fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2_k}(-1) \longrightarrow \mathcal{O}^3_{\mathbb{P}^2_k} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Compute $H^1(\mathcal{E}(-2))$ and show that $\mathcal{E}$ is not split.

The famous *Hartshorne's Conjecture* asks whether any locally free sheaf of rank 2 on $\mathbb{P}^n_k$ is split for $n \geqslant 7$.

Exercise 20.18.11 ($\mathbb{P}^2$ minus a point)**.** Consider $X = \mathbb{P}^2_k - \{(0:0:1)\}$, with the affine covering given by $U_0 = \operatorname{Spec} k[x, y]$ and $U_1 = \operatorname{Spec} k[x^{-1}, x^{-1}y]$. Show that $H^0(X, \mathcal{O}_X) = k$ and

$$H^1(X, \mathcal{O}_X) = \bigoplus_{b > -a,\, a < 0} k \cdot x^a y^b.$$

Deduce that $\mathbb{P}^2_k - \{(0:0:1)\}$ is not an affine scheme.

Exercise 20.18.12. Continuing the previous exercise, let C be the curve defined by $y^3 + x^2 + x$ in U_0 and $(x^{-1}y)^3 + x^{-2} + x^{-1}$ in U_1. Show that $H^0(C, \mathcal{O}_C) = k$ and

$$H^1(X, \mathcal{O}_C) = k \cdot x^{-1}y^2.$$

More generally, if C is the curve defined by $y^d + x^{d-1} + x$ in U_0, and $(x^{-1}y)^d + x^{-d+1} + x^{-1}$ in U_1, then

$$H^1(C, \mathcal{O}_C) \simeq k^{\binom{d-1}{2}}.$$

Exercise 20.18.13 (Interpolation problems)**.** Let $Z = \{p_1, \ldots, p_r\} \subset \mathbb{P}^n_k$ be a finite set of points, with the reduced scheme structure.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that the space of degree d polynomials passing through $p_1, \dots, p_r$ has dimension equal to

$$\dim_k H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) = \binom{n+d}{d} - r + \dim_k H^1(\mathbb{P}^n_Z(d)).$$

b) Show that for two points in $\mathbb{P}^2_2$ there is a unique line passing through them.
c) Let $Z = \{p_1, p_2, p_3\}$ be three distinct points in $\mathbb{P}^2_k$, show that $H^1(\mathbb{P}^2_k, \mathcal{I}_Z(1)) \neq 0$ if and only if p_1, p_2, p_3 lie on a line.
d) Let $Z = \{p_1, \dots, p_5\}$ in $\mathbb{P}^2_k$. What possible values can $h^1(\mathbb{P}^2_k, \mathcal{I}_Z(2))$ take?

Exercise 20.18.14. Let $\mathcal{F}$ be a sheaf on a topological space X and let $\mathcal{U} = \{U_i\}_{i=1}^r$ be an open cover. Let $\sigma \in S_r$ be a permutation and consider the permuted cover $\mathcal{U}' = \{U_{\sigma(i)}\}_{i=1}^r$.

For each p, define a map $\phi^p : C^p(\mathcal{U}', \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$ by

$$\phi^p(\alpha)_{\sigma(i_0),\sigma(i_1),\dots,\sigma(i_p)} = \operatorname{sgn}(\sigma|_{i_0,\dots,i_p}) \cdot \alpha_{i_0,i_1,\dots,i_p}$$

where $\operatorname{sgn}(\sigma|_{i_0,\dots,i_p})$ is the sign of the permutation σ restricted to the indices $\{i_0, i_1, \dots, i_p\}$.

a) Show that each σ^p is bijective.
b) Show that ϕ^p is a map of complexes, i.e., $\phi^{p+1} \circ d^p = d'^p \circ \phi^p$.
c) Conclude that that $\phi^\bullet$ induces a canonical isomorphism $H^p(\mathcal{U}, \mathcal{F}) \simeq H^p(\mathcal{U}', \mathcal{F})$

Exercise 20.18.15. Let X be the scheme obtained by gluing two copies U, V of $\operatorname{Spec}(\mathbb{Z}_{(2)})$ to itself over the generic point. Compute $H^1(\mathcal{U}, \mathcal{O}_X)$ for the covering $\{U, V\}$.

Exercise 20.18.16. Give a second proof of Corollary 10.11, using the fact that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$ on $X = \operatorname{Spec} A$.

Exercise 20.18.17 (Cohomology of $\mathbb{P}^3_k$ minus a line)**.** Consider the scheme $X = \mathbb{P}^3_k - V_+(x_0, x_1)$. Compute the cohomology groups of $\mathcal{O}_X$.

Exercise 20.18.18. Let X be the gluing of two copies U, V of $\operatorname{Spec} \mathbb{Z}$ along the open set $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$. Show that X is not separated and describe the diagonal map. Compute the Čech cohomology groups $H^i(\mathcal{U}, \mathcal{O}_X)$ with respect to the cover $\mathcal{U} = \{U, V\}$.

Exercise 20.18.19. Let X and Y be closed subschemes of $\mathbb{P}^n_k$ with $X \subset Y$. Show that if $\chi(\mathcal{O}_X(m)) = \chi(\mathcal{O}_Y(m))$ for all $m \gg 0$, then $X = Y$. HINT: Consider the ideal sheaf of X in Y.

Exercise 20.18.20. Let $X \subset \mathbb{P}^n_k$ be a hypersurface of degree d. Show that X is not isomorphic to $\mathbb{P}^{n-1}_k$ for $d \geqslant n + 1$.

Exercise 20.18.21 (The $AF + BG$ theorem)**.** Let $F, G, H \in R = k[x_0, x_1, x_2]$ be homogeneous polynomials of degrees d, e, and h respectively. Suppse that for each $P \in V(F, G)$, we have $H_P \in (F_P, G_P) \subset \mathcal{O}_{\mathbb{P}^2_k, P}$. Then the *AF+BG theorem* states that there exist homogeneous polynomials A and B of degrees $h - d$ and $h - e$ respectively such that $H = AF + BG$. Prove this theorem by considering the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(m - d - e) \to \mathcal{O}_{\mathbb{P}^2_k}(m - d) \oplus \mathcal{O}_{\mathbb{P}^2_k}(m - e) \to \mathcal{I}_{C \cap D}(m) \to 0.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 20.18.22 (Čech cohomology in the non-separated case)**.** Let X be the affine plane with two origins, obtained by gluing two copies U, V of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ using the identity map on $\mathbb{A}^2_k - \{(0,0)\}$.

a) Compute $H^0(\mathcal{U}, \mathcal{O}_X)$, $H^1(\mathcal{U}, \mathcal{O}_X)$ and $H^2(\mathcal{U}, \mathcal{O}_X)$ for the cover $\mathcal{U} = \{U, V\}$.
b) Let $\mathcal{V} = \{U_0, U_1, U_2, U_3\}$, where $U_0 = U$, and the following subsets of V:

$$U_1 = D(x), \qquad U_2 = D(y), \qquad U_3 = D(1+x).$$

c) Which of the intersections $U_i \cap U_j$ are affine?
d) Define $c \in \mathcal{C}^2(\mathcal{V}, \mathcal{O}_X)$ by $c_{012} = x^{-1}y^{-1}$, $c_{123} = x^{-1}y^{-1}$, and $c_{ijk} = 0$ for all other i, j, k. Show that $d^2(c) = 0$, so $[c] \in H^2(\mathcal{V}, \mathcal{O}_X)$ is well-defined.
e) Show that $\mathcal{O}_X(U_0 \cap U_1) = k[x^{\pm 1}, y]$, $\mathcal{O}_X(U_0 \cap U_2) = k[x, y^{\pm 1}]$, and $\mathcal{O}_X(U_1 \cap U_2) = k[x^{\pm 1}, y^{\pm 1}]$.
f) Show that $[c] \neq 0$ in $H^2(\mathcal{V}, \mathcal{O}_X)$ and hence $H^2(\mathcal{U}, \mathcal{O}_X) \neq H^2(\mathcal{V}, \mathcal{O}_X)$

Exercise 20.18.23 (Serre's criterion for affineness)**.** Let X be a Noetherian scheme. In this exercise, we will prove that the following are equivalent:

(i) X is affine.
(ii) $H^i(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and all $i > 0$.
(iii) $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$.

The following steps show the non-trivial implication (iii) $\Rightarrow$ (i). Let $A = \Gamma(X, \mathcal{O}_X)$.

a) Let $P \in X$ be a closed point and let $U \subset X$ be an affine open set containing P. Let $Y = X - U$ with its reduced closed subscheme structure, $\mathcal{I}_Y$ its ideal sheaf, and $\mathcal{I}_P$ the ideal sheaf of the reduced point $\{P\}$. Define $\mathcal{I} = \mathcal{I}_Y \cap \mathcal{I}_P$. Show that $\mathcal{I}$ is a quasi-coherent ideal sheaf with corresponding closed subscheme $Z = Y \sqcup \{P\}$ and $\Gamma(Z, \mathcal{O}_Z) \simeq \Gamma(Y, \mathcal{O}_Y) \times \kappa(P)$.
b) Using $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ and assumption (iii), prove that the restriction $A \to \Gamma(Z, \mathcal{O}_Z)$ is surjective. Conclude there exists $f \in A$ such that $f|_Y = 0$ and $f(P) = 1$.
c) Prove that $X_f = \{x \in X \mid f(x) \neq 0\}$ is an open set contained in U and $X_f = D(f) \subset U$. So X_f is affine and contains P.
d) Using quasi-compactness and the previous step, find $f_1, \ldots, f_n \in A$ such that $X = X_{f_1} \cup \cdots \cup X_{f_n}$ and each X_{f_i} is affine.
e) Let $\theta\colon X \to \operatorname{Spec} A$ be the canonical morphism. Using the isomorphisms $A_f = \Gamma(X_f, \mathcal{O}_X)$, conclude that θ restricts to an isomorphism $X_{f_i} \to D(f_i)$ for each i. Show that the $f_1, \ldots, f_n$ from *d*) generate the unit ideal in A. Deduce that θ is an isomorphism and that X is affine.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

21

Projective embeddings and ampleness

21.1 Very ample and ample invertible sheaves

Given an invertible sheaf L on X and sections $s_0, \ldots, s_n \in \Gamma(X, L)$ that generate L, we obtain a morphism $X \to \mathbb{P}^n_A$. It is a natural question when L and the s_i produce an embedding of X into projective space. This leads to the notions of very ampleness and ampleness.

We begin with very ampleness, which is the more concrete of the two. A very ample sheaf is one that literally comes from an embedding into projective space. Ampleness is a more flexible variant, and the main theorem of this chapter is that the two notions agree after passing to a high tensor power.

Very ampleness

The prototype example of a very ample invertible sheaf is $\mathcal{O}_{\mathbb{P}^n_A}(1)$ on projective space. In general, a very ample invertible sheaf is a restriction of this sheaf.

Definition 21.1. Let X be a scheme over a Noetherian ring A. An invertible sheaf L on X is called *very ample over* A if there exists a locally closed embedding $\iota\colon X \hookrightarrow \mathbb{P}^n_A$ such that $L \simeq \iota^*\mathcal{O}_{\mathbb{P}^n_A}(1)$.

When X is proper over A, any such locally closed embedding is automatically closed, since X is proper and $\mathbb{P}^n_A$ is separated over A.

Lemma 21.2. Let X be a scheme over A. If L is very ample over A and M is globally generated, then $L \otimes M$ is very ample over A.

Proof Choose a locally closed embedding $i\colon X \hookrightarrow \mathbb{P}^a$ with $L \simeq i^*\mathcal{O}_{\mathbb{P}^a}(1)$, and let $g\colon X \to \mathbb{P}^b$ be the morphism defined by global sections generating M, so that $M \simeq g^*\mathcal{O}_{\mathbb{P}^b}(1)$. Then $L \otimes M \simeq (i,g)^*\mathcal{O}_{\mathbb{P}^a\times\mathbb{P}^b}(1,1)$. The morphism $(i,g)\colon X \to \mathbb{P}^a \times \mathbb{P}^b$ is a locally closed embedding: it factors as the graph embedding $X \hookrightarrow X \times \mathbb{P}^b$ followed by $i \times \mathrm{id}_{\mathbb{P}^b}\colon X \times \mathbb{P}^b \hookrightarrow \mathbb{P}^a \times \mathbb{P}^b$. By the Segre embedding, $\mathcal{O}_{\mathbb{P}^a\times\mathbb{P}^b}(1,1)$ is very ample, so $L \otimes M$ is very ample. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Ampleness

Very ampleness depends on the structure of X as an A-scheme. We now introduce a more flexible notion, called ampleness, which is an intrincic property on X.

The motivation for the definition comes from a different property of $\mathcal{O}_{\mathbb{P}^n_A}(1)$ on $\mathbb{P}^n_A$: Corollary 12.12, which says that for any coherent sheaf $\mathcal{F}$, there is a positive integer d_0 such that $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_A}(d)$ is globally generated for all $d \geqslant d_0$. That is, tensoring $\mathcal{F}$ by a sufficiently high power of $\mathcal{O}_{\mathbb{P}^n_A}(1)$, we obtain enough global sections to generate the sheaf.

Definition 21.3. Let X be a Noetherian scheme. An invertible sheaf L on X is called *ample* if for each coherent sheaf $\mathcal{F}$ there is an integer $m_0 > 0$ such that $\mathcal{F} \otimes L^{\otimes m}$ is globally generated for all $m \geqslant m_0$.

Taking $\mathcal{F} = \mathcal{O}_X$ shows that some tensor power $L^{\otimes m}$ is already globally generated when L is ample.

Lemma 21.4. Let L be an invertible sheaf on a Noetherian scheme X. The following are equivalent.

(i) L is ample.
(ii) $L^{\otimes m}$ is ample for every $m > 0$.
(iii) $L^{\otimes m}$ is ample for some $m > 0$.

Proof For (i) $\Rightarrow$ (ii), fix $r > 0$ and a coherent sheaf $\mathcal{F}$. Since L is ample, there is N with $\mathcal{F} \otimes L^{\otimes n}$ globally generated for all $n \geqslant N$. Hence $\mathcal{F} \otimes (L^{\otimes r})^{\otimes q} \simeq \mathcal{F} \otimes L^{\otimes rq}$ is globally generated for all $q \geqslant N$, so $L^{\otimes r}$ is ample.

The implication (ii) $\Rightarrow$ (iii) is trivial.

Assume (iii), and choose $m > 0$ with $L^{\otimes m}$ ample. Let $\mathcal{F}$ be coherent. For each $i = 0, \dots, m-1$ the sheaf $\mathcal{F} \otimes L^{\otimes i}$ is coherent, so there is N_i with $\mathcal{F} \otimes L^{\otimes i} \otimes (L^{\otimes m})^{\otimes q} \simeq \mathcal{F} \otimes L^{\otimes (mq+i)}$ globally generated for all $q \geqslant N_i$. Choose $N \geqslant mN_i + i$ for every i. Then for every $n \geqslant N$, writing $n = mq + i$ with $0 \leqslant i < m$, we have $q \geqslant N_i$, so $\mathcal{F} \otimes L^{\otimes n}$ is globally generated. Hence L is ample. □

It turns out that allowing tensor powers in the definition is convenient in these arguments and ampleness has better formal properties than very ampleness. So L is ample if and only if some tensor power is ample. This is not true in general for very ampleness. Moreover, ampleness behaves well under finite pullbacks, whereas very ampleness does not.

Proposition 21.5. Let $f\colon X \to Y$ be a finite morphism of schemes proper over a Noetherian ring A, and let L be an ample invertible sheaf on Y. Then f^*L is ample on X.

Proof Fix a coherent sheaf $\mathcal{F}$ on X, and set $\mathcal{G}_m = \mathcal{F} \otimes (f^*L)^{\otimes m}$. Since f is finite, $f_*\mathcal{F}$ is coherent on Y. Because L is ample, the sheaf $f_*\mathcal{F} \otimes L^{\otimes m}$ is generated by global sections for all $m \gg 0$. By the projection formula, $f_*\mathcal{G}_m \simeq f_*\mathcal{F} \otimes L^{\otimes m}$. Hence for all $m \gg 0$ there is a surjection $\mathcal{O}_Y^{\oplus I} \twoheadrightarrow f_*\mathcal{G}_m$ for some index set I. Pulling back along f, we obtain a surjection $\mathcal{O}_X^{\oplus I} \twoheadrightarrow f^*f_*\mathcal{G}_m$. Since f is affine, the adjunction map $f^*f_*\mathcal{G}_m \to \mathcal{G}_m$ is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

surjective. Therefore $\mathcal{G}_m$ is generated by global sections for all $m \gg 0$. As this holds for every coherent sheaf $\mathcal{F}$, it follows that f^*L is ample. □

In particular, if L is ample and if $Z \subset X$ is a closed subscheme, then $L|_Z$ is ample on Z.

Proposition 21.6. Let X be a proper scheme over a Noetherian ring A, and let L be a very ample invertible sheaf on X. Then L is ample.

Proof Choose a locally closed embedding $i\colon X \hookrightarrow \mathbb{P}^N_A$ with $i^*\mathcal{O}_{\mathbb{P}^N_A}(1) \simeq L$. Since X is proper over A and $\mathbb{P}^N_A$ is separated over A, the morphism i is a closed embedding.

Let $\mathcal{F}$ be a coherent sheaf on X and set $\mathcal{G} = i_*\mathcal{F}$. By Lemma **??**, the sheaf $\mathcal{G}(q)$ is globally generated for all $q \gg 0$. By the projection formula, $i_*(\mathcal{F} \otimes L^{\otimes q}) \simeq \mathcal{G}(q)$. Pulling back a surjection $\mathcal{O}_{\mathbb{P}^N_A}^{\oplus m} \twoheadrightarrow \mathcal{G}(q)$ along i gives a surjection $\mathcal{O}_X^{\oplus m} \twoheadrightarrow \mathcal{F} \otimes L^{\otimes q}$ for all $q \gg 0$. Thus L is ample. □

The heart of the matter is a criterion that manufactures a very ample power out of a cover of X by distinguished affine opens $D(s)$. Recall that for a section $s \in H^0(X, L^{\otimes n})$ we write $D(s) = \{\, x \in X \mid s(x) \neq 0 \,\}$ for the open locus where s does not vanish.

Proposition 21.7. Let X be a proper scheme over a ring A, and let L be an invertible sheaf on X. Suppose there are finitely many sections $s_i \in H^0(X, L^{\otimes n_i})$, with $n_i > 0$, such that the affine opens $D(s_i)$ cover X. Then $L^{\otimes m}$ is very ample over A for some $m > 0$. In particular, L is ample.

Proof Choose a common multiple n of the n_i and replace each s_i by s_i^{n/n_i}. This does not change $D(s_i)$, and we may therefore assume that $s_i \in H^0(X, L^{\otimes n})$ for every i.

Let $R = \bigoplus_{d \geqslant 0} H^0(X, L^{\otimes nd})$. By Lemma 12.7, there is an isomorphism $\Gamma(D(s_i), \mathcal{O}_X) \simeq (R_{s_i})_0$. Since $D(s_i)$ is an affine, hence quasi-compact, open subset of the finite type A-scheme X, it is of finite type over A. So $\Gamma(D(s_i), \mathcal{O}_X)$ is a finitely generated A-algebra. Choose homogeneous elements $t_{ij} \in R$ of degrees d_{ij} such that the fractions $t_{ij}/s_i^{d_{ij}}$ generate $\Gamma(D(s_i), \mathcal{O}_X)$ as an A-algebra. Also choose an integer N such that $N \geqslant d_{ij}$ for every i, j, and replace t_{ij} by $t_{ij}s_i^{N-d_{ij}}$. We then have $t_{ij} \in H^0(X, L^{\otimes nN})$, and the fractions t_{ij}/s_i^N still generate $\Gamma(D(s_i), \mathcal{O}_X)$.

Consider the sections s_i^N and t_{ij} of $L^{\otimes nN}$. Since the opens $D(s_i^N) = D(s_i)$ cover X, the sections s_i^N alone generate $L^{\otimes nN}$. Hence all these sections define an A-morphism $f\colon X \longrightarrow \mathbb{P}^M_A$ such that $f^*\mathcal{O}_{\mathbb{P}^M_A}(1) \simeq L^{\otimes nN}$. Let x_i and y_{ij} be the homogeneous coordinates corresponding to s_i^N and t_{ij}, respectively. Then $f^{-1}(D_+(x_i)) = D(s_i)$. On this open, the map $D(s_i) \to D_+(x_i)$ corresponds to $\Gamma(D_+(x_i), \mathcal{O}_{\mathbb{P}^M_A}) \to \Gamma(D(s_i), \mathcal{O}_X)$ sending y_{ij}/x_i to t_{ij}/s_i^N. This ring map is surjective because the elements generate $\Gamma(D(s_i), \mathcal{O}_X)$ as an A-algebra, so each $D(s_i) \to D_+(x_i)$ is a closed embedding. Let $V = \bigcup_i D_+(x_i)$. Since $f^{-1}(V) = X$, the morphism f factors through V, and $X \to V$ is a closed embedding. It follows that $f\colon X \to \mathbb{P}^M_A$ is a locally closed embedding. In fact, f is a closed embedding because X is proper over A. Hence $L^{\otimes nN} \simeq f^*\mathcal{O}_{\mathbb{P}^M_A}(1)$ is very ample. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Theorem 21.8. Let X be a proper scheme over a Noetherian ring A, and let L be an invertible sheaf on X. The following are equivalent.
(i) L is ample.
(ii) $L^{\otimes r}$ is very ample for some $r > 0$.

Proof (ii)⇒(i): if $L^{\otimes r}$ is very ample then it is ample by Proposition 21.6, hence L is ample by Lemma 21.4.

(i)⇒(ii): choose a finite cover of X by affine opens U_α with $L|_{U_\alpha}$ trivial, and let $Z_\alpha = X - U_\alpha$ with its reduced structure and ideal sheaf $\mathcal{I}_{Z_\alpha}$. Since L is ample, there is $m_\alpha > 0$ such that $\mathcal{I}_{Z_\alpha} \otimes L^{\otimes m_\alpha}$ is globally generated. For each $x \in U_\alpha$, choose a global section of $\mathcal{I}_{Z_\alpha} \otimes L^{\otimes m_\alpha}$ nonvanishing at x, and let $s \in H^0(X, L^{\otimes m_\alpha})$ be its image. Then s vanishes on Z_α, so $D(s) \subset U_\alpha$, while $s(x) \neq 0$, so $x \in D(s)$. As $D(s)$ is a principal open of the affine scheme U_α, it is affine. Letting x range over U_α and using quasi-compactness, finitely many such opens cover U_α; collecting them over all α gives finitely many sections $s_i \in H^0(X, L^{\otimes m_i})$ with $D(s_i)$ affine and covering X. By Proposition 21.7, some power $L^{\otimes m}$ is very ample. □

So for a proper scheme over A, an invertible sheaf is ample if and only if some high tensor power embeds the scheme into projective space. In fact, every large power is very ample:

Corollary 21.9. Let X be a proper scheme over a Noetherian ring A, and let L be an invertible sheaf on X. Then L is ample if and only if $L^{\otimes m}$ is very ample for all $m \gg 0$.

Proof If $L^{\otimes m}$ is very ample for some m, then L is ample by Theorem 21.8. Conversely, suppose L is ample and choose $r > 0$ with $L^{\otimes r}$ very ample. Applying the definition of ampleness to $\mathcal{F} = \mathcal{O}_X$, the sheaf $L^{\otimes j}$ is globally generated for all $j \geqslant j_0$. For $m \geqslant j_0 + r$, write $L^{\otimes m} = L^{\otimes r} \otimes L^{\otimes(m-r)}$; since $L^{\otimes(m-r)}$ is globally generated, Lemma 21.2 shows $L^{\otimes m}$ is very ample. □

Example 21.10. For $X = \mathbb{P}^1_A \times_A \mathbb{P}^1_A$, write $\mathcal{O}(a,b) = p_1^*\mathcal{O}_{\mathbb{P}^1_A}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_A}(b)$ for $a, b \in \mathbb{Z}$. Then $\mathcal{O}(a,b)$ is globally generated if $a, b \geqslant 0$, and very ample if $a, b > 0$. Indeed $\mathcal{O}(1,1)$ is very ample, giving the Segre embedding $\mathbb{P}^1_A \times_A \mathbb{P}^1_A \hookrightarrow \mathbb{P}^3_A$; so for $a, b > 0$ the sheaf $\mathcal{O}(a,b) = \mathcal{O}(1,1) \otimes \mathcal{O}(a-1, b-1)$ is very ample by Lemma 21.2, since $\mathcal{O}(a-1, b-1)$ is globally generated.

On the other hand, for every $m > 0$ the morphism defined by $\mathcal{O}(m, 0)$ factors through the first projection $X \to \mathbb{P}^1_A$, so no power of $\mathcal{O}(1,0)$ is very ample. By Theorem 21.8, $\mathcal{O}(1,0)$ is not ample. △

A cohomological criterion for ampleness

The next result shows that ampleness is equivalent to high tensor powers killing the higher cohomology groups of any coherent sheaf, just like $\mathcal{O}(1)$ on projective space.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Theorem 21.11 (Serre vanishing). Let X be a proper scheme over a Noetherian ring A, and let L be an invertible sheaf on X. The following are equivalent.

(i) L is ample.

(ii) For every coherent sheaf $\mathcal{F}$ on X, one has $H^q(X, \mathcal{F} \otimes L^{\otimes m}) = 0$ for all $m \gg 0$ and $q > 0$.

Proof (i)⇒(ii): by Theorem 21.8, choose $r > 0$ and a closed embedding $i\colon X \hookrightarrow \mathbb{P}^N_A$ with $i^*\mathcal{O}_{\mathbb{P}^N_A}(1) \simeq L^{\otimes r}$. Let $\mathcal{F}$ be coherent, and for $t = 0, \dots, r-1$ let $\mathcal{G}_t = i_*(\mathcal{F} \otimes L^{\otimes t})$. If $m = rq + t$, the projection formula gives $i_*(\mathcal{F} \otimes L^{\otimes m}) \simeq \mathcal{G}_t(q)$, and so

$$H^p(X, \mathcal{F} \otimes L^{\otimes m}) \simeq H^p(\mathbb{P}^N_A, \mathcal{G}_t(q)).$$

By Serre's theorem on projective space, for each t the sheaves $\mathcal{G}_t(q)$ have no higher cohomology for $q \gg 0$, so we obtain (ii).

(ii)⇒(i): we produce a finite affine cover of X by opens $D(s)$ and apply Proposition 21.7. Cover X by finitely many affine opens U with $L|_U$ trivial. Fix one such open and set $Z = X - U$ with its reduced structure and ideal sheaf $\mathcal{I}_Z$. Let $x \in U$. Since $U \cap \overline{\{x\}}$ is a nonempty scheme of finite type over the Noetherian ring A, it has a closed point y. For $m \gg 0$, the exact sequence

$$0 \to \mathcal{I}_{Z\cup\{y\}} \otimes L^{\otimes m} \to \mathcal{I}_Z \otimes L^{\otimes m} \to \kappa(y) \to 0$$

together with $H^1(X, \mathcal{I}_{Z\cup\{y\}} \otimes L^{\otimes m}) = 0$ from (ii) shows that $H^0(X, \mathcal{I}_Z \otimes L^{\otimes m}) \to \kappa(y)$ is surjective. Choose a section mapping to a nonzero element, with image $s \in H^0(X, L^{\otimes m})$. Then s vanishes on Z, so $D(s) \subset U$, and $s(y) \neq 0$, so $y \in D(s)$. Since $y \in \overline{x}$, we also have $x \in D(s)$. As $D(s)$ is a principal open of the affine U, it is affine. Hence every point of X lies in an affine open of the form $D(s)$ with s a section of some power of L. By quasi-compactness finitely many of these cover X, and Proposition 21.7 shows that some power of L is very ample, as required. □

Ampleness on a curve

As an application we show that on a smooth projective curve any single point gives an ample invertible sheaf.

Proposition 21.12. Let X be a proper curve over a field k. Then there is a finite morphism $f\colon X \to \mathbb{P}^1_k$. In particular, X is projective.

Proof Let $S \subset X$ be the set of singular points. Choose an affine open subset $U \subset X$ containing S. Since U is an affine curve, we may choose a nonconstant function $t \in \mathcal{O}_X(U)$. It gives a morphism $U \to \mathbb{A}^1_k \subset \mathbb{P}^1_k$.

We regard t as a rational function on X. If $P \in X - U$, then P is a regular point of X, so $\mathcal{O}_{X,P}$ is a DVR. Hence either $t \in \mathcal{O}_{X,P}$ or $t^{-1} \in \mathcal{O}_{X,P}$. Thus the rational map $X \dashrightarrow \mathbb{P}^1_k$ defined by t is defined at every point of $X - U$ as well. It is already defined on U, so it gives a morphism $f\colon X \to \mathbb{P}^1_k$. Since t is nonconstant, f is nonconstant. Hence every fibre of f is zero-dimensional, and therefore finite. Since X is proper, f is proper, and a proper morphism with finite fibers is finite.

Finally, $f^*\mathcal{O}_{\mathbb{P}^1_k}(1)$ is ample on X, so X is projective because it contains an ample invertible sheaf. □

Exercise 21.1.1. Prove the converse of Proposition 21.5. That is, if f is finite and f^*L is ample, then L is ample.

21.2 A criterion for being a closed embedding

For a morphism $f\colon X \to Y$, it can easily happen that f is a homeomorphism without being an isomorphism. For instance, the normalization map

$$f\colon \operatorname{Spec}\mathbb{C}[t] \longrightarrow \operatorname{Spec}\mathbb{C}[x,y]/(y^2-x^3), \qquad t \mapsto (t^2,t^3),$$

is a homeomorphism, but not an isomorphism. At the origin, the induced map on local rings is not surjective: x and y maps to t^2 and t^3, but there is no element that maps to t. Alternatively, the induced map on cotangent spaces is not surjective.

The next proposition shows that, for finite maps between varieties over an algebraically closed field, this failure on cotangent spaces is the only obstruction to being a closed embedding.

Proposition 21.13. Let $f\colon X \to Y$ be a finite morphism of schemes of finite type over an algebraically closed field k. Then f is a closed embedding if and only if

(i) f is injective on k-points; and

(ii) for each k-point $P \in X(k)$, the induced map on cotangent spaces

$$f_P^*\colon \mathfrak{m}_{f(P)}/\mathfrak{m}_{f(P)}^2 \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2 \tag{21.1}$$

is surjective.

Proof Closed embeddings clearly satisfy (i) and (ii), so we prove the converse.

The statement is local on Y, so we may assume that $Y = \operatorname{Spec} A$. As f is a finite morphism, $X = \operatorname{Spec} B$ for some finitely generated A-module B and f is induced by a ring map $\phi\colon A \to B$. We must show that ϕ is surjective. Since B is a finite A-module, it is enough to show that $\phi_{\mathfrak{m}}\colon A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m} \subset A$.

Fix a maximal ideal $\mathfrak{m} \subset A$. If $\mathfrak{m}$ is not in the image of f, then the fiber over $\mathfrak{m}$ is empty, so $B/\mathfrak{m}B = 0$. By Nakayama's lemma this implies $B_{\mathfrak{m}} = 0$, and there is nothing to prove.

Now suppose that $\mathfrak{m}$ lies in the image of f. Because B is of finite type over the algebraically closed field k, every maximal ideal of B corresponds to a k-point of X. Condition (i) therefore implies that there is a unique maximal ideal $\mathfrak{n} \subset B$ lying over $\mathfrak{m}$. Then $B_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{n}B_{\mathfrak{m}}$. Replacing A and B by their localizations at $\mathfrak{m}$ and $\mathfrak{n}$, we may assume that $\phi\colon (A,\mathfrak{m}) \to (B,\mathfrak{n})$ is a map of local rings.

We claim that $\mathfrak{n} = \phi(\mathfrak{m})B$. Let $J = \phi(\mathfrak{m})B \subset B$. Since ϕ is local, we have $J \subset \mathfrak{n}$. Condition (ii) says exactly that the induced map $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2$ is surjective, so the image of J generates $\mathfrak{n}/\mathfrak{n}^2$. Equivalently, $J + \mathfrak{n}^2 = \mathfrak{n}$. From this we see that $\mathfrak{n}(\mathfrak{n}/J) = \mathfrak{n}/J$. By Nakayama's lemma applied to the finitely generated B-module $\mathfrak{n}/J$, we obtain $\mathfrak{n}/J = 0$, so $J = \mathfrak{n}$. It follows that $B/\mathfrak{m}B = B/J = B/\mathfrak{n} \simeq k$, which is a 1-dimensional k-vector

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

space. By Nakayama's Lemma again, we see that the A-module B can be generated by one element, which means that ϕ is surjective. □

The condition (21.1) holds if and only if the induced map on tangent spaces

$$df_P \colon T_P X \longrightarrow T_{f(P)} Y \tag{21.2}$$

is injective. This is an analogue of the inverse function theorem, that an injective map with invertible jacobian has an inverse.

Corollary 21.14 (Cohomological criterion for closed embeddings). Let X be a proper scheme of finite type over an algebraically closed field k, let L be a globally generated invertible sheaf on X, and let $f \colon X \to \mathbb{P}^n_k$ be the morphism defined by sections of L. Assume that

(i) $H^1(X, L \otimes \mathfrak{m}_P \mathfrak{m}_Q) = 0$ for every pair of distinct points $P, Q \in X(k)$;
(ii) $H^1(X, L \otimes \mathfrak{m}_P^2) = 0$ for every point $P \in X(k)$.

Then f is a closed embedding.

Here $\mathfrak{m}_P$ denotes the ideal sheaf of the point P. Since $\mathfrak{m}_P \mathfrak{m}_Q = \mathfrak{m}_P \cap \mathfrak{m}_Q$ for distinct P, Q, the two hypotheses can be combined into the single condition $H^1(X, L \otimes \mathcal{I}_Z) = 0$ for every 0-dimensional closed subscheme $Z \subset X$ of length 2.

Proof We verify the two conditions in Proposition 21.14: that f is injective on k-points and that for every $P \in X(k)$ the pullback map on cotangent spaces $f_P^* \colon \mathfrak{m}_{f(P)}/\mathfrak{m}_{f(P)}^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2$ is surjective.

Choose a basis $s_0, \ldots, s_n$ for $H^0(X, L)$. Then over the open set $D(s_0)$, the morphism f is induced by the ring map

$$k\Big[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\Big] \longrightarrow \Gamma(D(s_0), \mathcal{O}_X), \qquad \frac{x_i}{x_0} \longmapsto \frac{s_i}{s_0},$$

where s_i/s_0 is the usual regular function on the open set $D(s_0)$.

To show that f is injective, let $P, Q \in X(k)$ be distinct. It is enough to find a section $s = \sum a_i s_i$ with $s(P) = 0$ and $s(Q) \neq 0$, for then $f(P)$ lies in the hyperplane $V(\sum a_i x_i) \subset \mathbb{P}^n_k$ whereas $f(Q)$ does not. Consider the exact sequence

$$0 \to L \otimes \mathfrak{m}_P \mathfrak{m}_Q \to L \to L|_P \oplus L|_Q \to 0.$$

By assumption $H^1(X, L \otimes \mathfrak{m}_P \mathfrak{m}_Q) = 0$, so the map $H^0(X, L) \to L|_P \oplus L|_Q$ is surjective. Taking a lift of $(0, 1) \in L|_P \oplus L|_Q$, we see that such a section s exists, so f is injective on k-points.

Now let $P \in X(k)$. Since L is globally generated, after a linear change of coordinates we may assume that $f(P) = (1 : 0 : \cdots : 0)$ and $s_0(P) \neq 0$, hence $P \in D(s_0)$. For $i = 1, \ldots, n$, write $y_i = x_i/x_0$ for affine coordinates on $D(x_0)$. Then $f^\sharp(y_i) = s_i/s_0$, so in these coordinates $\mathfrak{m}_{f(P)}/\mathfrak{m}_{f(P)}^2$ is generated by the classes $\overline{y_1}, \ldots, \overline{y_n}$, and f_P^* sends $\overline{y_i}$ to the class of s_i/s_0 in $\mathfrak{m}_P/\mathfrak{m}_P^2$. To conclude that f_P^* is surjective, we need to show that every element of $\mathfrak{m}_P/\mathfrak{m}_P^2$ is a linear combination of the classes of $s_1/s_0, \ldots, s_n/s_0$. Consider the exact sequence

$$0 \to L \otimes \mathfrak{m}_P^2 \to L \to L \otimes (\mathcal{O}_X/\mathfrak{m}_P^2) \to 0.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

By assumption $H^1(X, L \otimes \mathfrak{m}_P^2) = 0$, so $H^0(X, L) \to H^0(X, L \otimes (\mathcal{O}_X/\mathfrak{m}_P^2))$ is surjective. Since $s_0(P) \neq 0$, multiplication by s_0^{-1} identifies $L \otimes (\mathcal{O}_X/\mathfrak{m}_P^2)$ with $\mathcal{O}_X/\mathfrak{m}_P^2$ near P. Therefore every class in $\mathfrak{m}_P/\mathfrak{m}_P^2$ is represented by s/s_0 for some $s \in H^0(X, L)$. Subtracting a multiple of s_0, we may assume $s(P) = 0$. Then writing $s = a_0 s_0 + \cdots + a_n s_n$, this forces $a_0 = 0$, since $s_0(P) \neq 0$ and $s_i(P) = 0$ for $i \geqslant 1$. Hence s/s_0 is a linear combination of $s_1/s_0, \ldots, s_n/s_0$, as required.

Finally, since X is proper, f is proper. As f is injective on k-points, each fiber over a closed point consists of at most one point, so f is finite. Proposition 21.14 now shows that f is a closed embedding. □

21.3 Extended example: The blow-up of the projective plane

Let X be the blow-up of $\mathbb{P}^2_k$ at the point $P = (0 : 0 : 1)$. We keep the notation of Example 19.42. We claim that the sheaf $\mathcal{O}_X(2H - E)$ is very ample. Indeed,

$$\mathcal{O}_X(2H - E) \simeq \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^2_k}(1, 1)|_X,$$

and the Segre embedding of $\mathbb{P}^1_k \times \mathbb{P}^2_k$ restricts to a closed embedding of X. From the relation $xv - yu = 0$ we see that the image is contained in a hyperplane $\mathbb{P}^4_k \subset \mathbb{P}^5_k$. Away from E, this is the map defined by the conics through P:

$$(x : y : z) \longmapsto (x^2 : xy : y^2 : xz : yz).$$

Writing the coordinates in this order as $(Z_0 : \cdots : Z_4)$, the image is cut out by the 2×2 minors of

$$\begin{pmatrix} Z_0 & Z_1 & Z_3 \\ Z_1 & Z_2 & Z_4 \end{pmatrix}.$$

This means that X embeds as a smooth surface in $\mathbb{P}^4_k$. In particular, $2H - E$ is very ample. The product model also gives

$$\mathcal{O}_X(E) \simeq \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^2_k}(-1, 1)|_X,$$

which again restricts to $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ on E.

More generally, we see that

(i) $\mathcal{O}_X(aH + bE)$ is globally generated if and only if and $a + b \geqslant 0$ and $b \leqslant 0$.

(ii) $\mathcal{O}_X(aH + bE)$ is very ample if and only if and $a + b > 0$ and $b < 0$.

Indeed, H and $H - E$ are both globally generated, since they define morphisms to $\mathbb{P}^2_k$ and $\mathbb{P}^1_k$. Hence the same applies to all positive linear combinations of them. Similarly, since $2H - E$ is very ample and H and $H - E$ are globally generated, we see that all divisors of the form $2H - E + sH + t(H - E)$ are very ample for all $s, t \geqslant 0$. On the other hand H and $H - E$ are not ample: $\mathcal{O}_X(H)$ restricts to $\mathcal{O}_E$ on E and $\mathcal{O}_X(H - E)$ restricts to $\mathcal{O}_C$ on $C = V(u) \simeq \mathbb{P}^1_k$. This covers all divisors in the region (ii).

21.4 Exercises

Exercise 21.4.1. Use Proposition 21.14 to show that the map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ defined by $(s : t) \mapsto (s^3 : s^2t : st^2 : t^3)$ is a closed embedding.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 21.4.2. Use Proposition 21.14 to show that the Segre map $\mathbb{P}^m_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^{(m+1)(n+1)-1}$ is a closed embedding.

Exercise 21.4.3. Show that a morphism $f\colon X \to Y$ of varieties over an algebraically closed field k is a closed embedding if and only if the induced maps $X(k) \to Y(k)$ and $X(R) \to Y(R)$ are injective, where $R = k[\epsilon]/\epsilon^2$ is the ring of dual numbers.

Exercise 21.4.4. Use Proposition 21.14 to show that $f\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$, $t \to (t + t^4, t^2)$ is a closed embedding. What is the equation defining the image?

Exercise 21.4.5. Let X be a proper scheme over a Noetherian ring A, and let L be an invertible sheaf on X. Show that L is ample on X if and only if $L|_{X_{\mathrm{red}}}$ is ample on X_{red}. HINT: Let $\mathcal{N} \subset \mathcal{O}_X$ be the nilradical, so $\mathcal{N}^r = 0$ for some $r \gg 0$. Use the filtration $0 = \mathcal{N}^r\mathcal{F} \subset \cdots \subset \mathcal{N}\mathcal{F} \subset \mathcal{F}$ for a coherent sheaf $\mathcal{F}$, and apply Theorem 21.11 to the quotients.

Exercise 21.4.6. Let X be a proper scheme over a Noetherian ring A, let $X_1, \ldots, X_r$ be the irreducible components of X_{red}, and let L be an invertible sheaf on X. Show that L is ample on X if and only if $L|_{X_i}$ is ample on X_i for every i.

Exercise 21.4.7. Let X be a proper scheme over an algebraically closed field, and let L be an invertible sheaf on X. Assume that L is globally generated and that $L|_C$ is ample for every integral curve $C \subset X$. Show that L is ample. HINT: Consider a morphism $f\colon X \to \mathbb{P}^n_A$ with $L \simeq f^*\mathcal{O}_{\mathbb{P}^n_A}(1)$. If some fiber had positive dimension, it would contain an integral curve C, and $L|_C$ would be trivial.

Exercise 21.4.8. Let X be a projective scheme over a Noetherian ring A, let L be an ample invertible sheaf on X, and let M be any invertible sheaf on X. Show that $M \otimes L^{\otimes n}$ is very ample for all $n \gg 0$.

Exercise 21.4.9. Let X and Y be proper schemes over a field k, let $p_1, p_2\colon X \times_k Y \to X, Y$ be the projections, and let D and D' be ample divisors on X and Y. Show that $p_1^*D + p_2^*D'$ is an ample divisor on $X \times_k Y$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

22

Curve theory

22.1 Sheaves on curves

When X is a regular curve, the local rings $\mathcal{O}_{X,P}$ at closed points $P \in X$ are discrete valuation rings, or equivalently PIDs. This fact makes the study of sheaves on X particularly simple.

Recall that for a ring A, an A-module M has a torsion submodule $T \subset M$ consisting of all torsion elements, i.e., the elements of M which are annihilated by a nonzerodivisor of A. The quotient M/T is a torsion free A-module. If the ring A is a discrete valuation ring and M is finitely generated, M/T is in fact free. This follows from the structure theorem for modules over a PID,see Section C.4.8.

This generalizes to sheaves on curves as follows. Let X be a curve and $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then $\mathcal{F}$ contains a torsion subsheaf $\mathcal{F}_{\mathrm{tors}}$, whose sections over an open set $U \subset X$ consists of the elements of $\mathcal{F}(U)$ which are killed by some nonzerodivisor of $\mathcal{O}_X(U)$ (see Exercise 10.13.30 on page 204). The quotient $\mathcal{F}_{\mathrm{tf}} = \mathcal{F}/\mathcal{F}_{\mathrm{tors}}$ is then torsion free in the sense that on open affine subsets U, $\mathcal{F}_{\mathrm{tf}}(U)$ (which equals $\mathcal{F}(U)/\mathcal{F}_{\mathrm{tors}}(U)$) is a torsion free module over $\mathcal{O}_X(U)$. For a point $P \in X$, the stalk of $\mathcal{F}_{\mathrm{tf}}$ is a torsion free module over $\mathcal{O}_{X,P}$. Since $\mathcal{O}_{X,P}$ is a PID, $(\mathcal{F}_{\mathrm{tf}})_P$ is free. As this holds for every P, $\mathcal{F}_{\mathrm{tf}}$ is locally free by Theorem 11.22.

As $\mathcal{F}$ is of finite type, it is locally free over an open subset, so the torsion part $\mathcal{F}_{\mathrm{tors}}$ is a sheaf supported on finitely many points $p_1, \ldots, p_r$. Moreover, $\mathcal{F}_{\mathrm{tors}}$ decomposes as a direct sum of skyscraper sheaves at these points.

Theorem 22.1. Let X be a regular curve and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then there is a decomposition

$$\mathcal{F} \simeq \mathcal{F}_{\mathrm{tors}} \oplus \mathcal{F}_{\mathrm{tf}}.$$

where $\mathcal{F}_{\mathrm{tors}} \subset \mathcal{F}$ is the torsion subsheaf and $\mathcal{F}_{\mathrm{tf}}$ is locally free.

Proof Write $\mathcal{T} = \mathcal{F}_{\mathrm{tors}}$ and $\mathcal{E} = \mathcal{F}/\mathcal{T}$. Then $\mathcal{E}$ is locally free and $\mathcal{T}$ is supported at finitely many closed points, say $P_1, \ldots, P_n$. These fit into the exact sequence

$$0 \longrightarrow \mathcal{T} \xrightarrow{\iota} \mathcal{F} \xrightarrow{q} \mathcal{E} \to 0.$$

We claim that this sequence splits. For each i, choose an affine open U_i containing P_i but none of the other points P_j. We may also assume that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^r$ is free. Since U_i is affine, $\mathcal{F}(U_i) \to \mathcal{E}(U_i)$ is surjective and since $\mathcal{E}(U_i)$ is free, it admits a section. So there is a map $s_i \colon \mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$ such that $q \circ s_i = \mathrm{id}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Let $U_0 = X - \{P_1, \ldots, P_r\}$. On U_0, the torsion sheaf $\mathcal{T}$ vanishes, so $q\colon \mathcal{F}|_{U_0} \to \mathcal{E}|_{U_0}$ is an isomorphism. Let $s_0 = q^{-1}\colon \mathcal{E}|_{U_0} \to \mathcal{F}|_{U_0}$ be the inverse.

We claim that the s_i glue to a splitting $s\colon \mathcal{E} \to \mathcal{F}$. If $i \neq j$, then $U_i \cap U_j$ does not contain any of the points $P_1, \ldots, P_n$, so $\mathcal{T}|_{U_i \cap U_j} = 0$. Therefore q is an isomorphism on $U_i \cap U_j$, so s_i and s_j must be equal to q^{-1}. Similarly, s_i and s_0 are equal on $U_i \cap U_0$. Hence the $s_0, s_1, \ldots, s_n$ glue to a global map $s\colon \mathcal{E} \to \mathcal{F}$ such that $q \circ s = \mathrm{id}_{\mathcal{E}}$, and hence $\mathcal{F} \simeq \mathcal{T} \oplus \mathcal{E} = \mathcal{F}_{\mathrm{tors}} \oplus \mathcal{F}_{\mathrm{tf}}$. □

Proposition 22.2. Let $f\colon X \to Y$ be a finite morphism of projective regular curves over a field k, and let $\mathcal{E}$ be a locally free sheaf of finite rank on X. Then $f_*\mathcal{E}$ is a locally free sheaf on Y and

$$\operatorname{rank}(f_*\mathcal{E}) = (\deg f)(\operatorname{rank}\mathcal{E}).$$

Proof Since f is finite, $f_*\mathcal{E}$ is coherent. We first treat the case $\mathcal{E} = \mathcal{O}_X$. Let $V = \operatorname{Spec} A \subset Y$ be an affine open subset and $f^{-1}(V) = \operatorname{Spec} B$, so that $(f_*\mathcal{O}_X)|_V = \widetilde{B}$ with B a finitely generated A-algebra. As X and Y are integral and f is dominant, $A \to B$ is injective, and since B is an integral domain it is torsion free as an A-module. Hence $f_*\mathcal{O}_X$ is locally free by Theorem 22.1. To compute its rank, pass to the generic point η of Y: the stalk $(f_*\mathcal{O}_X)_\eta = K(X)$, viewed as a vector space over $\mathcal{O}_{Y,\eta} = K(Y)$, gives

$$\operatorname{rank}(f_*\mathcal{O}_X) = [K(X) : K(Y)] = \deg f.$$

Now let $\mathcal{E}$ be locally free of rank r, and cover Y by affine open sets V with $\mathcal{E}|_{f^{-1}(V)} \simeq \mathcal{O}^{\oplus r}$. Applying f_* gives $(f_*\mathcal{E})|_V \simeq (f_*\mathcal{O}_X|_V)^{\oplus r}$, so $f_*\mathcal{E}$ is locally free of rank $r \deg f$. □

22.2 Splitting of locally free sheaves on $\mathbb{P}^1_k$

Theorem 22.3 (Locally free sheaves on $\mathbb{P}^1_k$). Let k be a field and let $\mathcal{E}$ be a locally free sheaf of rank r on $\mathbb{P}^1_k$. Then there exist unique integers $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$ such that

$$\mathcal{E} \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1_k}(a_i). \tag{22.1}$$

Proof Write $\mathcal{O}(a) = \mathcal{O}_{\mathbb{P}^1_k}(a)$. The proof uses induction on r. For $r = 1$, the statement follows from $\operatorname{Pic}(\mathbb{P}^1_k) \simeq \mathbb{Z}$.

Consider $\operatorname{Hom}(\mathcal{O}(n), \mathcal{E}) = \operatorname{Hom}(\mathcal{O}, \mathcal{E}(-n)) = H^0(\mathbb{P}^1_k, \mathcal{E}(-n))$. When n is very large, this group is zero, because $\mathcal{E}$ is coherent. When n is very negative, then $\mathcal{E}(-n)$ is globally generated, so the group is nonzero. Let $n \in \mathbb{Z}$ be the maximal integer with the property that $\operatorname{Hom}(\mathcal{O}(n), \mathcal{E}) \neq 0$. Choose a nonzero map $\phi\colon \mathcal{O}(n) \to \mathcal{E}$. Since ϕ is nonzero, it is injective (its kernel would be locally free of rank 0).

Let $\mathcal{F} = \operatorname{Coker}\phi$, and consider the exact sequence

$$0 \longrightarrow \mathcal{O}(n) \xrightarrow{\phi} \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $\mathcal{F}$ is coherent, we may decompose it as $\mathcal{F} = \mathcal{F}_{\mathrm{tf}} \oplus \mathcal{F}_{\mathrm{tors}}$, where $\mathcal{F}_{\mathrm{tf}}$ is locally free of rank $r - 1$ and $\mathcal{F}_{\mathrm{tors}}$ is a torsion sheaf.

We claim that $\mathcal{F}_{\mathrm{tors}} = 0$, so that $\mathcal{F}$ is actually locally free. Let $\pi\colon \mathcal{F} \to \mathcal{F}_{\mathrm{tf}}$ be the projection and $\psi\colon \mathcal{E} \to \mathcal{F}_{\mathrm{tf}}$ the composition $\mathcal{E} \to \mathcal{F} \to \mathcal{F}_{\mathrm{tf}}$. Define $L = \mathrm{Ker}(\psi)$. Then $\mathrm{Im}\,\phi \subset L$, and we have two exact sequences

$$0 \longrightarrow L \longrightarrow \mathcal{E} \xrightarrow{\psi} \mathcal{F}_{\mathrm{tf}} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{O}(n) \longrightarrow L \longrightarrow \mathcal{F}_{\mathrm{tors}} \longrightarrow 0. \qquad (*)$$

Since $\mathcal{E}$ and $\mathcal{F}_{\mathrm{tf}}$ are locally free, the kernel L is locally free of rank 1, say $L \simeq \mathcal{O}(b)$ for $b \in \mathbb{Z}$.

If $\mathcal{F}_{\mathrm{tors}} \neq 0$, then the map $\mathcal{O}(n) \to \mathcal{O}(b)$ is nonzero but not an isomorphism. As $\mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(b)) = H^0(\mathcal{O}(b-n))$, this implies $b > n$. But then $\mathcal{O}(b) \simeq L \to \mathcal{E}$ is nonzero, contradicting the maximality of n. Therefore $\mathcal{F}_{\mathrm{tors}} = 0$ and $b = n$.

Now, twisting the first sequence by $\mathcal{O}(-n)$, we obtain

$$0 \to \mathcal{O} \to \mathcal{E}(-n) \to \mathcal{F}(-n) \to 0.$$

By induction, there are integers $b_2 \geqslant \cdots \geqslant b_r$ with $\mathcal{F}(-n) \simeq \bigoplus_{i=2}^{r} \mathcal{O}(b_i)$.

We show that $b_i \leqslant 0$ for every i. Twisting the above sequence by $\mathcal{O}(-1)$ gives

$$0 \to \mathcal{O}(-1) \to \mathcal{E}(-n-1) \to \mathcal{F}(-n-1) \to 0.$$

By the choice of n, $H^0(\mathcal{E}(-n-1)) = 0$. Since $H^1(\mathcal{O}(-1)) = 0$, the long exact sequence in cohomology gives $H^0(\mathcal{F}(-n-1)) = 0$. Hence $H^0(\mathcal{O}(b_i - 1)) = 0$, and so $b_i \leqslant 0$, for every $i = 2, \ldots, r$.

Next we show that the sequence splits. Twisting by $\mathcal{O}(-b_i)$ gives

$$0 \to \mathcal{O}(-b_i) \to \mathcal{E}(-n-b_i) \to \mathcal{F}(-n-b_i) \to 0.$$

Since $b_i \leqslant 0$, we have $H^1(\mathcal{O}(-b_i)) = 0$, so $\mathcal{E}(-n-b_i) \to \mathcal{F}(-n-b_i)$ is surjective on global sections. The section of $\mathcal{F}(-n-b_i)$ corresponding to the inclusion $\mathcal{O}(b_i) \to \mathcal{F}(-n)$ therefore lifts to a section of $\mathcal{E}(-n-b_i)$, or equivalently to a morphism $\mathcal{O}(b_i) \to \mathcal{E}(-n)$. Taking the direct sum of these, we obtain a morphism $\mathcal{F}(-n) \to \mathcal{E}(-n)$ whose composition with $\mathcal{E}(-n) \to \mathcal{F}(-n)$ is the identity. Hence $\mathcal{E}(-n) \simeq \mathcal{O} \oplus \mathcal{F}(-n)$, and twisting back,

$$\mathcal{E} \simeq \mathcal{O}(n) \oplus \bigoplus_{i=2}^{r} \mathcal{O}(n + b_i).$$

This is of the form (22.1) with $a_1 = n$ and $a_i = n + b_i$ for $i \geqslant 2$. The inequalities $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$ hold because $b_i \leqslant 0$ and $b_2 \geqslant \cdots \geqslant b_r$.

Once we know that $\mathcal{E}$ splits as a direct sum of $\mathcal{O}(a_i)$'s, the integers a_i are unique. Indeed, $h^0(\mathcal{E}(m)) = \sum_i \max\{0, m + a_i + 1\}$ depends only on $\mathcal{E}$, and the a_i can be recovered from this function. □

22.3 Divisors on curves

Let X be a regular curve over an algebraically closed field k. The codimension 1-subsets of a curve are precisely the closed points, so that a Weil divisor is a finite formal combination $D = \sum_i n_i P_i$ of closed points $P_1, \ldots, P_r$ in X, where the coefficients are integers. We define the *degree* as

$$\deg D = \sum n_i$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

If D is given by the Cartier data (U_i, f_i) (see Definition 19.23), then

$$\deg D = \sum_{P \in X} \operatorname{ord}_P(f_i)$$

where we for each P choose an i so that $P \in U_i$.

To compute $\operatorname{ord}_P(f)$, choose a local parameter $t \in \mathcal{O}_{X,P}$ generating the maximal ideal (e.g., using the local model of Example 18.30), and write $f = u \cdot t^n$ where u is a unit in $\mathcal{O}_{X,P}$. Then $\operatorname{ord}_P(f) = n$.

Example 22.4. Consider the curve $X \subset \mathbb{A}^2_{\mathbb{Q}}$ defined by the equation $x^2 + y^2 = 1$, and let $P = (1, 0)$. Then the rational function $x - 1$ has order of vanishing 2 at P. To see this, note that y is a local parameter at P, because $\frac{\partial}{\partial x}(x^2 + y^2 - 1) = 2x$ is nonzero at P. Moreover, in $\mathcal{O}_{X,P}$, we have the relation $(x - 1)(x + 1) = -y^2$, and $x + 1$ is a unit in $\mathcal{O}_{X,P}$, and hence $\operatorname{ord}_P(x - 1) = 2$. △

If $f\colon X \to Y$ is a dominant morphism, we can understand the pullback map on divisors as follows. First of all, f induces a map of function fields

$$f^\sharp\colon K(Y) \hookrightarrow K(X)$$

Let $Q \in Y$ be a closed point, and choose a local parameter $t \in \mathcal{O}_{Y,Q}$. We define

$$f^*(Q) = \sum_{P \in f^{-1}(Q)} \operatorname{ord}_P(f^\sharp t) P.$$

The integer $e_P = \operatorname{ord}_P(f^\sharp t)$ is called the *ramification index* of f at P. It does not depend on the choice of t, because multiplying t by a unit in $\mathcal{O}_{Y,Q}$ does not change the order of vanishing $\operatorname{ord}_P(f^\sharp t)$ (a unit in $\mathcal{O}_{Y,Q}$ stays a unit in $\mathcal{O}_{X,P}$).

Extending by linearity, we obtain a well-defined map of groups

$$f^*\colon \operatorname{Div} X \longrightarrow \operatorname{Div} Y.$$

If D is represented by the Cartier data $\{(U_i, g_i)\}$, where $g_i \in K(Y)^\times$, then the pullback f^*D is represented by $\{(f^{-1}U_i, f^\sharp g_i)\}$ on X.

Note that $f^*(Q)$ is a divisor supported on the fiber $f^{-1}(Q)$. In fact, it has the same multiplicities as the scheme-theoretic fiber X_Q since $f^*(Q)$ is locally defined by the equations $f^\sharp(g_i)$ and these form local generators for the ideal sheaf of the closed subscheme X_Q.

Lemma 22.5. Let $f\colon X \to Y$ be a dominant morphism of regular curves. Then

$$\deg f^*D = \deg f \cdot \deg D$$

for every divisor D on Y.

Proof We first prove the formula for $D = Q$, where $Q \in Y$ is a closed point.

Choose an affine neighbourhood $V = \operatorname{Spec} A$ of Q and write $f^{-1}(V) = \operatorname{Spec} B$ and B is a free A-module of rank $\deg f$; this is possible because $f_*\mathcal{O}_X$ is locally free of rank $\deg f$. Let $t \in A$ be a local parameter at Q. Then the scheme-theoretic fiber X_Q is the spectrum of the ring $B \otimes_A \kappa(Q) \simeq B/tB$, so $\dim_{\kappa(Q)}(B/tB) = \deg f$.

On the other hand, B/tB is Artinian, hence

$$B/tB \simeq \prod_{P \in f^{-1}(Q)} (B/tB)_{\mathfrak{m}_P}.$$

For each $P \in f^{-1}(Q)$, if s is a local parameter at P, then by definition of the ramification index, $f^\sharp(t) = us^{e_P}$ for some unit $u \in \mathcal{O}_{X,P}^\times$. Therefore

$$(B/tB)_{\mathfrak{m}_P} \simeq \mathcal{O}_{X,P}/(s^{e_P}),$$

which has dimension e_P over $\kappa(P)$. Hence

$$\dim_{\kappa(Q)}(B/tB)_{\mathfrak{m}_P} = e_P[\kappa(P) : \kappa(Q)].$$

Summing over all $P \in f^{-1}(Q)$, we obtain

$$\deg f = \dim_{\kappa(Q)}(B/tB) = \sum_{P \in f^{-1}(Q)} e_P[\kappa(P) : \kappa(Q)] = \deg(f^*Q).$$

Hence $\deg(f^*Q) = \deg f$. By linearity, $\deg(f^*D) = \deg f \cdot \deg D$ for every divisor D on Y. □

Example 22.6 (Principal divisors)**.** Let $g \in K(X)^\times$ be a rational function. Then if $G\colon X \to \mathbb{P}^1_k$ denotes the corresponding morphism, we have

$$\operatorname{div}(g) = G^*((0:1) - (1:0)).$$

This follows because $G^\sharp(t_0/t_1) = g$ and $G^\sharp(t_1/t_0) = 1/g$ and and t_0/t_1 and t_1/t_0 are local parameters at $(0:1)$ and $(1:0)$ respectively. △

Lemma 22.7. For a nonzero $g \in K(Y)$ and a morphism $f\colon X \to Y$, we have

$$f^*(\operatorname{div} g) = \operatorname{div}(g \circ f).$$

Proof Let $G\colon Y \to \mathbb{P}^1$ be the morphism corresponding to g. Then $G \circ f$ is the morphism corresponding to $g \circ f$ and so by Example 22.6 above,

$$f^*(\operatorname{div} g) = f^*(G^*((0:1) - (1:0)) = (G \circ f)^*((0:1) - (1:0)) = \operatorname{div}(g \circ f).$$

□

Corollary 22.8. Let X be a regular projective curve. For a nonzero rational function $g \in K(X)$, we have $\deg(\operatorname{div} g) = 0$. Hence the degree map descends to a map

$$\deg\colon \operatorname{Cl}(X) \to \mathbb{Z}.$$

In other words, linearly equivalent divisors have the same degree.

Proof This is clear if g is a constant. If not, g defines a dominant morphism $G\colon X \to \mathbb{P}^1_k$ so that $\operatorname{div}(g) = G^*((1:0) - (0:1))$ and we are done by the above lemma. □

Corollary 22.9. Let X be a regular projective curve. If D is a divisor with $\deg(D) < 0$, then $\Gamma(X, \mathcal{O}_X(D)) = 0$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof The group $\Gamma(X, \mathcal{O}_X(D))$ consists of rational functions $f \in K(X)$ so that $\operatorname{div}(f) + D$ is effective. However, if f is nonzero, then $\operatorname{div}(f) + D$ has negative degree, whereas effective divisors have non-negative degree. □

Example 22.10. Let k be a field of characteristic 0 and consider the curve $X \subset \mathbb{P}^2_k = \operatorname{Proj} k[x_0, x_1, x_2]$ defined by

$$x_2^2 x_0 = x_1^3 + x_1^2 x_0 + x_0^3. \tag{22.2}$$

One checks that X is a regular curve. Let us compute $\operatorname{div}(f)$ for the rational function $f = x_2/x_0 + 1$.

On $D_+(x_0)$, with $x = x_1/x_0$, $y = x_2/x_0$, the curve is defined by $y^2 = x^3 + x^2 + 1$, and $f = y + 1$ is regular there, so there are no points of $D_+(x_0)$ where $\operatorname{ord}(f) < 0$. Rewriting the equation as $(y-1)(y+1) = x^2(x+1)$, we see that the zeros of $y+1$ are $P_1 = (1:0:-1)$ and $P_2 = (1:-1:-1)$. In the local ring at P_1, both $y-1$ and $x+1$ are invertible, so $y + 1 = x^2(x+1)(y-1)^{-1} \in (x^2)$. Since $\mathfrak{m}_{P_1} = (x, y+1)$, we get $\mathfrak{m}_{P_1} = (x)$, so x is a local parameter at P_1 and $\operatorname{ord}_{P_1}(f) = 2$. Near P_2, both $y-1$ and x are units, so $y + 1 = x^2(x+1)(y-1)^{-1} \in (x+1)$. Since $\mathfrak{m}_{P_2} = (x+1, y+1)$, this means that $x+1$ is a local parameter at P_2 and $\operatorname{ord}_{P_2}(f) = 1$.

It remains to consider $Y \cap V(x_0)$. Setting $x_0 = 0$, we see that $Y \cap V(x_0)$ is the single point $Q = (0:0:1)$. On $D_+(x_2)$, with $u = x_0/x_2$, $v = x_1/x_2$, the curve is given by $u = v^3 + v^2 u + u^3$, the point Q is the origin $(u, v) = (0, 0)$, and $f = u^{-1} + 1$. Writing the curve equation as $u(1 - v^2 - u^2) = v^3$, we see that $1 - v^2 - u^2$ is a unit near Q, so $u = v^3(1 - v^2 - u^2)^{-1} \in (v^3)$; since $\mathfrak{m}_Q = (u, v)$, this forces $\mathfrak{m}_Q = (v)$, so v is a local parameter at Q and $\operatorname{ord}_Q(u) = 3$. Since $1 + u$ is invertible at Q, $\operatorname{ord}_Q(f) = \operatorname{ord}_Q(1+u) - \operatorname{ord}_Q(u) = -3$.

In all, we find that

$$\operatorname{div}(f) = 2P_1 + P_2 - 3Q = 2(1:0:-1) + (1:-1:-1) - 3(0:0:1),$$

Note that this divisor has degree 0, in accordance with Corollary 22.8. △

22.4 The canonical divisor

Also in this section, we work with a smooth curve X defined over an algebraically closed field k. Then the sheaf of differentials $\Omega_{X/k}$ is an invertible sheaf.

Definition 22.11 (The canonical divisor)**.** Let X be a smooth curve over k. A *canonical divisor* is a divisor K_X such that

$$\mathcal{O}_X(K_X) \simeq \Omega_{X/k}. \tag{22.3}$$

The terminology is a little bit akward. Any two canonical divisors will be linearly equivalent, but there is no divisor which is *canonical*. It is the sheaf Ω_X and the class $[K_X]$ in the class gorup which are canonical.

One can describe canonical divisors as follows. The stalk of the sheaf $\Omega_{X/k}$ at the generic point of X is equal to $\Omega_{K(X)/k}$, which is a 1-dimensional $K(X)$-vector space. The elements are called *rational differential forms*.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Given a rational differential form $\omega \in \Omega_{K(X)/k}$, we can produce a canonical divisor $\operatorname{div}(\omega)$ as follows. Let $P \in X$ be a closed point and let t be a local paramter of the local ring $\mathcal{O}_{X,P}$. Then $\Omega_{\mathcal{O}_{X,P}/k}$ is generated as an $\mathcal{O}_{X,P}$-module by the element dt. We may therefore write $\omega = f\,dt$ for some $f \in K(X)$. We define $\operatorname{ord}_P(\omega) = \operatorname{ord}_P(f)$ and

$$\operatorname{div}(\omega) = \sum_P \operatorname{ord}_P(\omega) P$$

One checks that $\operatorname{ord}_P(\omega)$ is independent of the choices of the local parameters, and that there are only finitely many P for which $\operatorname{ord}_P(\omega) \neq 0$, so that the above divisor is well-defined.

We say that ω is regular if $\operatorname{ord}_P(\omega) \geqslant 0$. Each regular section defines a global section in $\Gamma(X, \Omega_{X/k})$.

If η is a local generator $\Omega_{X/k}$ at a point $P \in X$, it defines a generator for $\Omega_{K(X)/k}$ as a $K(X)$-vector space as well, and every rational differential form ω is of the form $\omega = g\eta$ for some rational function $g \in K(X)$. In fact, $\Omega_{K(X)/k} = \Omega_{\mathcal{O}_{X,P}/k} \otimes_{\mathcal{O}_{X,P}} K(X)$.

If ω and ω' are two rational differentials, then $\omega' = h\omega$ for some $h \in K(X)$. Therefore, for each $P \in X$ we have $\operatorname{ord}_P(hg_P) = \operatorname{ord}_P(h) + \operatorname{ord}_P(g_P)$, and by the formula (22.4), we get $\operatorname{div}(h\omega) = \operatorname{div} h + \operatorname{div} \omega$.

Now let $\omega \in \Omega_{K(X)/k}$ be a nonzero element. For each point $P \in X$ choose a generator η_P for $\Omega_{\mathcal{O}_{X,P}/k}$ and write $\omega = g_P \eta_P$ with $g_P \in K(X)$. Then define

$$\operatorname{div} \omega = \sum_{P \in X} \operatorname{ord}_P(g_P) P. \tag{22.4}$$

The expression on the right in (22.4) is independent of the choice of local generators. This is because two generators η'_P and η_P will be related by $\eta_P = u\eta'_P$ with u a unit in $\mathcal{O}_{X,P}$, and hence $\omega = g_P\eta_P = ug_P\eta'_P$, and $\operatorname{ord}_P(ug_P) = \operatorname{ord}_P(g_P)$. Note also that the sum in fact is finite, because any local generator η_P extends to a generator for $\Omega_{U/k}$ in some neighbourhood U of P, and the corresponding rational function g_P has only finitely many zeroes and poles (and $X - U$ is a finite set as well).

If ω and ω' are two rational differentials, then $\omega' = h\omega$ for some $h \in K(X)$. Therefore, for each $P \in X$ we have $\operatorname{ord}_P(hg_P) = \operatorname{ord}_P(h) + \operatorname{ord}_P(g_P)$, and by the formula (22.4), we get $\operatorname{div}(h\omega) = \operatorname{div} h + \operatorname{div} \omega$.

What we have done so far is valid over any field as long as $\Omega_{X/k}$ is invertible. When the ground field is algebraically closed, there is a local description of the rational differentials in terms of local parameters that make calculations easier.

Lemma 22.12. Assume that X is regular at the closed point $P \in X$ and that t is local parameter at P. Then every element of $\Omega_{K(X)/k}$ is of the form $g\,dt$ with $g \in K(X)$. In other words, $\Omega_{K(X)/k}$ is of rank 1 over $K(X)$ with dt as a basis.

Proof The Zariski cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ at P is generated by the class of a local parameter, and by Proposition 17.14, it follows that dt generates $\Omega_{\mathcal{O}_{X,P}/k}$ when X is regular at P. □

Note that $g\,dt$ defines a section of $\Omega_{U/k}$ over some open set $U \subset X$. In other words, ω is a rational section of $\Omega_{X/k}$. Using Proposition **??**, we obtain

Proposition 22.13. The invertible sheaf associated to $\operatorname{div}\omega$ is isomorphic to Ω_X.

Let X be a smooth projective geometrically irreducible curve over a field k. We define the *genus* of X is defined as the integer $g = h^0(X, \Omega_X)$. The genus is the most important invariant of a curve. In Section 23.1, we will see that $h^0(X, \Omega_X) = h^1(X, \mathcal{O}_X)$. We verify this in three examples.

Example 22.14. Let $X = \mathbb{P}^1_k$ with the usual covering $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$. The differential form dt is an element of $\Omega_{K(X)/k}$, which generates $\Omega_{X/k}|_{U_0}$. This means that $\operatorname{ord}_P(dt) = 0$ for every $P \in U_0$.

For the remaining point $(1:0)$ at infinity, note that $s = t^{-1}$ is a local parameter there, with $(1:0)$ corresponding to the origin in U_1. We have $ds = d(t^{-1}) = -t^{-2}dt$, and hence $dt = -(t^{-1})^{-2}ds$. This means that $\operatorname{ord}_{(1:0)} dt = -2$, so that $\operatorname{div} dt = -2(1:0)$. Hence

$$K_{\mathbb{P}^1_k} = -2(1:0).$$

This shows that $\Omega_X \simeq \mathcal{O}_{\mathbb{P}^1_k}(-2)$ and $\Gamma(\mathbb{P}^1_k, \Omega_{\mathbb{P}^1_k}) = 0$. △

Example 22.15. Let k be a field of characteristic different from 2, and let

$$X = V_+(x_1^2x_2 - x_0^3 + x_0x_2^2) \subset \mathbb{P}^2_k.$$

Let $u = x_0/x_2$ and $v = x_1/x_2$ be affine coordinates on $D_+(x_2)$. Here X is given by the affine equation $v^2 = u^3 - u$. We consider the rational differential $\omega = du$.

At k-points $P = (a, b)$ with $b \neq 0$, $u - a$ is a local parameter, so $\operatorname{ord}_P(\omega) = 0$. If $b = 0$, the points are $(0, 0)$, $(-1, 0)$, and $(1, 0)$. There v is a local parameter, and

$$(3u^2 - 1)\, du = 2v\, dv.$$

As $3u^2 - 1$ is a unit in $\mathcal{O}_{X,P}$, we see that $\operatorname{ord}_P(\omega) = 1$ at each point.

It remains to consider the unique point at infinity, $P = (0:1:0)$. On $D_+(x_1)$, set $s = x_0/x_1$ and $t = x_2/x_1$. The equation becomes $t = s^3 - st^2$. Rewriting this as $s^3 = t(1 + st)$ shows that s is a local parameter at P and $\operatorname{ord}_P(t) = 3$. Since $u = s/t$,

$$\omega = d\left(\frac{s}{t}\right) = t^{-1}\, ds - st^{-2}\, dt = \frac{t\, ds - s\, dt}{t^2}.$$

Differentiating the local equation gives

$$(1 + 2st)\, dt = (3s^2 - t^2)\, ds,$$

Solving for dt and substituting into the expression for ω, we obtain

$$\omega = -2(t + 2st^2)^{-1}\, ds.$$

and hence $\operatorname{ord}_P(\omega) = -\operatorname{ord}_P(t + 2st^2) = -3$. In all, we have

$$\operatorname{div}(\omega) = (0:0:1) + (-1:0:1) + (1:0:1) - 3(0:1:0).$$

This divisor is a principal divisor; it is the divisor associated to the rational function v. In particular, $\Omega_X \simeq \mathcal{O}_X$ and X has genus equal to $h^0(X, \Omega_X) = h^0(X, \mathcal{O}_X) = 1$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example 22.16 (The genus formula for plane curves)**.** Let $X \subset \mathbb{P}^2_k$ be a plane curve, defined by a homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. In Chapter 17, we computed that $\Omega_X \simeq \mathcal{O}_X(d-3)$, which, by the ideal sequence, has

$$h^0(X, \Omega_X) = h^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d-3)) = \frac{(d-1)(d-2)}{2}.$$

This is the same as the number obtained for $h^1(X, \mathcal{O}_X)$ in Chapter 20. △

Hyperelliptic curves

A regular projective curve X of genus $g \geqslant 2$ is said to be *hyperelliptic* if there is a finite degree 2 map $X \to \mathbb{P}^1_k$. There was already a notion of 'hyperelliptic curves' in Section 5.4, so we begin by showing that the two definitions agree. Throughout the section, we assume the characteristic of k is not 2.

Let X be a curve equipped with a degree 2 morphism $f\colon X \to \mathbb{P}^1_k$. Then the function field $K(X)$ is a degree 2 extension of $K(\mathbb{P}^1_k) = k(x)$. Let $1, y$ be a basis for $K(X)$ as a $k(x)$-vector space. The elements $1, y, y^2$ are $k(x)$-linearly dependent, so there is a relation of the form $y^2 + Ay + B = 0$ for some elements $A, B \in k(x)$. Replacing y by a suitable multiple $u(x)y$ for $u(x) \in k[x]$, we may take $A, B \in k[x]$. Because the characteristic of k is not equal to 2, we may complete the square to reduce to an equation $y^2 = p(x)$ with $p(x) \in k[x]$. Finally, by performing variable changes of the form $y = (x-a)\cdot v$, we may assume that $p(x)$ has no repeated roots. This shows that X has the same function field as the regular curve defined by $y^2 = p(x)$ in $\mathbb{A}^2_k$. Hence X is birational, and therefore isomorphic, to one of the hyperelliptic curves of Section 5.4.

Let X be a hyperelliptic curve defined by the affine equation

$$y^2 = p(x), \qquad p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and $n = 2g+1$ or $n = 2g+2$.

As we will show in Theorem 23.27, a hyperelliptic curve of genus $\geqslant 2$ cannot be embedded in $\mathbb{P}^2_k$. However, there is a useful projective model in the weighted projective plane $\mathbb{P}(1, 1, g+1)$ with variables X_0, X_1, Y of degrees 1, 1, and $g+1$. The distinguished open set $D_+(X_0)$ is isomorphic to $\mathbb{A}^2_k$ with coordinates $y = Y/X_0^{g+1}$, $x = X_1/X_0$. Therefore, the equation $y^2 = p(x)$ homogenizes to

$$Y^2 = a_n X_1^n + a_{n-1}X_1^{n-1}X_0 + \cdots + a_0 X_0^n$$

if $n = 2g+2$ is even, and

$$Y^2 = a_n X_1^n X_0 + a_{n-1}X_1^{n-1}X_0^2 + \cdots + a_0 X_0^{n+1}$$

if $n = 2g+1$ is odd. Hence X embeds as a divisor in $\mathbb{P}(1, 1, g+1)$.

Exercise 22.5.4 shows that hyperelliptic curves also admit projective models in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

Example 22.17 (Canonical divisors)**.** We can compute a canonical divisor of X by choosing $K_X = \operatorname{div}(\omega)$, where $\omega = y^{-1}dx$ (the appearance of the term y^{-1} will be explained below).

Consider the affine chart $U = \operatorname{Spec} k[x, y]/(y^2 - p(x))$. We claim that

$$K_X|_U = 0. \tag{22.5}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Indeed, if $P = (a, b) \in U$, where $b \neq 0$, then $x - a$ is a local parameter because $\frac{\partial}{\partial y}(y^2 - p(x)) = 2y$ is nonzero at P. Moreover,

$$\omega = y^{-1}dx = y^{-1}d(x - a).$$

so $\operatorname{ord}_P(\omega) = 0$ since y is a unit in $\mathcal{O}_{X,P}$.

Let $\lambda_1, \dots, \lambda_n$ be the roots of $p(x)$. We assume that X is smooth, so the λ_i are distinct. If $P = (\lambda_i, 0)$, then y is a local parameter, as $y^2 = (x - \lambda_i) \cdot (\text{unit})$ in $\mathcal{O}_{X,P}$. Here

$$\frac{dx}{y} = \frac{2\,dy}{p'(x)}.$$

As p has no repeated roots, $p'(x)$ is a unit in $\mathcal{O}_{X,P}$, so again $\operatorname{ord}_P(\omega) = 0$. This means that $K_X|_U = 0$.

The remaining points lie in the closed set $V(u)$ in $\operatorname{Spec} k[u, v]/(v^2 - u^{2g+2}p(u^{-1}))$. If n is even, then there are two points $P_1 = (0, 1)$ and $P_2 = (0, -1)$. For these points, u defines a local parameter and

$$\omega = \frac{dx}{y} = \frac{2\,dy}{p'(x)} = -\frac{u^{g-1}\,du}{v}.$$

As v is invertible at P_1 and P_2, we obtain

$$\operatorname{div}(\omega) = (g - 1)P_1 + (g - 1)P_2. \tag{22.6}$$

If n is odd, then $V(u)$ consists of a single point $P = (0, 0)$, and $v^2 = u\cdot(\text{unit})$ in $\mathcal{O}_{X,P}$. Therefore, v is a local parameter at P and $\operatorname{ord}_P(u) = 2$. Moreover, $2v\,dv = du\cdot(\text{unit})$, and from (22.6), we conclude

$$\operatorname{div}(\omega) = (g - 1)\operatorname{ord}_P(u)P = (2g - 2)P.$$

△

Proposition 22.18. Let X be a hyperelliptic curve of genus g defined by the affine equation $y^2 = p(x)$ where p has degree $2g + 1$ or $2g + 2$. Then

$$H^0(X, \Omega_X) = k\frac{dx}{y} \oplus k\frac{x\,dx}{y} \oplus \cdots \oplus k\frac{x^{g-1}\,dx}{y}$$

Let us consider some examples of divisors on these hyperelliptic curves. For simplicity, assume that $n = 2g + 1$ is odd, and X is defined by the affine equation

$$y^2 = p(x) = x^{2g+1} + x$$

That is, X is glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x, y]}{(-y^2 + x^{2g+1} + x)} \text{ and } B = \frac{k[u, v]}{(-v^2 + u + u^{2g+1})}$$

As before, we glue $D(x)$ to $D(u)$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

We compute the groups $\Gamma(X, \mathcal{O}_X(mP))$, where P is the 'point at infinity' of X, that is, the point corresponding to the maximal ideal $\mathfrak{m} = (u, v)$ in V. In other words, we look for

rational functions $f \in K(X)$ which are regular on $X - P = U$ and have a pole at P of order at most m.

To control the poles at P, we first compute the orders of vanishing of x, y, u, v at P. In the local ring $\mathcal{O}_{X,P} = B_{\mathfrak{m}}$, the defining relation allows us to express u as

$$u = v^2(1 + u^{2g})^{-1} = v^2(\text{unit}), \tag{22.7}$$

so v is a local parameter at P. Consequently, since $y = x^{g+1}v$,

$$\begin{gathered} \operatorname{ord}_P(v) = 1, \quad \operatorname{ord}_P(u) = 2 \quad \operatorname{ord}_P(x) = \operatorname{ord}_P(u^{-1}) = -2 \\ \operatorname{ord}_P(y) = \operatorname{ord}_P(x^{g+1}v) = (g+1)(-2) + 1 = -2g - 1. \end{gathered} \tag{22.8}$$

Identify $K(X)$ with the fraction field of A, so that f is expressed in terms of x, y. Note $P \notin U$, and in fact $X - P = U$, since $V(u) = V(u, v) = \{P\}$. As f is regular on $U = \operatorname{Spec} A$, this forces f to be a *polynomial* in x and y.

Viewing A as a $k[x]$-module, we may write $A = k[x] \oplus k[x]y$ and express f as $a(x) + b(x)y$ where $a(x), b(x) \in k[x]$. By (22.8), we have

$$\begin{aligned} \operatorname{ord}_P(a(x)) &= \operatorname{ord}_P(a(u^{-1})) = -2 \deg a \\ \operatorname{ord}_P(b(x)y) &= \operatorname{ord}_P(b(u^{-1})y) = -\deg b - 2g - 1. \end{aligned}$$

As $g \geqslant 1$, any nonconstant rational function with a pole at P satisfies $\operatorname{ord}_P(f) \leqslant -2$. Hence $\Gamma(X, \mathcal{O}_X(P)) = k$ consists only of constants.

For the divisor $2P$, we obtain a new section, namely x. As $-\deg b - 2g - 1 \leqslant -3$, additional sections with $b(x) \neq 0$ do not appear, and

$$\Gamma(X, \mathcal{O}_X(2P)) = k \oplus k\,x.$$

The invertible sheaf $L = \mathcal{O}_X(2P)$ is in fact globally generated by these two sections: the section '1' generates L over the open set $X - P$, and x generates L in a neighborhood of P. This is of course as expected: the morphism defined by $1, x$ is exactly the double cover morphism $X \to \mathbb{P}^1_k$.

For $D = 3P$, we allow poles of order 3 at P. We have $\operatorname{ord}_P(a(x) + b(x)y) \leqslant -4$ unless $g = 1$ and b is constant. This means that $\Gamma(X, \mathcal{O}_X(3p)) = k \oplus kx \oplus ky$ if $g = 1$, and $\Gamma(X, \mathcal{O}_X(3p)) = k \oplus kx$ for $g \geqslant 2$.

We treat the cases $g = 1$ and $g = 2$ in more detail.

Example 22.19 ($g = 1$)**.** When $g = 1$, the sections $s_0 = 1, s_1 = x, s_2 = y$ define a morphism $\phi\colon X \to \mathbb{P}^2_k$. which is a closed embedding. To see this, let t_0, t_1, t_2 be homogeneous coordinates on $\mathbb{P}^2_k$, so that $\phi^* t_i = s_i$ for $i = 0, 1, 2$. Over $U = D(s_0) = \phi^{-1}D_+(t_0)$, the morphism ϕ is given by

$$\phi|_U\colon U \longrightarrow D_+(t_0) \simeq \mathbb{A}^2_k, \qquad (x, y) \mapsto (1 : x : y) \tag{22.9}$$

which is identified with the morphism

$$\operatorname{Spec} k[x, y]/(-y^2 + x^3 + x) \longrightarrow \operatorname{Spec} k[x, y].$$

In particular, ϕ is a closed embedding over $\phi^{-1}D_+(t_0)$. Over $V = D(x) = \phi^{-1}D_+(t_1)$, rewrite the sections in terms of u and v:

$$(1 : x : y) = (x^{-1} : 1 : x^{-1}y) = (u : 1 : v),$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

so the morphism takes the form

$$\phi|_V \colon V \longrightarrow D(t_1) \subset \mathbb{P}^2_k, \qquad (u,v) \mapsto (u:1:v). \tag{22.10}$$

This shows that ϕ is also a closed embedding over $\phi^{-1}D_+(t_1)$. As these two open sets cover the image $\phi(X) \subset \mathbb{P}^2_k$, the morphism ϕ is a closed embedding. The relation $y^2 = x^3 + x$ gives the following defining equation for X in $\mathbb{P}^2_k$:

$$t_2^2 t_0 = t_1^3 + t_0^2 t_1.$$

That is, X embeds as a plane cubic curve. △

Example 22.20 ($g = 2$)**.** When $g = 2$, the divisor $3P$ does not give a projective embedding, as there are only the two sections 1 and x. For the divisor $4P$, we have three sections $1, x, x^2$ generating $\Gamma(X, \mathcal{O}_X(4P))$. Geometrically, the induced morphism $X \to \mathbb{P}^2_k$ maps onto the degree 2 curve $t_1^2 = t_0 t_2$, hence is not a closed embedding (degree 2 curves have genus 0). More directly, $X \to \mathbb{P}^2_k$ factors as $X \to \mathbb{P}^1_k \to \mathbb{P}^2_k$, where the first map is the double cover and $\mathbb{P}^1_k \to \mathbb{P}^2_k$ is the embedding of $\mathbb{P}^1_k$ as a conic.

For the divisor $5P$, we obtain

$$\Gamma(X, \mathcal{O}_X(5P)) = k \oplus k\,x \oplus k\,x^2 \oplus k\,y.$$

The sections $s_0 = 1, s_1 = x, s_2 = x^2$, and $s_3 = y$ globally generate $\mathcal{O}_X(5P)$, and define a morphism $\phi \colon X \longrightarrow \mathbb{P}^3_k$. Over the open set $U = D(s_0)$, ϕ is given by

$$\phi|_U \colon U \longrightarrow \mathbb{A}^3_k, \qquad (x,y) \mapsto (1 : x : x^2 : y)$$

Over $V = D(s_1)$, we rewrite the sections in terms of u, v as $1, u^{-1}, u^{-2}, u^{-3}v$, and after multiplying by the unit u^3,

$$\phi|_V \colon V \longrightarrow \mathbb{A}^3_k, \qquad (u,v) \mapsto (u^3 : u^2 : u : v).$$

This shows that ϕ is a closed embedding.

From the defining sections, we see that $s_1^2 - s_0 s_2 = 0$, so the image of X lies on the quadric surface $t_1^2 - t_0 t_2 = 0$. There are also further relations among the sections coming from the relation $y^2 = x^5 + x$, for example $t_3^2 t_0 - t_1 t_2^2 - t_1 t_0^2 = 0$. △

Exercise 22.4.1. Show that a regular projective curve is hyperelliptic if and only if it admits a divisor D of degree 2 with $h^0(D) = 2$.

22.5 Exercises

Exercise 22.5.1. Let k be an algebraically closed field. Show that any two curves over k are homeomorphic.

Exercise 22.5.2. Let $f \colon X \to Y$ be a birational morphism between curves, where Y is regular. Then $f(X)$ is open in Y, and f induces an isomorphism $X \simeq f(X)$. HINT: The complement of $f(X)$ is a finite set of points. For $x \in X$, show that $\mathcal{O}_{Y,f(x)}$ contains $\mathcal{O}_{X,x}$ inside $K(X)$ and deduce that $\mathcal{O}_{Y,f(x)} = \mathcal{O}_{X,x}$ by the properties of discrete valuation rings.

Exercise 22.5.3. Let $f \colon X \to Y$ be a morphism of curves over an algebraically closed field k.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

a) Show that the pushforward of a torsion coherent sheaf on X is a torsion coherent sheaf on Y.
b) Let X be a curve. Show that a coherent sheaf $\mathcal{F}$ on X is torsion if and only if it is supported on a finite (hence proper closed) subset of X.
c) Let X be a curve and $\mathcal{F}$ a torsion sheaf on X. Show that $H^0(X, \mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$.

Exercise 22.5.4. Let $X \subset \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ be a smooth divisor of bidegree $(g+1, 2)$.
a) Show that the first projection $X \to \mathbb{P}^1_k$ is finite of degree 2.
b) Show that $h^1(X, \mathcal{O}_X) = g$. HINT: Use an exact sequence and Exercise **??**.
c) Show that conversely any hyperelliptic curve embeds into $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. HINT: Write $y^2 = p(x)$ with $p(x) = a(x)b(x)$ with $\deg a, \deg b \leqslant g+1$ and set $v = y/a(x)$.

Exercise 22.5.5. Show that the integers appearing in Theorem 22.3 are unique. HINT: Consider the global sections of a suitable twist $\mathcal{E}(m)$.

Exercise 22.5.6 (Weak Riemann–Roch)**.** Let C be an integral projective curve over an algebraically closed field k, and let L be a *very ample* invertible sheaf on C. Show that for every integer $m \geqslant \deg L$ we have

$$h^0(C, L^{\otimes m}) \;\geqslant\; m \cdot \deg L + 1 - \binom{\deg L - 1}{2}.$$

HINT: Choose a finite birational projection $\pi\colon C \to C' \subset \mathbb{P}^2_k$ such that $L = \pi^*\mathcal{O}_{C'}(1)$. Use the inclusion $\mathcal{O}_{C'} \hookrightarrow \pi_*\mathcal{O}_C$ to compare $H^0(C, L^{\otimes m})$ and $H^0(C', \mathcal{O}_{C'}(m))$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

23

The Riemann–Roch theorem and applications

A fundamental problem in algebraic geometry is how to compute the spaces of global sections $H^0(X, \mathcal{O}_X(D))$ for some divisor D. The sections of $\mathcal{O}_X(D)$ allow us to define morphisms and rational maps to projective space $\mathbb{P}^n_k$, and in good cases, we even get a projective embedding. Then later, we we can ask about geometric information coming from the embedding, for instance the degree or the equations defining the image.

In this chapter we will introduce the most powerful tool to attack these questions, the Riemann–Roch theorem. This has many important consequences for the classification of algebraic curves.

23.1 The Riemann–Roch formula

In this chapter, X will denote a smooth projective variety of dimension 1 over an algebraically closed field k. When we say 'point', we will always mean a closed point, i.e., a k-point.

Since X is a curve, the interesting cohomology groups of a coherent sheaf are $H^0(X, \mathcal{F})$ and $H^1(X, \mathcal{F})$; the other cohomology groups are zero by Theorem 20.20. We will mostly be interested in the case when $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor D. In this case, we will occasionally write $h^i(X, D)$ or $h^i(D)$ for $h^i(X, \mathcal{O}_X(D))$ when the meaning is clear from context.

Our most basic tool for studying the cohomology groups $H^0(X, \mathcal{O}_X(D))$ is the ideal sheaf sequence of a point $P \in X$, which takes the form

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \kappa(P) \longrightarrow 0 \tag{23.1}$$

The first map is the inclusion and the second is restriction (evaluation) to P. Here we have identified the ideal sheaf $\mathfrak{m}_P \subset \mathcal{O}_X$ by the invertible sheaf $\mathcal{O}_X(-P)$, and the sheaf $i_*\mathcal{O}_P$ with the skyscraper sheaf with value $\kappa(P)$ at P.

If L is an invertible sheaf, we can tensor (23.1) by L and get the exact sequence

$$0 \longrightarrow L(-P) \longrightarrow L \longrightarrow \kappa(P) \longrightarrow 0, \tag{23.2}$$

where $L(-P)$ is the subsheaf of sections of L vanishing at P, and we also identify $L \otimes \kappa(P) \simeq \kappa(P)$. In particular, taking $L = \mathcal{O}_X(D + P)$ in (23.2) we get

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + P) \longrightarrow \kappa(P) \longrightarrow 0 \tag{23.3}$$

The long exact sequence in cohomology takes the form

$$0 \to H^0(X, D) \to H^0(X, D + P) \xrightarrow{\mathrm{ev}_P} k \longrightarrow H^1(X, D) \to H^1(X, D + P) \to 0. \tag{23.4}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This means that when replacing D by $D + P$, exactly one of the following occurs:

(A) $h^0(X, D + P) = h^0(X, D) + 1$ and $h^1(X, D + P) = h^1(X, D)$.
This happens if and only if some section of $\mathcal{O}_X(D + P)$ does not vanish at P.

(B) $h^0(X, D + P) = h^0(X, D)$ and $h^1(X, D + P) = h^1(X, D) + 1$.
This happens if and only if every section of $\mathcal{O}_X(D + P)$ vanishes at P.

This leads to the following basic bound:

Lemma 23.1. Let D be a divisor on X with $\deg D \geqslant 0$, then

$$h^0(X, D) \leqslant \deg D + 1.$$

Proof This is clear if $h^0(X, D) = 0$. If $h^0(X, D) > 0$, we may assume that $D = \sum n_P P$ is effective. Then the inequality follows by applying the two points (A) and (B) $\deg D$ times. □

Recall that the Euler characteristic $\chi(\mathcal{F})$ is defined as the alternating sum of the dimensions $h^i(X, \mathcal{F})$. Applying χ to (23.3), and using that χ is additive on exact sequences, we get

$$\chi(\mathcal{O}_X(D + P)) = \chi(\mathcal{O}_X(D)) + \chi(\kappa(P)) = \chi(\mathcal{O}_X(D)) + 1. \tag{23.5}$$

The next result is a very useful formula to compute $\chi(\mathcal{O}_X(D))$ for any divisor D:

Theorem 23.2 (Easy Riemann–Roch). Let X be a smooth projective curve of genus g and let D be a divisor on X. Then

$$h^0(\mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g \tag{23.6}$$

Proof Let $P \in X$ be a point. By (23.5), the left-hand side of the formula (23.6) increases by 1 if we replace D by $D + P$. On the other hand, also the right-hand side of the equation above increases by 1 by adding P to D, because $\deg(D + P) = \deg D + 1$. This means that the theorem holds for a divisor D if and only if it holds for $D + P$ for any closed point P. So by adding and subtracting points, we can reduce to the case when $D = 0$. But in that case the left-hand side of the formula is simply

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g$$

which equals the right-hand side, since g is defined to be $h^1(X, \mathcal{O}_X)$. □

As a first corollary, we prove that the divisor of a rational function always has degree 0. This explains the observations that the 'number of zeros' and 'number of poles' matched up in the examples in Chapter 22.

Corollary 23.3. If $f \in K(X) - 0$, then $\deg(\operatorname{div}(f)) = 0$.

Proof By Proposition 19.28, $\mathcal{O}_X(\deg f) \simeq \mathcal{O}_X$, so applying (23.6) to $D = \operatorname{div} f$, we find

$$1 - g = \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(\deg f)) = \deg f + 1 - g,$$

and so $\deg(\operatorname{div}(f)) = 0$. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The number we are really after is the number $h^0(X, D)$, the dimension of global sections of $\mathcal{O}_X(D)$. By the formula (23.6), the quantity $h^0(X, D) - h^1(X, D)$ is easy to compute, as $\deg D + 1 - g$ involves only the degree of D and the genus. This means that if we, for some reason, could argue that $H^1(X, D) = 0$, we would know that $h^0(X, D) = \deg D + 1 - g$.

Even without the vanishing of H^1, we always have $h^1(X, D) \geqslant 0$, so this leads to the following *lower bound* on $h^0(X, \mathcal{O}_X(D))$, which is often enough in applications.

Corollary 23.4. $h^0(X, \mathcal{O}_X(D)) \geqslant \deg D + 1 - g$.

Example 23.5. That the Riemann–Roch formula holds for $X = \mathbb{P}^1_k$ follows from the computations in Chapter 20. Indeed, as $\mathrm{Cl}(\mathbb{P}^1_k) = \mathbb{Z}$, it suffices to check that the formula holds for divisors of the form $D = dP$ where $P \in \mathbb{P}^1_k$ is a point and $d \in \mathbb{Z}$. In this case, the right-hand-side of the fomula equals $\deg D + 1 - 0 = d + 1$.

If $d \geqslant 0$, $H^0(\mathbb{P}^1_k, \mathcal{O}_X(dP))$ is identified with the space of homogenous degree d polynomials in x_0, x_1, while $H^1(\mathbb{P}^1_k, \mathcal{O}_X(dP)) = 0$. Hence $h^0(X, D) = d + 1$. If $d < 0$, we have $h^0(X, D) = 0$ and $h^1(X, D) = -d - 1$. △

Example 23.6. Consider again the case where X is a hyperelliptic curve of genus 2, as in Example 22.20. The computations in that example leads to the following table of the various cohomology groups $H^i(X, \mathcal{O}_X(nP))$ for the point $P = (u, v)$:

D	0	$1P$	$2P$	$3P$	$4P$	$5P$	$6P$	$7P$
$H^0(X, D)$	1	1	2	2	3	4	5	6
$H^1(X, D)$	2	1	1	0	0	0	0	0
$\chi(\mathcal{O}_X(D))$	-1	0	1	2	3	4	5	6

Note again that adding P either increases H^0 or decreases H^1 by 1. △

As the example shows, the cohomology groups $H^1(X, \mathcal{O}_X(D))$ vanish provided that the degree $\deg D$ is large enough. This is a general fact for all divisors on all projective curves, and it will be a consequence of the following theorem, known as *Serre duality*. Although we will only prove this when X is a curve, we include for reference a more general statement which holds in any dimension:

Theorem 23.7 (Serre duality). Let X be a smooth projective variety of dimension n over a field k. Then there is a canonical isomorphism $H^n(X, \Omega^n_X) \simeq k$, and for any locally free sheaf $\mathcal{E}$ on X and each $i = 0, 1, \ldots, n$ functorial perfect pairings

$$H^i(X, \mathcal{E}) \times H^{n-i}(X, \mathcal{E}^\vee \otimes \Omega^n_X) \longrightarrow H^n(X, \Omega^n_X) \simeq k \tag{23.7}$$

The theorem allows us to view the space of global sections $H^0(X, \mathcal{E})$ as the dual of the top cohomology group $H^n(X, \mathcal{E}^\vee \otimes \Omega^n_X)$ via a certain k-linear pairing. The important consequence of this is that $H^0(X, \mathcal{E})$ and $H^n(X, \mathcal{E}^\vee \otimes \Omega^n_X)$ have the same dimension.

When X is a curve, applying Serre duality to $\mathcal{E} = \mathcal{O}_X$ has the following consequence:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proposition 23.8. Let X be a smooth curve over an algebraically closed field k. Then
$$h^0(X, \Omega_X) = h^1(X, \mathcal{O}_X).$$

In other words, the genus of a curve can be interpreted either as the dimension of the first cohomology group $H^1(X, \mathcal{O}_X)$ or the space of global differential forms $H^0(X, \Omega_X)$.

By Serre duality, we may upgrade the Riemann–Roch formula (23.6) as follows. Choose a canonical divisor K_X so that $\Omega_X \simeq \mathcal{O}_X(K_X)$. Applying Serre duality to $\mathcal{E} = \mathcal{O}_X(K_X - D)$, we get an equality
$$h^1(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(K_X - D)),$$
and Theorem 23.2 takes the following form:

Theorem 23.9 (Riemann–Roch). Let X be a smooth projective curve of genus g and let D be a divisor on X. Then
$$h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - g.$$

This result is significantly more powerful than the Riemann–Roch formula we saw earlier. It gives stronger conclusions, because the group $H^0(X, \mathcal{O}_X(K_X - D))$ is easier to interpret: it is the space of global sections of the sheaf associated to the divisor $K_X - D$. Often, we can readily argue that no such global sections exist. For instance, in the case $\deg D > \dim K_X$ then $K_X - D$ cannot be effective, as effective divisors have non-negative degree.

In fact, we can use the Riemann–Roch formula to compute the degree of K_X:

Corollary 23.10. Let X be a smooth projective curve of genus g. Then
$$\deg K_X = 2g - 2.$$

Proof Applying the Riemann–Roch formula to $D = K_X$, we obtain
$$g - 1 = \dim_k H^0(X, \mathcal{O}_X(K_X)) - \dim_k H^0(X, \mathcal{O}_X(K_X - K_X)) = \deg K + 1 - g.$$
Solving for $\deg K$, we obtain $\deg K_X = 2g - 2$. □

Corollary 23.11. Let D be a divisor of degree $\geqslant 2g - 1$. Then
(i) $H^1(X, \mathcal{O}_X(D)) = 0$.
(ii) $h^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g$.
If $\deg D = 2g - 2$, then $H^1(X, \mathcal{O}_X(D)) \neq 0$ if and only if $D \sim K_X$.

Proof The first two items follow from the discussion above.

For the last part, note that $h^0(X, \mathcal{O}_X(K_X - D)) \neq 0$ if and only if $K_X - D$ is linearly equivalent to an effective divisor. The only effective divisor of degree 0 is the divisor 0, so this happens if and only if K_X and D are linearly equivalent. □

Therefore, in our path to computing $h^0(X, \mathcal{O}_X(D))$ for a divisor D, we are left with the 'intermediate cases' where D has degree between 0 and $2g - 2$. In this region, the behaviour of $H^0(X, \mathcal{O}_X(D))$ becomes more subtle, and it does not only depend on $\deg D$ and g.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\deg D$	< 0	0	1	$\dots$	$2g-2$	$\geqslant 2g-1$
$h^0(X,D)$	0	?	?	$\dots$	?	$\deg D + 1 - g$
$h^1(X,D)$	$-(\deg D + 1 - g)$	?	?	$\dots$	?	0

23.2 Base-point freeness and very ampleness on curves

Let X be a regular projective curve over an algebraically closed field k.

Definition 23.12. Let D be a divisor on X.

(i) We say that a point $P \in X$ is a *base point* of D, or a base point of $H^0(X, \mathcal{O}_X(D))$, if every section of $H^0(X, \mathcal{O}_X(D))$ vanishes at P.

(ii) If D has no base points, we say that D is *base-point free*. This is equialent to saying that the invertible sheaf $\mathcal{O}_X(D)$ is globally generated.

(iii) D is said to be *very ample* if the morphism induced by the sections of $H^0(X, \mathcal{O}_X(D))$ is a closed embedding $X \to \mathbb{P}^n_k$. D is *ample* if some positive multiple mD is very ample.

We say that $\mathcal{O}_X(D)$ is ample (resp. very ample) if D is.

In light of the isomorphism $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathfrak{m}_P \simeq \mathcal{O}_X(D-P)$, we see that P is a base-point if and only if the inclusion $H^0(X, \mathcal{O}_X(D-P)) \subset H^0(X, \mathcal{O}_X(D))$ is an isomorphism. In general, the dimension of $H^0(X, \mathcal{O}_X(D-P))$ can be at most 1 less than $H^0(X, \mathcal{O}_X(D))$, by the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D-P)) \longrightarrow H^0(X, \mathcal{O}_X(D)) \xrightarrow{\mathrm{ev}_P} \kappa(P) \tag{23.8}$$

Therefore, D is base-point free if and only if for every point $P \in X$,

$$h^0(D-P) = h^0(D) - 1.$$

If this condition is satisfied, we obtain a morphism $f\colon X \to \mathbb{P}^n_k$, by Theorem 12.20.

We now turn to the question whether f is a closed embedding.

Theorem 23.13 (Base-point free and very ample divisors on curves). For a divisor D,

(i) D is base point free if and only if

$$h^0(D-P) = h^0(D) - 1 \qquad \text{for every point } P \in X(k).$$

(ii) D is very ample if and only if

$$h^0(D-P-Q) = h^0(D) - 2 \qquad \text{for every two points } P, Q \in X(k)$$

(including the case $P = Q$)

Proof The first item (i) follows from the above discussion.

For (ii), we use the two conditions in Proposition 21.14. That is, f should be injective on k-points and the pullback map on cotangent spaces should be surjective.

Let us choose a basis $s_0, \dots, s_n$ for $H^0(X, \mathcal{O}_X(D))$. Then over the open set $D(s_0)$, f is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

induced by the ring map

$$k\Big[\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\Big] \longrightarrow \Gamma(D(s_0),\mathcal{O}_X) \tag{23.9}$$

sending x_i/x_0 to s_i/s_0. Here we recall that s_i/s_0 is viewed as regular function on $D(s_0)$ as in Section 12.6.

To show that f is injective, we want to show that for any two points $P,Q \in X(k)$, we have $f(P) \neq f(Q)$. This happens if we are able to find a section $s = \sum a_i s_i$ such that $s(P) = 0$ and $s(Q) \neq 0$. Because in that case, $f(P)$ lies in the hyperplane $V(\sum a_i x_i) \subset \mathbb{P}^n_k$, whereas $f(Q)$ does not.

Therefore, we want to show that there is a section s which lies in $H^0(X,\mathcal{O}_X(D-P))$, but not in $H^0(X,\mathcal{O}_X(D-P-Q))$. It is therefore enough to show that for any P and Q, we have a strict inequality $h^0(D-P-Q) < h^0(D-P)$. As $h^0(D-P) = h^0(D)-1$, by the above, this is equivalent to the condition

$$h^0(D-P-Q) = h^0(D) - 2 \quad \text{for every pair } P,Q \in X. \tag{23.10}$$

Next we would like to know whether the pullback map on cotangent spaces

$$f^*\colon \mathfrak{m}_{f(P)}/\mathfrak{m}^2_{f(P)} \longrightarrow \mathfrak{m}_P/\mathfrak{m}^2_P$$

is surjective. As X is a regular curve, $\mathfrak{m}_P/\mathfrak{m}^2_P$ is 1-dimensional vector space, so we just need f^* to be nonzero.

Let us for simplicitly assume that P maps to the point $(1:0:\cdots:0) \in \mathbb{P}^n_k$ (we may acheive this by a linear change of coordinates). This means that the morphism is given locally by $D(s_0) \to D(x_0)$ as in (23.9). More precisely, if $y_i = x_i/x_0$ $i = 1,\dots,n$ denote the affine coordinates on $D(x_0)$, we have $f^\sharp(y_i) = s_i/s_0 \in \Gamma(D(s_0),\mathcal{O}_X)$. In these coordinates, $\mathfrak{m}_{f(P)}/\mathfrak{m}^2_{f(P)}$ is the k-vector space generated by the classes $\overline{y_1},\dots,\overline{y_n}$, and f^* maps $\overline{y_i}$ to $\overline{s_i/s_0}$ in $\mathfrak{m}_P/\mathfrak{m}^2_P$. It is therefore enough to show that some linear combination of the y_i pulls back to an element in $\mathfrak{m}_P$ which vanishes to order 1 at P. Note that a linear combination $\sum a_i s_i/s_0$ vanishes to order 1 at P if and only if $\sum a_i s_i$ vanishes to order 1 at P, that is, lies in $H^0(X,\mathcal{O}_X(D-P))$ but not in $H^0(X,\mathcal{O}_X(D-2P))$.

Therefore, to show that such linear forms exist, we need to show that $H^0(X,\mathcal{O}_X(D-2P))$ is strictly smaller than $H^0(X,\mathcal{O}_X(D-P))$. As above, the dimension can drop by at most 1, so this happens if and only if

$$h^0(D-2P) = h^0(D-P) - 1 = h^0(D) - 2. \qquad \square$$

Corollary 23.14.
(i) If $\deg D \geqslant 2g$, then D is base point free and $h^1(X,D) = 0$.
(ii) If $\deg D \geqslant 2g+1$, then D is very ample.

Proof When $\deg D \geqslant 2g$, then $h^1(X,D-P) = h^1(X,D) = 0$, by Corollary 23.11, so

$$h^0(D-P) = \chi(\mathcal{O}_X(D-P)) = \chi(\mathcal{O}_X(D)) - 1 = h^0(D) - 1.$$

Similarly, when $\deg D \geqslant 2g+1$, we get $h^1(X,D-P-Q) = 0$ for all $P,Q \in X$, which implies $h^0(D-P-Q) = h^0(D) - 2$. $\square$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

In particular, the theorem implies that a divisor D on a curve is ample if and only if $\deg D > 0$.

23.3 Curves of genus 0

The results of the previous sections are particularly strong when the genus is small. In the case X is a curve of genus 0, a divisor D is base-point free if and only if $\deg D \geqslant 0$ and very ample if and only if $\deg D > 0$. If we apply this observation to a divisor of the form $D = P$ for a k-point P, we see that $h^0(D) = 1 + 1 - 0 = 2$ and D defines a closed embedding $X \to \mathbb{P}^1_k$. Therefore, we have proved the following remarkable theorem:

Theorem 23.15. Let X be a regular curve of genus 0 over an algebraically closed field k. Then X is isomorphic to $\mathbb{P}^1_k$.

Example 23.16. Consider the plane curve $X \subset \mathbb{P}^2_k$ defined by the homogeneous equation

$$x_0^2 + x_1^2 - x_2^2 = 0.$$

Then X as a k-point $P = (1 : 0 : 1)$. Let us find a basis for $H^0(X, \mathcal{O}_X(P))$.

We are looking for rational functions g which have a pole of order 1 at P and no poles at any other point. The linear form $x_0 - x_2$ vanishes at P, but it has multiplicity 2 there. To make up for this, we can consider a quotient $\ell/(x_0 - x_2)$ where ℓ is a linear form in x_0, x_1, x_2 vanishing with order 1 at P. For instance, we can take $\ell = x_1$. And indeed, the quotient

$$s = \frac{x_1}{x_0 - x_1}$$

satisfies

$$\operatorname{div}(s) = (1 : 0 : 1) + (1 : 0 : -1) - 2(1 : 0 : 1) = (1 : 0 : -1) - (1 : 0 : 1).$$

and so $s \in H^0(X, \mathcal{O}_X(P))$. As P has degree 1, we see that $H^0(X, \mathcal{O}_X(P)) = k \oplus k\, s$.

The morphism defined by the sections $1, s$ is

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1_k \\ (x_0 : x_1 : x_2) &\longmapsto (x_1 : x_0 - x_2) \end{aligned}$$

can be viewed as the projection from P. △

Remark 23.17 (Curves of genus 0 over general fields). For non-algebraically closed fields k, there exist curves of genus 0 which are not isomorphic to $\mathbb{P}^1_k$. For instance, the conic curve $X = V_+(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_{\mathbb{R}}$ has genus 0, but it has no $\mathbb{R}$-points at all. Similarly, the conic $X = V_+(x_0^2 + x_1^2 - 3x_2^2)$ has infinitely many $\mathbb{R}$-points, but no $\mathbb{Q}$-points. In these examples, X cannot be isomorphic to $\mathbb{P}^1_k$, but the base change to the algebraic closure $X_{\bar{k}} = X \times_k \overline{k}$ becomes isomorphic to $\mathbb{P}^1_{\overline{k}}$.

That being said, curves of genus 0 over arbitrary fields are well understood. Here is a brief explaination. If we assume that $H^0(X, \mathcal{O}_X) = k$ and $H^1(X, \mathcal{O}_X) = 0$ and $X(k) \neq \varnothing$, then $X \simeq \mathbb{P}^1_{\mathbb{Q}}$. Indeed, if X admits a k-point P, then one shows that the invertible sheaf $\mathcal{O}_X(P)$ is very ample by pulling back to $X_{\bar{k}}$, where one can apply Theorem 23.15. Therefore,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

we again find that $X \simeq \mathbb{P}^1_k$. A nice and surprising fact, which holds for any field k, is that any regular projective curve of genus 0 is isomorphic to plane curve of degree 2, $X = V_+(Q) \subset \mathbb{P}^2_k$. To prove this, one considers the anticanonical divisor $D = -K_X$ (which is always defined over k, regardless of whether X has a k-point or not). By working on $X_{\bar{k}}$, one shows that this is very ample, and embeds X as a degree 2 curve in $\mathbb{P}^2_k$.

We conclude by another characterisation of $\mathbb{P}^1$, which will be important for studying curves of higher genus as well.

Proposition 23.18. Let X be a smooth projective curve. Then:
(i) X has a point P with $h^0(X, P) = 2$ if and only if $X \simeq \mathbb{P}^1_k$.
(ii) For any divisor D of positive degree,

$$h^0(X, D) \leqslant \deg D + 1$$

with equality if and only if $X \simeq \mathbb{P}^1_k$.

Proof (i): Suppose $h^0(X, \mathcal{O}_X(P)) = 2$. If every global section of $\mathcal{O}_X(P)$ vanished at P then $h^0(X, \mathcal{O}_X(P))$ would be 1, so there exists a section not vanishing at P. Hence the divisor P is base-point free and defines a morphism $f\colon X \longrightarrow \mathbb{P}^1_k$, with $f^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_X(P)$. Taking degrees shows that $\deg f = 1$, so f is a finite birational morphism between smooth projective curves and hence an isomorphism. Therefore $X \simeq \mathbb{P}^1_k$.

Conversely, if $X = \mathbb{P}^1_k$, then any divisor is linearly equivalent to dP for some point $P \in X$ and $\mathcal{O}_X(dP) \simeq \mathcal{O}_{\mathbb{P}^1_k}(d)$. Hence any point $P \in \mathbb{P}^1_k$ satisfies $h^0(\mathbb{P}^1_k, P) = 2$. Moreover, we see that the inequality in (ii) is an equality for $X = \mathbb{P}^1_k$.

(ii): The inequality was proved in Lemma 23.1, so the only novel part is about the equality.

Suppose $h^0(X, \mathcal{O}_X(D)) = d + 1$ for some D of degree $d \geqslant 1$. Since $h^0(X, D) > 0$, we may assume that $D = P_1 + \cdots + P_d$ for some (possibly equal) points $P_1, \ldots, P_d \in X$. Then in the sequence

$$h^0(X, \mathcal{O}_X),\ h^0(X, P_1),\ h^0(X, P_1 + P_2), \ldots, h^0(X, D).$$

each step must increase the dimension by exactly 1. In particular $h^0(X, \mathcal{O}_X(P_1)) = 2$, so by part (i) we get $X \simeq \mathbb{P}^1_k$. □

Corollary 23.19. A smooth projective curve X over an algebraically closed field is isomorphic to $\mathbb{P}^1_k$ if and only if $\mathrm{Cl}(X) \simeq \mathbb{Z}$.

Proof By Theorem **??**, the Picard group of any projective space is isomorphic to $\mathbb{Z}$. Conversely, suppose X is a curve with $\mathrm{Cl}(X) \simeq \mathbb{Z}$. This implies that the degree map $\deg\colon \mathrm{Cl}(X) \to \mathbb{Z}$ is an isomorphism (it is clearly surjective because a point has degree 1).

Let P, Q be two distinct points on X. Because $\mathrm{Cl}(X) = \mathbb{Z}$, the two divisors P and Q are linearly equivalent, so there is a nonzero rational function $f \in K(X)$ such that $\mathrm{div}(f) = Q - P$, hence $\mathrm{div}(f) + P = Q \geqslant 0$. Therefore, $1, f$ give two 2 linearly independent global sections of $\mathcal{O}_X(P)$ and hence $X \simeq \mathbb{P}^1_k$ by Proposition 23.18. □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

23.4 Curves of genus 1

A plane curve $X \subset \mathbb{P}^2_k$ of degree 3 has genus 1, by the genus formula $g = (d-1)(d-2)/2$.

Using Riemann–Roch, we show that in fact *every* curve of genus 1 can be realized as a plane cubic:

Theorem 23.20. Any projective curve X of genus 1 over an algebraically closed field can be embedded as a plane cubic curve in $\mathbb{P}^2_k$.

Proof Pick a point $P \in X$ and consider the divisor $D = 3P$. Then D has degree $3 \geqslant 2g+1$, so it is very ample. Furthermore, by Riemann–Roch, $h^0(3P) = 3$, so there is a projective embedding $\phi : X \to \mathbb{P}^2_k$. The image $\phi(X)$ is a smooth curve of degree equal to $\deg \phi^* \mathcal{O}_{\mathbb{P}^2_k}(1) = \deg D = 3$. □

We can make this a little bit more explicit as follows. For a point $P \in X$ and consider divisors of the form mP. By Riemann–Roch, we have $h^0(X, mP) = m$ for every $m \geqslant 1$. Therefore, we may pick sections $x \in H^0(X, 2P)$ and $y \in H^0(X, 3P)$ as in the following table

	h^0	basis
$\mathcal{O}_X(P)$	1	1
$\mathcal{O}_X(2P)$	2	$1, x$
$\mathcal{O}_X(3P)$	3	$1, x, y$
$\mathcal{O}_X(4P)$	4	$1, x, y, x^2$
$\mathcal{O}_X(5P)$	5	$1, x, y, x^2, xy$
$\mathcal{O}_X(6P)$	6	$1, x, y, x^2, xy, x^3, y^2$

Now, as $H^0(X, \mathcal{O}_X(6P))$ is 6-dimensional, there must be a linear relation between the 7 sections $1, x, y, x^2, xy, x^3, y^2$. It is not hard to see that, after a coordinate change, we can take it to be the following cubic

$$y^2 + a_1 xy + a_2 y = x^3 + a_3 x^2 + a_4 x + a_5$$

where $a_1, a_2, a_3, a_4, a_5 \in k$. When the characteristic of k is not 2 or 3, the equation can be further simplified to

$$y^2 = x^3 + ax + b.$$

This is known as the *Weierstrass form* of the curve X. Any such equation defines a regular genus 1 curve, provided that $4a^3 + 27b^2 \neq 0$ (Exercise 16.5.15).

In contrast to the genus 0 case, there are many non-isomorphic genus 1 curves. For instance, among the curves $X_t = V(x_1^2 x_2 + x_0^3 + x_0 x_2^2 + t x_2^3) \subset \mathbb{P}^2_{\mathbb{C}}$, where $t \in \mathbb{C}$, a given curve X_a is isomorphic to at most a finite number of other X_t's.

The group law

One of the most important facts about genus 1 curves X is that the set of k-points $X(k)$ can be given the structure of an abelian group. That is, there is a natural way of 'adding' k-points on X. This happens only for curves of genus 1.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The key point in understanding the group structure is to consider divisors on X. Consider the subgroup $\mathrm{Cl}^0(X) \subset \mathrm{Cl}(X)$ consisting of divisor classes of degree 0, that is, the kernel of the degree map $\deg\colon \mathrm{Cl}(X) \to \mathbb{Z}$.

Fix a point $O \in X(k)$ and define the following map:

$$\begin{aligned} \xi\colon X(k) &\longrightarrow \mathrm{Cl}^0(X) \\ P &\longmapsto [P - O]. \end{aligned}$$

Lemma 23.21. The map ξ is a bijection.

Proof ξ is injective: $\xi(P) = \xi(Q)$ implies that $P \sim Q$. Since X has genus 1, $h^0(X, P) = 1$ by Riemann–Roch, which implies that $P = Q$.

ξ is surjective: Let $D = \sum n_i P_i$ be a divisor of degree 0. Then $D' = D + O$ has degree 1, so by Riemann–Roch, $H^0(X, \mathcal{O}_X(D'))$ is 1-dimensional. Hence there exists an effective divisor of degree 1 which is linearly equivalent to D'. In other words, $D' \sim Q$ for some point $Q \in X$. This means that $D + O \sim Q$, so $[D] \sim [Q - O] = \xi(Q)$ is in the image of ξ. □

Using this bijection, we can put a group structure on the set $X(k)$ by transporting the group structure of $\mathrm{Cl}^0(X)$:

Theorem 23.22. Let X be a smooth curve of genus 1 and let $O \in X(k)$ be a point. Then the set of k-points $X(k)$ on a genus 1 has the structure of an abelian group with O as the identity element.

The group law has the following famous geometric interpretation. To make the discussion a bit more concrete, we consider the curve $X \subset \mathbb{P}^2_k$ defined by the equation

$$y^2 z = x^3 + axz^2 + bz^3$$

and the 'point at infinity' $O = (0 : 1 : 0)$.

If $L \subset \mathbb{P}^2_k$ is a line, we get a divisor $L|_X$ on X, supported on the finite set $L \cap X$. More formally, we take a section $s \in \mathcal{O}_{\mathbb{P}^2_k}(1)$ defining L and restrict it to X. The restricted divisor on X consists of three points P, Q, R (counted with multiplicity). Since any two lines in $\mathbb{P}^2_k$ are linearly equivalent, the divisors coming from any two lines are linearly equivalent divisors on X. Hence if L and L' meet X in P, Q, R, and P', Q', R' respectively, then

$$P + Q + R \sim P' + Q' + R'.$$

The chosen point $O = (0 : 1 : 0)$ is an *inflection point*, in the sense that there is a line $L = V(z) \subset \mathbb{P}^2_k$ which intersects X with multiplicity three at O. Hence any three collinear points P, Q, R in X satisfy

$$P + Q + R \sim 3O.$$

Now, given two k-points $P, Q \in X$, we draw the line L they span (see Figure 23.1). This line intersects X in a third point R, which satisfies $P + Q + R = 3O$. To define the 'sum' $P + Q$ (which should be a new k-point of X), we then look for a point S so that in $\mathrm{Cl}^0(X)$

$$S - O = (P - O) + (Q - O)$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

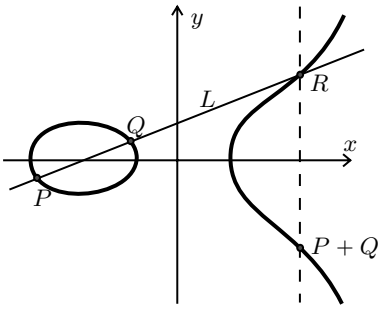


Figure 23.1 The group law on X

or equivalently $S + O \sim P + Q$. Using the above relation this becomes $S + O \sim 3O - R$, or $R + S + O = 3O$. Therefore, draw the line L' from O to R (shown as the dotted line in the figure), and define S to be the third intersection point of L' with X. By construction, we get $(P - O) + (Q - O) = (S - O)$ in $\mathrm{Cl}^0(X)$, and we define $P + Q$ to be S.

The whole discussion above can be carried out for any genus 1 equipped with a k-point $O \in X(k)$. A genus 1 curve together with the choice of a k-point is often called an *elliptic curve*. For each pair of k-points P and Q, the coordinates of the sum $S = P + Q$ be be expressed explicitly as rational functions of the coordinates of P and Q. Moreover, S again a k-point (this uses the fact that if two of intersection points of a line with a cubic curve are defined over k, then so is the third). When k is a finite field, this construction plays a central role in elliptic curve cryptography.

23.5 Curves of genus 2

Let X be a smooth projective curve of genus 2. We have already seen examples of such curves, namely hyperelliptic curves obtained by gluing two copies of the affine curve $y^2 = p(x)$, where $p(x)$ is a polynomial of degree 5 or 6. In fact, every genus 2 curve arises in this way. Indeed, the canonical divisor K_X is base-point free and defines a morphism $f\colon X \to \mathbb{P}^1_k$. Since $\deg K_X = 2g - 2 = 2$, f has degree 2, so we obtain:

Proposition 23.23. Every curve of genus 2 is hyperelliptic.

While Proposition **??** showed that a genus 2 curve cannot be embedded in $\mathbb{P}^2_k$, every genus 2 curve can be embedded as a divisor of bidegree $(2, 3)$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

23.6 Curves of genus 3

We next classify curves of genus 3. Here are the two prototype examples:

Example 23.24. Let $X \subset \mathbb{P}^2_k$ be a plane quartic curve, i.e., a curve of degree $d = 4$. By the genus formula, its genus is $\frac{1}{2}(d-1)(d-2) = 3$. Note that

$$\Omega_X = \mathcal{O}_{\mathbb{P}^2_k}(d-3)|_X = \mathcal{O}_X(1).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Hence the canonical divisor K_X is very ample, being the restriction of the hyperplane divisor on $\mathbb{P}^2_k$. Note that the morphism defined by K_X is exactly the embedding $X \to \mathbb{P}^2_k$. △

Example 23.25. Let $X \subset \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ be a smooth divisor of bidegree $(2, 4)$. Then as we saw in Section 22.4, X is hyperelliptic of genus 3. △

These two examples are a bit different. In the first case, the canonical divisor K_X is very ample, and embeds X into $\mathbb{P}^2_k$. In the second, K_X is not very ample: instead it defines a degree 2 map to $\mathbb{P}^1_k$. This distinction is a general phenomenon for curves of genus 3:

Proposition 23.26. Let X be a smooth curve of genus 3. Then either:
(i) K_X is very ample, and the canonical embedding realizes X as a smooth plane quartic; or
(ii) K_X is not very ample, in which case X is hyperelliptic and can be embedded as a divisor of type $(2, 4)$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

In fact, this is a consequence of a more general theorem of Max Noether:

Theorem 23.27 (Noether). Let X be a smooth projective curve of genus $g \geqslant 1$. Then:
(i) K_X is base-point free.
(ii) If $g \geqslant 2$, then K_X is very ample if and only if X is not hyperelliptic.

Proof (i): If $g = 1$, then $\mathcal{O}_X(K_X) = \mathcal{O}_X$, so K_X hence base-point free.

Next we assume that $g \geqslant 2$. We cannot apply Theorem 23.13 (i) directly to prove this, because the degree is too small. However, we can use Riemann–Roch to check directly that the conditions in Theorem 23.13 apply. We need to check that for every point $P \in X$,

$$h^0(X, K_X - P) = h^0(X, K_X) - 1 = g - 1$$

This follows by applying Riemann–Roch to the divisor $D = P$; we get $h^0(X, P) - h^0(X, K_X - P) = 1 + 1 - g = 2 - g$. As P is effective and $X \not\simeq \mathbb{P}^1_k$, we have $h^0(X, P) = 1$, so we get $h^0(X, K_X - P) = g - 1$.

Proof of (ii): By Theorem 23.13, K_X is very ample if and only if for all $P, Q \in X$,

$$h^0(K_X - P - Q) = h^0(K_X) - 2 = g - 2$$

Applying Riemann–Roch to the divisor $P + Q$, we find

$$h^0(P + Q) - h^0(K_X - P - Q) = 2 + 1 - g = 3 - g$$

From these two equations, we see that K_X is very ample if and only if $h^0(P + Q) = 1$ for every $P, Q \in X(k)$.

In general, if X is a curve of genus $g \geqslant 1$, and $P, Q \in X$ are two points, then $h^0(P + Q)$ takes either the value 1 or 2. This follows from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(P) \longrightarrow \mathcal{O}_X(P + Q) \longrightarrow \kappa(Q) \longrightarrow 0$$

along with the fact that $h^0(P) = 1$ since $X \not\simeq \mathbb{P}^1_k$. Therefore, K_X is not very ample if and only if $h^0(P + Q) = 2$ for some $P, Q \in X$. This in turn is equivalent to X being hyperelliptic (see Exercise 23.9.1). □

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

By the theorem, the curves of genus $\geqslant 2$ fall into two types: the hyperelliptic curves, which we know well, and the non-hyperelliptic curves. For the curves of the latter type, K_X is very ample, and there is a 'canonical embedding' defined by the sections of $\mathcal{O}_X(K_X)$

$$\iota\colon X \longrightarrow \mathbb{P}^{g-1}.$$

The next step then is to describe the equations defining the image of X in $\mathbb{P}^{g-1}$.

23.7 Curves of genus 4

In this section, we classify curves of genus 4. By the genus formula $g = (d-1)(d-2)/2$, there cannot be any smooth genus 4 curves in $\mathbb{P}^2_k$. However, there are two families of genus 4 curves in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$:

- A curve of bidegree $(2,5)$ in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ is hyperelliptic of genus 4.
- A curve of bidegree $(3,3)$ is also of genus 4. Geometrically, this curve arises as the intersection of the quadric surface $Q = \mathbb{P}^1_k \times_k \mathbb{P}^1_k \subset \mathbb{P}^3_k$ with a surface $S \subset \mathbb{P}^3_k$ of degree 3, so it is a complete intersection.

We will prove that any curve of genus 4 is of one of these two types.

Theorem 23.28. Let X be a regular curve of genus 4. Then one of the following holds:
(i) X is hyperelliptic (in which case X embeds as a $(2,5)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1$);
(ii) $X = V_+(Q) \cap V_+(C)$ is the complete intersection of a quadric surface and a cubic surface in $\mathbb{P}^3_k$.

Proof By Theorem 23.27, a genus 4 is either hyperelliptic and K_X defines the degree 2 cover to $\mathbb{P}^1_k$; or it is non-hyperelliptic and K_X is very ample. We will assume that we are in the latter case, so that K_X defines an embedding

$$\phi\colon X \longrightarrow \mathbb{P}^{g-1}_k = \mathbb{P}^3_k.$$

The degree of the embedded curve is $\deg K_X = 2g-2 = 6$.

To find the equations of X in $\mathbb{P}^3_k$, we use a very powerful technique in curve theory, namely combining Riemann–Roch with the sheaf cohomology of projective space.

Twisting the ideal sheaf sequence of X by $\mathcal{O}_{\mathbb{P}^3_k}(2)$, we get

$$0 \longrightarrow \mathcal{I}_X(2) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(2)|_X \longrightarrow 0.$$

Taking global sections, we obtain

$$0 \to H^0(\mathbb{P}^3_k, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(2)) \to H^0(X, \mathcal{O}_X(2))$$

In this sequence $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(2))$ is the 10-dimensional space of quadrics in $x_0, \dots, x_3$ and $H^0(\mathbb{P}^3, \mathcal{I}_X(2))$ is the space of quadrics in the ideal of X.

Note that $\mathcal{O}_{\mathbb{P}^3}(1)|_X = \mathcal{O}_X(K_X)$. Applying Riemann-Roch to the divisor $2K_X$, we see that $h^1(2K_X) = h^0(-K_X) = 0$, and

$$h^0(X, \mathcal{O}_{\mathbb{P}^3}(2)|_X) = \deg 2K_X + 1 - g = 12 + 1 - 4 = 9.$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Since $h^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$, we see that the map $H^0(\mathcal{O}_{\mathbb{P}^3_k}(2)) \to H^0(\mathcal{O}_X(2))$ must have a nontrivial kernel, so $H^0(\mathbb{P}^3_k, \mathcal{I}_X(2)) \neq 0$.

The conclusion is that X lies in some surface of degree 2, say $Q \subset \mathbb{P}^3_k$ is defined by $q \in k[x_0, x_1, x_2, x_3]$. Note that X does not lie in a plane (then X could not have genus 4). This implies that q must be irreducible. So X lies on either a singular quadric cone $V_+(x_0x_2 - x_1^2)$ or the regular quadric surface $V_+(x_0x_3 - x_1x_2)$. The quadric Q is moreover unique, because if $X \subset Q \cap Q'$, then X would have degree $\leqslant 4$.

We next study cubics in the ideal I_X. Consider the sequence

$$0 \to H^0(\mathcal{O}_X(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_X(3))$$

Here $h^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(3)) = 20$. Using Riemann–Roch as above, we find that $h^0(X, \mathcal{O}_X(3)) = 15$, and hence $h^0(\mathcal{O}_X(3)) \geqslant 5$. This implies that there is a cubic $c \in k[x_0, x_1, x_2, x_3]$ which is not in the span of $x_0q, \dots, x_3q$, hence $c \notin (q)$. Note that $V_+(q, c)$ is a 1-dimensional closed subscheme of $\mathbb{P}^3_k$ of degree 6 containing X as a component, and as X is also of degree 6, we must have $X = V_+(q, c)$. Hence X is a complete intersection of a quadric $V_+(q)$ and a cubic surface $V_+(c)$. □

In Exercises 23.9.8 and 23.9.9, we will go on to classify curves of genus 5 and 6.

23.8 Proof of Serre duality

The aim of the next few sections is to prove the following:

Theorem 23.29 (Serre duality). Let X be a projective curve over an algebraically closed field k. Then there exist a coherent $\mathcal{O}_X$-module ω_X, together with an isomorphism $H^1(X, \omega_X) \simeq k$, such that for any locally free sheaf $\mathcal{F}$ on X, there is a perfect pairing

$$H^0(X, \mathcal{F}) \times H^1(X, \omega_X \otimes \mathcal{F}^\vee) \longrightarrow H^1(X, \omega_X) \simeq k \tag{23.11}$$

In particular, $H^0(X, \mathcal{F})$ and $H^1(X, \omega_X \otimes \mathcal{F}^\vee)$ have the same dimension.

The sheaf ω_X is called a *dualizing sheaf*. It is unique up to isomorphism (see Exercise 23.9.20). The mere existence of ω_X is usually not enough for applications or explicit computations. The cruicial point is that, when X is smooth, the dualizing sheaf is isomorphic to the cotangent sheaf:

Theorem 23.30. If X is a smooth projective curve, the dualizing sheaf ω_X is isomorphic to the cotangent sheaf Ω_X.

These results are fundamental in algebraic geometry, and several proofs can be found in the literature (see for instance, Hartshorne (2013), Liu (2002), Serre (1955), Serre (2013), Kempf (1993), Vakil (2024)). The proof presented follows ideas from Kempf (1993) and Vakil (1999). It has the advantage of using very little machinery, but it is less conceptual than the other approaches.

We will prove the two theorems in three steps:

(i) We note that both theorems hold for $X = \mathbb{P}^1_k$, where $\omega_X = \mathcal{O}_{\mathbb{P}^1_k}(-2)$ serves as a dualizing sheaf (and we know this coincides with $\Omega_{\mathbb{P}^1_k}$).

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

(ii) Then we prove the existence of ω_X for a general curve, by using a finite map $f\colon X \to \mathbb{P}^1_k$. The sheaf ω_X is constructed just to satisfy the formal properties of Serre duality.
(iii) We finally prove that $\omega_X \simeq \Omega_X$.

The first step, Serre duality for $\mathbb{P}^1_k$, is the easiest:

Lemma 23.31 (Serre duality for $\mathbb{P}^1_k$). Serre duality holds for $\mathbb{P}^1_k$ with $\omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$.

Proof We showed in Corollary 20.22 that Serre duality holds for any projective space for the sheaves $\mathcal{F}$ of the form $\mathcal{O}_{\mathbb{P}^1_k}(d)$. Recalling that any locally free sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$ is a direct sum of invertible sheaves, we get the pairing in general.

□

The dualizing sheaf

In this section, A will be a Noetherian ring and B will be an A-algebra which is finitely generated as an A-module.

For an A-module N, we consider the dual module $\operatorname{Hom}_A(B, N)$, which is a priori an A-module. The key observation is that this can also be viewed as a B-*module*, via the rule

$$b \cdot \psi(y) = \psi(b \cdot y), \quad y \in B$$

for each A-linear map $\psi\colon B \to N$. If N is finitely generated over A and B is finite over A, then $\operatorname{Hom}_A(B, N)$ is finitely generated as a B-module.

The main property we need is the following:

Lemma 23.32. Let L be a B-module and N be an A-module. There is a natural isomorphism of A-modules

$$\operatorname{Hom}_A(L_A, N) \xrightarrow{\sim} \operatorname{Hom}_B\left(L, \operatorname{Hom}_A(B, N)\right)_A, \tag{23.12}$$

where L_A denotes L regarded as an A-module.

Proof Define the map in (23.12) by sending $g\colon L \to N$ to $h\colon L \to \operatorname{Hom}_A(B, N)$ given by $h(\ell)(b) = g(b \cdot \ell)$ for $\ell \in L$, $b \in B$. One checks that this assignment is B-linear in ℓ, and that the map $g \mapsto h$ is A-linear.

Conversely, given a B-linear $h\colon L \to \operatorname{Hom}_A(B, N)$, define $g\colon L_A \to N$ by $g(\ell) = h(\ell)(1)$. Then g is A-linear, and one checks that these constructions are inverse to each other. □

The operation $N \mapsto \operatorname{Hom}_A(B, N)$ defines a functor from A-modules to B-modules. It behaves well with respect to localization: if $S \subset A$ is a multiplicative set, then since B is finitely presented as an A-module, there is a natural isomorphism of $S^{-1}B$-modules

$$S^{-1}\operatorname{Hom}_A(B, N) \simeq \operatorname{Hom}_{S^{-1}A}(S^{-1}B, S^{-1}N). \tag{23.13}$$

Because of this, the construction generalizes well to a version for quasi-coherent sheaves.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For a finite morphism $f\colon X \to Y$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, we define a sheaf $f^!\mathcal{G}$ by setting for each open affine $U \subset \operatorname{Spec} A$,

$$\Gamma(f^{-1}U, f^!\mathcal{G}) = \operatorname{Hom}_{\mathcal{O}_Y(U)}(\mathcal{O}_X(f^{-1}U), \mathcal{G}(U)) \tag{23.14}$$

and gluing the corresponding sheaves on each affine. The sheaf $f^!\mathcal{G}$ is quasi-coherent, as it coincides with the sheaf associated to $\operatorname{Hom}_A(B, N)$ ovover each affine open. Moreover, as f is finite, $f^!\mathcal{G}$ is of finite type whenever $\mathcal{G}$ is.

By (23.14), we obtain

$$f_* f^!\mathcal{G} = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}).$$

More generally, if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, we have a natural isomorphism

$$f_*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \simeq \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}). \tag{23.15}$$

This follows from the adjunction isomorphism (23.12) and the fact that these isomorphisms are compatible with localization.

Definition 23.33. Let X be a smooth projective curve over k, and let $f\colon X \to \mathbb{P}^1_k$ be a finite morphism. We define the *dualizing sheaf* of X to be the $\mathcal{O}_X$-module

$$\omega_X = f^!\omega_{\mathbb{P}^1_k}.$$

where $\omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$.

So far we haven't used the fact that X is smooth; any projective curve admits a dualizing sheaf ω_X. In the smooth case, we will prove in Section 23.8 that $\omega_X \simeq \Omega_X$. In particular, the sheaf is independent of the chosen finite map to $\mathbb{P}^1_k$. As a first step towards this, we next show that ω_X is invertible.

Proposition 23.34. Let X be a smooth projective curve. Then ω_X is an invertible sheaf.

Proof Since X is a smooth curve, ω_X is locally free if and only if it is torsion free (Theorem 22.1). Let $\mathcal{T}$ denote the torsion subsheaf and $\mathcal{E}$ is the torsion free part, so that $\omega_X = \mathcal{E} \oplus \mathcal{T}$ by Theorem 22.1. Applying f_*, we get

$$f_*\omega_X = f_*\mathcal{E} \oplus f_*\mathcal{T}.$$

Applying the formula (23.15), shows that $f_*\omega_X = \mathcal{H}om(f_*\mathcal{O}_X, \omega_{\mathbb{P}^1_k})$. As f is finite and surjective, $f_*\mathcal{O}_X$ is locally free (Proposition 22.2). Since $\omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$ is also invertible, we find that $f_*\omega_X$ is also locally free (as the sheaf $\mathcal{H}om$ of two locally free sheaves is locally free). Note that $f_*\mathcal{T}$ is again a torsion sheaf on $\mathbb{P}^1_k$. As $f_*\mathcal{T}$ is a direct summand of a locally free sheaf, we must have $f_*\mathcal{T} = 0$. This implies that $\Gamma(X, \mathcal{T}) = \Gamma(\mathbb{P}^1_k, f_*\mathcal{T}) = 0$. On a curve, the only torsion sheaf with no global sections is the zero sheaf (any nonzero torsion coherent sheaf is supported at finitely many closed points, and admits a nonzero map to a skyscraper $\kappa(P)$, giving a nonzero global section), so $\mathcal{T} = 0$, and hence ω_X is locally free.

We next show that ω_X has rank 1. To show this, it suffices to compute the stalk at the generic point $\xi = \operatorname{Spec} K(X)$. We can compute this using any affine open sets $\operatorname{Spec} B \subset X$

and $\operatorname{Spec} A \subset \mathbb{P}^1_k$. Then with $S = B - 0$, the stalk $\omega_{X,\xi}$ is given by

$$S^{-1} \operatorname{Hom}_A(B, L) = \operatorname{Hom}_{K(\mathbb{P}^1_k)}(K(X), K(\mathbb{P}^1_k))$$

where the equality uses the localization compatibility (23.14) with $S = B - \{0\}$. This is a $K(\mathbb{P}^1_k)$-vector space of dimension equal to the degree of the field extension $K(X)/K(\mathbb{P}^1_k)$, i.e., the degree of f. Hence, as a $K(X)$-vector space it has dimension 1. Therefore, ω_X is an invertible sheaf. □

Having established this fact, we can finish the proof of Serre duality for this ω on X. For any locally free sheaf $\mathcal{F}$ on X, we have a a chain of natural bijections:

$$\begin{aligned}
H^1(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X) &= H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)) && (23.16)\\
&= H^1(\mathbb{P}^1_k, f_* \mathcal{H}om(\mathcal{F}, \omega_X)) && (23.17)\\
&= H^1(\mathbb{P}^1_k, \mathcal{H}om(f_*\mathcal{F}, \omega_{\mathbb{P}^1_k})) && (23.18)\\
&= H^1(\mathbb{P}^1_k, (f_*\mathcal{F})^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \omega_{\mathbb{P}^1_k}) && (23.19)\\
&= H^0(\mathbb{P}^1_k, f_*\mathcal{F})^\vee && (23.20)\\
&= H^0(X, \mathcal{F})^\vee. && (23.21)
\end{aligned}$$

Many of the steps here are non-trivial: (23.17) and (23.21) use Lemma 20.18. (23.16) and (23.19) follow by Proposition 11.1 and the fact that $\mathcal{F}$, $f_*\mathcal{F}$, ω_X and $\omega_{\mathbb{P}1}$ are locally free. (23.18) uses the adjoint property (23.15) and finally (23.20) is Serre duality on $\mathbb{P}^1_k$.

The dualizing sheaf is the canonical sheaf

The goal of this section is to show that the dualizing sheaf ω_X is isomorphic to the cotangent sheaf Ω_X. Both of these sheaves are locally free: the first by Corollary 23.34, and Ω_X because X is smooth. However, at this point, we do not even have a map between them.

The proof makes use of the following two basic properties of the dualizing sheaf:

(i) $H^1(X, \omega_X) \neq 0$.
(ii) $H^1(X, \omega_X \otimes \mathcal{O}_X(P)) = 0$ for every point $P \in X$.

These are consequences of Serre duality applied to $\mathcal{F} = \mathcal{O}_X$ and $\mathcal{F} = \mathcal{O}_X(-P)$ respectively. Note that $H^1(X, \omega_X(P)) \simeq H^0(X, \mathcal{O}_X(-P))^\vee = 0$, since a global section of $\mathcal{O}_X(-P)$ would be a regular function on the projective curve X vanishing at P, hence zero.

In fact, we will see that these properties characterize ω_X uniquely, and it is this which in turn allows us to conclude that $\Omega_X = \omega_X$.

Proof of Theorem 23.30 We will work with the fiber product $X \times X$ (taken over k) with the two projections $p, q \colon X \times X \to X$. Consider the diagonal $\Delta \subset X \times X$ (which we view as a closed subscheme, because X is separated over k). Since $X \times X$ is a smooth surface, Δ is a divisor on $X \times X$, and the corresponding ideal sheaf $\mathcal{I}$ is the invertible sheaf $\mathcal{O}_{X\times X}(-\Delta)$. Consider the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_{X\times X}(-\Delta) \longrightarrow \mathcal{O}_{X\times X} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

where $i \colon X \to X \times X$ is the diagonal embedding. Tensoring the sequence by $\mathcal{O}_{X\times X}(\Delta)$,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

we obtain

$$0 \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}(\Delta) \longrightarrow \mathcal{O}_{X\times X}(\Delta) \otimes_{\mathcal{O}_{X\times X}} i_*\mathcal{O}_X \longrightarrow 0 \tag{23.22}$$

By the projection formula, the sheaf on the right is isomorphic to $i_*(i^*\mathcal{O}_{X\times X}(\Delta))$. Furthermore, by the definition of the cotangent sheaf, the pullback $i^*\mathcal{O}_{X\times X}(-\Delta)$ is given by Ω_X. Dually, this means that the sheaf on the right-hand side in (23.22) is isomorphic to $i_*\mathcal{T}_X$, and the sequence takes the form

$$0 \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}(\Delta) \longrightarrow i_*\mathcal{T}_X \longrightarrow 0. \tag{23.23}$$

We now tensor this sequence by the invertible sheaf $q^*\omega_X$, to get a sequence

$$0 \longrightarrow q^*\omega_X \longrightarrow q^*\omega_X(\Delta) \longrightarrow i_*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{T}_X) \longrightarrow 0. \tag{23.24}$$

Here we have used the projection formula again:

$$i_*\mathcal{T}_X \otimes_{\mathcal{O}_{X\times X}} q^*\omega_X = i_*(\mathcal{T}_X \otimes_{\mathcal{O}_X} i^*q^*\omega_X) = i_*(\mathcal{T}_X \otimes_{\mathcal{O}_X} \omega_X).$$

If we restrict the sequence (23.24) to the open set $V \times X$, where V is affine, and take the long exact sequence in cohomology, we get an exact sequence

$$\Gamma(V\times X, i_*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{T}_X)) \longrightarrow H^1(V\times X, q^*\omega_X) \longrightarrow H^1(V\times X, q^*\omega_X(\Delta)). \tag{23.25}$$

By Lemma 20.18, we may identify the first group with $\Gamma(V, \omega_X \otimes_{\mathcal{O}_X} \mathcal{T}_X)$. The next lemma identifies the middle group.

Lemma 23.35. Let V and X be varieties over k with V affine and so that X can be covered by two affines U_0, U_1. Let $q\colon V\times X \to X$ denote the second projection. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms

$$\begin{aligned} H^0(V\times X, q^*\mathcal{F}) &= \mathcal{O}_V(V) \otimes_k H^0(X,\mathcal{F}) \\ H^1(V\times X, q^*\mathcal{F}) &= \mathcal{O}_V(V) \otimes_k H^1(X,\mathcal{F}). \end{aligned} \tag{23.26}$$

Proof Recall the exact sequence of k-vector spaces

$$0 \to H^0(X,\mathcal{F}) \to \mathcal{F}(U_0)\times\mathcal{F}(U_1) \xrightarrow{d^0} \mathcal{F}(U_0\cap U_1) \to H^1(X,\mathcal{F}) \to 0. \tag{23.27}$$

Tensoring by $\mathcal{O}_V(V)$ over k preserves exactness, so we get an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_V(V) \otimes_k H^0(X,\mathcal{F}) \to \mathcal{O}_V(V)\otimes_k \mathcal{F}(U_0) \times \mathcal{O}_V(V)\otimes_k \mathcal{F}(U_1) \\ \to \mathcal{O}_V(V)\otimes_k \mathcal{F}(U_0\cap U_1) \to \mathcal{O}_V(V)\otimes_k H^1(X,\mathcal{F}) \to 0. \end{aligned}$$

Note that the map in the middle is exactly the Cech complex of the sheaf $q^*\mathcal{F}$ with respect to the two affine open sets $V\times U_0$ and $V\times U_1$. By exactness, we get the desired isomorphisms. □

By the lemma, we may identity the middle cohomology group in (23.25) with $\Gamma(V,\mathcal{O}_X)\otimes_k H^1(X,\omega_X)$. Below, we will show that

$$H^1(V\times X, q^*\omega_X(\Delta)) = 0. \tag{23.28}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Given this, the sequence (23.25) can be written

$$\Gamma(V, \omega_X \otimes \mathcal{T}_X) \longrightarrow H^1(X, \omega_X) \otimes_k \mathcal{O}_X(V) \longrightarrow 0. \tag{23.29}$$

These identifications are compatible with restriction maps, i.e., if $V' \subset V$ is a smaller affine open set, the following diagram commutes

$$\begin{array}{ccccc} \Gamma(V, \omega_X \otimes \mathcal{T}_X) & \xrightarrow{\delta} & H^1(V \times X, q^*\omega_X) & = & H^1(X, \omega_X) \otimes_k \mathcal{O}_X(V) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(V', \omega_X \otimes \mathcal{T}_X) & \xrightarrow{\delta} & H^1(V' \times X, q^*\omega_X) & = & H^1(X, \omega_X) \otimes_k \mathcal{O}_X(V') \end{array}$$

Therefore, these maps glue to define a map of *sheaves*

$$\omega_X \otimes \mathcal{T}_X \longrightarrow \mathcal{O}_X.$$

By (23.29), this map is surjective. As both sheaves are locally free, this must be an isomorphism (the kernel of a surjective map of invertible sheaves is locally free of rank 0, hence 0), so $\omega_X \otimes \mathcal{T}_X \simeq \mathcal{O}_X$. But this means that $\omega_X \simeq \Omega_X$.

To conclude, we need to show that (23.28) holds. Let $y \in V$ be a k-point and consider the subscheme $\iota\colon y \times X \to V \times X$. Identifying $y \times X$ with X, we have

$$\iota^*(q^*\omega_X) = \omega_X$$

and

$$\iota^*(\mathcal{O}_{X\times X}(\Delta)|_{V\times X}) = \mathcal{O}_X(y)$$

Hence $\iota^*(q^*\omega_X(\Delta)) = \omega_X(y)$. This sheaf has an H^1 which is zero, by the dualizing property:

$$H^1(X, \omega_X(y)) = H^0(X, \mathcal{O}_X(-y))^\vee = 0 \tag{23.30}$$

For the last step we use the fact that X is projective, since $\Gamma(X, \mathcal{O}_X(-y))$ is the group of global regular functions vanishing at y, but all regular functions are constants.

We will need the following lemma, which basically says that if the pullback of a sheaf to a fiber has trivial H^1, then the original sheaf has trivial H^1.

Lemma 23.36 (H^1 and base change). Let A be a Noetherian ring and let $Y \subset \mathbb{P}^n_A$ be a projective scheme. Assume Y is covered by two affine opens U_0, U_1. Let $h\colon Y_B = Y \times_A \operatorname{Spec} B \to Y$ be a morphism induced by a ring map $A \to B$. Then for any quasi-coherent sheaf $\mathcal{F}$ on Y, there is a natural isomorphism of B–modules

$$H^1(Y_B, h^*\mathcal{F}) \simeq H^1(Y, \mathcal{F}) \otimes_A B.$$

Proof As Y is Noetherian and separated, $H^1(Y, \mathcal{F})$ is computed by the Čech complex

$$\mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_{01}), \tag{23.31}$$

and there is a right-exact sequence

$$\mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_{01}) \longrightarrow H^1(Y, \mathcal{F}) \longrightarrow 0. \tag{23.32}$$

Tensoring (23.31) by B, we obtain the Cech complex of $h^*\mathcal{F}$ with respect to the open affine covering $V_i = h^{-1}(U_i) = U_i \times_A \operatorname{Spec} B$ for $i = 0, 1$. This means that $H^1(Y_B, h^*\mathcal{F})$ is the cokernel of $d \otimes \mathrm{id}_B$. Since tensoring is right-exact, this cokernel is obtained by tensoring (23.32) by B, which gives $H^1(Y_B, h^*\mathcal{F}) \simeq H^1(Y, \mathcal{F}) \otimes_A B$. □

We apply the lemma to $Y = V \times X$, $S = V$, $f = p$, $B = A/\mathfrak{m}$ and $\mathcal{F} = q^*\omega_X(\Delta)$. Since $H^1(X, \iota^*(q^*\omega_X(\Delta))) = 0$, we find that the A-module $M = H^1(V \times X, q^*\omega_X(\Delta))$ satisfies $M \otimes_A A/\mathfrak{m} = 0$. Since this happens for every maximal ideal $\mathfrak{m}$ in A, Nakayama's lemma implies that $M = 0$. This concludes the proof of Serre duality. □

23.9 Exercises

Exercise 23.9.1. Show that a curve X is hyperelliptic if and only if there exists an effective divisor D of degree 2 with $h^0(X, \mathcal{O}_X(D)) = 2$.

Exercise 23.9.2. Let X be a regular quasi-projective curve over an algebraically closed field k. Show that if $\Gamma(X, \mathcal{O}_X) \neq k$, then X is affine. HINT: Embed X in a projective regular curve $\overline{X}$ and consider divisors supported on the closed subset $\overline{X} - X$.

Exercise 23.9.3. Let $f\colon X \to Y$ be a non-constant morphism of curves. If D is a divisor on Y, show that $h^0(X, f^*\mathcal{O}_Y(D)) \geqslant h^0(Y, \mathcal{O}_Y(D))$.

Exercise 23.9.4 (Riemann–Roch for higher rank sheaves)**.** Let X be a regular projective curve and let $\mathcal{E}$ be a locally free sheaf of rank r. Show that

$$\chi(X, \mathcal{E}) = \deg(\det \mathcal{E}) + r\chi(\mathcal{O}_X) \tag{23.33}$$

Exercise 23.9.5. Find an example of an $\mathcal{O}_X$-module $\mathcal{F}$ of finite type so that the equality in Lemma (23.36) does not hold if we replace H^1 by H^0.

Exercise 23.9.6. Let $X \subset \mathbb{P}^2_k$ be a plane curve of degree $\leqslant 3$. Show that the automorphism group of X is infinite.

Exercise 23.9.7. Let $X \subset \mathbb{P}^2_k$ be a plane curve of degree 4. Show that any automorphism $\phi : X \to X$ is induced by a linear transformation $\mathbb{P}^2_k \to \mathbb{P}^2_k$. Deduce that the general plane quartic has trivial automorphism group.

Exercise 23.9.8 (Curves of genus 5)**.**
a) Let X be a complete intersection of 3 quadric hypersurfaces in $\mathbb{P}^4_k$. Show that X is non-hyperelliptic of genus 5.
b) Conversely, show that any non-hyperelliptic curve of genus 5 is contained in the intersection of 3 quadric hypersurfaces $Q_0 \cap Q_1 \cap Q_2$ in $\mathbb{P}^4_k$.
c) * Show that there exist non-hyperelliptic genus 5 curves which are not complete intersections of 3 quadrics (it can happen that $Q_0 \cap Q_1 \cap Q_2$ has dimension $\geqslant 2$).

Exercise 23.9.9 (Curves of genus 6)**.** Show that the following varieties define genus 6 curves:
a) A degree 5 curve in $\mathbb{P}^2_k$
b) A $(2, 7)$-divisor in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.
c) A $(3, 4)$-divisor in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.
d) * Show that the classes of curves constructed in a), b), c) are disjoint.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise 23.9.10. Let X be a smooth hyperelliptic curve of genus g. Show that if $f\colon X \to \mathbb{P}^1_k$ is a finite morphism of degree at most g, then f factors via the degree 2 covering $\sigma\colon X \to \mathbb{P}^1_k$. HINT: Consider a map $X \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

Exercise 23.9.11 (Curves of degree 3)**.** Let $X \subset \mathbb{P}^n_k$ be a regular curve of degree 3. Show that X has genus $\leqslant 1$ and is either a twisted cubic or a plane cubic.

Exercise 23.9.12 (Curves of degree 4)**.** Let $X \subset \mathbb{P}^n_k$ be a regular curve of degree 4. Show that X has genus $0, 1$ or 3. Find a projective models in each case.

Exercise 23.9.13 (Hyperelliptic curves)**.** Let X be a hyperelliptic curve of genus g, that is, admitting a base-point free divisor D of degree 2.

a) Show that if $g \geqslant 2$, then D is unique up to linear equivalence, and that $K_X \sim (g-1)D$. POSSIBLE HINT: If $f, g\colon X \to \mathbb{P}^1_k$ are two degree 2 morphisms, consider a point $P \in X$ such that $f^{-1}(f(P)) = P + Q$ and $g^{-1}(g(P)) = P + Q$ for $P, Q, R \in X$ distinct. Then consider the divisor $P + Q + R$.
b) Show that if $g = 1$, then X contains infinitely many such divisors D which are non-linearly equivalent.

Exercise 23.9.14. Let X be a curve over a field k and let $\mathcal{F}$ be a coherent sheaf on X. Show that $H^i(X_{\overline{k}}, q^*\mathcal{F}) \simeq H^i(X, \mathcal{F}) \otimes_k \overline{k}$ for all i, where $q\colon X_{\overline{k}} \to X$ is the base change morphism.

Exercise 23.9.15 (Trace map)**.**

a) Let $A \to B$ be an A-algebra finite over A. Define a map $\mathrm{Tr}_{B/A}\colon B \to A$ by sending b to the trace of the A-linear map $L_b\colon B \to B$, $x \mapsto bx$. Show that this is a map of A-modules.
b) Let $f\colon X \to Y$ be a finite morphism of regular projective curves over a field k. Show that there is a natural map $\mathrm{Tr}\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$
c) Assuming k has characteristic 0, show that Tr splits the usual map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Deduce that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{E}$ for some locally free sheaf $\mathcal{E}$.
d) Let $f\colon X \to Y$ be a dominant morphism of regular projective curves. Show that $g(X) \geqslant g(Y)$ HINT: Apply Exercise 23.9.15.
e) (Lüroth's theorem): Let $f\colon \mathbb{P}^1_k \to X$ be a dominant morphism of regular projective curves. Show that $X \simeq \mathbb{P}^1_k$.

Exercise 23.9.16 (Fermat's Last Theorem for polynomials)**.** As an application of Exercise *c*), show that if $n \geqslant 3$, there are no coprime polynomials $P, Q, R \in \mathbb{C}[t]$ such that

$$P^n + Q^n = R^n.$$

What happens when $n = 1$ or 2?

HINT: Construct a morphism to a Fermat curve in $\mathbb{P}^2_{\mathbb{C}}$.

Exercise 23.9.17.

a) Show that any curve of genus 1 can be embedded as a curve in $\mathbb{P}^{d-1}_k$ of degree d, for any $d \geqslant 3$.
b) Show that a complete intersection of two degree 2 surfaces in $\mathbb{P}^3_k$ is a curve of genus 1.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

c) Show that a genus 1 curve in $\mathbb{P}^{d-1}$ of degree d is not a complete intersection for $d \geqslant 4$. HINT: Consider the canonical bundle of a complete intersection.

Exercise 23.9.18. Let X be a hyperelliptic curve of genus 2 and let $\pi\colon X \to \mathbb{P}^1_k$ be the double cover morphism.

a) Let $\sigma\colon X \to X$ be an automorphism. Show that $\sigma \circ \pi = \pi \circ \sigma$.
b) Show that σ acts on the set of 6 branch points in $\mathbb{P}^1_k$.
c) Show that if an automorphism of $\mathbb{P}^1_k$ fixes 6 points, then it is the identity. Conclude that the automorphism group of X is finite.
d) Generalize this to any hyperelliptic curve of genus $\geqslant 2$. HINT: Consider the map $f\colon X \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ given by $f = (\pi, \pi \circ \sigma)$.

Exercise 23.9.19. Let X be a smooth projective curve of genus g over an infinite field k. Show that if $n \geqslant g+1$, the divisor $D = P_1 + \cdots + P_n$ is base-point free for 'most' k-points $P_1, \ldots, P_n$ and $H^1(X, D) = 0$. HINT: Consider the divisor $K_X - D$.

Exercise 23.9.20. Show that a dualizing sheaf ω is unique up to isomorphism. HINT: Consider $\operatorname{Hom}_{\mathcal{O}_X}(\omega, \omega')$.

Exercise 23.9.21. Let X be a smooth curve of genus g. Show that the projections induce a map

$$\operatorname{Pic}(X) \times \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X \times X).$$

Show that this is an isomorphism when $g = 0$, but not when $g \geqslant 1$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Appendix A

Further constructions and examples

A.1 Cyclic covers

If $f \in \mathcal{O}_X(X)$ is a global section, we can form the $\mathcal{O}_X$-algebra

$$\mathcal{A} = \mathcal{O}_X[t]/(t^m - f)$$

The morphism $\mathbf{Spec}\,\mathcal{A} \to X$ is finite of degree m, and fails to be m-to-1 over the zero set $V(f)$.

More generally, let L be an invertible sheaf and let $s \in \Gamma(X, L^{\otimes m})$ be a nonzero global section. The total space of the line bundle L is

$$\mathbf{V}(L^{-1}) = \mathbf{Spec}(\operatorname{Sym} L^{-1}).$$

If $U \subset X$ is an open set on which L is trivial, then after choosing a generator e of $L|_U$ we may write $L|_U \simeq \mathcal{O}_U e$, and $s|_U = f\, e^{\otimes m}$ for some $f \in \mathcal{O}_X(U)$. Over U, the total space $\mathbf{V}(L^{-1})$ is just $U \times \mathbb{A}^1$, with fiber coordinate t, and inside it we have the closed subscheme defined by the equation $t^m - f = 0$. On such an open set U, the structure sheaf of Y is therefore $\mathcal{O}_U[t]/(t^m - f)$. If we choose a different generator $e' = ue$ with $u \in \mathcal{O}_X(U)^\times$, then f is replaced by $u^{-m}f$, while the corresponding fiber coordinate is replaced by $u^{-1}t$; hence the equation $t^m = f$ is unchanged. Hence these local closed subschemes glue to a well-defined closed subscheme $Y \subset \mathbf{V}(L^{-1})$.

The X-scheme Y can also be described by the $\mathcal{O}_X$-algebra

$$\mathcal{A} = \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-m+1}.$$

where the multiplication is defined as follows. If $a + b < m$, then $L^{-a} \otimes L^{-b} \simeq L^{-(a+b)}$, while if $a + b \geqslant m$, then multiplication is obtained from the section $s\colon \mathcal{O}_X \to L^{\otimes m}$:

$$L^{-a} \otimes L^{-b} \simeq L^{-a-b} \otimes \mathcal{O}_X \xrightarrow{\mathrm{id}\otimes s} L^{-a-b} \otimes L^m \simeq L^{-(a+b-m)}.$$

Then the relative spectrum $Y = \mathbf{Spec}\,\mathcal{A}$, together with its projection to X, is exactly the closed subscheme of $\mathbf{V}(L^{-1})$ described above. It is called the *ramified cyclic cover* associated to s. The morphism π is finite of degree m, generically m-to-1, and fails to be m-to-1 over the zero locus $V(s)$.

Example A.1 (Double covers of $\mathbb{P}^n_A$)**.** Let A be a ring, let $R = A[x_0, \ldots, x_n]$ with its usual grading, and let $f \in R_{2d}$ be a homogeneous polynomial of degree $2d$. Then f defines a global section of $\mathcal{O}_{\mathbb{P}^n_A}(2d)$. Taking $L = \mathcal{O}_{\mathbb{P}^n_A}(d)$ and $m = 2$ in the preceding construction, we obtain a double cover $\pi\colon X \to \mathbb{P}^n_A$.

We can describe X using the standard affine cover of $\mathbb{P}^n_A$. For each i, let $U_i = D_+(x_i) =$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\operatorname{Spec} R_i$, where $R_i = A[x_0/x_i, \dots, x_n/x_i]$. Over U_i, the line bundle $\mathcal{O}_{\mathbb{P}^n_A}(d)$ is trivialized by x_i^d, so the section f is represented by the function $f_i = f(x_0/x_i, \dots, x_n/x_i) \in R_i$. Hence

$$\pi^{-1}(U_i) = \operatorname{Spec} S_i, \qquad S_i = R_i[t_i]/(t_i^2 - f_i).$$

On the overlap $U_i \cap U_j$, the coordinates are related by $t_j = (x_i/x_j)^d t_i$. Therefore $t_j^2 = (x_i/x_j)^{2d} f_i = f_j$, so S_i and S_j become isomorphic after inverting x_j/x_i. Therefore the affine schemes $\operatorname{Spec} S_i$ glue to a scheme X, and the inclusions $R_i \hookrightarrow S_i$ glue to a finite morphism $\pi\colon X \to \mathbb{P}^n_A$ of degree 2.

Equivalently, if $L = \mathcal{O}_{\mathbb{P}^n_A}(d)$, then $X = \mathbf{Spec}(\mathcal{O}_{\mathbb{P}^n_A} \oplus L^{-1})$, where the algebra structure is determined by the section $f \in \Gamma(\mathbb{P}^n_A, L^{\otimes 2})$; on each U_i this is exactly the algebra $R_i[t_i]/(t_i^2 - f_i)$ above.

If A is a field of characteristic different from 2, then π is 2-to-1 away from the hypersurface $V(f) \subset \mathbb{P}^n_A$ and 1-to-1 along $V(f)$. For $n = 1$, this recovers the usual construction of hyperelliptic curves. △

Exercise A.1.1. Assume $\operatorname{char} k \neq 2$ and let $X \subset \mathbb{P}^2_k$ be the curve $x_1^2 x_2 = x_0^3 + x_0 x_2^2$, with $P = (0:0:1)$, $O = (0:1:0)$, $x = x_0/x_2$, $y = x_1/x_2$ and $L = \mathcal{O}_X(P - O)$.

a) Show that $\operatorname{div}(x) = 2P - 2O$, so that $L^{\otimes 2} \simeq \mathcal{O}_X$. Let s be the section corresponding to 1 and $\pi\colon Y \to X$ the associated cyclic cover. The multiplication $L^{-1} \otimes L^{-1} \to \mathcal{O}_X$ is then multiplication by $1/x$.

b) On $U = X - \{O\} = \operatorname{Spec} k[x,y]/(y^2 - x^3 - x)$, show that $L^{-1}|_U$ is the ideal $I = (x, y)$ and that $I^2 = (x)$. Let $u, v \in \mathcal{A}(U) = \mathcal{O}_X(U) \oplus I$ be the elements x, y of the second summand. Show that $u^2 = x$, $uv = y$, $v^2 = x^2 + 1$, and that $\mathcal{A}(U) \simeq k[u,v]/(v^2 - u^4 - 1)$.

c) Show that $(x, y) \mapsto (1/x, y/x^2)$ is an automorphism of X swapping P and O, and that $\mathcal{A}(V) \simeq k[r,w]/(w^2 - r^4 - 1)$ for $V = X - \{P\}$.

d) Show that the two charts are glued by $r = u^{-1}$ and $w = vu^{-2}$, so that Y is the quartic $z_2^2 = z_0^4 + z_1^4$ in $\mathbb{P}(1,1,2)$. Show that π is 2-to-1 everywhere.

A.2 Relative Proj

The construction of relative spectra has a natural analogue for projective spectra. Let X be a scheme and let $\mathcal{R} = \bigoplus_{d \geqslant 0} \mathcal{R}_d$ be a quasi-coherent graded $\mathcal{O}_X$-algebra such that $\mathcal{R}_0 = \mathcal{O}_X$. So for each open set $U \subset X$, the group $\mathcal{R}(U)$ is a graded ring with degree 0 isomorphic to $\mathcal{O}_X(U)$, and the restriction maps are morphisms of graded rings.

For an open affine $U \subset X$, we can form the projective spectrum $\operatorname{Proj} \mathcal{R}(U)$ and its structure morphism $\pi_U\colon \operatorname{Proj} \mathcal{R}(U) \to U$. If $V \subset U$ is another affine, the restriction map $\mathcal{R}(U) \to \mathcal{R}(V)$ is a map of graded rings which one checks induces a morphism $\operatorname{Proj} \mathcal{R}(V) \to \operatorname{Proj} \mathcal{R}(U)$. As in the relative Spec situation, this morphism is an isomorphism onto the open set $\pi_U^{-1}(V)$. Using similar arguments as in the relative Spec case, one checks that the schemes $\operatorname{Proj} \mathcal{R}(U)$ glue to a scheme $\mathbf{Proj}\, \mathcal{R}$ and the morphisms π_U glue to a morphism $\mathbf{Proj}\, \mathcal{R} \to X$. The resulting scheme $\mathbf{Proj}\, \mathcal{R}$ is called the *relative Proj* of $\mathcal{R}$'. It comes equipped with a structure morphism $\pi\colon \mathbf{Proj}\, \mathcal{R} \to X$.

Each $\operatorname{Proj} \mathcal{R}(U)$ comes with a quasi-coherent sheaf $\mathcal{O}(1)$. One can check that these

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

sheaves glue to a quasi-coherent sheaf $\mathcal{O}(1)$ on $\mathbf{Proj}\,\mathcal{R}$. If $\mathcal{R}$ is locally generated by $\mathcal{R}_1$ as an $\mathcal{O}_X$-algebra, then $\mathcal{O}(1)$ is invertible.

Example A.2 (Projective bundles)**.** Let $\mathcal{E}$ be a locally free sheaf of rank r and consider the symmetric algebra $\mathrm{Sym}^*(\mathcal{E})$, which we write as

$$\mathrm{Sym}^*(\mathcal{E}) = \mathcal{O}_X \oplus \mathcal{E}t \oplus \mathrm{Sym}^2(\mathcal{E})t^2 \oplus \cdots$$

The scheme $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}(\mathrm{Sym}^*(\mathcal{E}))$ is usually[1] is called a *projective bundle*. The fibers of the morphism $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ are projective spaces of dimension $r-1$. We also have

$$\Gamma(\mathbf{Proj}\,\mathcal{R}, \mathcal{O}(1)) \simeq \mathcal{E}.$$

△

Example A.3 (Hirzebruch surfaces)**.** An interesting class of examples is provided by the *Hirzebruch surfaces*. Consider a locally free sheaf of the form $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-d)$ on $\mathbb{P}^1_k$, where $d \in \mathbb{Z}$. The corresponding projective bundle $\mathbb{P}(\mathcal{E})$ is exactly the Hirzebruch surface $\mathbb{F}_d$ from Example A.22. The fibers of $\mathbb{F}_d \to \mathbb{P}^1_k$ are projective lines. △

Exercise A.2.1. Let $m, n \geqslant 1$ be integers and let $R = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ with the $\mathbb{Z}^2$-grading given by $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$.

a) Compute $R(\mathbf{w})$ and $I_{\mathbf{w}}$ for the vector $\mathbf{w} = (1,1)$.
b) Show that the corresponding scheme is $\mathbb{P}^m_k \times_k \mathbb{P}^n_k$, and that the $D_+(x_iy_j)$ reduces to the standard affine cover.

Exercise A.2.2. Consider the polynomial ring $R = k[x_0, x_1, y_0, y_1]$ with the $\mathbb{Z}^2$-grading defined by $\deg(x_0) = (1,0)$, $\deg(x_1) = (1,0)$, $\deg(y_0) = (0,1)$ and $\deg(y_1) = (-1,1)$.

a) Compute the $R(\mathbf{w})$ and $I_{\mathbf{w}}$ for the vector $w = (1,1)$.
b) For each monomial x_iy_j with $0 \leqslant i,j \leqslant 1$, compute the localized ring $(R_{x_iy_j})_{\mathbf{0}}$. Show that the scheme $\mathbf{w}$-Proj(R) is isomorphic to the blow-up of $\mathbb{P}^2_k$ at a point.
c) Show that the exceptional divisor E is the closed subscheme cut out by the equation $y_1 = 0$.

Exercise A.2.3 (Hirzebruch surfaces)**.** Let $r \geqslant 0$, and glue the following affine schemes through their common localizations in $K = k(x,y)$:

$$U_{00} = \operatorname{Spec} k[x,y], \qquad U_{01} = \operatorname{Spec} k[x,y^{-1}],$$
$$U_{10} = \operatorname{Spec} k[x^{-1}, x^r y], \qquad U_{11} = \operatorname{Spec} k[x^{-1}, x^{-r}y^{-1}].$$

Denote the resulting scheme by X. Consider the Cartier data

$$F_0 = (x,x,1,1), \qquad F_\infty = (1,1,x^{-1},x^{-1}),$$
$$E = (y,1,x^r y,1), \qquad S = (1,y^{-1},1,x^{-r}y^{-1}),$$

where the entries correspond to $U_{00}, U_{01}, U_{10}, U_{11}$.

a) Show that X is a regular integral surface and that $X \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$ when $r = 0$.

[1] Warning: There are different notations in literature: in many texts $\mathbb{P}(\mathcal{E})$ refers to $\mathbf{Proj}(\mathrm{Sym}^*(\mathcal{E}^\vee))$. The differences in notation is rooted in the two viewpoints of denoting $\mathbb{P}^n_k$ by lines or 1-dimensional quotients. The notation here is the so-called 'Grothendieck notation' for projective bundles.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

b) Show that F_0, F_∞, E, and S are well-defined effective Cartier divisors, and identify F_0 and E with $\mathbb{P}^1_k$. Prove that

$$\operatorname{div}(x) = F_0 - F_\infty, \qquad \operatorname{div}(y) = E - S + rF_\infty,$$

and deduce that $\operatorname{Cl}(X) \simeq \operatorname{Cl}(X) \simeq \mathbb{Z}F_0 \oplus \mathbb{Z}E$. HINT: $X - (F_0 \cup E) = U_{11}$. For independence, restrict a relation to F_0 and E.

c) Let x_1, x_0, y_1, y_0 be the canonical sections defining F_0, F_∞, E, S respectively. Show that $\mathcal{O}_X(F_0)$ is globally generated by the sections x_0, x_1. Show that the fibers of the morphisim $X \to \mathbb{P}^1_k$ are projective lines and that E and S meet each fiber in one point.

d) Assume that $r \geqslant 1$. Show that $\mathcal{O}_X(S)$ is globally generated by y_0 together with the sections $x_0^r y_1, x_0^{r-1} x_1 y_1, \ldots x_1^r y_1$. Show that the corresponding morphism $\varphi\colon X \to \mathbb{P}^{r+1}_k$ is birational onto its image and contracts E to a point.

e) Consider the k-algebra

$$R = \bigoplus_{a,b \in \mathbb{Z}} H^0(X, \mathcal{O}_X(aF_0 + bE)).$$

Show that $R \simeq k[x_0, x_1, y_0, y_1]$ where $\deg x_0 = \deg x_1 = (1,0)$ $\deg y_1 = (0,1)$, $\deg y_0 = (r,1)$. Under this isomorphism, check that $x = x_1/x_0$ and $y = x_0^r y_1/y_0$, and identify the coordinate rings of the four charts with the degree-$(0,0)$ parts of the localizations $R_{x_i y_j}$. Determine $H^0(X, \mathcal{O}_X(F_0))$ and $H^0(X, \mathcal{O}_X(S))$.

f) Assume that $r \geqslant 1$. Show that $H^0(X, \mathcal{O}_X(nE)) = k \cdot y_1^n$ for every $n \geqslant 0$. Show also that $\mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-r)$.

Exercise A.2.4. With the notation of the preceding exercise, show that $X - S$ is obtained by gluing $U_{00} = \operatorname{Spec} k[x, y]$ and $U_{10} = \operatorname{Spec} k[x^{-1}, x^r y]$ along their common distinguished open subsets. Deduce that $X - S$ is the line bundle constructed in Section 5.4 for $n = r$. So the Hirzebruch surface X compactifies this line bundle by adding the section S at infinity.

A.3 Blow-ups

One of the main applications of the Proj construction is to define blow-ups. Blow-ups are birational morphisms used for extending rational maps and resolving singularities.

To motivate the construction, let A be a k-algebra and choose elements $f_0, \ldots, f_n \in A$. These determine a rational map

$$F\colon \operatorname{Spec} A \dashrightarrow \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n],$$

informally given by the tuple $(f_0 : \cdots : f_n)$. Equivalently, viewing $\operatorname{Spec} A$ as $\mathbb{P}^0_A = \operatorname{Proj} A[t]$, the map comes from the map of graded k-algebras

$$\phi\colon k[x_0, \ldots, x_n] \longrightarrow A[t], \qquad x_i \mapsto f_i t.$$

Its *base locus* is the closed subset $V(\mathfrak{a}) \subset \operatorname{Spec} A$, where $\mathfrak{a} = (f_0, \ldots, f_n)$. The map is everywhere defined precisely when $\mathfrak{a} = A$, i.e. when the f_i generate the unit ideal.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example A.4. If $\mathfrak{a} = (f_0, \dots, f_n) = (g)$ is principal, write $f_i = gc_i$. Then $(f_0 : \cdots : f_n)$ and $(c_0 : \cdots : c_n)$ define the same rational map, so the common factor g may be removed. In particular, a principal base ideal produces no genuine indeterminacy. $\triangle$

To resolve the indeterminacy, one forms the *Rees algebra* (or *blow-up algebra*)

$$R(\mathfrak{a}) = \bigoplus_{d \geqslant 0} \mathfrak{a}^d t^d \subset A[t],$$

graded by $\deg t = 1$ and $\deg a = 0$ for $a \in A$. So $R(\mathfrak{a})$ consists of the polynomials $\sum_{d \geqslant 0} a_d t^d$ with $a_d \in \mathfrak{a}^d$.

Since $R(\mathfrak{a})_0 = A$, the scheme $\operatorname{Proj} R(\mathfrak{a})$ comes equipped with a natural morphism

$$\pi \colon \operatorname{Proj} R(\mathfrak{a}) \longrightarrow \operatorname{Spec} A.$$

This is an isomorphism away from the center $V(\mathfrak{a})$. Indeed, if $f \in \mathfrak{a}$, then $\mathfrak{a}A_f = A_f$, hence $\mathfrak{a}^d A_f = A_f$ for all $d \geqslant 0$. Therefore

$$\pi^{-1}(D(f)) = \operatorname{Proj} R(\mathfrak{a})_f = \operatorname{Proj} A_f[t] = \operatorname{Spec} A_f = D(f).$$

Since the opens $D(f)$ with $f \in \mathfrak{a}$ cover $\operatorname{Spec} A - V(\mathfrak{a})$, the restriction

$$\pi^{-1}(\operatorname{Spec} A - V(\mathfrak{a})) \to \operatorname{Spec} A - V(\mathfrak{a})$$

is an isomorphism.

Moreover, the degree-1 elements $f_0 t, \dots, f_n t$ generate the irrelevant ideal of $R(\mathfrak{a})$, so they define a morphism

$$\widetilde{F} \colon \operatorname{Proj} R(\mathfrak{a}) \longrightarrow \mathbb{P}^n_k$$

that fits into the commutative diagram

$$\begin{array}{ccc} & \operatorname{Proj} R(\mathfrak{a}) & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \widetilde{F}} \\ \operatorname{Spec} A & \xrightarrow{\quad F \quad} & \mathbb{P}^n_k. \end{array}$$

Hence $\widetilde{F}$ is a morphism which extends the original rational map F. The scheme $\operatorname{Proj} R(\mathfrak{a})$ is called the *blow-up of* $\operatorname{Spec} A$ *along* $\mathfrak{a}$, and is denoted $\operatorname{Bl}_{\mathfrak{a}}(\operatorname{Spec} A)$.

The standard affine charts are the basic opens $D_+(f_i t) \subset \operatorname{Proj} R(\mathfrak{a})$, corresponding to the rings

$$\big(R(\mathfrak{a})_{f_i t}\big)_0 = A\Big[\frac{f_0}{f_i}, \dots, \frac{f_n}{f_i}\Big] \subset A_{f_i},$$

and these charts glue in the evident way.

The blow-up has the following explicit description. The map

$$\sigma \colon A[u_0, \dots, u_n] \longrightarrow R(\mathfrak{a}), \qquad u_i \mapsto f_i t$$

is a surjective map of graded rings, so

$$\operatorname{Bl}_{\mathfrak{a}}(\operatorname{Spec} A) \simeq V_+(\operatorname{Ker} \sigma) \subset \mathbb{P}^n_A.$$

The kernel always contains the obvious relations $u_i f_j - u_j f_i$. In general there may be more.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For example, if $\mathfrak{a} = (x^2, xy, y^2) \subset k[x, y]$, then $u_0u_2 - u_1^2$ lies in $\operatorname{Ker}\sigma$, although it is not generated by the linear relations.

However, if $f_0, \dots, f_n$ form a regular sequence, then by (Fulton, 2013, Lemma A.6.1) these are the only relations:

$$\operatorname{Bl}_{\mathfrak{a}}(\operatorname{Spec} A) = \operatorname{Proj} \frac{A[u_0, \dots, u_n]}{(u_i f_j - u_j f_i : 0 \leqslant i < j \leqslant n)}. \tag{A.1}$$

Proposition A.5. Let $\mathfrak{a} \subset A$ be an ideal, $X = \operatorname{Spec} A$, and $\widetilde{X} = \operatorname{Bl}_{\mathfrak{a}}(X)$. Then:
(i) If X is irreducible, reduced, or integral, then so is $\widetilde{X}$.
(ii) The scheme-theoretic fiber $E = \pi^{-1}(V(\mathfrak{a}))$ is a Cartier divisor on $\widetilde{X}$.
(iii) π is surjective and induces an isomorphism $\widetilde{X} - E \simeq X - V(\mathfrak{a})$.

Proof (i): The Rees algebra $R(\mathfrak{a})$ is a graded subring of $A[t]$. If A is irreducible, reduced, or integral, then the same is true of $A[t]$, and therefore of $R(\mathfrak{a})$. This yields (1).

(ii): on the chart $D_+(f_i t)$ the ideal $\mathfrak{a}\mathcal{O}_{\widetilde{X}}$ is generated by f_i, since each f_j can be written as $f_i(f_j/f_i)$. Hence $E = V(\mathfrak{a}\mathcal{O}_{\widetilde{X}})$ is locally principal, hence an effective Cartier divisor.

(iii): This follows from the isomorphism established above over $\operatorname{Spec} A - V(\mathfrak{a})$. □

The exceptional divisor can be understood as a Proj as well. Since $\mathfrak{a}^d \otimes_A A/\mathfrak{a} \simeq \mathfrak{a}^d/\mathfrak{a}^{d+1}$, the scheme theoretic fiber over $V(\mathfrak{a})$ is given by

$$E = \operatorname{Proj}\left(\bigoplus_{d \geqslant 0} \mathfrak{a}^d/\mathfrak{a}^{d+1}\right),$$

We now have

$$\mathcal{O}_{\widetilde{X}}(1) \simeq \mathfrak{a}\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}(-E)$$

so $\mathcal{O}_X(E) \simeq \mathcal{O}_X(-1)$. Moreover, the normal bundle of the exceptional divisor is

$$\mathcal{O}_{\widetilde{X}}(E)|_E \simeq \mathcal{O}_E(-1).$$

The blow-up satisfies the following universal property. The morphism $\pi\colon \operatorname{Bl}_{\mathfrak{a}}(\operatorname{Spec} A) \to \operatorname{Spec} A$ is universal among morphisms $Y \to \operatorname{Spec} A$ for which the pulled-back ideal $\mathfrak{a} \cdot \mathcal{O}_Y$ is invertible. In this sense, the blow-up is the initial way of making any ideal $\mathfrak{a}$ locally principal.

Example A.6 (The blow-up of the affine plane)**.** The most classical example of a blowup arises from the rational map

$$\pi\colon \mathbb{A}^2_k \dashrightarrow \mathbb{P}^1_k, \qquad (a, b) \mapsto (a : b).$$

Geometrically, a k-point of $\mathbb{P}^1_k$ corresponds to a line through the origin in $\mathbb{A}^2(k) = k^2$, and for $(a, b) \neq (0, 0)$ the point $\pi(a, b)$ is precisely the line through $(0, 0)$ and (a, b). The map is undefined at the origin, since there is no unique such line there. The blow-up resolves this indeterminacy by replacing the origin with a copy of $\mathbb{P}^1_k$, whose points parametrize all possible directions through the origin.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

In more detail, let $\mathfrak{a} = (x, y) \subset k[x, y]$. Since x, y is a regular sequence, we have

$$\mathrm{Bl}_{(x,y)}\mathbb{A}^2_k = V_+(xu_1 - yu_0) \subset \mathbb{A}^2_k \times_k \mathbb{P}^1_k.$$

On the chart $D_+(u_0)$, writing $v = u_1/u_0$ gives $y = xv$, so

$$D_+(u_0) \simeq \operatorname{Spec} k[x, v].$$

On the chart $D_+(u_1)$, writing $w = u_0/u_1$ gives $x = yw$, so

$$D_+(u_1) \simeq \operatorname{Spec} k[y, w].$$

On the overlap one has $v = w^{-1}$ and $y = xv$.

The exceptional divisor over the origin is a copy of $\mathbb{P}^1_k$, given in these charts by the equations $x = 0$ and $y = 0$, respectively. △

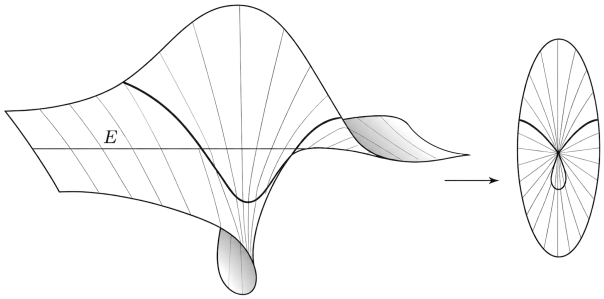


Figure A.1 *The blow-up of the plane at a point*

Example A.7 (Blowing up a coordinate subspace)**.** Let $\mathfrak{a} = (x_1, \dots, x_r) \subset k[x_1, \dots, x_n]$, where $r \leqslant n$. Again the generators form a regular sequence, so

$$\mathrm{Bl}_{\mathfrak{a}}\mathbb{A}^n_k = V_+(x_i u_j - x_j u_i : 1 \leqslant i < j \leqslant r) \subset \mathbb{A}^n_k \times_k \mathbb{P}^{r-1}_k.$$

The second projection extends the rational map

$$\mathbb{A}^n_k \dashrightarrow \mathbb{P}^{r-1}_k, \qquad (x_1, \dots, x_n) \mapsto (x_1 : \cdots : x_r).$$

△

Example A.8 (Blowing up the cusp)**.** Let

$$X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$$

and let $\mathfrak{a} = (x, y)$ be the maximal ideal at the origin. We compute the blow-up of X by working inside the blow-up of the ambient plane from Example A.6.

On the chart $D_+(u_0) \simeq \operatorname{Spec} k[x, u]$, where $u = u_1/u_0$ and $y = xu$, the pullback of the cusp equation becomes $(xu)^2 - x^3 = x^2(u^2 - x)$. The factor x^2 comes from the exceptional divisor, so the strict transform is

$$\widetilde{X} \cap D_+(u_0) = \operatorname{Spec} k[x, u]/(u^2 - x).$$

On the chart $D_+(u_1) \simeq \operatorname{Spec} k[y,w]$, where $w = u_0/u_1$ and $x = yw$, we get $y^2-(yw)^3 = y^2(1-yw^3)$, hence

$$\widetilde{X} \cap D_+(u_1) = \operatorname{Spec} k[y,w]/(1-yw^3).$$

Both charts are smooth, so the blow-up is a desingularization of the cusp. The exceptional divisor meets $\widetilde{X}$ in a single point. △

Example A.9 (The quadric cone)**.** Let $X = \operatorname{Spec} A$, where $A = k[x,y,z]/(xz-y^2)$, and let $L = V(x,y) \subset X$. Then the line L is a prime divisor, which is not Cartier.

Consider the blow-up $\pi\colon \widetilde{X} = \operatorname{Bl}_{(x,y)}X \to X$ along the ideal $\mathfrak{a} = (x,y)$. This blow-up is covered by the two standard charts. On $D_+(xt)$, set $u = y/x$. Then $y = xu$ and $z = xu^2$, so $D_+(xt) \simeq \operatorname{Spec} k[x,u]$. On $D_+(yt)$, set $v = x/y$. Since $v = x/y = y/z$ in A_y, we get $y = zv$ and $x = zv^2$, and therefore $D_+(yt) \simeq \operatorname{Spec} k[z,v]$. Hence $\widetilde{X}$ is covered by two affine planes, so in particular it is smooth.

The inverse image of D is easy to read off from these charts. On $D_+(xt)$ it is cut out by $x = 0$. On $D_+(yt)$ it is cut out by $y = zv = 0$ so it has two components there: $z = 0$ and $v = 0$. The component $z = 0$ belongs to the exceptional fiber over the singular point, while $v = 0$ is the strict transform of D. In particular, the strict transform of D is principal on $\widetilde{X}$, hence Cartier.

In general, blowing up along a non-Cartier divisor replaces it by a Cartier divisor on a less singlar scheme. △

Example A.10 (The Atiyah flop)**.** Consider the 3-dimensional quadric cone

$$X = \operatorname{Spec} k[x,y,z,w]/(xw-yz).$$

We will compare three blow-ups of X. First let $X_{xy} = \operatorname{Bl}_{(x,y)}X$. It is defined in $X \times_k \mathbb{P}^1_k$ by the equations

$$yu_0 - xu_1 = 0, \qquad wu_0 - zu_1 = 0.$$

On the chart $D_+(u_0)$, setting $u = u_1/u_0$, we get $y = xu$ and $w = zu$, so this chart is $\operatorname{Spec} k[x,z,u]$. On $D_+(u_1)$, setting $v = u_0/u_1$, we get $x = yv$ and $z = wv$, so this chart is $\operatorname{Spec} k[y,w,v]$. Hence X_{xy} is smooth. Its exceptional curve $C_{xy} \subset X_{xy}$ is the fiber over the origin. On the first chart it is given by $x = z = 0$, hence is an affine line with coordinate u, and on the second chart it is given by $y = w = 0$, hence is an affine line with coordinate v. These glue by $u = v^{-1}$, so we see that $C_{xy} \simeq \mathbb{P}^1_k$.

Similarly, the blowup $X_{xz} = \operatorname{Bl}_{(x,z)}X$, is smooth with exceptional curve $C_{xz} \simeq \mathbb{P}^1_k$.

Now consider the blowup of X at the origin $(0,0,0,0) \in X$, $Y = \operatorname{Bl}_{(x,y,z,w)}X$.. Here the exceptional divisor is given by

$$E = \operatorname{Proj}\left(\bigoplus_{d\geqslant 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}\right) \simeq \operatorname{Proj} k[x,y,z,w]/(xw-yz),$$

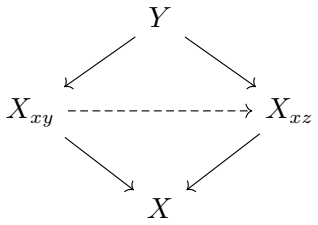
where $\mathfrak{m} = (x,y,z,w)$. Therefore, E is isomorphic to a smooth quadric surface in $\mathbb{P}^3_k$, i.e., $E \simeq \mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

Geometrically, the first two blow-ups are the two 'small resolutions' of the node: they replace the singular point by a curve $\mathbb{P}^1_k$. The third blow-up is not small; it replaces the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

singular point by the quadric surface $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. The blow-up Y dominates the two small resolutions:



Over the exceptional divisor $E \simeq \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, the two morphism $Y \to X_{xy}$ sends E onto the curve $C_{xy} \simeq \mathbb{P}^1_k$, and $Y \to X_{xz}$ sends E to the curve $C_{xz} \simeq \mathbb{P}^1_k$.

An interesting point is that the rational map $X_{xy} \dashrightarrow X_{xz}$ is an isomorphism away from the exceptional curves C_{xy} and C_{xz}, and it replaces one exceptional $\mathbb{P}^1_k$ by the other. This birational transformation is called the *Atiyah flop*. △

Example A.11. Let $\mathfrak{m} = (7, t) \subset \mathbb{Z}[t]$. Since $7, t$ is a regular sequence, the blow-up is

$$\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^1_{\mathbb{Z}} = V_+(tu_1 - 7u_0) \subset \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}.$$

Over $D(t)$ and $D(7)$ the ideal $\mathfrak{m}$ becomes principal, so the blow-up is an isomorphism over these subsets. Indeed, π induces isomorphisms

$$\pi^{-1}(D(t)) = \mathrm{Proj}\left(R \otimes_A A[\tfrac{1}{t}]\right) = \mathrm{Proj}(A[u_0, \tfrac{1}{t}]) \simeq D(t)$$
$$\pi^{-1}(D(7)) = \mathrm{Proj}\left(R \otimes_A A[\tfrac{1}{7}]\right) = \mathrm{Proj}(A[u_1, \tfrac{1}{7}]) \simeq D(7)$$

Therefore π is an isomorphism away from the closed point $\mathfrak{m}$.

On the other hand, the schreme-theoretic fiber of $\mathfrak{m}$ is given by

$$\mathrm{Proj}(A[u_0, u_1] \otimes_A A/\mathfrak{m}) = \mathrm{Proj}(A[u_0, u_1]/(7, t)) = \mathbb{P}^1_{\mathbb{F}_7}.$$

So geometrically, the blow-up is an isomorphism over $\mathbb{A}^1 - \{\mathfrak{m}\}$ and replaces $\mathfrak{m}$ with a projective line $\mathbb{P}^1_{\mathbb{F}_7}$. △

For 1-dimensional rings, blow-ups are closely related to normalization.

Example A.12. Let $A = \mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[\alpha]/(\alpha^2 + 3)$ and $\mathfrak{m} = (2, 1 + \alpha)$. In the Rees algebra write $u = 2t$ and $v = (1 + \alpha)t$. Then the two affine charts are

$$D_+(u) \simeq \mathrm{Spec}\, A\Big[\frac{(1+\alpha)}{2}\Big], \qquad D_+(v) \simeq \mathrm{Spec}\, A\Big[\frac{2}{1+\alpha}\Big].$$

In fact, since $(1 + \alpha)(1 - \alpha) = 4$, one has $2/(1 + \alpha) = (1 - \alpha)/2 = 1 - (1 + \alpha)/2$, the two chart rings coincide. Hence

$$\mathrm{Bl}_{\mathfrak{m}}(\mathrm{Spec}\, A) \simeq \mathrm{Spec}\, \mathbb{Z}\Big[\frac{1+\sqrt{-3}}{2}\Big].$$

Hence the blow-up is affine and equal to the normalization $\overline{A}$ of A. The fiber over $\mathfrak{m}$ is a single point, since (2) remains prime in $\overline{A}$. △

Example A.13. Let $A = \mathbb{Z}[\sqrt{-7}] = \mathbb{Z}[\beta]/(\beta^2+7)$ and consider the blow-up of $\operatorname{Spec} A$ along $\mathfrak{m} = (2, 1+\beta)$. If $\omega = (1+\beta)/2$. One chart is $D_+(u) \simeq \operatorname{Spec} \mathbb{Z}[\omega]$, which is the normalization of A. On the other chart, writing $x = u/v = 2/(1+\beta)$ gives $\beta = 1-4x$ and $2x^2 - x + 1 = 0$, so $D_+(v) \simeq \operatorname{Spec} \mathbb{Z}[x]/(2x^2 - x + 1)$. The blow-up is obtained by gluing these two affine charts. On the first chart the inverse image of $\mathfrak{m}$ is $V(2)$, and $(2) = \mathfrak{p}_+ \cap \mathfrak{p}_-$ with $\mathfrak{p}_\pm = \big(2, \frac{1\pm\beta}{2}\big)$. Therefore the point $\mathfrak{m}$ splits into two points on the blow-up. △

A.3.0 Exercises. Here are some blow-up exercises. Below, k is always a field.

Exercise A.3.1. Let $X = \mathbb{P}(1,1,d) = \operatorname{Proj} k[x_0, x_1, x_2]$, where $\deg x_0 = \deg x_1 = 1$ and $\deg x_2 = d$, and let $P = V_+(x_0, x_1) \in X$ be the singular point. Show that $\operatorname{Bl}_P X \simeq \mathbb{P}_{\mathbb{P}^1_k}(\mathcal{O} \oplus \mathcal{O}(-d))$. For $d = 2$, we recover the desingularization of the quadric cone.

Exercise A.3.2. Let $C \subset \mathbb{P}^3_k$ be the twisted cubic. Show that the blow-up of $\mathbb{P}^3_k$ along C is

$$X = V_+(x_1u_0 - x_2u_1 + x_3u_2,\ x_0u_0 - x_1u_1 + x_2u_2) \subset \mathbb{P}^3_k \times_k \mathbb{P}^2_k.$$

Let $p\colon X \to \mathbb{P}^3_k$ and $q\colon X \to \mathbb{P}^2_k$ be the projections. What does the exceptional divisor look like? Show that all fibers of q are isomorphic to projective lines. Show that $\mathcal{E} = q_* p^* \mathcal{O}_{\mathbb{P}^3_k}(1)$ is a locally free sheaf of rank 2 on $\mathbb{P}^2_k$, and that $X \simeq \mathbb{P}(\mathcal{E})$. The sheaf $\mathcal{E}$ is called the *Bordiga bundle*.

Exercise A.3.3. Let $X = \operatorname{Spec} k[x,y,z]/(xy - z^3)$, and let $\mathfrak{m} = (x,y,z)$ be the maximal ideal at the origin. Show that the blow-up $\widetilde{X} = \operatorname{Bl}_{\mathfrak{m}} X$ is covered by three affine charts, each of which is smooth over k. Describe the exceptional divisor explicitly. Generalize the computation to $\operatorname{Spec} k[x,y,z]/(xy - z^n)$ for any $n \geqslant 2$.

Exercise A.3.4. Let X be the blow-up of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ along the ideal $\mathfrak{a} = (x^2, xy, y^2) = (x,y)^2$. Show that X is isomorphic to $\operatorname{Bl}_{(x,y)} \mathbb{A}^2_k$.

Show that the blow-up of $\mathbb{A}^2_k$ along the ideal $\mathfrak{a} = (x^2, y^2) \subset k[x,y]$ has an affine chart isomorphic to $\operatorname{Spec} k[x,y,v]/(y^2 - x^2 v)$. Deduce that blowing up along a nonreduced subscheme need not produce a smooth scheme. Over $k = \mathbb{R}$, the hypersurface $y^2 = x^2 v$ is called the *Whitney umbrella*. Compute its singular locus.

Show that the blow-up of $\mathbb{A}^2_k$ along the ideal (x^2, y) is singular, and compute its singular locus.

Exercise A.3.5. Consider the following blow-ups of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$.

a) Let $\mathfrak{a} = (x^2, xy, y^2) = (x,y)^2$. Show that $\operatorname{Bl}_{\mathfrak{a}} \mathbb{A}^2_k \simeq \operatorname{Bl}_{(x,y)} \mathbb{A}^2_k$.
b) Let $\mathfrak{a} = (x^2, y^2)$. Show that $\operatorname{Bl}_{\mathfrak{a}} \mathbb{A}^2_k$ has an affine chart isomorphic to $\operatorname{Spec} k[x,y,v]/(y^2 - x^2 v)$. Deduce that blowing up along a nonreduced subscheme need not produce a smooth scheme. Over $k = \mathbb{R}$, the hypersurface $y^2 = x^2 v$ is called the *Whitney umbrella*. Compute its singular locus.
c) Let $\mathfrak{a} = (x^2, y)$. Show that $\operatorname{Bl}_{\mathfrak{a}} \mathbb{A}^2_k$ is singular, and determine its singular locus.

Exercise A.3.6. Let $X = \mathbb{A}^3_k$ with coordinates x, y, z, and let $Z \subset X$ be the closed subscheme defined by (x, yz). Show that the blow-up $\operatorname{Bl}_Z X$ is covered by two affine charts. Determine which charts are smooth, and describe the singular locus of $\operatorname{Bl}_Z X$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise A.3.7. Let $X = \operatorname{Spec} k[x,y,z]/(z^2 - xy^2)$, and let $\mathfrak{m} = (x,y,z)$ be the maximal ideal at the origin. Compute the blow-up $\operatorname{Bl}_{\mathfrak{m}} X$ by writing down affine charts. Study the singular locus of the blowup.

Exercise A.3.8. Let $X = \mathbb{A}^3_k$ and let $L \subset X$ be the line defined by (x,y). Let $P \in L$ be the origin, and let $\widetilde{X} = \operatorname{Bl}_L X$. Show that the strict transform of the plane $V(x)$ in $\widetilde{X}$ is isomorphic to the blow-up of $\mathbb{A}^2_k$ at a point. Describe also the exceptional divisor of $\widetilde{X} \to X$.

Exercise A.3.9. Let $X = \mathbb{A}^3_k$ and let $Z \subset X$ be the union of the two coordinate axes defined by $(x,y) \cap (x,z) = (x,yz)$. Show that the blow-up $\operatorname{Bl}_Z X$ is singular. Determine the exceptional divisor over the origin, and compare with the blow-up along each axis separately.

Exercise A.3.10. Let $X = \mathbb{P}^2_k$ and let $P, Q \in X$ be two distinct closed points. Show that $\operatorname{Bl}_{\{P,Q\}} X$ is isomorphic to the blow-up of $\operatorname{Bl}_P X$ at the point lying over Q. Describe the exceptional divisors and compute their self-intersection numbers.

Exercise A.3.11. Let $X = \mathbb{P}^2_k$, and let $P \in X$ be a closed point. Let $\widetilde{X} = \operatorname{Bl}_P X$. Show that the strict transform of a plane curve of degree d having multiplicity m at P is linearly equivalent to $dH - mE$, where H is the pullback of a line and E is the exceptional divisor.

Exercise A.3.12. Let $X = \operatorname{Spec} k[x,y,z]/(x^2+y^2+z^2)$ over a field k of characteristic different from 2, and let $\mathfrak{m} = (x,y,z)$. Compute the blow-up $\operatorname{Bl}_{\mathfrak{m}} X$. Describe the exceptional divisor and determine whether the blow-up is smooth.

Exercise A.3.13. Let k be a field.

a) Consider $\mathbb{P}^2_k$ and the point $P = (1:0:0)$. Show that $\operatorname{Bl}_P \mathbb{P}^2_k$ is isomorphic to the closed subscheme of $\mathbb{P}^2 \times_k \mathbb{P}^1_k$ defined by the bihomogneeous equation $x_1 u - x_2 v = 0$. Describe the exceptional divisor and the fibers of the projection to $\mathbb{P}^1_k$.

b) Let X be the blow-up of $\mathbb{P}^3_k$ along the line $L = V_+(x_0, x_1)$. Show that X admits a natural morphism to $\mathbb{P}^1_k$ whose fibers are isomorphic to projective planes. Identify $\operatorname{Bl}_L X$ as a projective bundle over $\mathbb{P}^1_k$.

Exercise A.3.14. Let $Q = V_+(x_0x_2 - x_1^2) \subset \mathbb{P}^3_k$ be a quadric cone and $P = (0:0:0:1)$. Show that the blow-up of Q at P is a Hirzebruch surface $\mathbb{F}_2$.

Exercise A.3.15. Consider the standard Cremona transformation $\phi\colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ defined by $(x:y:z) \mapsto (yz:xz:xy)$. Show that the base locus of ϕ consists of the three coordinate points. Let $\pi\colon X \to \mathbb{P}^2_k$ be the blow-up of $\mathbb{P}^2_k$ at these three points. Show that X is isomorphic to the subscheme of $\mathbb{P}^2_{x,y,z} \times \mathbb{P}^2_{u,v,w}$ defined by

$$X = \{((x:y:z),(u:v:w)) \mid xu = yv = zw\}.$$

Show that ϕ lifts to an automorphism $X \to X$.

Exercise A.3.16. Consider the two rings $A = \mathbb{Z}[x]/(x^2+4)$ and $B = \mathbb{Z}[x]/(x^2+7)$.

a) Compute the blow-up of $\operatorname{Spec} A$ at the ideal $\mathfrak{m} = (2,x)$

b) Compute the blow-up of $\operatorname{Spec} B$ at the ideal $\mathfrak{m} = (2, 1+x)$.

In each case, compare the blow-up morphim with the normalization and compute the fiber over $\mathfrak{m}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

A.4 Toric varieties

In this section, we explore a large class of examples, known as *toric varieties*. Like projective spaces, toric varieties are defined in terms of algebras generated by monomials. The correspondence between monomials and lattice points in $\mathbb{R}^n$ sets up an interesting interplay between algebra and combinatorics, which allows us to do computations with toric varieties using combinatorial methods (and in some cases, solve combinatorial problems using tools from algebraic geometry). We will confine ourselves to discussing affine and projective toric varieties. We refer to the standard texts Fulton (1993) and Cox et al. (2024) for more information about this rich subject.

A.4.0 Affine toric varieties. Let k be a field and consider the ring of Laurent polynomials $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We will use multiindex notation for monomials, that is, if $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, then $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$.

Given a subset $S \subset \mathbb{R}^n$, we define the *cone* of S to be the non-negative span of S, that is,

$$\operatorname{Cone}(S) = \left\{ \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s \;\mid\; \mathbf{v}_i \in S, \text{ and } \lambda_i \geqslant 0 \,\forall\, i \right\}.$$

For a cone C, we define the following subalgebra generated by the monomials corresponding to the lattice points in C:

$$k[C] = \bigoplus_{\mathbf{m} \in C \cap \mathbb{Z}^n} k\, \mathbf{x}^{\mathbf{m}}. \tag{A.2}$$

The *toric variety* associated to C is the affine scheme $X = \operatorname{Spec} k[C]$.

We will usually assume that C is *rational polyhedral*, meaning that it is generated by a finite set of integer points $S \subset \mathbb{Z}^n$. This implies that there are finitely many lattice points $\mathbf{m}_1, \ldots, \mathbf{m}_r \in C$ so that every point in $C \cap \mathbb{Z}^n$ can be expressed as a non-negative $\mathbb{Z}$-linear combination of the $\mathbf{m}_1, \ldots, \mathbf{m}_r$. On the level of rings, this means that the following map of k-algebras

$$\phi\colon k[y_1, \ldots, y_r] \longrightarrow k[C], \qquad y_i \mapsto \mathbf{x}^{\mathbf{m}_i}. \tag{A.3}$$

is surjective, and hence induces a closed embedding

$$\iota\colon X \longrightarrow \mathbb{A}_k^r.$$

To describe X inside $\mathbb{A}_k^r$, we need to find the kernel of ϕ. Write $\mathcal{A}$ for the $n \times r$ matrix with the vectors $\mathbf{m}_1, \ldots, \mathbf{m}_r$ as column vectors.

Proposition A.14. Let $\mathcal{A}$ be the $n \times r$ matrix whose columns are $\mathbf{m}_1, \ldots, \mathbf{m}_r$. Then

$$\operatorname{Ker} \phi = I_{\mathcal{A}} = (\mathbf{y}^u - \mathbf{y}^v \mid u, v \in \mathbb{N}^r,\ \mathcal{A}u = \mathcal{A}v).$$

Proof Every binomial $\mathbf{y}^u - \mathbf{y}^v$ with $\mathcal{A}u = \mathcal{A}v$ clearly lies in $\operatorname{Ker} \phi$, so $I_{\mathcal{A}} \subseteq \operatorname{Ker} \phi$.

For the reverse inclusion, we use a monomial order argument. Consider the lexicographic ordering $\prec$ on monomials in $k[y_1, \ldots, y_r]$, so that for instance $y_1^2 > y_1 y_2 > y_1 y_3 > y_2^2 > y_2 y_3 > y_3^2$. If $g \in \operatorname{Ker} \phi$, we can write it as $g = c_u \mathbf{y}^u + \sum_{v \prec u} c_v \mathbf{y}^v$. where $c_u \mathbf{y}^u \neq 0$ is

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

the leading term of g. Applying ϕ, we get

$$0 = \phi(g) = c_u \mathbf{x}^{\mathcal{A}u} + \sum_{v \prec u} c_v \mathbf{x}^{\mathcal{A}v}.$$

This is an identity of polynomials in $k[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$, so there must be cancellations between the monomials. In particular, the term $c_u \mathbf{x}^{\mathcal{A}u}$ must cancel with some other terms in the sum. This means that there exists some v with $v \prec u$ such that $\mathcal{A}u = \mathcal{A}v$, or in other words, $u - v \in \operatorname{Ker} \mathcal{A}$. Replacing g with $g - c_u(\mathbf{x}^u - \mathbf{x}^v)$, we obtain a polynomial which has a leading term which is strictly smaller than that of g with respect to $\prec$. Note that $\mathbf{x}^u - \mathbf{x}^v$ belongs to the ideal $I_{\mathcal{A}}$. Continuing in this manner, we eventually obtain the zero polynomial, which means that g is an element of $I_{\mathcal{A}}$. □

Hence $I_{\mathcal{A}}$ is a prime ideal defined by binomials. To find a *finite* generating set, a few more computations are usually needed. A basic algorithm is to keep adding binomials $\mathbf{y}^u - \mathbf{y}^v$ until one obtains a prime ideal of the right codimension.

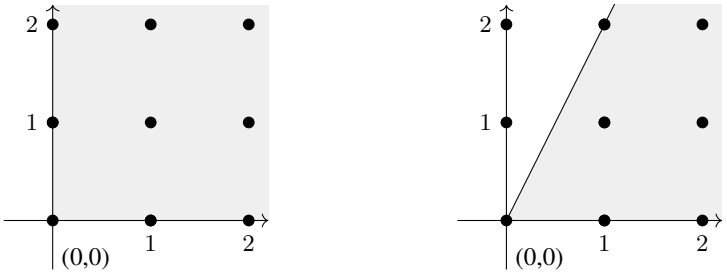


Figure A.2 Two cones in $\mathbb{R}^2$. On the left, the cone generated by $(1,0)$ and $(0,1)$. On the right, the cone generated by $(1,0)$ and $(1,2)$.

Example A.15. Let C be the first quadrant, generated by $(1,0)$, $(0,1)$. Then $k[C] = k[x,y]$ and the corresponding toric variety is $\mathbb{A}^2_k$. △

Example A.16. Let C be the cone generated by $(1,0)$, $(1,2)$. Here

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

which has kernel generated by the vector $(1,-2,1)$. This gives $\mathcal{I}_{\mathcal{A}} = (v^2 - uw)$, and $k[x, xy, xy^2] \simeq k[u,v,w]/(v^2 - uw)$. Therefore, the corresponding toric variety is the quadric surface $V(v^2 - uw)$ in $\mathbb{A}^3_k$. △

Example A.17. Note that $C = \mathbb{R}^n$ itself is a cone. The corresponding toric variety is the *algebraic torus*

$$T = \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

The reason behind the name is the following. When $k = \mathbb{C}$, the set of $\mathbb{C}$-points $T(\mathbb{C})$ is identified with $(\mathbb{C}^\times)^n$, which in the Euclidean topology is homotopy equivalent to a product $(S^1)^n$. For $n = 2$, this is the classical torus $S^1 \times S^1$.

If X is any toric variety, then T embeds as a distinguished open subscheme in X. Indeed,

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

if $\mathbf{v} \in C$ is any vector in the interior of C, then the localization $k[C]_{\mathbf{x}^{\mathbf{v}}}$ is isomorphic to $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. In this way, toric varieties may be viewed as 'compactifications' of the torus T. △

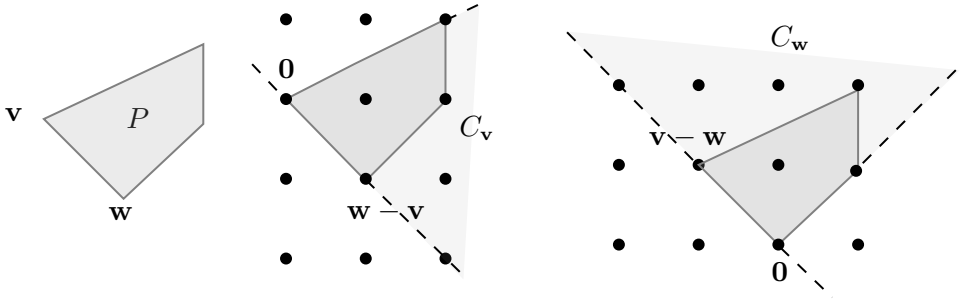
A.4.0 Projective toric varieties. For projective toric varieties the natural input is a lattice polytope. By definition, a *lattice polytope* $P \subset \mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{Z}^n$. In other words, there are $\mathbf{m}_1, \dots, \mathbf{m}_r \in \mathbb{Z}^n$ such that

$$P = \left\{ \lambda_1 \mathbf{m}_1 + \cdots + \lambda_r \mathbf{m}_r \;\mid\; \sum_{i=1}^{r} \lambda_i = 1 \text{ and } \lambda_1, \dots, \lambda_r \geqslant 0 \right\}.$$

Starting with a lattice polytope P, we can define a collection of cones as follows. For each $\mathbf{v} \in P \cap \mathbb{Z}^n$, let $P - \mathbf{v} = \{\, x \in \mathbb{R}^n \mid x + \mathbf{v} \in P \,\}$ denote the translate of P by $-\mathbf{v}$, so that the vertex $\mathbf{v}$ ends up at the origin. Then the cone

$$C_{\mathbf{v}} = \mathrm{Cone}(P - \mathbf{v})$$

is a cone spanned by the elements of P as viewed from the vertex $\mathbf{v}$. This is pictured in the figure below.



In particular, to the polytope P, we can attach k-algebras $k[C_{\mathbf{v}}]$, one for each $\mathbf{v} \in P \cap \mathbb{Z}^n$. As P has only finitely many vertices, the cones $C_{\mathbf{v}}$ are each spanned by finitely many points with integer coordinates. This implies that each $k[C_{\mathbf{v}}]$ is a finitely generated k-algebra.

Lemma A.18. If $\mathbf{v}$ and $\mathbf{w}$ are two vertices of P, then

$$k\left[C_{\mathbf{v}}\right]_{\mathbf{x}^{\mathbf{w}-\mathbf{v}}} = k\left[C_{\mathbf{w}}\right]_{\mathbf{x}^{\mathbf{v}-\mathbf{w}}}. \tag{A.4}$$

Proof Let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be the vertices of P, ordered so that $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{w} = \mathbf{v}_2$. Then the cone $C_{\mathbf{v}}$ is the non-negative span of the vectors $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_r - \mathbf{v}_1$, and $C_{\mathbf{w}}$ is the non-negative span of the vectors $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \dots, \mathbf{v}_r - \mathbf{v}_2$. Clearly, these spans are the same up to translation by $\mathbf{v}_2 - \mathbf{v}_1$, so we get

$$C_{\mathbf{v}} + \mathbb{R}(\mathbf{w} - \mathbf{v}) = C_{\mathbf{w}} + \mathbb{R}(\mathbf{w} - \mathbf{v}). \tag{A.5}$$

The equality (A.4) follows, because both sides are generated by the same monomials. □

Consider the affine schemes

$$X_{\mathbf{v}} = \mathrm{Spec}(k[C_{\mathbf{v}}])$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

where $v \in P$ is a vertex. If $\mathbf{w} \in P$ is another vertex, the affine scheme

$$X_{\mathbf{v},\mathbf{w}} = \operatorname{Spec}(k[C_{\mathbf{v}}]_{\mathbf{x}^{\mathbf{w}-\mathbf{v}}})$$

is identified with a distinguished open set in $X_{\mathbf{v}}$ (defined by the monomial $\mathbf{x}^{\mathbf{w}-\mathbf{v}}$). The gluing conditions for gluing the $X_{\mathbf{v}}$ are then satisfied. The resulting scheme, X, is called the *toric variety* associated to the polytope P.

Example A.19 (The projective line)**.** Let $P = [0, 1]$ be the unit interval in $\mathbb{R}$. Then P has two vertices 0 and 1. For the vertex 0, which is already at the origin, the cone S_0 is the non-negative span of P, that is, $S_0 = [0, \infty)$. Likewise $S_1 = (-\infty, 0]$. This gives the following two k-algebras:: $A_0 = k[x]$ and $A_1 = k[x^{-1}]$. The corresponding toric variety is $\mathbb{P}^1_k$. △

Example A.20 (Projective space)**.** Let $P \subset \mathbb{R}^2$ be the triangle with vertices at $0, e_1$ and e_2. There are three cones: $S_1 = \{x \geqslant 0, y \geqslant 0\}$, $S_2 = \{y \geqslant 0, x + y \leqslant 0\}$ and $S_3 = \{x \geqslant 0, x + y \leqslant 0\}$. These are depicted in Figure A.3.

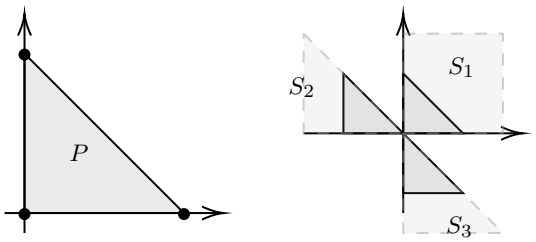


Figure A.3 Constructing the projective plane

The cones S_1, S_2, S_3 give the following three k-algebras

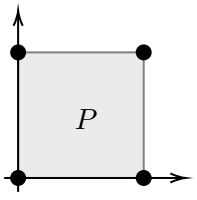
$$A_1 = k[x, y], \quad A_2 = k[x^{-1}, x^{-1}y], \quad A_3 = k[y^{-1}, xy^{-1}].$$

By Example **??**, we recognize the corresponding toric variety as the projective plane $\mathbb{P}^2_k$.

More generally, if $P \subset \mathbb{R}^n$ is the *standard n-simplex*, that is, the polytope with the $n + 1$ vertices $0, e_1, \ldots, e_n$, then the corresponding toric variety is the projective space $\mathbb{P}^n_k$.

△

Example A.21. Let P be the unit square, that is, the convex hull of the four points $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$.

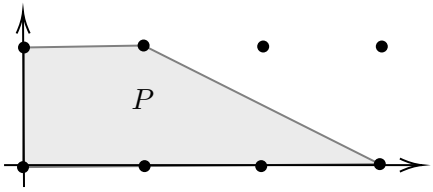


The four cones $S_1, \ldots, S_4$ are the four quadrants in $\mathbb{R}^2$. The corresponding k-algebras are

$$A_1 = k[x, y] \qquad A_2 = k[x^{-1}, y]$$
$$A_3 = k[x^{-1}, y^{-1}] \qquad A_4 = k[x, y^{-1}]$$

The corresponding toric variety is $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. Note that the polytope P is itself a product: $P = [0,1] \times [0,1]$, and $[0,1]$ is the polytope defining $\mathbb{P}^1_k$. This is a general fact about products of toric varieties (see Exercise ???). △

Example A.22 (Hirzebruch surfaces)**.** Let $r \in \mathbb{N}$ and let P be the convex hull of the four points $(0,0)$, $(0,1)$, $(1,0)$ and $(r+1,0)$. This is pictured below for $r = 2$. The case $r = 0$ was considered in the previous example.



This gives the following k-algebras:

$$A_1 = k[x,y] \qquad A_2 = k[x^{-1}, x^{-r}y]$$
$$A_3 = k[x^{-1}, x^r y^{-1}] \qquad A_4 = k[x, y^{-1}]$$

The corresponding toric variety $\mathbb{F}_r$ is called a *Hirzebruch surface*. △

Each of the rings $k[C_{\mathbf{v}}]$ and their localizations are subrings of $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and the morphisms $T \to \operatorname{Spec} k[C_{\mathbf{v}}]$ glue to an open embedding $T \to X$.

The toric variety X admits a closed embedding into projective space. More precisely, let $\mathbf{m}_1, \dots, \mathbf{m}_r$ be the lattice points of P, and assume that P satisfies the following property: for each i, the semigroup $C_{\mathbf{m}_i} \cap \mathbb{Z}^n$ is generated by the elements $\mathbf{m}_1 - \mathbf{m}_i, \mathbf{m}_2 - \mathbf{m}_i, \dots, \mathbf{m}_r - \mathbf{m}_i$. This condition is not too restrictive, as it is satisfied for sufficiently large scalings of P, and X remains unchanged under scaling P. Moreover, it is automatically satisfied if P has dimension at most 2.

In this setting, $k[C_{\mathbf{m}_i}]$ is generated by the monomials $\mathbf{x}^{\mathbf{m}_1 - \mathbf{m}_i}, \dots, \mathbf{x}^{\mathbf{m}_r - \mathbf{m}_i}$, and we can define a surjective map of k-algebras

$$\phi_i \colon k\left[\frac{y_1}{y_i}, \dots, \frac{y_r}{y_i}\right] \longrightarrow k[C_{\mathbf{m}_i}], \qquad \frac{y_j}{y_i} \mapsto \mathbf{x}^{\mathbf{m}_j - \mathbf{m}_i} \tag{A.6}$$

and hence a closed embedding $\operatorname{Spec} k[C_{\mathbf{m}_i}] \to U_i = \operatorname{Spec} k[\frac{y_1}{y_i}, \dots, \frac{y_r}{y_i}]$. It is straightforward to check that these embeddings glue to an embedding

$$f \colon X \longrightarrow \mathbb{P}^{r-1}_k.$$

Alternatively, if $T = \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is the dense torus, the monomials $\mathbf{x}^{\mathbf{m}_1}, \dots, \mathbf{x}^{\mathbf{m}_r}$ define a morphism $T \longrightarrow \mathbb{P}^{r-1}_k$ and X is naturally isomorphic to the Zariski closure of its image in $\mathbb{P}^{r-1}_k$.

Another nice property of the toric varieties X is that they provide economical ways to compactify hypersurfaces of the torus T. If $f \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a nonzero Laurent polynomial, we can consider the associated *Newton polytope* $P_f \subset \mathbb{R}^n$ by taking the convex hull of the points in $\mathbb{Z}^n$ corresponding to monomials appearing in f.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example A.23. Consider the polynomial

$$f = 1 + x^2y + xy^2$$

Then the Newton polytope $P = P_f$ is the convex hull of the vertices $(0,0)$, $(2,1)$, $(1,2)$. The toric variety associated to P is covered by the three affine charts

$$\begin{aligned} R_1 &= k[x^2y, xy, xy^2] = k[u_1, u_2, u_3]/(u_2^2 - u_1u_3) \\ R_2 &= k[x^{-2}y^{-1}, x^{-1}, x^{-1}y] = k[v_1, v_2, v_3]/(v_2^3 - v_1v_3) \\ R_3 &= k[x^{-1}y^{-2}, y^{-1}, xy^{-1}] = k[w_1, w_2, w_3]/(w_2^3 - w_1w_3) \end{aligned}$$

The hypersurface given by $V(f) \subset T = \operatorname{Spec} k[x^{\pm 1}, y^{\pm 1}]$ is an open set in a hypersurface $Y \subset X$. Over the open set $U_1 = \operatorname{Spec} R_1$ this is simply given by the linear equation $1 + u_1 + u_3 = 0$. Over $U_2 = \operatorname{Spec} R_2$ it is given by the 'dehomogenization' $x^{-2}y^{-1} + 1 + x^{-1}y = v_1 + 1 + v_3$ and so on. △

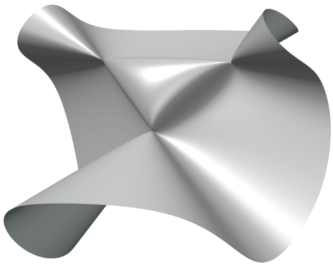


Figure A.4 A cubic surface.

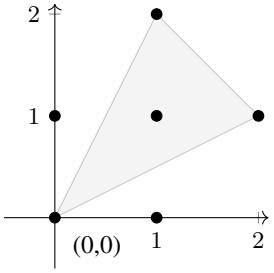


Figure A.5 The polytope P

Exercise A.4.1 (Products of toric varieties)**.** Let $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ be full-dimensional lattice polytopes. Show that for vertices $\mathbf{v}$ of P and $\mathbf{w}$ of Q one has $C_{(\mathbf{v},\mathbf{w})} = C_{\mathbf{v}} \times C_{\mathbf{w}} \subset \mathbb{R}^{m+n}$. Moreover, show that the toric variety associated to $P \times Q$ is isomorphic to $X_P \times_k X_Q$.

Exercise A.4.2. In the notation of Proposition A.14, let $T = \{u_1, \dots, u_r\}$ be a $\mathbb{Z}$-basis for $\operatorname{Ker} \mathcal{A}$ and let

$$I'_{\mathcal{A}} = (\mathbf{y}^{u_+} - \mathbf{y}^{u_-} \mid u \in S)$$

where we decompose $u = u_+ - u_- \in \mathbb{Z}^n$ in terms of its non-negative and non-positive entries. Show that $I_{\mathcal{A}}$ is the saturation of $I'_{\mathcal{A}}$ with respect to the maximal ideal at the origin, i.e., $I_{\mathcal{A}} = I'_{\mathcal{A}} : (y_1, \dots, y_m)^{\infty}$.

Exercise A.4.3. Let $C \subset \mathbb{R}^n$ be a rational polyhedral cone.

a) Show that the semigroup $C \cap \mathbb{Z}^n$ is finitely generated, and conclude that the k-algebra $k[C]$ is finitely generated. HINT: Let $v_1, \dots, v_r$ be generators of C. Write any $w \in C \cap \mathbb{Z}^n$ as $w = \sum_{i=1}^r n_i v_i + \sum_{i=1}^r a_i v_i$ with $n_i \in \mathbb{N}$ and $0 \leqslant a_i \leqslant 1$, and use compactness.

b) Let $P \subset \mathbb{R}^n$ be a lattice polytope. Show that there exists an integer $N_0 \geqslant 1$ such that for every $N \geqslant N_0$ and every vertex $\mathbf{v}$ of P, the semigroup $\operatorname{Cone}(NP -$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$N\mathbf{v}) \cap \mathbb{Z}^n$ is generated by the elements $\mathbf{m} - N\mathbf{v}$ with $\mathbf{m} \in NP \cap \mathbb{Z}^n$. HINT: Apply a) to the cone $\mathrm{Cone}(NP - N\mathbf{v}) = \mathrm{Cone}(P - \mathbf{v})$.

c) Let $P \subset \mathbb{R}^3$ be the convex hull of the lattice points $0, e_1, e_2, e_1 + e_2 + 3e_3$. Show that P has only 4 lattice points. Show that $2P$ contains the lattice point $e_1 + e_2 + 2e_3$ and that this is not a sum of two lattice points from P. Hence P does not satisfy the condition mentioned on page 444.

Exercise A.4.4 (Faces)**.** Let P be a lattice polytope and let $Q \subset P$ be a *face* of P, that is, there is a linear form ℓ on $\mathbb{R}^n$ so that $P \subset \{\ell \geqslant 0\}$ and $P \cap \{\ell = 0\} = Q$. Show that there is a closed embedding $X_Q \to X_P$. Describe the corresponding embeddings for the examples $\mathbb{P}^1_k$ and $\mathbb{P}^2_k$.

Exercise A.4.5 (Divisors on toric varieties)**.** Let $P \subset \mathbb{R}^n$ be a full-dimensional lattice polytope, and let X be the corresponding projective toric variety. Let $F_1, \ldots, F_s$ be the facets of P. For each i, choose a primitive integral linear form $n_i \in (\mathbb{Z}^n)^\vee$ and an integer a_i such that $F_i = \{\langle \mathbf{m}, n_i\rangle = -a_i\}$ and $P \subset \{\langle \mathbf{m}, n_i\rangle \geqslant -a_i\}$.

Fix i. For each vertex $\mathbf{v}$ of P, define $D_{i,\mathbf{v}} \subset X_\mathbf{v}$ as follows. If $\mathbf{v} \notin F_i$, set $D_{i,\mathbf{v}} = \varnothing$. If $\mathbf{v} \in F_i$, let $C_{i,\mathbf{v}} = C_\mathbf{v} \cap \{\, \mathbf{m} \in \mathbb{R}^n \mid \langle \mathbf{m}, n_i\rangle = 0 \,\}$, and let $D_{i,\mathbf{v}} \subset X_\mathbf{v} = \mathrm{Spec}(k[C_\mathbf{v}])$ be the closed subscheme defined by the kernel of the surjection $k[C_\mathbf{v}] \to k[C_{i,\mathbf{v}}]$ that sends $\mathbf{x}^\mathbf{m}$ to $\mathbf{x}^\mathbf{m}$ if $\mathbf{m} \in C_{i,\mathbf{v}} \cap \mathbb{Z}^n$ and to 0 otherwise.

a) Show that if $\mathbf{v} \in F_i$, then the defining ideal of $D_{i,\mathbf{v}}$ is the monomial ideal generated by the $\mathbf{x}^\mathbf{m}$ with $\langle \mathbf{m}, n_i\rangle > 0$.
b) Show that $D_{i,\mathbf{v}}$ has codimension 1 in $X_\mathbf{v}$ whenever $\mathbf{v} \in F_i$.
c) Show that the subschemes $D_{i,\mathbf{v}}$ glue to a prime divisor $D_i \subset X$.
d) Show that $X - (D_1 \cup \cdots \cup D_s) \simeq \mathrm{Spec}(k[\mathbb{Z}^n])$.
e) For $\mathbf{m} \in \mathbb{Z}^n$, regard $\mathbf{x}^\mathbf{m}$ as a rational function on X and show that $\mathrm{div}(\mathbf{x}^\mathbf{m}) = \sum_{i=1}^s \langle \mathbf{m}, n_i\rangle D_i$.
f) Deduce that there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \bigoplus_{i=1}^s \mathbb{Z}D_i \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Exercise A.4.6. Let $P \subset \mathbb{R}^n$ be a full-dimensional lattice polytope, and let X be the associated projective toric variety. For vertices $\mathbf{v}, \mathbf{w}$ of P, define the affine toric varieties $X_\mathbf{v}$ and $X_{\mathbf{v},\mathbf{w}} \subset X_\mathbf{v}$. Consider the invertible sheaf L obtained by gluing the structure sheaves $\mathcal{O}_{X_\mathbf{v}}$ using the transition functions $g_{\mathbf{v},\mathbf{w}} = \mathbf{x}^{\mathbf{w}-\mathbf{v}}$ on $X_{\mathbf{v},\mathbf{w}}$.

a) Show that the functions $\mathbf{x}^{\mathbf{w}-\mathbf{v}}$ satisfy the cocycle condition.
b) Show that a lattice point $\mathbf{m} \in \mathbb{Z}^n$ lies in P if and only if $\mathbf{m} - \mathbf{v} \in C_\mathbf{v}$ for every vertex $\mathbf{v} \in P$.
c) For each lattice point $\mathbf{m} \in P \cap \mathbb{Z}^n$, show that the monomials $\mathbf{x}^{\mathbf{m}-\mathbf{v}} \in k[C_\mathbf{v}]$ glue to a global section $s_\mathbf{m} \in \Gamma(X, L)$.
d) Show that the sections $s_\mathbf{m}$, for $\mathbf{m} \in P \cap \mathbb{Z}^n$, form a basis of $\Gamma(X, L)$. Show that L is globally generated, and that the associated morphism $X \to \mathbb{P}^N$ is the projective embedding defined in the text.

Exercise A.4.7 (Smoothness of toric varieties)**.** Let $C \subset \mathbb{R}^n$ be a full-dimensional rational polyhedral cone, and write $S = C \cap \mathbb{Z}^n$, $X = \mathrm{Spec}\, k[S]$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_r \in \mathbb{Z}^n$ be the

primitive generators of the rays of C. Let $\mathfrak{m} \subset k[S]$ be the ideal generated by all monomials $\mathbf{x}^{\mathbf{u}}$ with $0 \neq \mathbf{u} \in S$, and let $x \in X$ be the corresponding closed point. We will show that X is smooth if and only if the primitive generators of C form a basis of $\mathbb{Z}^n$.

a) Show that if $r = n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form a basis of $\mathbb{Z}^n$, then $k[S] \simeq k[t_1, \ldots, t_n]$, so X is smooth.
b) Call an element $\mathbf{u} \in S - \{0\}$ *indecomposable* if it cannot be written as $\mathbf{u} = \mathbf{u}' + \mathbf{u}''$ with nonzero $\mathbf{u}', \mathbf{u}'' \in S$. Show that each primitive generator $\mathbf{u}_i$ is indecomposable.
c) Show that the classes of the monomials $\mathbf{x}^{\mathbf{u}}$, for $\mathbf{u}$ indecomposable, form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Deduce that $\dim T_x X$ is equal to the number of indecomposable elements of S.
d) Now assume that X is smooth. Since $\dim X = n$, deduce that S has exactly n indecomposable elements. Conclude that $r = n$ and that the only indecomposable elements of S are $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Hence show that $S = \mathbb{Z}_{\geqslant 0}\mathbf{u}_1 + \cdots + \mathbb{Z}_{\geqslant 0}\mathbf{u}_n$. Show that this implies that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ form a basis of $\mathbb{Z}^n$. HINT: If they span a proper sublattice, look for a nonzero lattice point in $\{\, a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \mid 0 \leqslant a_i < 1 \,\}$.
e) Now let $P \subset \mathbb{R}^n$ be a full-dimensional lattice polytope. Show that the associated projective toric variety X is smooth if and only if, for every vertex $\mathbf{v}$ of P, the primitive generators of the rays of $C_{\mathbf{v}}$ form a basis of $\mathbb{Z}^n$.

Exercise A.4.8 (Canonical divisor of a toric variety)**.** Let X be a smooth projective toric variety defined by a lattice polytope $P \subset \mathbb{R}^n$. For each codimension 1 face F, let D_F be the corresponding prime divisor. We will show that $K_X = -\sum_F D_F$.

a) Let $T = \operatorname{Spec} k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \subset X$. Show that the rational n-form
$$\eta = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$
restricts to a nowhere vanishing regular n-form on T.
b) Fix a vertex v of P, and let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ be the primitive generators of C_v. Then $k[C_v] = k[y_1, \ldots, y_n]$ for $y_i = \mathbf{x}^{\mathbf{m}_i}$. Show that the faces containing v may be numbered $F_1, \ldots, F_n$ such that $D_{F_i} \cap X_v = V(y_i)$ for each i.
c) Show that
$$\eta = \pm\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n} = \pm\frac{dy_1 \wedge \cdots \wedge dy_n}{y_1 \cdots y_n}.$$
HINT: If $y_j = \prod_i x_i^{a_{ij}}$, then $\det(a_{ij}) = \pm 1$ and $\frac{dy_j}{y_j} = \sum_i a_{ij}\frac{dx_i}{x_i}$.
d) Show that $\operatorname{div}(\eta)|_{X_v} = -(D_{F_1} + \cdots + D_{F_n})|_{X_v}$.
e) Show that if a facet F does not contain v, then $D_F \cap X_v = \varnothing$. Conclude that $\operatorname{div}(\eta) = -\sum_F D_F$, and hence that $K_X = -\sum_F D_F$.

A.5 Grassmannians

Grassmannian varieties are important examples in algebraic geometry. Just like projective space parameterizes 1-dimensional subspaces of a vector space, Grassmannians parameterize

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

linear subspaces of any fixed dimension. For integers d and n with $0 \leqslant d \leqslant n$, there is a scheme $\mathrm{Gr}(d,n)$ of finite type over $\mathbb{Z}$ such that for any field k, the k-points are given by

$$\mathrm{Gr}(d,n)(k) = \big\{ L \subset k^n \;\mid\; \dim_k L = d \big\}. \tag{A.7}$$

For $d = 1$, this is just projective space $\mathbb{P}^{n-1}$.

Let k be a field. Fix integers d and n with $0 \leqslant d \leqslant n$ and set $r = n - d$. There are two equivalent ways of thinking about a d-dimensional linear space $L \subset k^n$. One may regard it as a subspace, or as the kernel of a surjective linear map $k^n \to k^r$. In the present discussion, the latter perspective will be turn out to be more useful, because such a surjection can be represented by a $r \times n$-matrix A of rank r. Thus A defines the subspace $L = \mathrm{Ker}(A)$. This description is not unique: if $g \in \mathrm{GL}_r(k)$ then gA has the same kernel, so the kernel depends only on the $\mathrm{GL}_r(k)$-orbit of A. The following proposition makes this precise.

Proposition A.24. The assignment $A \mapsto \mathrm{Ker}(A)$ defines a bijection between

$$\{r \times n \text{ matrices of rank } r \text{ over } k\} \big/ \mathrm{GL}_r(k)$$

and the set of d-dimensional subspaces of k^n.

Proof Every d-dimensional subspace $L \subset k^n$ is the kernel of a surjective map $k^n \twoheadrightarrow k^r$, so the map is surjective. If A and B have rank r and $\mathrm{Ker}(A) = \mathrm{Ker}(B) = L$, then both maps factor through isomorphisms $\overline{A}, \overline{B} \colon k^n/L \simeq k^r$. Hence $B = gA$, where $g = \overline{B}\,\overline{A}^{-1} \in \mathrm{GL}_r(k)$. Hence two full-rank matrices define the same d-plane if and only if they differ by left multiplication by an element of $\mathrm{GL}_r(k)$. □

Example A.25. The two matrices

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \tag{A.8}$$

give rise to the same 2-dimensional subspace in k^4, as the second matrix is obtained from the first by row operations. △

Based on this, we would like to view the Grassmannian as a sort of quotient space

$$(\mathbb{A}^{rn} - Z) \,/\, \mathrm{GL}_r(k) \tag{A.9}$$

where $\mathbb{A}^{rn}$ parameterizes the $r \times n$-matrices and Z is the closed subset of matrices of rank less than r.

One can understand this quotient space a bit better using row reduction. The basic idea is that any matrix A of rank r will be equivalent to some matrix with the identity matrix in r of its n columns. That means that we may write the quotient space A.9 as a union of quotients U_I/GL_r where $U_I \subset \mathbb{A}^{rn}$ denotes the set of matrices with identity matrix in the columns indexed by I. Note that U_I is naturally an affine space of dimension $nr - r^2 = r(n-r)$. It therefore makes sense to try to construct $\mathrm{Gr}(d,n)$ by gluing together such affine spaces.

A.5.0 The gluing construction of the Grassmannian. In the discussion above, we considered linear spaces over a fixed field k. To get a scheme which solves our problem for all fields simoultaneously, we work over the base ring $\mathbb{Z}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Let

$$\mathbb{A}^{rn} = \operatorname{Spec} \mathbb{Z}[x_{ij} \mid 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n].$$

We think of $\mathbb{A}^{rn}$ as the affine space parameterizing $r \times n$-matrices.

Write $M = (x_{ij})$ for the $r \times n$-matrix of indeterminates. If $I \subset \{1, \dots, n\}$ is a subset of size r and A is an $r \times n$-matrix, we let A_I denote the $r \times r$ submatrix of A given by the columns in I.

For each subset $I = \{j_1 < \cdots < j_r\}$ of size r, let $U_I \subset \mathbb{A}^{rn}$ be the closed subscheme defined by the r^2 linear equations $x_{a,j_b} - \delta_{a,b} = 0$ for $1 \leqslant a, b \leqslant r$. That is U_I parameterizes the matrices with the identity matrix in the I-columns. Since these are r^2 independent linear equations, U_I is an affine space of dimension

$$rn - r^2 = r(n-d) = d(n-d).$$

There are $\binom{n}{r}$ affine spaces in total, one for each size r subset I.

We write $M^{(I)}$ for the universal matrix on U_I, i.e. the $r \times n$ matrix obtained from M by setting the I-columns equal to the identity.

Example A.26. When $n = 4$ and $r = 2$, the Grassmannian $\operatorname{Gr}(2,4)$ is covered by the six affine spaces, each isomorphic to $\mathbb{A}^4$. In suggestive notation:

$$U_{12} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{bmatrix}, \quad U_{13} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} 1 & x_{12} & 0 & x_{14} \\ 0 & x_{22} & 1 & x_{24} \end{bmatrix},$$
$$U_{14} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} 1 & x_{12} & x_{13} & 0 \\ 0 & x_{22} & x_{23} & 1 \end{bmatrix}, \quad U_{23} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} x_{11} & 1 & 0 & x_{14} \\ x_{21} & 0 & 1 & x_{24} \end{bmatrix},$$
$$U_{24} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ x_{21} & 0 & x_{23} & 1 \end{bmatrix}, \quad U_{34} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & x_{22} & 0 & 1 \end{bmatrix}.$$

△

For two subsets of size r, I and J, define the open subscheme

$$U_{I,J} = D(\det M_J^{(I)}) \subset U_I.$$

An R-point of $U_{I,J}$ is an $r \times n$ matrix A over R whose I-columns form the identity matrix and whose J-minor A_J is invertible. For such a matrix A, the matrix $A_J^{-1}A$ has the identity matrix in the J-columns, so it defines an R-point of $U_{J,I}$. Therefore, for every ring R we get a bijection

$$U_{I,J}(R) \longrightarrow U_{J,I}(R), \quad A \mapsto (A_J)^{-1}A. \tag{A.10}$$

This construction is functorial in R, hence by Yoneda's Lemma it comes from an isomorphism of affine schemes

$$\tau_{IJ} \colon U_{I,J} \simeq U_{J,I}.$$

These isomorphisms moreover satisfy the gluing condition

$$\tau_{IK} = \tau_{JK} \circ \tau_{IJ}. \tag{A.11}$$

Indeed, both sides of the equation, when applied to a matrix A with the identity matrix in the I-columns, row-reduce A to a matrix with the identity matrix in the K-columns. The

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

left-hand side does this directly, while the right-hand side does so in two steps: first to the identity in columns J, then to the identity in columns K.

More algebraically, for $A \in U_{I,J}(R) \cap U_{I,K}(R)$, we have

$$\tau_{JK}(\tau_{IJ}(A)) = \tau_{JK}(A_J^{-1}A) = \big((A_J^{-1}A)_K\big)^{-1}(A_J^{-1}A).$$

Since $(A_J^{-1}A)_K = A_J^{-1}A_K$, this equals $(A_J^{-1}A_K)^{-1}(A_J^{-1}A) = A_K^{-1}A = \tau_{IK}(A)$.

It follows that the affine spaces U_I glue to a scheme, which we call the *Grassmannian* $\mathrm{Gr}(d,n)$.

Let $Z \subset \mathbb{A}^{rn}$ be the closed subset defined by all the $r \times r$-minors of the matrix of variables M. There is a 'quotient morphism'

$$\pi\colon \mathbb{A}^{rn} - Z \longrightarrow \mathrm{Gr}(r,n). \tag{A.12}$$

This is constructed by gluing together local morphisms

$$\pi_I\colon D(\det M_I) \longrightarrow U_I, \tag{A.13}$$

which are induced by the natural transformation

$$D(\det M_I)(R) \longrightarrow U_I(R), \quad A \mapsto A_I^{-1}A$$

Here, an element of $D(\det M_I)(R)$ is a matrix A with $\det A_I$ invertible in R, so the map is well defined.

The morphisms π_I are moreover compatible with the gluing isomorphisms. If $A \in \mathbb{A}^{rn}(R)$ has both A_I and A_J invertible, then

$$\tau_{IJ}\left(A_I^{-1}A\right) = \big((A_I^{-1}A)_J\big)^{-1}(A_I^{-1}A) = (A_I^{-1}A_J)^{-1}(A_I^{-1}A) = A_J^{-1}A = \pi_J(A)$$

Hence the π_I glue to a global morphism π in (A.12).

Proposition A.27. For each I, the map $A \mapsto A_I,\ A_I^{-1}A$ defines an isomorphism

$$D(\det M_I) \simeq \mathrm{GL}_r \times U_I, \qquad A \longmapsto (A_I,\ A_I^{-1}A)$$

Proof The inverse is the multiplication map

$$\mathrm{GL}_r \times U_I \longrightarrow D(\det M_I), \qquad (g,B) \longmapsto gB.$$

Indeed, if $B \in U_I(R)$ then $B_I = I_r$, so $(gB)_I = g$, which is invertible. The two formulas are inverse to one another. Indeed, one composition is $A \mapsto (A_I,\ A_I^{-1}A) \to A_I(A_I^{-1}A) = A$ and the other is $(g,B) \mapsto gB \mapsto ((gB)_I,\ (gB)_I^{-1}gB) = (g,B)$. □

This means that the map $\pi\colon \mathbb{A}^{rn} - Z \to \mathrm{Gr}(d,n)$ is a so-called GL_r-*bundle* over $\mathrm{Gr}(d,n)$. It behaves exactly like the quotient space by the action of GL_r.

Corollary A.28. If k is a field, then the k-points of $\mathrm{Gr}(d,n)$ are naturally identified with the d-dimensional linear subspaces of k^n.

Proof By Proposition A.27, two k-points of $\mathbb{A}^{rn} - Z$ map to the same point of $\mathrm{Gr}(d,n)$ if and only if they differ by left multiplication by an element of $\mathrm{GL}_r(k)$. Now apply Proposition A.24. □

Hence, for a field k, the k-points are precisely the equivalence classes $[M]$ of $r \times n$-matrices, where we say $[M] = [M']$ if $M' = gM$ for some $g \in \mathrm{GL}_r(k)$. For instance, for the two matrices in (A.8) we have

$$\left[\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}\right].$$

In this way, the variables x_{ij}, $i = 1, \dots, r$, $j = 1, \dots, n$ can be considered as *generalized homogeneous coordinates* on $\mathrm{Gr}(d, n)$.

A.5.0 The Plucker embedding. Let $\mathbb{P}^{\binom{n}{r}-1}$ denote the projective space with homogeneous coordinates y_I, one for each size r subset I. We will construct a closed embedding

$$\rho\colon \mathrm{Gr}(d, n) \longrightarrow \mathbb{P}^{\binom{n}{r}-1}. \tag{A.14}$$

Locally, consider the morphism $\rho_I\colon U_I \to \mathbb{P}^{\binom{n}{r}-1}$ defined by sending an R-point $A \in U_I(R)$ to the projective point

$$\rho_I(A) = (\det A_J)_{J \subset \{1,\dots,n\}}.$$

That is, ρ_I sends a matrix to all of its $r \times r$-minors. This is well-defined because $\det A_I = 1$, so the image is contained in the open set $D_+(y_I)$.

This assignment is compatible with the gluing isomorphisms τ_{IJ}. If $A \in U_{I,J}(A)$, then the determinant $\det A_J$ is invertible, and since the determinants of $\tau_{IJ}(A) = A_J^{-1}A$ all get multiplied with the invertible factor $\det A_J$, we have $\rho_J(\tau_{IJ}(A)) = \rho_J(A)$. Therefore, the morphisms ρ_I glue to a morphism ρ as in (A.14).

Proposition A.29. The morphism ρ is a closed embedding.

Proof It suffices to show that for each I, the restricted morphism ρ_I is a closed embedding. After renumbering the coordinates, we may assume that $I = \{1, \dots, r\}$. Then an R-point $A \in U_I(R)$ is represented by a matrix of the form

$$A = \begin{pmatrix} 1 & & & x_{1,r+1} & \dots & x_{1,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & x_{r,r+1} & \dots & x_{r,n} \end{pmatrix}$$

and the morphism $\mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1}$ sends A to all of its $r \times r$-minors (excluding the minor of A_I, which is equal to 1). It is clear that the set of all minors contains (up to ± 1) all the variables x_{ij} with $j \notin I$. Hence $\rho_I\colon \mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1}$ is (up to ± 1) given by the graph of a morphism $\mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1-r(n-r)}$, and hence is a closed embedding (Example 4.23). □

This means that $\mathrm{Gr}(r, n)$ embeds as a closed subscheme of $\mathbb{P}^{\binom{n}{r}-1}$. Its defining ideal consists of all the polynomial relations between the $r \times r$-minors of the generic matrix (x_{ij}). This ideal is generated by quadratic polynomials, the *Plücker quadrics* (see (Dolgachev, 2003, Corollary 2.5)). We will not need this explicit description in what follows. In fact, the Grassmannian is probably the best example of a projective variety that is not best studied by its equations in some projective space.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example A.30. The Grassmannian $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ is defined by the quadric

$$y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23} = 0.$$

This identity comes from the following relation between the minors of a 2×4 matrix:

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} = 0.$$

There are no further relations because $\mathrm{Gr}(2,4)$ and the quadric above are integral and of the same dimension. △

If R is a ring, we define $\mathrm{Gr}_R(d,n) \simeq \mathrm{Gr}_{\mathbb{Z}}(d,n) \times_{\mathbb{Z}} \operatorname{Spec} R$. The main case of interest is when $R = k$ is a field.

Proposition A.31. For a field k, the Grassmannian $\mathrm{Gr}_k(d,n)$ is a smooth projective variety of dimension

$$\dim \mathrm{Gr}_k(d,n) = d(n-d).$$

Proof By construction, $\mathrm{Gr}_k(d,n)$ has an affine covering of affine spaces $\mathbb{A}_k^{r(n-r)}$, so it is smooth of dimension $r(n-d) = d(n-d)$. □

A.5.0 The universal quotient bundle. On each chart U_I the matrix $M^{(I)}$ defines an $\mathcal{O}_{U_I}$-linear map $q_I \colon \mathcal{O}_{U_I}^{\oplus n} \to \mathcal{O}_{U_I}^{\oplus r}$ Because the I-columns form the identity matrix, q_I is surjective. We will glue these to a surjective map of locally free sheaves

$$q \colon \mathcal{O}_{\mathrm{Gr}(d,n)}^{\oplus n} \longrightarrow \mathcal{Q}^{\oplus r}$$

To do this, restrict q_I to the open set $U_{I,J}$ and transport using the isomorphism $\tau_{JI}^\sharp \colon \mathcal{O}(U_{I,J}) \simeq \mathcal{O}(U_{J,I})$. This gives a surjective $\mathcal{O}(U_{J,I})$-linear map

$$\tau_{JI}^\sharp(q_I) \colon \mathcal{O}(U_{J,I})^{\oplus n} \longrightarrow \mathcal{O}(U_{J,I})^{\oplus r}.$$

represented by the matrix $\tau_{JI}^\sharp(M^{(I)})$. The key point is that the isomorphism τ_{JI} is determined by the entries of the matrix $\left(M_I^{(J)}\right)^{-1} M^{(J)}$. More precisely, we have the following relation

$$\tau_{JI}^\sharp(M^{(I)}) = \left(M_I^{(J)}\right)^{-1} M^{(J)}.$$

Let $\phi_{IJ} \colon \mathcal{O}(U_{J,I})^{\oplus r} \simeq \mathcal{O}(U_{J,I})^{\oplus r}$ be multiplication by the matrix $M_I^{(J)}$. This is related to the map q_J by $q_J = \phi_{IJ} \circ \tau_{JI}^\sharp(q_I)$. Equivalently, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}(U_{J,I})^{\oplus n} & \xrightarrow{\tau_{JI}^\sharp(q_I)} & \mathcal{O}(U_{J,I})^{\oplus r} \\ \| & & \downarrow{\scriptstyle \phi_{IJ}} \\ \mathcal{O}(U_{J,I})^{\oplus n} & \xrightarrow[q_J]{} & \mathcal{O}(U_{J,I})^{\oplus r} \end{array}$$

We next check that the ϕ_{IJ} satisfy the cocycle relation. On the triple overlap $U_{K,I} \cap U_{K,J}$, the composite $\phi_{JK} \circ \tau_{KJ}^\sharp(\phi_{IJ})$ is multiplication by $M_K^{(J)}\, \tau_{KJ}^\sharp(M_I^{(J)})$, while ϕ_{IK} is multiplication by $M_I^{(K)}$. Since the I-columns of $M^{(J)}$ are $M_I^{(J)}$, the identity $\tau_{KJ}^\sharp(M^{(J)}) =$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\left(M_K^{(J)}\right)^{-1} M^{(K)}$ implies

$$\tau_{KJ}^\sharp(M_I^{(J)}) = \left(M_K^{(J)}\right)^{-1} M_I^{(K)}.$$

Multiplying by $M_K^{(J)}$ gives

$$\phi_{JK} \circ \tau_{KJ}^\sharp(\phi_{IJ}) = \phi_{IK}.$$

From this, we see that the sheaves $\mathcal{O}_{U_I}^{\oplus r}$ can be glued to a locally free sheaf $\mathcal{Q}$ using the isomorphisms ϕ_{IJ} on the overlaps, and that the maps q_I glue to an $\mathcal{O}_{\mathrm{Gr}(d,n)}$-linear map

$$q\colon \mathcal{O}_{\mathrm{Gr}(d,n)}^{\oplus n} \longrightarrow \mathcal{Q}.$$

Since $q|_{U_I} = q_I$ for each I, the sheaf $\mathcal{Q}$ is locally free of rank r and the map q is surjective.

Define $\mathcal{S} = \mathrm{Ker}(q)$. Then $\mathcal{S}|_{U_I} = \mathrm{Ker}(q_I)$, so $\mathcal{S}$ is locally free of rank d. Thus we obtain a short exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathrm{Gr}(d,n)}^{\oplus n} \xrightarrow{q} \mathcal{Q} \longrightarrow 0.$$

The sheaf $\mathcal{Q}$ is called the *universal quotient bundle*, and $\mathcal{S}$ the *universal subbundle*.

A.5.0 The Grassmannian functor. Let r and n be non-negative integers. Define for each scheme T, the set

$$\mathbf{G}_{r,n}(T) = \left\{ \begin{array}{l} \text{quotients } q\colon \mathcal{O}_T^{\oplus n} \to \mathcal{Q} \to 0 \\ \text{where } \mathcal{Q} \text{ is locally free of rank } r \end{array} \right\} / \sim$$

where two quotients $q\colon \mathcal{O}_T^{\oplus n} \to \mathcal{Q}$ and $q'\colon \mathcal{O}_T^{\oplus n} \to \mathcal{Q}'$ are defined to be equivalent if they have the same kernel, as subsheaves of $\mathcal{O}_T^{\oplus n}$.

If $f\colon S \to T$ is a morphism, then pullback defines a map of sets

$$\mathbf{G}_{r,n}(T) \longrightarrow \mathbf{G}_{r,n}(S), \qquad [\mathcal{O}_T^n \to \mathcal{Q}] \mapsto [\mathcal{O}_S^n \to f^*\mathcal{Q}].$$

This makes sense because $f^*\mathcal{O}_T = \mathcal{O}_S$ and f^* is right-exact and preserves being locally free. In this way, $\mathbf{G}_{r,n}(T)$ is a functor from $\mathsf{Sch}^{\mathrm{op}}$ to Sets.

Lemma A.32. Let T a scheme, let Q be a locally free sheaf of rank r and let $\mathcal{O}_T^n \to Q$ be a surjective map of $\mathcal{O}_X$-modules. Then for any point $t \in T$, there exists an open neighborhood V of t and a subset $I \subset \{1, \ldots, n\}$ of size r such that the induced map $\mathcal{O}_V^r \to Q|_V$ via the I-th coordinate inclusion $\mathcal{O}_V^r \to \mathcal{O}_V^n$ is an isomorphism.

Proof Since Q is locally free of rank r, after shrinking around t, we may assume $Q|_V = \mathcal{O}_V^r$. The quotient map $\mathcal{O}_V^n \to Q|_V = \mathcal{O}_V^r$ is given by an $r \times n$ matrix A with entries in $\mathcal{O}_V(V)$.

Since the map is surjective and $Q|_V$ has rank r, the matrix A has rank r at every point of V. By shrinking V if necessary, there exists a subset I of size r such that the $r \times r$ submatrix A_I (columns indexed by I) is invertible on V. The composition $\mathcal{O}_V^r \xrightarrow{\iota_I} \mathcal{O}_V^n \xrightarrow{A} \mathcal{O}_V^r$ is then given by the matrix A_I, which is invertible. □

The next theorem says that the Grassmannian $\mathrm{Gr}(d,n)$ represents the functor $\mathbf{Gr}_{d,n}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Theorem A.33. For every scheme T there is a natural bijection

$$\mathrm{Hom}(T, \mathrm{Gr}(d,n)) = \mathbf{Gr}_{d,n}(T).$$

Proof Given a morphism $\phi\colon T \to \mathrm{Gr}(r,n)$, we obtain a quotient $\mathcal{O}_T^n \to \phi^*\mathcal{Q} \to 0$ where $\phi^*\mathcal{Q}$ is locally free of rank r. This defines a map

$$\Phi\colon \mathrm{Hom}(T, \mathrm{Gr}(r,n)) \longrightarrow \mathbf{G}_{r,n}(T).$$

To define the inverse, consider a quotient $q\colon \mathcal{O}_T^n \to Q$. We construct a morphism $f\colon T \to \mathrm{Gr}(r,n)$ as follows.

By Lemma A.32, we may cover T by open affine subschemes $V = \operatorname{Spec} R$ such that $Q|_V \simeq \mathcal{O}_V^r$, and such that for some size r subset $I = I(V)$, the composition $\mathcal{O}_V^I \hookrightarrow \mathcal{O}_V^n \to Q|_V \simeq \mathcal{O}_V^r$ is an isomorphism. Let $\mathscr{B}$ denote the collection of all such open subsets.

Note that the composition $\mathcal{O}_V^n \to Q|_V \simeq \mathcal{O}_V^r$ is defined by an $r \times n$-matrix with entries in R. Moreover, our assumption implies that $\det A_I$ is invertible in R. Therefore, the normalized matrix $B = A_I^{-1}A$ determines a morphism $f_{V,I}\colon V \to U_I \subset \mathrm{Gr}(d,n)$. Note that $f_{V,I}$ is in fact the unique morphism which pulls back the canonical surjection $\mathcal{O}_{U_I}^n \to \mathcal{O}_{U_I}^r$ to the quotient $\mathcal{O}_V^n \to \mathcal{O}_V^r$ defined by the matrix B. We will need this fact below.

The morphism $f_{V,I}$ is independent of the choice of isomorphism $Q|_V \simeq \mathcal{O}_V^r$: a different isomorphism gives rise to a matrix A' which is related to A by $A' = gA$ where $g \in \mathrm{GL}_r(R)$. The corresponding normalized is then left unchanged:

$$(gA)_I^{-1}(gA) = A_I^{-1}A = B.$$

The morphism $f_{V,I}\colon V \to \mathrm{Gr}(d,n)$ is also independent of the choice of index set I. If J is another size r subset so that both A_I and A_J are invertible, then

$$A_J^{-1}A = (A_I^{-1}A_I)(A_I^{-1}A) = B_I^{-1}B$$

On the level of morphisms, this implies that $f_{V,J} = \tau_{IJ} \circ f_{V,I}$, and so the two morphisms $V \to U_I$ and $V \to U_J$ are identified when mapping into $\mathrm{Gr}(d,n)$. We will therefore write $f_V\colon V \to \mathrm{Gr}(d,n)$ for the morphism $f_{V,I}$.

The morphisms f_V glue to a morphism $f\colon T \to \mathrm{Gr}(d,n)$. Indeed, for two $V, V' \in \mathscr{B}$, we may cover $V \cap V'$ by open affines $W \in \mathscr{B}$ which are distinguished in both V and V'. For such a W, the map $\mathcal{O}_V^n \to \mathcal{O}_V^r \to \mathcal{Q}|_V$ restricts to $\mathcal{O}_W^n \to \mathcal{O}_W^r \to \mathcal{Q}|_W$, so the above receipe for constructing f_W implies that $f_W = f_V|_W$. Since this happens for any W, we have $f_V|_{V\cap V'} = f_{V'}|_{V\cap V'}$ and so the morphisms glue to a morphism $f\colon T \to \mathrm{Gr}(d,n)$. This defines the inverse map

$$\Psi\colon \mathbf{G}_{r,n}(T) \longrightarrow \mathrm{Hom}(T, \mathrm{Gr}(r,n)), \quad \Psi(q) = f.$$

Let us check that these assignments indeed are inverses:

$\Phi \circ \Psi = \mathrm{id}$: Starting with a quotient $q\colon \mathcal{O}_T^n \to Q$, we construct the morphism $f\colon T \to \mathrm{Gr}(d,n)$ and we need to check that q is equivalent to the pulled back quotient $\mathcal{O}_T^n \to f^*\mathcal{Q}$. That is, we need to show that $\mathrm{Ker}(\mathcal{O}_T^n \to Q) = \mathrm{Ker}(\mathcal{O}_T^n \to f^*\mathcal{Q})$. Since these are subsheaves of the same sheaf $\mathcal{O}_T^n$, it suffices to show that they agree over an affine covering. Taking an affine V as above, which maps into $U_I \subset \mathrm{Gr}(d,n)$ for some I, the universal

quotient $\mathcal{O}_{U_I}^n \to \mathcal{Q}|_{U_I} = \mathcal{O}_{U_I}^r$ pulls back to the quotient $\mathcal{O}_V^n \to \mathcal{O}_V^r$ defined by the matrix $B = A_I^{-1}A$. As A_I is invertible, we have $\operatorname{Ker} B = \operatorname{Ker} A$, and hence

$$\operatorname{Ker}(\mathcal{O}_T^n \to f^*\mathcal{Q})|_V = \widetilde{\operatorname{Ker} B} = \widetilde{\operatorname{Ker} A} = \operatorname{Ker}(\mathcal{O}_T^n \to Q)|_V.$$

$\Psi \circ \Phi = \mathrm{id}$: Starting with a morphism $f\colon T \to \operatorname{Gr}(d,n)$, consider the quotient $q\colon \mathcal{O}_T^n \to f^*\mathcal{Q}$. Let g denote the morphism constructed by q; we need to show that $f = g$. It suffices to show that $f|_V = g|_V$ for the affines $V = \operatorname{Spec} R$ in an open covering of T. We may therefore assume that $f(V) \subset U_I$ for some I. In that case, f is of the form $V \to U_I$, i.e., defined by a unique $r \times n$-matrix B with entries in R and $B_I = \mathrm{id}$.

On the other hand, B is also the same matrix which defines the kernel of q as $\operatorname{Ker}(q_V) = \operatorname{Ker} B$. Thus by the above receipe for constructing g, we have $f|_V = g|_V$ since the two are determined by the same matrix, and hence $f = g$. □

A.5.0 Lines on hypersurfaces. Grassmannians are important in classical algebraic geometry, especially in questions related to linear spaces contained in subvarieties. Here is one classical application of the Grassmannian $\operatorname{Gr}(2,4)$ to lines on hypersurfaces.

Theorem A.34. Let k be an algebraically closed field and let $d \geqslant 1$. A general degree d hypersurface in $\mathbb{P}^3$

(i) contains no lines if $d > 3$,
(ii) contains only finitely many lines if $d = 3$,
(iii) contains positive-dimensional families of lines if $d \leqslant 2$.

The space parameterizing degree d hypersurfaces in $\mathbb{P}_k^3$ is a projective space $\mathbb{P}_k^N$ of dimension

$$N \;=\; \binom{d+3}{3} - 1.$$

To prove Theorem A.34, we work with the Grassmannian $\operatorname{Gr}(2,4)$ over k, which is a smooth projective variety of dimension 4. The key idea is to use the *incidence variety*

$$V \subset \operatorname{Gr}(2,4) \times_k \mathbb{P}_k^N, \tag{A.15}$$

which parameterizes the pairs $([\ell],[f]) \in \operatorname{Gr}(2,4) \times \mathbb{P}^N$ such that $\ell \subset V_+(f)$, i.e., f vanishes identically on ℓ.

The scheme structure on V is defined as follows. Consider the affine chart $U_{12} \subset \operatorname{Gr}(2,4)$ with affine coordinates u_2, u_3, v_2, v_3. A line ℓ in $\mathbb{P}_k^3$ corresponding to $[\ell] \in U_{12}$ has the form

$$x_0 + u_2x_2 + u_3x_3 = 0, \qquad x_1 + v_2x_2 + v_3x_3 = 0.$$

Consider the restriction map

$$\rho\colon H^0(\mathbb{P}_k^3, \mathcal{O}(d)) \longrightarrow H^0(\ell, \mathcal{O}_\ell(d)) = H^0(\mathbb{P}_k^1, \mathcal{O}(d)),$$

Then saying that $\ell \subset V_+(f)$, means that $f \in \operatorname{Ker}\rho$. Because $H^0(\mathbb{P}_k^1, \mathcal{O}(d))$ has dimension $d+1$, this imposes $d+1$ linear constraints on the coefficients of f. Concretely, if $f = \sum a_{e_0\cdots e_3} x_0^{e_0} \cdots x_3^{e_3}$, then the condition that $f|_\ell$ is identically zero means that the polynomial

$$\sum_{e_0+\cdots+e_3=d} a_{e_0\cdots e_3}(-u_2s - u_3t)^{e_0}(-v_2s - v_3t)^{e_1} s^{e_2} t^{e_3}$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

is the zero polynomial in s and t. Collecting the coefficients, we get a system of $d+1$ equations which are linear in the coefficients $a_{e_0\cdots e_3}$ and of degree d in u_i, v_j. These constraints are independent: after a change of coordinates we may assume that ℓ is given by $x_0 = x_1 = 0$; then restriction sends $f(x_0, x_1, x_2, x_3) \mapsto f(0, 0, x_2, x_3)$ and every homogeneous polynomial of degree d in x_2, x_3 occurs as such a restriction. One checks that these local equations glue on overlaps to define V as a closed subscheme of $\mathrm{Gr}(2,4) \times_k \mathbb{P}^N_k$. By construction, the k-points of V correspond exactly to the pairs $(\ell, [f])$ as in (A.15).

We next show that V is irreducible by considering the projection

$$\pi_1 \colon V \longrightarrow \mathrm{Gr}(2,4).$$

From what we saw above, the fibers of π_1 can be identified with a codimension $d+1$ linear subspace $\mathbb{P}^{N-d-1}$ in $\mathbb{P}^N_k$. In fact, $\pi_1 \colon V \to \mathrm{Gr}(2,4)$ is a Zariski locally trivial projective $\mathbb{P}^{N-d-1}$-fibration over $\mathrm{Gr}(2,4)$. By Exercise 8.11.16, this implies that V is irreducible of dimension

$$\dim V \;=\; \dim \mathrm{Gr}(2,4) + (N-d-1) \;=\; 4 + (N-d-1) \;=\; N-d+3.$$

If we now consider $\pi_2 \colon V \to \mathbb{P}^N_k$, the general fiber has dimension

$$\dim V - \dim \mathbb{P}^N_k \;=\; (N-d+3) - N \;=\; 3-d.$$

From this, we have three cases:

- If $d > 3$ then $\dim V < \dim \mathbb{P}^N_k$. Hence the image $\pi_2(V)$ is a proper closed subset of $\mathbb{P}^N_k$. This means that the general degree-d hypersurface contains no line; containing a line is a codimension $d-3$ condition in the space of hypersurfaces.
- If $d = 3$ then $\dim V = \dim \mathbb{P}^N_k$, so a general fiber of π_2 is 0-dimensional. This means that every cubic surface contains a line. Moreover, π_2 is generically finite, so there is a dense set of hypersurfaces with precisely d lines, where $d = \deg \pi_2$. In fact, the degree is equal to 27; see below.
- If $d < 3$ then $\dim V > \dim \mathbb{P}^N_k$, so π_2 has fibers of positive dimension. For $d = 1$, a plane contains a 2-dimensional family of lines, and for $d = 2$, a smooth quadric surface contains a 1-dimensional family of lines.

Example A.35 (Lines on the Fermat cubic)**.** Assume $\mathrm{char}(k) \neq 3$ and consider the surface

$$X = V_+(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3_k.$$

On the chart $U_{12} \subset \mathrm{Gr}(2,4)$, a line has equations

$$x_0 + u_2 x_2 + v_2 x_3 = 0, \qquad x_1 + u_3 x_2 + v_3 x_3 = 0.$$

Substituting $x_0 = -u_2 s - v_2 t$, $x_1 = -u_3 s - v_3 t$, $x_2 = s$, $x_3 = t$ into the equation gives

$$\begin{aligned} u_2^3 + u_3^3 = 1, \quad u_2^2 v_2 + u_3^2 v_3 = 0, \\ u_2 v_2^2 + u_3 v_3^2 = 0, \qquad v_2^3 + v_3^3 = 1. \end{aligned}$$

One checks that the solutions force either $u_2 = 0$ or $u_3 = 0$. If $u_2 = 0$ then $v_3 = 0$ and $v_2^3 = u_3^3 = 1$, giving $3 \times 3 = 9$ solutions on this chart. Also, the case $u_3 = 0$ gives another 9. Permuting coordinates and repeating the same analysis on the other charts produces a total

of 27 lines on the Fermat cubic. Explicitly, if ω is a primitive cube root of unity, then for each pair (j,k) with $0 \leqslant j,k \leqslant 2$ one obtains three lines:

$$\begin{aligned} x_0 + \omega^k x_1 &= 0, \qquad & x_2 + \omega^j x_3 &= 0, \\ x_0 + \omega^k x_2 &= 0, \qquad & x_1 + \omega^j x_3 &= 0, \\ x_0 + \omega^k x_3 &= 0, \qquad & x_1 + \omega^j x_2 &= 0. \end{aligned}$$

△

A more detailed study of the indicence correspondence shows that $\pi_2 \colon V \to \mathbb{P}^N_k$ is actually finite over the entire open set $U \subset \mathbb{P}^N_k$ consisting of smooth cubic surfaces. Since since the fiber over the Fermat cubic consists of 27 reduced points, this implies that *any* smooth cubic surface contains exactly 27 lines.

Theorem A.36 (Cayley–Salmon). Over an algebraically closed field, every smooth cubic surface $X \subset \mathbb{P}^3$ contains exactly 27 lines.

Exercises

Exercise A.5.1 (Local coordinates)**.** Just as the ratios $x_0/x_i, \ldots, x_n/x_i$ form local affine coordinates on $\mathbb{P}^n$, the affine coordinates of U_I are given by the entries of the matrix $M_I^{-1}M$ where $M = (x_{ij})$ is the generic $r \times n$-matrix of variables. Write $R_I = \mathbb{Z}[M_I^{-1}M] \subset \mathbb{Q}(x_{ij})$ for the $\mathbb{Z}$-algebra generated by the entries in the matrix $M_I^{-1}M$.

a) Show that we may identify $\mathcal{O}_{\mathrm{Gr}(d,n)}(U_I) = R_I$.
b) Show that there is an equality of subrings of $\mathbb{Q}(x_{ij})$,

$$(R_I)_{\det(M_I^{-1}M_J)} = (R_J)_{\det(M_J^{-1}M_I)}.$$

Show that $\mathrm{Gr}(d,n)$ is obtained by gluing the $\operatorname{Spec} R_I$ using these equalities.
c) Consider the R_I-linear map $R_I^{\oplus n} \to R_I^{\oplus r}$ defined by multiplication by $M_I^{-1}M$. Show that the induced maps $\mathcal{O}_{\mathrm{Gr}(d,n)}(U_I)^n \to \mathcal{O}_{\mathrm{Gr}(d,n)}(U_I)^r$ are surjective and glue to the universal quotient $\mathcal{O}^n_{\mathrm{Gr}(d,n)} \to \mathcal{Q}$.

Exercise A.5.2. Let k be a field and write $\mathrm{Gr}(d,n)$ for the Grassmannian over k. Write $\mathcal{S}$ and $\mathcal{Q}$ for the universal sheaves. We will compute the sheaf of differentials $\Omega = \Omega_{\mathrm{Gr}(d,n)/k}$.

a) For $I = \{j_1 < \cdots < j_r\}$, write $K = \{1, \ldots, n\} - I = \{k_1 < \cdots < k_d\}$. Show that $\mathcal{S}|_{U_I}$ is freely generated by the sections

$$s_b = e_{k_b} - \sum_{a=1}^{r} x_{a,k_b} e_{j_a}, \qquad b = 1, \ldots, d,$$

while $\mathcal{Q}|_{U_I}$ is freely generated by the classes $\overline{e}_{j_1}, \ldots, \overline{e}_{j_r}$.
b) Let $d \colon \mathcal{O}_{\mathrm{Gr}(d,n)} \to \Omega$ be the universal derivation and define σ by

$$\mathcal{S} \hookrightarrow \mathcal{O}^{\oplus n}_{\mathrm{Gr}(d,n)} \xrightarrow{d^{\oplus n}} \mathcal{O}^{\oplus n}_{\mathrm{Gr}(d,n)} \otimes \Omega \to \mathcal{Q} \otimes \Omega.$$

Show that while d is not $\mathcal{O}_{\mathrm{Gr}(d,n)}$-linear, the map σ *is* $\mathcal{O}_{\mathrm{Gr}(d,n)}$-linear. HINT: Apply the product rule and use that $\mathcal{S} = \operatorname{Ker}(\mathcal{O}^{\oplus n}_{\mathrm{Gr}(d,n)} \to \mathcal{Q})$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

c) Use the local bases from *a*) to show that $\sigma_{U_I}(s_b) = -\sum_{a=1}^{r} \overline{e}_{j_a} \otimes dx_{a,k_b}$.

d) Viewing σ as a section of $\mathcal{H}om(\mathcal{S}, \mathcal{Q}\otimes\Omega)$, define a map $\widetilde{\sigma}\colon \mathcal{H}om(\mathcal{Q}, \mathcal{S}) \to \Omega$ as follows. Given $\phi\colon \mathcal{Q} \to \mathcal{S}$, compose with σ to obtain $\mathcal{Q} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\sigma} \mathcal{Q}\otimes\Omega$. This gives a section of $\mathcal{H}om(\mathcal{Q}, \mathcal{Q}\otimes\Omega) \simeq \mathcal{Q}^\vee \otimes \mathcal{Q} \otimes \Omega$. Define $\widetilde{\sigma}(\phi)$ by applying $\mathrm{tr}\otimes\mathrm{id}_\Omega$, where $\mathrm{tr}\colon \mathcal{Q}^\vee \otimes \mathcal{Q} \to \mathcal{O}_{\mathrm{Gr}(d,n)}$ is the evaluation map.

Using *c*), show that on U_I a local section $\phi = \sum_{a,b} c_{b,a}\, \overline{e}^{\vee}_{j_a} \otimes s_b$ maps to $\widetilde{\sigma}(\phi) = -\sum_{a,b} c_{b,a}\, dx_{a,k_b}$. Since $\{dx_{a,k_b}\}$ is a basis of $\Omega_{U_I/k}$, conclude that $\widetilde{\sigma}$ is an isomorphism. Hence

$$\Omega_{X/k} \simeq \mathcal{H}om(\mathcal{Q}, \mathcal{S}) \simeq \mathcal{Q}^\vee \otimes \mathcal{S}.$$

Exercise A.5.3 (Linear spaces on a quadric)**.** Let $Q \subset \mathbb{P}^n_k$ be a smooth quadric hypersurface, and for $m \geqslant 0$ let $F_m(Q) \subset \mathrm{Gr}(m+1, n+1)$ be the closed subscheme parametrizing projective m-planes contained in Q.

a) Show that if $Q = V_+(q)$ for a nondegenerate quadratic form q on k^{n+1}, then $F_m(Q)$ is the zero locus of the induced section of $\mathrm{Sym}^2(\mathcal{S}^\vee)$, where $\mathcal{S}$ is the universal subbundle on $\mathrm{Gr}(m+1, n+1)$.

b) Deduce that every irreducible component of $F_m(Q)$ has dimension at least

$$(m+1)(n-m) - \binom{m+2}{2}.$$

c) Let $S \subset \mathbb{P}^3$ be a smooth quadric surface and $Q \subset \mathbb{P}^4$ be a smooth quadric threefold. Show that $F_1(S) \simeq \mathbb{P}^1 \sqcup \mathbb{P}^1$ and $F_1(Q) \simeq \mathbb{P}^3$.

Exercise A.5.4 (Lines on a cubic threefold)**.** Let $X \subset \mathbb{P}^4_k$ be a smooth cubic hypersurface and let $F_1(X) \subset \mathrm{Gr}(2,5)$ be the closed subscheme parametrizing lines contained in X.

a) Show that the cubic equation defining X induces a global section of $\mathrm{Sym}^3(\mathcal{S}^\vee)$ on $\mathrm{Gr}(2,5)$ whose zero locus is precisely $F_1(X)$.

b) Deduce that every irreducible component of $F_1(X)$ has dimension at least 2.

c) * Prove that for every smooth cubic threefold the scheme $F_1(X)$ is integral.

Exercise A.5.5 (Four lines in $\mathbb{P}^3$)**.** Fix a line $\ell \subset \mathbb{P}^3_k$, and consider the closed subset $\Sigma(\ell) = \{\, [\ell'] \in \mathrm{Gr}(2,4) \mid \ell' \cap \ell \neq \varnothing \,\}$. with the reduced scheme structure.

a) Use the Plücker embedding to show that $\Sigma(\ell)$ is a hyperplane section of $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$.

b) Let $\ell_1, \ldots, \ell_4$ be four lines in general position in $\mathbb{P}^3_k$. Show that the set of lines meeting all four is the intersection of $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ with four independent hyperplanes. Deduce from Example A.30 that there are exactly two lines in $\mathbb{P}^3$ meeting all four of the lines $\ell_1, \ldots, \ell_4$.

Exercise A.5.6. The Grassmannnian is a good example of a *moduli space*, which classifies geometric objects (locally free quotients) up to isomorphism. Show that the more naive functor $F\colon \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(T) = \{\text{locally free sheaves of rank } r \text{ on } T\}/\text{isomorphism}$$

is not representable by a scheme. HINT: Show that F does not satisfy the Locality axiom.

Exercise A.5.7 (Locally free sheaves on glued schemes)**.** Let X be obtained by gluing schemes X_i along open subschemes $X_{ij} \subset X_i$ via isomorphisms $\tau_{ij}\colon X_{ij} \simeq X_{ji}$. Assume the cocycle condition: for all i, j, k, the open set in X_i corresponding to the triple intersection is $X_{ijk} = X_{ij} \cap X_{ik} = X_{ij} \cap \tau_{ij}^{-1}(X_{jk})$, and on this open set one has $\tau_{ik} = \tau_{jk} \circ \tau_{ij}$.

Let $g_{ij} \in \mathrm{GL}_r(\mathcal{O}_{X_i}(X_{ij}))$ be matrices such that for all i, j, k, we have $g_{ik} = g_{ij}\tau_{ij}^\sharp(g_{jk})$ on X_{ijk}, Here $\tau_{ij}^\sharp(g_{jk})$ means that every entry of g_{jk} is transported to $X_{ijk} \subset X_i$ using τ_{ij}.

For an open subset $W \subset X$, write $W_i = \iota_i^{-1}(W)$ where $\iota_i\colon X_i \to X$ is the open embedding. Define $\mathcal{E}(W)$ to be the set of tuples $(s_i)_i$, with $s_i \in \mathcal{O}_{X_i}^r(W_i)$, such that on $W_i \cap X_{ij}$ one has $s_i = g_{ij}\tau_{ij}^\sharp(s_j)$.

a) Show that $\mathcal{E}$ is a sheaf, and has a natural structure of an $\mathcal{O}_X$-module.
b) Show that for each i, the projection $(s_k)_k \mapsto s_i$ identifies $\mathcal{E}|_{X_i}$ with $\mathcal{O}_{X_i}^r$. Show that for these trivializations, the g_{ij} are the transition functions of $\mathcal{E}$.

A.6 Gluing together many DVRs

Semi-local rings are rings with finitely many maximal ideals. In the next two examples we describe several such rings and explain how they can be obtained by gluing local schemes together.

Example A.37 (Semi-local rings)**.** Consider the two rings $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$. These are both discrete valuation rings with common fraction field $\mathbb{Q}$ and maximal ideals (2) and (3), respectively. As described in Example 2.29 on page 29, the scheme $X_1 = \operatorname{Spec}\mathbb{Z}_{(2)}$ consists of two points: the closed point x_1 corresponding to (2) and the generic point η_1 corresponding to (0). Likewise, $X_2 = \operatorname{Spec}\mathbb{Z}_{(3)}$ consists of two points x_2 and η_2. The generic points are identified with $\operatorname{Spec}\mathbb{Q}$ via the open immersions $\operatorname{Spec}\mathbb{Q} \to X_i$ for $i = 1, 2$. We may therefore glue X_1 and X_2 along their generic points to obtain a scheme X with one generic point η and two closed points x_1 and x_2.

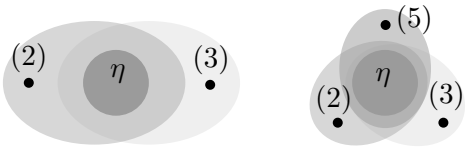
By (5.1), we can compute $\mathcal{O}_X(X)$ as

$$\mathcal{O}_X(X) = \mathcal{O}_X(X_1) \cap \mathcal{O}_X(X_2) \simeq \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}.$$

This intersection is a semi-local ring, with maximal ideals (2) and (3). By the Main Theorem of Affine Schemes (Theorem 4.11) there is a morphism

$$X \to \operatorname{Spec}\big(\mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}\big),$$

and it is left as an exercise to show that this morphism is an isomorphism.



More generally, let $\mathcal{P} = \{p_1, \ldots, p_r\}$ be a finite set of prime numbers. For each $p \in \mathcal{P}$, consider the scheme $X_p = \operatorname{Spec}\mathbb{Z}_{(p)}$. Each X_p has a generic point η_p, namely the image of the canonical open immersion $\operatorname{Spec}\mathbb{Q} \to X_p$. By identifying η_p and η_q via the identity map on $\operatorname{Spec}\mathbb{Q}$, we may glue the schemes X_p together to form a new scheme X.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

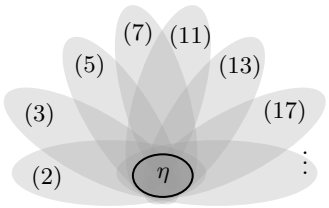


Figure A.6 Gluing together all the schemes $\operatorname{Spec}\mathbb{Z}_{(p)}$

We may compute the global sections of the structure sheaf of the new scheme using the sheaf sequence

$$\begin{array}{ccccc} 0 \to \Gamma(X,\mathcal{O}_X) & \longrightarrow & \prod_{p\in\mathcal{P}}\Gamma(X_p,\mathcal{O}_X) & \longrightarrow & \prod_{p,q\in\mathcal{P}}\Gamma(X_p\cap X_q,\mathcal{O}_X) \\ & & \| \downarrow & & \downarrow \| \\ & & \prod_{p\in\mathcal{P}}\mathbb{Z}_{(p)} & \xrightarrow[\rho]{} & \prod_{p,q\in\mathcal{P}}\mathbb{Q}. \end{array}$$

Here, the map ρ sends an r-tuple $(a_p)_{p\in\mathcal{P}}$ to the sequence $(a_p - a_q)_{p,q\in\mathcal{P}}$. It follows that the kernel of ρ is the intersection

$$A_{\mathcal{P}} = \bigcap_{p\in\mathcal{P}} \mathbb{Z}_{(p)}.$$

This is a semi-local ring whose maximal ideals are $(p)A_{\mathcal{P}}$ for $p \in \mathcal{P}$. There is a canonical morphism $X \to \operatorname{Spec} A_{\mathcal{P}}$, and again we leave it to the reader to verify that this morphism is an isomorphism. △

Example A.38. In Example A.37, we glued together the affine schemes $X_p = \operatorname{Spec}\mathbb{Z}_{(p)}$ for p in a finite set $\mathcal{P}$ of prime numbers. The gluing construction for schemes imposes no finiteness condition on the number of pieces, however, so we may equally well take $\mathcal{P}$ to be the set of all prime numbers.

The resulting scheme $X_{\mathcal{P}}$ is rather peculiar: it is neither affine nor Noetherian (although every point lies in some affine Noetherian neighbourhood). As a scheme over $\mathbb{Z}$, the canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ is bijective and continuous, but it is not a homeomorphism. Moreover, for every open subset $U \subset \operatorname{Spec}\mathbb{Z}$, the induced map on sections $\pi^\sharp\colon \Gamma(U,\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}) \to \Gamma(\pi^{-1}U,\mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism. Equivalently, $\pi^\sharp\colon \mathcal{O}_{\operatorname{Spec}\mathbb{Z}} \to \pi_*(\mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism of sheaves.

As in Example A.37, the scheme $X_{\mathcal{P}}$ is constructed by gluing the schemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ together along their generic points. The computation of global sections, however, changes substantially. The global sections are computed with the help of the sheaf sequence as above, and the kernel of ρ is still $\bigcap_{p\in\mathcal{P}}\mathbb{Z}_{(p)}$, but this intersection is now equal to $\mathbb{Z}$. Indeed, a rational number $\alpha = a/b$ lies in $\mathbb{Z}_{(p)}$ precisely when the denominator b is not divisible by p. So, if α lies in every $\mathbb{Z}_{(p)}$, then b has no nontrivial prime factor. Hence $b = \pm 1$, and therefore $\alpha \in \mathbb{Z}$.

The canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ may be described as follows. Each scheme $\operatorname{Spec}\mathbb{Z}_{(p)}$ maps naturally to $\operatorname{Spec}\mathbb{Z}$ via the inclusion $\mathbb{Z} \subset \mathbb{Z}_{(p)}$. Under this map, the generic point of $\operatorname{Spec}\mathbb{Z}_{(p)}$ goes to the generic point of $\operatorname{Spec}\mathbb{Z}$, and the closed point goes

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

to $(p) \in \operatorname{Spec}\mathbb{Z}$. Since these maps agree on the generic points, they glue to the canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$. This map is a continuous bijection by construction, but it is not a homeomorphism. Indeed, each $\operatorname{Spec}\mathbb{Z}_{(p)}$ is open in $X_{\mathcal{P}}$ by construction, whereas its image in $\operatorname{Spec}\mathbb{Z}$ is not open, since its complement is infinite.

The underlying topological space of $X_{\mathcal{P}}$ is not Noetherian, since the open subsets $\operatorname{Spec}\mathbb{Z}_{(p)}$ form an open cover with no finite subcover. For each prime p, the open subset $U_p = X_{\mathcal{P}} - \{(p)\}$ maps bijectively to $D(p) \subset \operatorname{Spec}\mathbb{Z}$, and $\Gamma(U_p, \mathcal{O}_{X_{\mathcal{P}}}) = \mathbb{Z}[1/p]$. Nevertheless, U_p and $D(p)$ are not isomorphic. △

Exercise A.6.1. Verify the claims in Examples A.37 and A.37 above that X is isomorphic respectively to $\operatorname{Spec}\mathbb{Z}_2 \cap \mathbb{Z}_3$ and to $\operatorname{Spec}A_P$. HINT: Use the uniqueness statement in Proposition 5.4 on page 80.

Exercise A.6.2. Glue $\operatorname{Spec}\mathbb{Z}_{(2)}$ to itself along the generic point to obtain a scheme X. Show that X is not affine. HINT: Show that $\mathcal{O}_X(X) = \mathbb{Z}_{(2)}$.

A.7 The ring $\mathbb{Z}[\sqrt{-5}]$

The ring $A = \mathbb{Z}[x]/(x^2+5)$ has appeared already a few times in the text. In this section, we will compute the class group of $\operatorname{Spec}A$. We consider the ideal $\mathfrak{a} = (2, 1+x) \subset A$. This ideal is prime because

$$A/\mathfrak{a} \simeq \mathbb{Z}[x]/(2, 1+x, x^2+5) \simeq \mathbb{Z}/2, \tag{A.16}$$

which is an integral domain.

Lemma A.39. The ideal $\mathfrak{a} = (2, 1+x)$ is not principal.

Proof Define the *norm* $N\colon A \to \mathbb{Z}_{\geqslant 0}$ by $N(a+bx) = a^2 + 5b^2 = (a+bx)(a-bx)$. It satisfies $N(uv) = N(u)N(v)$ for all $u, v \in A$, and $N(u) = 1$ if and only if u is a unit. There is no element of norm 2 in A, as the equation $a^2 + 5b^2 = 2$ has no integer solutions.

Suppose that $\mathfrak{a} = (f)$ with $f = a + bx$. Multiplication by f on the basis $(1, x)$ has matrix

$$M = \begin{pmatrix} a & -5b \\ b & a \end{pmatrix},$$

whose determinant is $N(f) = a^2 + 5b^2$. Hence as a subgroup, $(f) = fA \subset A$ has index $|\det(M)| = N(f)$, so $|A/(f)| = N(f)$. However, (A.16), $|A/\mathfrak{a}| = 2$, so $N(f) = 2$, which is not possible. Therefore $\mathfrak{a}$ is not principal. □

Lemma A.40. Let $p\colon A^2 \to \mathfrak{a}$ and $s\colon \mathfrak{a} \to A^2$ be defined by

$$p(a,b) = 2a + (1+x)b, \qquad s(y) = \Big(-y,\ y\frac{1-x}{2}\Big).$$

Then $p \circ s = \mathrm{id}$ and hence $A^2 \simeq \mathfrak{a} \oplus \operatorname{Ker} p$.

Proof If $y \in \mathfrak{a}$, then $y = 2u + (1+x)v$ for some $u, v \in A$. A simple substition shows that $s(y) = (-2u - (1-x)v, u(1-x) + 3v)$, so $s(y)$ is a well-defined element of

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

A. Moreover, composing the two maps shows that $p(s(y)) = y$. Therefore the sequence $0 \to \operatorname{Ker} p \to A^2 \xrightarrow{p} \mathfrak{a} \to 0$ splits, giving $A^2 \simeq \mathfrak{a} \oplus \operatorname{Ker} p$. □

These two lemmas together imply that the A-module $\mathfrak{a}$ is projective but not free.

Proposition A.41. The ring $A = \mathbb{Z}[x]/(x^2+5)$ is normal.

Proof Let $\alpha = r + sx \in \mathbb{Q}(x)/(x^2+5)$ with $r, s \in \mathbb{Q}$, and assume that α is integral over A. Then α satisfies a monic polynomial $f(t) \in \mathbb{Z}[t]$. The minimal polynomial of α over $\mathbb{Q}$ is

$$q(t) = t^2 - 2r\,t + (r^2 + 5s^2).$$

Since $q(t)$ divides $f(t)$ in $\mathbb{Q}[t]$, Gauss's Lemma implies that $q(t) \in \mathbb{Z}[t]$. Hence $2r \in \mathbb{Z}$ and $r^2 + 5s^2 \in \mathbb{Z}$. Write $r = m/2$ and $s = n/2$ for integers m, n. Then $r^2 + 5s^2 = (m^2 + 5n^2)/4 \in \mathbb{Z}$ which implies that $m^2 + 5n^2 \equiv 0 \pmod 4$. Since $5 \equiv 1 \pmod 4$, this implies that both m, n are even, hence $r, s \in \mathbb{Z}$. Therefore $\alpha \in A$ and A is integrally closed. □

As we have seen, A is not a UFD, so $\operatorname{Cl}(\operatorname{Spec} A) \neq 0$. We now compute the class group:

Proposition A.42. The class group of $A = \mathbb{Z}[x]/(x^2+5)$ is

$$\operatorname{Cl}(\operatorname{Spec} A) \simeq \mathbb{Z}/2,$$

generated by the class of the prime divisor $Y = V(2, 1+x)$.

Proof Since A is a normal integral domain of dimension 1, the prime divisors $V(\mathfrak{p})$ correspond to the nonzero prime ideals $\mathfrak{p}$.

First, the equation (A.16) shows that $(2, 1+x)$ is prime, so $Y = V(2, 1+x)$ is a prime divisor. Y is moreover nonzero in $\operatorname{Cl}(\operatorname{Spec} A)$ because it is not principal.

Let $\mathfrak{p}$ be a nonzero prime ideal. We claim that $V(\mathfrak{p})$ is either principal or equivalent to Y in $\operatorname{Cl}(\operatorname{Spec} A)$. To see this, we use the morphism $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ induced by the integral extension $\mathbb{Z} \subset A$. Note that π is surjective, by the Going-Up Theorem.

The point $\mathfrak{p}$ lies over a nonzero prime $(p) \in \operatorname{Spec} \mathbb{Z}$. There are three cases to consider:

(i) (p) remains prime in A.
(ii) (p) is a square of a prime ideal in A.
(iii) (p) decomposes as

$$(p) = (p, x-a) \cap (p, x+a) \tag{A.17}$$

where $a \in \mathbb{Z}$ is an integer such that $a^2 + 5 \equiv 0 \pmod p$.

In case (i), pA is prime in A, and so $V(\mathfrak{p}) = V(pA)$ is principal. The case (ii) happens if and only if the minimal polynomial $x^2 + 5 = 0$ has a multiple root modulo p. This happens only for $p = 2$ and $p = 5$. For $p = 2$, we have $(2) = (1+x)^2$. Therefore, $\operatorname{div}(2) = 2V(2, 1+x) = 2Y$, and hence

$$2[Y] = 0 \text{ in } \operatorname{Cl}(\operatorname{Spec} A).$$

For $p = 5$, we have $(5) = (x^2)$, and so $V(\mathfrak{p}) = V(5, x) = V(x)$, which is principal.

We therefore reduce to the case (iii). By adjusting a by a multiple of p, we may assume $0 \leqslant$

$a < p$. From (A.17), $\operatorname{div}(p) = V(p, x+a) + V(p, x-a)$, so $V(p, x+a) = -V(p, x-a)$ in $\operatorname{Cl}(\operatorname{Spec} A)$. Hence it suffices to analyze $V(p, x+a)$.

Consider more generally ideals of the form $(n, x+a)$ where $n \geqslant 2$ and $0 \leqslant a < n$ and $a^2 + 5 \equiv 0 \pmod{n}$.

If $n = 2$, we recover $Y = V(2, 1+x)$.

If $n = 3$, then we must have $a = 1$. As $(1+x, x^2+5) = (1+x, 6)$, we get

$$\operatorname{div}(1+x) = V(3, 1+x) + V(2, 1+x).$$

Hence $V(3, 1+x) = -Y$ in $\operatorname{Cl}(\operatorname{Spec} A)$.

If $n = 4$, no solutions exist, as $a^2 + 1 \equiv 0 \pmod{4}$ has no integer solutions.

If $n = 5$, then $a = 0$, and $(5, x) = (x)$, which is principal.

If $n \geqslant 6$, write $a^2 + 5 = bn$. Then we get

$$\begin{aligned} \operatorname{div}(x+a) &= V(x+a, x^2+5) \\ &= V(a+x, bn) = V(a+x, b) + V(a+x, n) \end{aligned}$$

But as $a^2 + 5 = bn$ and $n \geqslant 6$, we must have $b < n$. Therefore, by induction on n, the class of $V(b, a+x)$ can be written as a multiple of Y in the class group.

From this, we see that every height 1 prime of A is either principal or equivalent to $\pm Y$. Since $[Y] \neq 0$ and $2[Y] = 0$, we conclude that $\operatorname{Cl}(\operatorname{Spec} A) = \mathbb{Z}/2$. □

More generally, for the quadratic number field $K = \mathbb{Q}(\sqrt{d})$ for a square-free integer d, the ring of integers O_K (that is, the integral closure of $\mathbb{Z}$ in K) is a normal domain of dimension one (see Exercise 18.6.4). The above argument can be extended to show:

Theorem A.43. The class group of O_K is finite.

When $d < 0$, it is known that the class group $\operatorname{Cl}(O_K)$ is trivial if and only if

$$d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

It is currently unknown whether there exist infinitely many positive values of d for which $\operatorname{Cl}(O_K) \neq 0$.

A.7.1 Exercises.

Exercise A.7.1 (The Gaussian integers)**.** For a complex number $z = a + ib$, we let $N(z) = a^2 + b^2$ denote the *norm* of z.

a) Show that for any $\alpha, \beta \in \mathbb{Z}[i]$ with $\beta \neq 0$, there exists μ and ρ in $\mathbb{Z}[i]$ such that $\alpha = \mu\beta + \rho$ where $N(\beta) > N(\rho)$. HINT: Consider the lattice generated by $\beta, i\beta$ and consider the square nearest α.

b) Show that $\mathbb{Z}[i]$ is a PID. HINT: If $\mathfrak{a} \subset \mathbb{Z}[i]$ is a nonzero ideal, show that $\mathfrak{a}$ is generated by a nonzero element in $\mathfrak{a}$ of minimal norm.

c) Conclude that $\mathbb{Z}[i]$ is a UFD.

Exercise A.7.2. Let d be a square free integer and assume that $d \not\equiv 1 \mod 4$ so that $\mathbb{Z}[\sqrt{d}]$ is normal. Show that the class group of $\mathbb{Z}[\sqrt{d}]$ is finite. HINT: Imitate the technique used for the example with $\mathbb{Z}[\sqrt{-5}]$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise A.7.3. Show that the rings $\mathbb{Z}[\sqrt{-5}]_2$ and $\mathbb{Z}[\sqrt{-5}]_3$ are unique factorization domains.

Exercise A.7.4. Consider the ring $A = \mathbb{Z}[\sqrt{-7}]$ and set $\alpha = \frac{1}{2} + \frac{\sqrt{-7}}{2} \in \mathbb{Q}(\sqrt{-7})$.

a) Show that α satisfies the integral equation $x^2 - 2x + 8 = 0$, yet $\alpha \notin A$. Conclude that A is not normal.
b) Show that $A = \mathbb{Z}[2\alpha]$ and the normalization map is given by $\pi\colon \operatorname{Spec}\mathbb{Z}[\alpha] \to \operatorname{Spec}\mathbb{Z}[2\alpha]$.
c) Consider the two prime ideals $\mathfrak{p}_1 = (2, \alpha)$ and $\mathfrak{p}_2 = (2, \alpha - 1)$ in $\operatorname{Spec}\mathbb{Z}[\alpha]$. Show that $\pi(\mathfrak{p}_1) = \pi(\mathfrak{p}_2) = (2, 2\alpha)$ and that π is an isomorphism away from these two points. Compare this example with the nodal cubic example.

Exercise A.7.5. Let $A = \mathbb{Z}[\sqrt{8}]$. Show that the integral closure of A is given by $\overline{A} = \mathbb{Z}[\sqrt{2}]$. Show that the normalization map $\pi\colon \operatorname{Spec}\mathbb{Z}[\sqrt{2}] \to \operatorname{Spec}\mathbb{Z}[\sqrt{8}]$ is bijective on points, but it is not an isomorphism. Compare this with the normalization of the cuspidal cubic.

A.8 Pushouts of affine schemes

Gluing schemes along open subschemes have been a central theme in this book. In some cases, we can also glue two schemes along a common *closed* subscheme. For affine schemes, this is obtained using the fiber product of rings.

Let A, B, C be rings and let $f\colon A \to C$, $g\colon B \to C$ be surjective ring maps. From this data, we can form the *pullback ring* $A \times_C B$, defined by

$$A \times_C B = \{\, (a, b) \in A \times B \mid f(a) = g(b) \,\}.$$

This ring fits in a Cartesian diagram of rings

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\ f\ } & C \end{array}$$

Applying Spec, we get a pushout diagram of affine schemes

$$\begin{array}{ccc} \operatorname{Spec} C & \hookrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow{\scriptstyle i_B} \\ \operatorname{Spec} A & \xrightarrow{i_A} & \operatorname{Spec}(A \times_C B) \end{array} \tag{A.18}$$

This means that $\operatorname{Spec}(A \times_C B)$ satisfies a universal property dual to that of the fiber product: it is universal among diagrams of the form (A.18) with $\operatorname{Spec}(A \times_C B)$ replaced by some other affine scheme.

Proposition A.44. There is a natural homeomorphism

$$\operatorname{Spec}(A \times_C B) \;\simeq\; \big(\operatorname{Spec} A\big) \cup_{\operatorname{Spec} C} \big(\operatorname{Spec} B\big).$$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof Write $R = A \times_C B$. Since the projection maps π_A and π_B are surjective, the maps i_A and i_B are closed embeddings onto the closed subsets $V(K_A)$ and $V(K_B)$ where

$$K_A = \operatorname{Ker}(\pi_A) = 0 \times \operatorname{Ker}(g), \qquad K_B = \operatorname{Ker}(\pi_B) = \operatorname{Ker}(f) \times 0.$$

Note that $K_A \cap K_B = (0)$. Therefore, $\operatorname{Spec} R = V(K_A \cap K_B) = V(K_A) \cup V(K_B)$, and hence the images of i_A and i_B cover $\operatorname{Spec} R$.

The intersection $V(K_A) \cap V(K_B)$ is defined by the ideal $K_A + K_B = \{(a, b) \in R \mid a \in \operatorname{Ker}(f),\ b \in \operatorname{Ker}(g)\}$. There is an isomorphism of rings

$$R/(K_A + K_B) \xrightarrow{\simeq} C, \qquad (a, b) \mapsto f(a) = g(b),$$

Therefore, $V(K_A) \cap V(K_B) \simeq \operatorname{Spec} C$. It follows that i_A maps $\operatorname{Spec} A$ homeomorphically onto $V(K_A)$ and $V(\operatorname{Ker} f)$ homeomorphically onto $V(K_A) \cap V(K_B)$ (and similarly for i_B). Therefore i_A, i_B induce a homeomorphism

$$\big(\operatorname{Spec} A\big) \cup_{\operatorname{Spec} C} \big(\operatorname{Spec} B\big) \simeq \operatorname{Spec} R.$$

□

The same ring construction also describes a slightly more general operation called a *pinch*. This is the form that we will use to construct a proper non-projective variety.

Proposition A.45. Let $A \to C$ be a surjective ring map, let $B \to C$ be a finite ring map, and set $R = A \times_C B$. Then

$$\begin{array}{ccc} \operatorname{Spec} C & \hookrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow p \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R \end{array}$$

is a pushout in the category of affine schemes, and p is finite. If $B \to C$ is injective, then p is also surjective. Moverover, if A, B, C is a finitely generated algebra over a field k, then so is R.

Proof The pushout statement is the universal property of R. The map $R \to B$ is surjective. If A' is the image of $R \to A$, then $A/A' \simeq C/\operatorname{im}(B \to C)$ as R-modules. This module is finite over R, so lifting generators and adjoining $1 \in A$ shows that A is a finite R-module. Thus p is finite.

If $B \to C$ is injective, then $R \to A$ is injective. Hence A is an integral extension of the subring R, and the lying-over theorem shows that $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective.

For the last statement, choose algebra generators x_i of A over k and module generators e_j of A over R. Let $R_0 \subset R$ be the k-subalgebra generated by the coefficients expressing the $x_i e_j$ and 1 in terms of the e_j. Then A is a finite R_0-module. Since R_0 is Noetherian and $R \subset A$, the R_0-module R is finite. Hence R is a finitely generated k-algebra. □

Example A.46. Let k be a field of characteristic not equal to 2. Consider the ring maps $f\colon k[t] \to k \times k$ and $g\colon k \to k \times k$ defined by $f(p) = (p(1), p(-1))$ and $g(c) = (c, c)$,

respectively. The pullback ring is given by

$$\begin{aligned} k[t] \times_{k\times k} k &= \{\, (p(t), c) \in k[t] \times k \mid p(1) = p(-1) = c \,\} \\ &= \{\, p(t) \in k[t] \mid p(1) = p(-1) \,\} \\ &= k[t^2 - 1, t^3 - t]. \end{aligned}$$

To see the last equality: if $p(t)$ satisfies $p(1) = p(-1)$, then subtracting elements of the form $(t^2-1)^m(t^3-t)^n$ for $m, n \in \mathbb{N}$, we may reduce to the case where $p(t) = at + b$, in which case $a = 0$ and $b \in k$.

We recognize the ring $k[t^2 - 1, t^3 - t]$ as the coordinate ring of the nodal cubic curve:

$$k[t^2 - 1, t^3 - t] \simeq k[x, y]/(y^2 - x^2(x + 1)).$$

We have therefore shown that the nodal cubic curve X can be obtained by identifying two points of $\mathbb{A}^1_k$. Moreover, the quotient map $\mathbb{A}^1_k \to X$ is the normalization of X. △

Example A.47. Consider now the scheme X obtained by identifying two points in $\mathbb{A}^2_k$. Formally, X is the affine scheme given by the k-algebra

$$A = \{f \in k[x, y] \mid f(0, 0) = f(0, 1)\}.$$

This algebra A is generated by the four polynomials $x, xy, y^2 - y, y^3 - y$. To see this, note that if $f(x, y)$ is any polynomial satisfying $f(0, 0) = f(0, 1)$, we may subtract products of the form $b(x)(y^2 - y)^k$ or $b(x)(y^2 - y)^k(y^3 - y)$ until the y-degree of f is at most 1. Then the remaining polynomials can be written as polynomials in x and xy, and all such satisfy the condition.

The four polynomials define a closed embedding of X as a non-normal surface in $\mathbb{A}^4_k$ with a unique singular point. In fact, the normalization is given by the quotient map $\mathbb{A}^2_k \to \overline{X}$, induced by the inclusion $A \subset k[x, y]$. △

Example A.48 ($\operatorname{Spec} \mathbb{Z}\sqrt{-3}$)**.** The spectrum of the ring $A = \mathbb{Z}[\sqrt{-3}]$ is rather interesting. It can be viewed as a sort of singular curve over $\operatorname{Spec} \mathbb{Z}$. As such it shares many properties with the nodal cubic curve of Example A.46.

Note first that A is not a unique factorization domain. For example,

$$2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3}).$$

It is also not normal, because the element $\omega = (1 + \sqrt{-3})/2$ satisfies the monic equation $x^2 + x + 1 = 0$, but $\omega \notin A$. In fact, the integral closure of A is given by the ring the ring of Eisenstein integers $\mathbb{Z}[\omega]$, and $\mathbb{Z}[\omega]$ *is* a unique factorization domain. In particular, $\mathbb{Z}[\omega]$ is normal, and equals the integral closure of A inside the fraction field $\mathbb{Q}(\sqrt{-3})$. The normalization is a morphism

$$\operatorname{Spec} \mathbb{Z}[\omega] \to \operatorname{Spec} \mathbb{Z}[\sqrt{-3}],$$

induced by the inclusion $\mathbb{Z}[2\omega] \to \mathbb{Z}[\omega]$.

One may understand this morphism geometrically as follows. There are two ring maps $\phi, \psi \colon \mathbb{Z}[\omega] \to \mathbb{F}_2[x]/(x^2 + x + 1) = \mathbb{F}_4$, one sending ω to x, and the other sending ω to

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$x+1$. The subring of $\mathbb{Z}[\omega]$ where these coincide is exactly $\mathbb{Z}[2\omega] = A$. We get a pullback diagram of rings

$$\begin{array}{ccc} \mathbb{Z}[\sqrt{-3}] & \longrightarrow & \mathbb{Z}[\omega] \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ \mathbb{Z}[\omega] & \xrightarrow{\phi} & \mathbb{F}_4 \end{array}$$

This induces a homeomorphism between $\operatorname{Spec}\mathbb{Z}[\sqrt{-3}]$ and $\operatorname{Spec}\mathbb{Z}[\omega]$ with two points identified. Hence $\operatorname{Spec} A$ is obtained by identifying two points in the spectrum of the Eisenstein integers. △

Exercise A.8.1. Let $A = k[x]$, $B = k[y]$ and $C = k[x]/x = k[y]/y = k$, where k is a field. Show that $k[x] \times_k k[y]$ is isomorphic to $k[u,v]/uv$. What gluing does this example represent?

A.9 A proper, non-projective 3-fold

In this section, we sketch a construction of a proper 3-dimensional variety which is not projective. Let k be a field and consider $\mathbb{P}^3_k$ with homogeneous coordinates x, y, z, w. Let $L = V(x,z)$ and $C = V(w, y^2 - xz)$. Then L is a line, C is a smooth plane conic, and $L \cap C = \varnothing$. Consider the isomorphism

$$\iota\colon L \xrightarrow{\sim} C, \qquad (0:s:0:t) \mapsto (s^2 : st : t^2 : 0)$$

In this set-up, it is possible to glue $\mathbb{P}^3_k$ along itself by identifying L and C using ι. Formally, this means that there is a pushout diagram

$$\begin{array}{ccc} L \sqcup C & \xhookrightarrow{\alpha} & \mathbb{P}^3_k \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[j]{} & Y \end{array} \tag{A.19}$$

where α is the closed embedding of L and C and β equals ι on L and id_C on C.

It is not too difficult to verify that Y is a scheme. Such pushouts always exist when the involved schemes are affine (see Section A.8), and one proceeds by gluing together the 4 affine pushouts from the standard covering of $\mathbb{P}^3_k$.

The resulting scheme Y looks like a $\mathbb{P}^3_k$ which is 'pinched' along a smooth curve $C \subset Y$. Indeed, the morphism $f\colon \mathbb{P}^3_k \to Y$ is an isomorphism over the dense open $\mathbb{P}^3_k - (L \cup C)$, and it identifies a point $x \in L$ with exactly one point on C. From the gluing, we also see that Y is an integral 3-fold, of finite type over k.

The scheme Y is also proper over k. To see separatedness, consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle\Delta_X}\downarrow & & \downarrow{\scriptstyle\Delta_Y} \\ X \times_k X & \xrightarrow{f\times f} & Y \times_k Y. \end{array}$$

Since f is surjective, we have

$$\Delta_Y(Y) = \Delta_Y(f(X)) = (f \times f)(\Delta_X(X))$$

which is closed because $\Delta_X(X)$ is closed and f is finite, hence closed. Since $X \to \operatorname{Spec} k$ and $f\colon X \to Y$ are proper, the morphism $Y \to \operatorname{Spec} k$ is also proper.

Proposition A.49. The 3-fold Y is not projective.

Proof Assume for contradiction that there exists a projective embedding $Y \subset \mathbb{P}^n_k$. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_k}(1)|_Y$. We may assume that Y is not contained in any hyperplane, we have

$$\dim_k H^0(Y, \mathcal{L}) = n + 1. \tag{A.20}$$

Now write $\mathcal{L}|_C \simeq \mathcal{O}_{\mathbb{P}^1_k}(a)$ for some $a \in \mathbb{Z}$, using the identification $C \simeq \mathbb{P}^1_k$. Then $\mathcal{L}|_C$ has degree a. Since β restricts to an isomorphism on both components of $L \sqcup C$, the invertible sheaf $\beta^* j^* \mathcal{L}$ has degree a on both L and C.

On the other hand, by the commutative diagram (A.19), $\beta^* j^* \mathcal{L} = \alpha^* f^* \mathcal{L}$. Since $\operatorname{Pic}(\mathbb{P}^3_k) \simeq \mathbb{Z}$, we have $f^*\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^3_k}(d)$ for some $d \in \mathbb{Z}$. Restricting to L and C, we obtain invertible sheaves of degrees d and $2d$, respectively, because L is a line and C is a conic. As these degrees must both equal a, we must have $d = a = 0$. Therefore

$$f^*\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^3_k}. \tag{A.21}$$

We will obtain a contradiction from this by computing $H^0(X, f^*\mathcal{L})$. By the projection formula,

$$H^0(\mathbb{P}^3_k, f^*\mathcal{L}) = H^0\big(Y, \mathcal{L} \otimes f_* \mathcal{O}_{\mathbb{P}^3_k}\big).$$

Since f is finite and dominant, the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_{\mathbb{P}^3_k}$ is injective. Tensoring with $\mathcal{L}$ gives an injection $\mathcal{L} \hookrightarrow \mathcal{L} \otimes f_*\mathcal{O}_{\mathbb{P}^3_k}$. Taking global sections, we obtain an injection

$$H^0(Y, \mathcal{L}) \hookrightarrow H^0\big(Y, \mathcal{L} \otimes f_* \mathcal{O}_{\mathbb{P}^3_k}\big) = H^0(\mathbb{P}^3_k, f^*\mathcal{L}).$$

By (A.20) the left-hand side has dimension at least 2, whereas the right-hand side has dimension 1 by (A.21). This is a contradiction. Hence Y is not projective. □

Exercise A.9.1. For the proper threefold X as defined in Section A.9, show that $\operatorname{Cl}(X) \simeq \mathbb{Z}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Appendix B

Some category theory

B.0.1 Categories, functors and natural transformations. A *category* C consists of:

(i) A collection $\mathrm{Obj}(\mathsf{C})$ of *objects*.

(ii) For each pair of objects $A, B \in \mathrm{Obj}(\mathsf{C})$, a set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ of *morphisms* (or *arrows*) from A to B. $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is usually written $f\colon A \to B$.

(iii) For every triple of objects A, B, C, there is a composition map

$$\circ : \mathrm{Hom}_{\mathsf{C}}(A, B) \times \mathrm{Hom}_{\mathsf{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, C).$$

The composite of the pair (f, g) is written $g \circ f$.

These must satisfy the following two conditions:

- (Associativity): For all $f\colon A \to B$, $g : B \to C$, and $h\colon C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$.
- (Identity): For every object A, there exists a morphism $\mathrm{id}_A\colon A \to A$ called the identity morphism, such that for any $f\colon A \to B$, $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.

In the above definition we have not made precise what we mean by 'collection'. The reader may safely think of the objects of C as sets, but the notion of a category is more general. We will not discuss these foundational matters here.

Example B.1. Some familiar examples include: the category of sets, Sets, whose objects are sets and the morphisms are functions betweeen them; the category of groups, Grp whose objects are groups and the morphisms are group homomorphisms; the category of A-modules (A a ring) whose objects are A-modules M and morphisms are A-linear maps $M \to N$. △

Example B.2. For a category C there is also the *opposite category*, C^{op}, whose objects are the same as those as in C, but where the morphisms satisfy $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A)$. That is, the morphisms $A \to B$ in C^{op} exactly correspond to morphisms $B \to A$ in C. △

Example B.3. Let X be a topological space. Then the open sets $U \subset X$ form a category $\mathsf{Open}(X)$, where the morphisms $V \to U$ are the inclusions $V \subset U$. Here $\mathrm{Hom}_{\mathsf{Open}}(V, U) = \{*\}$ if $V \subset U$ and $\mathrm{Hom}_{\mathsf{Open}}(V, U) = \varnothing$ if $V \not\subset U$. △

A *functor* F from a category C to a category D consists of:

(i) A map $\mathrm{Obj}(\mathsf{C}) \to \mathrm{Obj}(\mathsf{D})$.

(ii) For each morphism $f\colon A \to B$ in C, a morphism $F(A) \to F(B)$ in D. These are required to preserve identity morphisms and compositions: $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(fg) = F(f) \circ F(g)$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

One also talks about *contravariant functors* F from C to D. The main difference is that for each morphism $f\colon A \to B$ in C the morphism $F(B) \to F(A)$ goes in the opposite way than ordinary functors. Formally, this just means that F is a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

Example B.4. In the category of rings, Rings, we define a functor $\mathrm{GL}_n\colon \mathsf{Rings} \to \mathsf{Groups}$ which sends a ring R to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$-matrices with coefficients in R. For a ring map $\phi\colon R \to S$, the there is an associated group homomorphism $\mathrm{GL}_n(\phi)\colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ such that

$$\mathrm{GL}_n(\phi)(M)_{i,j} = \phi(M_{i,j}).$$

In the case $n = 1$, this functor is usually denoted by $(-)^\times$; it sends a ring R to $R^\times$, the multiplicative group of invertible elements of R. △

Example B.5. For a topological space X, the set $C(U, \mathbb{R})$ of continuous maps $f\colon U \to \mathbb{R}$ defines a functor $\mathsf{Open}^{\mathrm{op}} \to \mathsf{Sets}$. More generally, any presheaf on X (of sets, groups, ..) can be regarded as a functor from $\mathsf{Open}^{\mathrm{op}}$ into (sets, groups, ..). △

A functor $F\colon \mathsf{C} \to \mathsf{D}$ is *faithful* if for all objects $A, B \in \mathsf{C}$, the function $\mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{D}}(F(A), F(B))$ is injective, and *full* if it is surjective. We say that F is *fully faithful* if it is full and faithful.

One also have the following notion of morphisms between two functors. If $F, G\colon \mathsf{C} \to \mathsf{D}$ are two functors, a *natural transformation* between them is a collection of morphisms

$$\eta_T\colon F(T) \longrightarrow G(T), \tag{B.1}$$

one for each object T in C, such that whenever $h\colon S \to T$ is a morphism in C there is a commutative diagram of sets

$$\begin{array}{ccc} F(S) & \xrightarrow{\eta_S} & G(S) \\ \downarrow & & \downarrow \\ F(T) & \xrightarrow{\eta_T} & G(T) \end{array}.$$

We say that two functors are *isomorphic* if there are natural transformations $\eta\colon F \to G$ and $\nu\colon G \to F$ such that both $\eta \circ \nu$ and $\nu \circ \eta$ are the identity natural transformations.

A functor $F\colon \mathsf{C} \to \mathsf{D}$ is said to define an *equivalence of categories* if there exists a functor $G\colon \mathsf{D} \to \mathsf{C}$ such that FG and GF are both isomorphic to the identity functors.

B.0.2 Yoneda's Lemma. We will in particular be interested in functors of the following form. For an object A in C, we define a functor from C to Sets by

$$h_A(-) = \mathrm{Hom}_{\mathsf{C}}(A, -)$$

If $f\colon B \to A$ is a morphism in C, there is for each T an induced map $f_T\colon h_A(T) \to h_B(T)$ defined by sending $g\colon A \to T$ to $g \circ f$. This induces a natural transformation of functors $\eta(f)\colon h_A \to h_B$. A fundamental fact is that every natural transformation is of this form:

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Lemma B.6 (Yoneda's Lemma). Let C be a category, let A be an object of C and let $F\colon \mathsf{C} \to \mathsf{Sets}$. Then there is a bijection

$$\operatorname{Hom}(h_A, F) = F(A). \tag{B.2}$$

In particular, for $F = h_B$, there is a bijection $\operatorname{Hom}(h_A, h_B) = \operatorname{Hom}_{\mathsf{C}}(B, A)$.

Proof We construct a bijection $\Phi\colon \operatorname{Hom}(h_A, F) \to F(A)$ as follows. For a natural transformation $\eta\colon h_A \to F$, define

$$\Phi(\eta) = \eta_A(\mathrm{id}_A),$$

where $\mathrm{id}_A \in h_A(A) = \operatorname{Hom}_{\mathsf{C}}(A, A)$ is the identity morphism.

For the inverse, let $u \in F(A)$ and define $\eta^u\colon h_A \to F$ componentwise by

$$\eta^u_T(f) = F(f)(u),$$

for each $f \in h_A(T) = \operatorname{Hom}_{\mathsf{C}}(A, T)$.

Let us check that η^u is a natural transformation. If $g\colon S \to T$ is a morphism in C, we need to show that the following diagram is commutative:

$$\begin{array}{ccc} h_A(S) & \xrightarrow{\eta^u_S} & F(S) \\ {\scriptstyle h_A(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ h_A(T) & \xrightarrow{\eta^u_T} & F(T) \end{array}$$

This follows because the counterclockwise composition sends $f \in h_A(S)$ to

$$\eta^u_T(g \circ f) = F(g \circ f)(u) = F(g)(F(f)(u)) = F(g)(\eta^u_S(f)),$$

which equals the clockwise composition. Therefore η^u is a natural transformation.

We next check that Φ and Ψ are mutually inverse. For any $u \in F(A)$,

$$\Phi(\Psi(u)) = \Phi(\eta^u) = \eta^u_A(\mathrm{id}_A) = F(\mathrm{id}_A)(u) = \mathrm{id}_{F(A)}(u) = u,$$

hence $\Phi \circ \Psi = \mathrm{id}$.

Next, for any natural transformation $\eta\colon h_A \to F$, we need to show that the transformation $\Psi(\Phi(\eta)) = \Psi(\eta_A(\mathrm{id}_A))$ is equal to η. For this, it suffices to show that the two transformations have the same components for each T in C. By definition, for any $f \in h_A(T) = \operatorname{Hom}_{\mathsf{C}}(A, T)$,

$$\Psi(\eta_A(\mathrm{id}_A))_T(f) = F(f)(\eta_A(\mathrm{id}_A)).$$

By the naturality of η, the following diagram commutes:

$$\begin{array}{ccc} h_A(A) & \xrightarrow{\eta_A} & F(A) \\ {\scriptstyle h_A(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ h_A(T) & \xrightarrow{\eta_T} & F(T) \end{array}$$

Applying this to $\mathrm{id}_A \in h_A(A)$, we find that

$$F(f)(\eta_A(\mathrm{id}_A)) = \eta_T(h_A(f)(\mathrm{id}_A))$$

But $h_A(f)(\mathrm{id}_A) = \mathrm{id}_A \circ f = f$, so $F(f)(\eta_A(\mathrm{id}_A)) = \eta_T(f)$. In all, we have shown that $\Psi(\eta_A(\mathrm{id}_A))_T(f) = \eta(f)$, and so Ψ is a right inverse of Φ. □

There is also the following contravariant version of Yoneda's Lemma, with a very similar proof, which allows us to study the contravariant functors $h^A(-) = \mathrm{Hom}_{\mathsf{C}}(-, A)$.

Lemma B.7 (Contravariant Yoneda's Lemma). Let C be a category, let A be an object of C and let $F\colon \mathsf{C} \to \mathsf{Sets}$ be a contravariant functor. Then there is a bijection

$$\mathrm{Hom}(h^A, F) = F(A). \tag{B.3}$$

In particular, for $F = h^B$, there is a bijection $\mathrm{Hom}(h^A, h^B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$.

A functor $F\colon \mathsf{C} \to \mathsf{Sets}$ is said to be *representable* if there exists an object A in C such that $F \simeq h_A$.

Example B.8. In the category of rings Rings, the functor defined by $F(R) = R$ taking a ring R to its underlying set is representable by $\mathbb{Z}[t]$. Indeed, $\mathrm{Hom}_{\mathsf{Rings}}(\mathbb{Z}[t], R) = R$ for every ring R. △

B.0.2 Adjoint functors. Let C and D be categories and let $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ be two functors. We say that F is a *left-adjoint to* G and that G is a *right-adjoint to* F, and that (F, G) form an *adjoint pair of functors* if there exist a collection of functorial bijections

$$\mathrm{Hom}_{\mathsf{D}}(F(A), B) = \mathrm{Hom}_{\mathsf{C}}(A, G(B))$$

for each pair of objects (A, B) where A is in C and B is in D.

If a left-adjoint exists or right-adjoint, it is unique up to unique isomorphism (e.g., by the Yoneda Lemma).

Example B.9. The forgetful functor $G\colon \mathsf{Mod}_A \to \mathsf{Sets}$ which sends an A-module M to its underlying set has a left adjoint defined by $F(S)$ be the free A-module $\bigoplus_{s\in S} Ae_s$. Then the bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(S, G(M)) = \mathrm{Hom}_{\mathsf{Mod}_A}(F(A), M)$$

is defined by sending a map of sets $\phi\colon S \to M$ to the A-linear map $\bigoplus_{s\in S} Ae_s \to M$, $e_s \mapsto \phi(s)$. △

Every functor F which has a right adjoint commutes with direct limits. If F is a functor with a right adjoint between abelian categories, then it is left exact.

Every functor G which has a left adjoint commutes with inverse limits. If G is a functor with a left adjoint between abelian categories, then it is right exact.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Appendix C

Some results from Commutative Algebra

C.1 Rings and Algebras

All rings are commutative with 1. Ring maps $\phi\colon A \to B$ are required to send 1 to 1.

An *A-algebra* is a ring B equipped with a ring map $\phi\colon A \to B$. The map ϕ allows us to multiply elements of B by elements of A via $a \cdot b = \phi(a)b$; we often suppress ϕ from the notation. For instance, $B = A[x_1, \dots, x_n]$ is an A-algebra. An A-algebra is *finitely generated* if it is isomorphic to one of the form $A[x_1, \dots, x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset A[x_1, \dots, x_n]$.

C.2 Ideals and Prime Ideals

The *radical* of an ideal $\mathfrak{a}$ is

$$\sqrt{\mathfrak{a}} = \{\, a \in A \mid a^n \in \mathfrak{a} \text{ for some } n \geqslant 1 \,\}.$$

The following formula is useful:

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}, \tag{C.1}$$

where the intersection runs over all prime ideals $\mathfrak{p}$ containing $\mathfrak{a}$. Indeed, if $f \in \sqrt{\mathfrak{a}}$, then $f^r \in \mathfrak{a} \subset \mathfrak{p}$ for every such $\mathfrak{p}$, so $f \in \mathfrak{p}$. Conversely, if $f^r \notin \mathfrak{a}$ for all $r \geqslant 1$, then $A_f \neq 0$ and $\mathfrak{a}_f$ is a proper ideal in A_f; any maximal ideal of A_f containing $\mathfrak{a}_f$ pulls back to a prime $\mathfrak{p} \supset \mathfrak{a}$ with $f \notin \mathfrak{p}$, so f does not lie in the right-hand side of (C.1).

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, the *ideal quotient* is

$$(\mathfrak{a} : \mathfrak{b}) = \{\, a \in A \mid a\mathfrak{b} \subset \mathfrak{a} \,\}.$$

The ideal $(0 : x)$ is called the *annihilator* of x.

An element $a \in A$ is a *zerodivisor* if there exists a nonzero $b \in A$ with $ab = 0$; a *unit* if $ab = 1$ for some b; *nilpotent* if $a^n = 0$ for some $n \geqslant 1$; and *irreducible* if it is not a unit and $a = xy$ implies x or y is a unit.

C.2.1 Prime and maximal ideals. An ideal $\mathfrak{p}$ is *prime* if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$; equivalently, $A/\mathfrak{p}$ is an integral domain. An ideal $\mathfrak{m}$ is *maximal* if there is no ideal strictly between $\mathfrak{m}$ and A; equivalently, $A/\mathfrak{m}$ is a field. Every maximal ideal is prime.

C.2.2 Ideals in quotient rings. For an ideal $\mathfrak{a} \subset A$, there is a natural bijection

$$\{\text{ideals in } A/\mathfrak{a}\} \longleftrightarrow \{\text{ideals in } A \text{ containing } \mathfrak{a}\}, \tag{C.2}$$

given by $J \mapsto \pi^{-1}(J)$ and $\mathfrak{b} \mapsto \mathfrak{b}/\mathfrak{a}$, where $\pi\colon A \to A/\mathfrak{a}$ is the quotient map. This bijection preserves inclusions and restricts to a bijection between prime ideals: $\mathfrak{b} \supset \mathfrak{a}$ is prime in A if and only if $\mathfrak{b}/\mathfrak{a}$ is prime in $A/\mathfrak{a}$, since $A/\mathfrak{b} \simeq (A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$.

C.3 Localization

A nonempty subset S of a commutative ring A is a *multiplicative set* if it is closed under multiplication and contains 1.

The *localization* of A with respect to a multiplicative set S, denoted $S^{-1}A$, is the set of fractions a/s with $a \in A$ and $s \in S$. Formally, $S^{-1}A$ is constructed by defining an equivalence relation on $A \times S$ by $(a, s) \sim (a', s')$ if there exists an element $t \in S$ such that $t(as' - a's) = 0$ in A. The elements of $S^{-1}A$ are denoted by a/s or $\frac{a}{s}$. There is a natural ring structure on $S^{-1}A$ and a canonical localization map

$$\ell\colon A \longrightarrow S^{-1}A, \qquad x \mapsto x/1.$$

The kernel of ℓ consists of those elements which are annihilated by some element of S, i.e., $sx = 0$ for some $s \in S$. In particular, the map ℓ is injective if A contains no zerodivisors. The ring $S^{-1}A$ is zero if and only if $0 \in S$.

If M is an A-module, one defines $S^{-1}M$ analogously as fractions m/s with $m \in M$, $s \in S$, subject to $(m, s) \sim (m', s')$ if $t(ms' - m's) = 0$ for some $t \in S$. Then $S^{-1}M$ is an $S^{-1}A$-module, and there is a canonical map $\ell\colon M \to S^{-1}M$.

The localization $S^{-1}A$ and the map ℓ satisfy the following universal property: For every ring B and any ring map $f\colon A \to B$ such that $f(s) \in B$ is invertible for all $s \in S$, there exists a *unique* ring map $g\colon S^{-1}A \to B$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \ell}\big\downarrow & \nearrow_{\exists! g} & \\ S^{-1}A & & \end{array} \tag{C.3}$$

The map g is uniquely determined by $g(a/s) = f(a) \cdot f(s)^{-1}$ for all $a \in A$ and $s \in S$.

Example C.1. The first basic example is $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$. The localization $A_\mathfrak{p} = S^{-1}A$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$; the elements of $A - \mathfrak{p}$ become units. △

Example C.2. The second basic example is $S = \{1, f, f^2, \dots\}$ for some $f \in A$. The localization $A_f = S^{-1}A$ consists of fractions a/f^n. There is an isomorphism $A[t]/(1 - ft) \xrightarrow{\sim} A_f$ sending $t \mapsto 1/f$. △

If $S \subset T$ are multiplicative sets, there is a canonical map $S^{-1}M \to T^{-1}M$. In particular, if $\mathfrak{q} \subset \mathfrak{p}$ are prime ideals, there is a localization map $A_\mathfrak{p} \to A_\mathfrak{q}$.

The functor $M \mapsto S^{-1}M$ is exact and commutes with direct sums, direct limits, and tensor products.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

C.3.1 Ideals in localizations. If $\mathfrak{a} \cap S \neq \varnothing$, then $S^{-1}\mathfrak{a} = (1)$ in $S^{-1}A$. For prime ideals there is a bijection

$$\{\text{prime ideals } P \text{ in } S^{-1}A\} \longleftrightarrow \{\text{prime ideals } \mathfrak{p} \text{ in } A \text{ with } \mathfrak{p} \cap S = \varnothing\}, \tag{C.4}$$

given by $P \mapsto \ell^{-1}(P)$ and $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$. These are mutual inverses: for a prime $P \subset S^{-1}A$ one has $a/s \in P \Leftrightarrow a/1 \in P \Leftrightarrow a \in \ell^{-1}(P)$, giving $S^{-1}(\ell^{-1}(P)) = P$; and for a prime $\mathfrak{p} \cap S = \varnothing$, if $a \in \ell^{-1}(S^{-1}\mathfrak{p})$ then $a/1 = c/s$ in $S^{-1}\mathfrak{p}$ for some $c \in \mathfrak{p}$, $s \in S$, so $ta \in \mathfrak{p}$ for some $t \in S$, and since $t \notin \mathfrak{p}$ we get $a \in \mathfrak{p}$.

C.3.2 Local rings. A ring A is *local* if it has exactly one maximal ideal. The following conditions are equivalent:

(i) A is local with maximal ideal $\mathfrak{m}$.
(ii) Every element of $A - \mathfrak{m}$ is a unit.
(iii) $\mathfrak{m}$ is maximal and every element of the form $1 + x$ with $x \in \mathfrak{m}$ is a unit.

Fields are local rings (with maximal ideal (0)). Further examples include discrete valuation rings and localizations $A_{\mathfrak{p}}$ (with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$).

C.3.3 Integral domains. If A is an integral domain, we write $K = K(A)$ for its fraction field, the localization $S^{-1}A$ with $S = A - \{0\}$. The map $a \mapsto a/1$ embeds $A \subset K$, and the localizations A_f, $A_{\mathfrak{p}}$, $A_{\mathfrak{m}}$ all sit as subrings of K.

Lemma C.3. Let A be an integral domain. Inside $K = K(A)$,

$$A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}},$$

where the intersection runs over all maximal ideals of A.

Proof The containment $A \subset \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ is clear. Conversely, let $f \in K$ lie in every $A_{\mathfrak{m}}$. The ideal $I = \{\, a \in A \mid af \in A \,\}$ is proper if $f \notin A$, hence contained in some maximal ideal $\mathfrak{m}$. But $f \in A_{\mathfrak{m}}$ means $sf \in A$ for some $s \notin \mathfrak{m}$, so $s \in I \subset \mathfrak{m}$, a contradiction. □

C.4 Modules

C.4.1 Finiteness conditions. An A-module M is *free* if $M \simeq \bigoplus_{i \in I} A$ for some index set I; the *rank* of M is $|I|$. We write A^n when $|I| = n$.

M is *finitely generated* if there exist $m_1, \ldots, m_n \in M$ generating M as an A-module; equivalently, there is a surjection $A^n \to M$.

M is of *finite presentation* if there is an exact sequence $A^m \to A^n \to M \to 0$ for some integers $m, n \geqslant 0$. If A is Noetherian, every finitely generated module is of finite presentation.

C.4.2 Tensor products. Let M and N be A-modules. The *tensor product* $M \otimes_A N$ is the quotient of the free module with basis $\{e_{m,n}\}_{m \in M,\, n \in N}$ by the relations

$$e_{m_1+m_2,n} - e_{m_1,n} - e_{m_2,n} = 0, \quad e_{m,n_1+n_2} - e_{m,n_1} - e_{m,n_2} = 0, \quad ae_{m,n} - e_{am,n} = ae_{m,n} - e_{m,an} = 0,$$

for all $m \in M$, $n \in N$, $a \in A$. We write $m \otimes n$ for the class of $e_{m,n}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

The map $(m, n) \mapsto m \otimes n$ is bilinear and satisfies the universal property: for every bilinear map $\phi\colon M \times N \to P$, there is a unique A-linear map $\overline{\phi}\colon M \otimes_A N \to P$ with $\phi = \overline{\phi} \circ (m, n \mapsto m \otimes n)$.

Example C.4. If A and B are k-algebras over a field k, then $A \otimes_k B$ is a k-algebra via $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$. △

Example C.5. For any A-module M and ideal $\mathfrak{a} \subset A$, there is a canonical isomorphism $M \otimes_A A/\mathfrak{a} \simeq M/\mathfrak{a}M$. △

C.4.3 Tensor product identities. In the formulas below, M, N, L denote A-modules and $S \subset A$ is a multiplicative set. Each '=' denotes a canonical isomorphism.

(i) $A \otimes_A M = M$.
(ii) $M \otimes_A N = N \otimes_A M$.
(iii) $M \otimes_A (N \otimes_A L) = (M \otimes_A N) \otimes_A L$.
(iv) $\left(\bigoplus_{i\in I} M_i\right) \otimes_A N = \bigoplus_{i\in I}(M_i \otimes_A N)$.
(v) $M \otimes_A A/\mathfrak{a} = M/\mathfrak{a}M$ for every ideal $\mathfrak{a} \subset A$.
(vi) If $A \to B$ is a ring map, M is an A-module, and N is a B-module, then $M \otimes_A N = (M \otimes_A B) \otimes_B N$.
(vii) $\mathrm{Hom}_A(M \otimes_A N, L) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, L))$ *(tensor-hom adjunction).*
(viii) $S^{-1}M = M \otimes_A S^{-1}A$.
(ix) $S^{-1}(M \otimes_A N) = S^{-1}M \otimes_{S^{-1}A} S^{-1}N$.
(x) If k is a field and M, N are k-vector spaces, then $\dim_k(M \otimes_k N) = (\dim_k M) \cdot (\dim_k N)$.

C.4.4 Hom identities.

(i) $\mathrm{Hom}_A(A^n, M) = M^n$.
(ii) $\mathrm{Hom}_A(A/I, M) = \{\, m \in M \mid Im = 0 \,\}$.

C.4.5 Exactness properties. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then:

- $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is exact for every multiplicative set S;
- $0 \to \mathrm{Hom}_A(L, M') \to \mathrm{Hom}_A(L, M) \to \mathrm{Hom}_A(L, M'')$ is exact for every A-module L;
- $0 \to \mathrm{Hom}_A(M'', L) \to \mathrm{Hom}_A(M, L) \to \mathrm{Hom}_A(M', L)$ is exact for every A-module L;
- $M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$ is exact for any A-module N.

Proposition C.6.

(i) For an A-module M, $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$.
(ii) A sequence $0 \to M' \to M \to M'' \to 0$ is exact if and only if $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}} \to 0$ is exact for all maximal ideals $\mathfrak{m}$.
(iii) A map $\phi\colon M \to N$ is injective (resp. surjective) if and only if $\phi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (resp. surjective) for all maximal ideals $\mathfrak{m}$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Proof If $m \in M$ is nonzero, then $\mathrm{Ann}(m) = \{\, a \in A \mid am = 0 \,\}$ is a proper ideal, contained in some maximal ideal $\mathfrak{m}$. If $M_{\mathfrak{m}} = 0$, then $sm = 0$ for some $s \in A - \mathfrak{m}$, but then $s \in \mathrm{Ann}(m) \subset \mathfrak{m}$, a contradiction. This proves (1). Parts (2) and (3) follow by applying (1) to the kernel and cokernel of the relevant maps. □

Proposition C.7. If M is finitely presented, then $\mathrm{Hom}_A(M, -)$ commutes with localization. That is, there is an isomorphism of $S^{-1}A$-modules

$$S^{-1}\,\mathrm{Hom}_A(M, N) = \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \tag{C.5}$$

Proof The isomorphism is defined as follows: sending $\phi\colon M \to N$ to the $S^{-1}A$-linear map $m/s \mapsto \phi(m)/s$, defines a map of A-modules

$$\mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

Since multiplication by an element $s \in S$ is an isomorphism on the A-module on the right, the map extends to a map (C.5).

For $M = A$, we have $\mathrm{Hom}_A(A, N) = N$ and (C.5) is the isomorphism $S^{-1}\,\mathrm{Hom}_A(A, N) = S^{-1}N \to \mathrm{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N) = S^{-1}N$. More generally, it is an isomorphism for any $M = A^N$, $N \in \mathbb{N}$.

The map (C.5) is functorial in M. Applying $\mathrm{Hom}_A(M, -)$ to a presentation $A^q \to A^p \to N \to 0$, we obtain

$$\begin{array}{ccccccc}
0 & \longrightarrow & S^{-1}\mathrm{Hom}_A(M,N) & \longrightarrow & S^{-1}\mathrm{Hom}_A(A^p,N) & \longrightarrow & S^{-1}\mathrm{Hom}_A(A^q,N) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N) & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^p,S^{-1}N) & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^q,S^{-1}N)
\end{array}$$

The two right-most vertical maps are isomorphisms (as shown above). By the five lemma, the left-most vertical map is an isomorphism as well. □

C.4.6 Change of rings. For a ring map $\phi\colon A \to B$, there are three natural operations on modules. Here M is an A-module, and N is a B-module:

- $\phi_! M = M \otimes_A B$ is the *extension of scalars* (B-module).
- $\phi^* N = N_A$ is the *restriction of scalars* (A-module).
- $\phi_* M = \mathrm{Hom}_A(B, M)$ is the *coinduced module* (A-module).

These satisfy the adjunctions

$$\mathrm{Hom}_B(\phi_! M, N) = \mathrm{Hom}_A(M, \phi^* N), \qquad \mathrm{Hom}_A(\phi^* N, M) = \mathrm{Hom}_B(N, \phi_* M).$$

C.4.7 Nakayama's Lemma. The following is a basic, but extremely useful lemma.

Lemma C.8 (Nakayama's lemma). Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module. If $\mathfrak{m}M = M$, then $M = 0$.

Proof Suppose $M \neq 0$ and let $m_1, \ldots, m_r$ ($r \geqslant 1$) be a minimal generating set. Since $M = \mathfrak{m}M$, we may write $m_r = a_1 m_1 + \cdots + a_r m_r$ with $a_i \in \mathfrak{m}$. Then $(1 - a_r)m_r =$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$a_1m_1 + \cdots + a_{r-1}m_{r-1}$. Since $1 - a_r$ is a unit in A, this expresses m_r as a combination of $m_1, \ldots, m_{r-1}$, contradicting minimality. □

Corollary C.9. Let A be a local ring and $\varphi\colon M \to N$ a map of finitely generated A-modules such that $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective. Then φ is surjective.

Proof If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective, then $N = \varphi(M) + \mathfrak{m}N$. Hence

$$\mathfrak{m}(N/\varphi(M)) = (\mathfrak{m}N + \varphi(M))/\varphi(M) = N/\varphi(M).$$

Applying Nakayama's Lemma to $N/\varphi(M)$, we get $N/\varphi(M) = 0$, so φ is surjective. □

The above result can be helpful also in the case A is not local, as by Proposition C.6, a map of A-modules $\phi\colon M \to N$ is surjective if and only if the localization $\phi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is surjective for each maximal ideal $\mathfrak{m}$.

C.4.8 Modules over principal ideal domains. A ring R is a *principal ideal domain* (PID) if it is an integral domain in which every ideal is generated by one element.

For a PID R there is the following structure result for modules. Let R be a PID and M a finitely generated R-module. Then there exist elements $d_1, \ldots, d_m \in R$ with $d_i \mid d_{i+1}$ for each i, and an integer $r \geqslant 0$, such that

$$M \simeq R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_m).$$

In particular, every submodule of a free R-module of finite rank n is free of rank at most n.

Example C.10. For $R = \mathbb{Z}$ this is the classification of finitely generated abelian groups: every such group is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_m)$, where each $d_i \geqslant 2$ and $d_i \mid d_{i+1}$. △

C.4.9 Symmetric and exterior powers. For an A-module M, the *tensor algebra* $T(M)$ is defined as

$$T(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n}$$

where $M^{\otimes n}$ is the n-fold tensor product $M \otimes_A \cdots \otimes_A M$. $T(M)$ has a multiplication given by the tensor product, but this is not commutative except in a few special cases. However, we can define $\mathrm{Sym}(M)$ as the quotient of $T(M)$ by the two-sided ideal generated by all elements of type $x \otimes y - y \otimes x$ for $x, y \in M$. Then $\mathrm{Sym}(M)$ is a graded, commutative, A-algebra. For $n \in \mathbb{N}$, we define $\mathrm{Sym}^n(M)$ to be the n-th graded piece of $\mathrm{Sym}(M)$. As the multiplication is commutative, we write $x_1x_2 \cdots x_n$ for the image of $x_1 \otimes \cdots \otimes x_n$ in $\mathrm{Sym}(M)$. So, explicitly, $\mathrm{Sym}^n(M)$ is generated as an A-module by all symbols $x_1x_2 \cdots x_n$ where $x_1, \ldots, x_n \in M$, modulo the relations $x_1x_2 \cdots x_n = x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}$ where $\sigma \in S_n$ is a permutation.

Example C.11. $\mathrm{Sym}(A^r) = A[x_1, \ldots, x_r]$ and $\mathrm{Sym}^n(A^r)$ is the degree-n piece, which is free of rank $\binom{n+r-1}{r-1}$, with basis the monomials $x_1^{e_1} \cdots x_r^{e_r}$ with $e_1 + \cdots + e_r = n$. △

The *exterior algebra* $\bigwedge M$ is defined as the quotient of $T(M)$ by the two-sided ideal generated by products $x \otimes x$, where $x \in M$. We let $x_1 \wedge \cdots \wedge x_n$ denote the image of $x_1 \otimes \cdots \otimes x_n$ in $\wedge(M)$. In this notation, the multiplication is skew-commutative in

the sense that $x \wedge y = (-1)^{kl} y \wedge x$ for $x \in \bigwedge^k M$, $y \in \bigwedge^l M$, and more generally $x_{\sigma(1)} \wedge x_{\sigma(2)} \cdots \wedge x_{\sigma(n)} = (\text{sign}(\sigma)) x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for a permutation $\sigma \in S_n$.

Example C.12. If M is free of rank r, then $\bigwedge^n M$ is free of rank $\binom{r}{n}$, with basis $\{x_{i_1} \wedge \cdots \wedge x_{i_n} : 1 \leqslant i_1 < \cdots < i_n \leqslant r\}$. In particular, $\bigwedge^n M = 0$ for $n > r$. △

Example C.13. An important special case is when $M = A^r$ is a free module of rank r. Then the A-module $\bigwedge^r M$ is called the *determinant* of M. The name comes from the following: $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ are two bases of M, related by $(y_1, \ldots, y_r) = (x_1, \ldots, x_r) \cdot S$ for some matrix S, then $y_1 \wedge \cdots y_r = (\det S) x_1 \wedge \cdots \wedge x_r$. △

C.5 Integral ring extensions

Let $A \to B$ be a ring map. An element $b \in B$ is *integral over* A if it satisfies a monic polynomial relation

$$b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 = 0, \quad a_i \in A.$$

Example C.14. The element $i \in \mathbb{C}$ is integral over $\mathbb{Z}$, satisfying $x^2 + 1 = 0$. More generally, $\mathbb{Z}[i]$ is an integral extension of $\mathbb{Z}$. For any field k, the element $t \in k[t]$ is integral over $k[t^2]$, satisfying $x^2 - t^2 = 0$, so $k[t^2] \subset k[t]$ is an integral extension. △

Theorem C.15 (Cayley-Hamilton). Let A be a ring, $\mathfrak{a} \subset A$ an ideal and let M be a finitely generated A-module. Let $\varphi \colon M \to M$ be an A-module homomorphism with $\varphi(M) \subset \mathfrak{a}M$. Then there exists a monic polynomial

$$x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in A[x]$$

with $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ such that

$$\varphi^n + a_{n-1} \varphi^{n-1} + \cdots + a_0 \text{id} = 0 \in \text{Hom}_A(M, M),$$

Here φ^i denotes the composition of φ with itself i times.

Let $T = (b_{ij})_{1 \leqslant i,j \leqslant n}$ be an $(n \times n)$-matrix with entries in A, and let T_{ij} be the $(n \times n)$-matrix obtained by deleting row i and column j. The adjoint of T is the matrix $\text{adj}(T) = (c_{ij})_{1 \leqslant i,j \leqslant n}$, where

$$c_{ij} = (-1)^{i+j} \det(T_{ji}).$$

The adjoint matrix has the property that $T \cdot \text{adj}(T) = \text{adj}(T) \cdot T = \det(T) I_n$, where I_n is the $n \times n$ identity matrix. This follows from the usual Laplace expansion of the determinant.

Proof Let $m_1, \ldots, m_n$ be generators for M. By the assumption on ϕ, we have $\varphi(m_i) \in \mathfrak{a}M$ for all $i = 1, \ldots, n$. Therefore, for each i, we may express

$$\varphi(m_i) = \sum_{j=1}^{n} b_{ij} m_j$$

for some elements $b_{ij} \in \mathfrak{a}$. Next, consider M as a module over the ring $A[x]$ by setting

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$x^i \cdot m := \varphi^i(x)$ for all $m \in M$. If I denotes the $n \times n$ identity matrix, and $T = (b_{ij})$, then the above equation can be written as

$$(xI - T) \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

If we multiply both sides by the adjoint matrix of $(xI - T)$, we obtain

$$\det(xI - T) \cdot I \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

Let $P(x)$ be the monic polynomial $\det(xI - T)$. Then $P(x)$ anihilates all the m_i, and hence $P(\phi)$ acts as the zero endomorphism on M. By the Laplace expansion of the determinant, we also see that the coefficient of x^j of P belongs to $\mathfrak{a}^j$. □

Theorem C.16. Let $A \to B$ be a ring map and $b \in B$. The following are equivalent:
(i) b is integral over A.
(ii) $A[b]$ is a finitely generated A-module.
(iii) There is a ring C with $A[b] \subset C \subset B$ such that C is a finitely generated A-module.

Proof (i)⇒(ii): If $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$, then $b^n = -(a_{n-1}b^{n-1} + \cdots + a_0)$, and by induction every power b^k with $k \geqslant n$ lies in the A-span of $\{1, b, \ldots, b^{n-1}\}$.

(ii)⇒(iii): Take $C = A[b]$.

(iii)⇒(i): Apply the Cayley–Hamilton theorem (Theorem C.15) with $M = C$, $\mathfrak{a}$ equal to the image of A in B, and $\varphi\colon C \to C$ given by multiplication by b. This yields a monic polynomial P with coefficients in A such that $P(\varphi) = 0$; evaluating at $1 \in C$ gives $P(b) \cdot 1 = 0$, so b is integral over A. □

Corollary C.17. Any finite ring extension $A \subset B$ is integral.

Proof Take $C = B$ in condition (iii): every $b \in B$ is integral over A. □

For a ring extension $A \subset B$, the *integral closure* of A in B is the subring $\{\, b \in B \mid b \text{ is integral over } A \,\} \subset B$. Using Theorem C.16, one shows that this ring is itself integrally closed in B.

An integral domain A is *normal* if it is integrally closed in its fraction field $K = K(A)$. Every unique factorization domain is normal (see Section C.11).

Lemma C.18. Let $A \subset B$ be an integral ring extension.
(i) If $\mathfrak{b} \subset B$ is an ideal, then $A/(\mathfrak{b} \cap A) \subset B/\mathfrak{b}$ is integral.
(ii) If $S \subset A$ is a multiplicative set, then $S^{-1}A \subset S^{-1}B$ is integral.

Proof (i): Let $b \in B$. Since b is integral over A, there is a relation relation $b^n + a_{n-1}b^{n-1} +$

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$\cdots + a_0 = 0$ with $a_i \in A$. Reducing this modulo $\mathfrak{b}$ shows that $\overline{b}$ is integral over $A/(\mathfrak{b} \cap A)$, as required.

(ii): Let $b/s \in S^{-1}B$. Choosing a monic relation for b over A as above and dividing by s^n, gives $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + (a_0/s^n) = 0$, so b/s is integral over $S^{-1}A$. □

Proposition C.19. Let $A \subset B$ be an integral ring extension with B an integral domain. Then A is a field if and only if B is a field.

Proof Suppose A is a field and let $0 \neq b \in B$. Then b satisfies a monic relation $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$. Since B is a domain we may assume $a_0 \neq 0$. Rearranging gives $b(b^{n-1} + \cdots + a_1) = -a_0$, and multiplying by $(-a_0)^{-1}$ shows b is a unit. Hence B is a field.

Conversely, assume B is a field and let $0 \neq a \in A$. Choose $b \in B$ with $ab = 1$. Since b is integral over A, it satisfies a monic relation $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$. Multiplying by a^n gives $1 + a_{n-1}a + \cdots + a_0a^n = 0$. Rearranging shows that a is invertible in A, so A is a field. □

Lemma C.20. Let $A \subset B$ be integral and $\mathfrak{q} \subset B$ a prime ideal. Then $\mathfrak{q}$ is maximal in B if and only if $\mathfrak{q} \cap A$ is maximal in A.

Proof By Lemma C.18(i), the extension $A/(\mathfrak{q} \cap A) \subset B/\mathfrak{q}$ is integral. Since $B/\mathfrak{q}$ is a domain, Proposition C.19 shows that $B/\mathfrak{q}$ is a field if and only if $A/(\mathfrak{q} \cap A)$ is a field. □

Theorem C.21. Let $A \subset B$ be an integral ring extension and consider the map

$$f\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad \mathfrak{q} \mapsto \mathfrak{q} \cap A.$$

(i) (Surjectivity): For each $\mathfrak{p} \in \operatorname{Spec} A$, there exists $\mathfrak{q} \in \operatorname{Spec} B$ with $f(\mathfrak{q}) = \mathfrak{p}$.
(ii) (Uniqueness): If $\mathfrak{q}_0 \subset \mathfrak{q}_1$ in B and $f(\mathfrak{q}_0) = f(\mathfrak{q}_1)$, then $\mathfrak{q}_0 = \mathfrak{q}_1$.
(iii) ('Going-Up'): Given a chain of prime ideals in A

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r,$$

and a $\mathfrak{q}_0 \in \operatorname{Spec} B$ with $f(\mathfrak{q}_0) = \mathfrak{p}_0$, there exists a chain of prime ideals in B,

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_r,$$

such that $f(\mathfrak{q}_i) = \mathfrak{p}_i$ for all i.

Proof (i): Let $S = A - \mathfrak{p}$. Then $S^{-1}A$ is a local ring with maximal ideal $S^{-1}\mathfrak{p}$. By Lemma C.18, the extension $S^{-1}A \subset S^{-1}B$ is integral. Let $\mathfrak{m} \subset S^{-1}B$ be a maximal ideal. By Lemma C.20, $\mathfrak{m} \cap S^{-1}A$ is maximal in $S^{-1}A$, hence equals $S^{-1}\mathfrak{p}$.

Let $\mathfrak{q}$ be the inverse image of $\mathfrak{m}$ under the localization map $B \to S^{-1}B$. Then $\mathfrak{q}$ is prime

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

in B and we see that $\mathfrak{q} \cap A = \mathfrak{p}$ by commutativity of the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \longrightarrow & B_{\mathfrak{q}} \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} B \\ \uparrow & & \uparrow \\ \operatorname{Spec} A_{\mathfrak{p}} & \longleftarrow & \operatorname{Spec} B_{\mathfrak{q}} \end{array}$$

(ii): Replacing A by $A/(\mathfrak{q}_0 \cap A)$ and B by $B/\mathfrak{q}_0$, we may assume that B is an integral domain and $\mathfrak{q}_0 = (0)$, $\mathfrak{q}_1 = \mathfrak{q}$, with $\mathfrak{q} \cap A = (0)$. We must show that $\mathfrak{q} = (0)$.

Let $S = A - \{0\}$. Then $S^{-1}A = \operatorname{Frac}(A)$ is a field. By Lemma C.18, the extension $S^{-1}A \subset S^{-1}B$ is integral. As $S^{-1}B$ is an integral domain, Proposition C.19 therefore implies that $S^{-1}B$ is a field. Therefore $S^{-1}B$ has a unique prime ideal (0). Now prime ideals of $S^{-1}B$ are in one-to-one correspondence with prime ideals of B disjoint from S. As both (0) and $\mathfrak{q}$ are disjoint from $S = A - \{0\}$, we must have $\mathfrak{q} = (0)$.

(iii): By induction, it suffices to consider the case $r = 1$. By Lemma C.18, the extension $A/\mathfrak{p}_0 \subset B/\mathfrak{q}_0$ is integral. Applying (i) to this extension and the prime ideal $\mathfrak{p}_1/\mathfrak{p}_0$ in $A/\mathfrak{p}_0$, we find a prime ideal $\overline{\mathfrak{q}}_1 \subset B/\mathfrak{q}_0$ such that $\overline{\mathfrak{q}}_1 \cap (A/\mathfrak{p}_0) = \mathfrak{p}_1/\mathfrak{p}_0$. Let $\mathfrak{q}_1 \subset B$ be the inverse image of $\overline{\mathfrak{q}}_1$ under the quotient map $B \to B/\mathfrak{q}_0$. Then $\mathfrak{q}_1$ is a prime ideal which contains $\mathfrak{q}_0$ and it satisfies $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. □

Proposition C.22. Let $A \subset B$ be an integral ring extension and $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ as above. Then:
(i) f is surjective.
(ii) f is a closed map: $f(V(\mathfrak{b})) = V(\mathfrak{b} \cap A)$ for every ideal $\mathfrak{b} \subset B$.

Proof Surjectivity is Theorem C.21(i). For closedness, the inclusion $f(V(\mathfrak{b})) \subset V(\mathfrak{b} \cap A)$ is clear. Applying surjectivity to the integral extension $A/(\mathfrak{b} \cap A) \to B/\mathfrak{b}$ (Lemma C.18) gives the opposite inclusion. □

C.6 Noetherian Rings

A ring A is *Noetherian* if every ideal of A is finitely generated, or equivalently, if every ascending chain of ideals $I_1 \subset I_2 \subset \cdots$ stabilizes.

Example C.23. Fields and $\mathbb{Z}$ are Noetherian. The polynomial ring $k[x_1, x_2, \dots]$ in countably many variables is *not* Noetherian: the chain $(x_1) \subset (x_1, x_2) \subset \cdots$ does not stabilize. △

Theorem C.24 (Hilbert's Basis Theorem). If A is Noetherian, then $A[x_1, \dots, x_n]$ is also Noetherian.

A ring A is *Artinian* if every descending chain of ideals $I_1 \supset I_2 \supset \cdots$ stabilizes. Every Artinian ring is Noetherian, has finitely many prime ideals (all of which are maximal), and has Krull dimension zero. The natural map $A \to \prod_{\mathfrak{m}} A_{\mathfrak{m}}$ is an isomorphism, expressing A as a product of Artinian local rings.

One also talks about Noetherian *modules*. An A-module M is called *Noetherian* if every

finitely generated submodule is finitely generated. Equivalently, M is Noetherian if every ascending chain of summodules $M_1 \subset M_2 \subset \cdots$ stabilizes.

If M is Noetherian, then every submodule of M and every quotient module M/N is also Noetherian. M is Noetherian if and only if N and M/N and both Noetherian. This implies that any module of the form $(M_1 \oplus \cdots \oplus M_r)/N$ where the M_i are Noetherian, is again Noetherian. In particular, if A is Noetherian, then finitely generated module over a Noetherian ring is a Noetherian module.

If A is a Noetherian ring and M is a finitely generated A-module, then M admits a filtration of the form

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M \tag{C.6}$$

where each successive quotient $M_i/M_{i-1} \simeq A/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \subset A$.

C.7 Some definitions from Field Theory

A *field extension* K/k is a ring map $k \to K$ between fields. The *degree* $[K : k] = \dim_k K$ is the dimension of K as a k-vector space. For a tower $k \subset K \subset L$ one has $[L : k] = [L : K][K : k]$.

The extension is *finite* if $[K : k] < \infty$; *finitely generated* if $K = k(a_1, \ldots, a_n)$ (the fraction field of $k[a_1, \ldots, a_n]$) for some $a_i \in K$. An element $a \in K$ is *algebraic over* k if $P(a) = 0$ for some nonzero $P \in k[t]$; the extension is *algebraic* if every element is algebraic. A field extension is finite if and only if it is algebraic and finitely generated.

Elements $a_1, \ldots, a_r \in K$ are *algebraically independent over* k if no nonzero polynomial $P \in k[x_1, \ldots, x_r]$ satisfies $P(a_1, \ldots, a_r) = 0$; equivalently, $k[a_1, \ldots, a_r] \simeq k[t_1, \ldots, t_r]$. A set $\{a_1, \ldots, a_n\}$ of algebraically independent elements is a *transcendence basis* for K/k if K is algebraic over $k(a_1, \ldots, a_n)$. All transcendence bases have the same cardinality, the *transcendence degree* $\operatorname{trdeg}_k K$. For a tower $k \subset K \subset L$,

$$\operatorname{trdeg}_k L = \operatorname{trdeg}_k K + \operatorname{trdeg}_K L.$$

If $A \subset B$ are integral domains with B finitely generated as an A-algebra, then $K(B)/K(A)$ is a finitely generated field extension of finite transcendence degree.

A field k is *algebraically closed* if every nonconstant polynomial in $k[x]$ has a root in k. An *algebraic closure* $\bar{k}$ is an algebraically closed field containing k, algebraic over k; it is unique up to isomorphism.

C.7.1 Separable extensions. A polynomial $f \in k[x]$ is *separable* if it has no repeated roots in $\bar{k}$. A field extension K/k is *separable* if every element of K algebraic over k is the root of a separable polynomial; equivalently, $K \otimes_k L$ is reduced for every extension L/k. Every extension in characteristic 0 is separable. Purely transcendental extensions are always separable.

A field k is *perfect* if every algebraic extension of k is separable; equivalently, $\operatorname{char}(k) = 0$, or $\operatorname{char}(k) = p > 0$ and every element of k is a p-th power. Examples of perfect fields include $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, all finite fields, and all algebraically closed fields.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Theorem C.25 (Primitive Element Theorem). Let k be a field and let K/k be a finite separable extension. Then there exists $\theta \in K$ such that $K = k(\theta)$.

Proof If k is finite, then so is K. Write $K = \mathbb{F}_{p^n}$. Any proper subfield of K has size p^s for some $s < n$, and its elements are roots of $T^{p^s} - T = 0$. The union of all proper subfields of K has at most $p + p^2 + \cdots + p^{n-1} < p^n$ elements. Therefore, there exists some $\theta \in K$ such that $\mathbb{F}_p(\theta) = K$, and hence $k(\theta) = K$. We may therefore assume that k is infinite.

By induction, it suffices to show: if α, β are algebraic over k and separable over k, then for all but finitely many $t \in k$, $k(\alpha - t\beta) = k(\alpha, \beta)$. Fix such α, β and put $E = k(\alpha, \beta)$. Let $f, g \in k[x]$ be the minimal polynomials of α, β over k. In $E[y]$ write

$$f(\alpha + y) = yF(y), \qquad g(\beta + y) = yG(y)$$

for some $F, G \in E[y]$ (this is possible since $f(\alpha) = g(\beta) = 0$). Because β is separable, it is a *simple* root of g, which implies $G(0) \neq 0$.

Applying Lemma C.26 below in the field E to $A = F$ and $B = G$, we may choose a $t \in k^\times$ such that the polynomials $F(ty)$ and $G(y)$ are coprime. Let $\theta = \alpha - t\beta$ and $L = k(\theta) \subset E$. We claim that $L = E$.

Consider the polynomial $h(x) = f(\theta + tx) \in L[x]$. With $x = \beta + y$ we obtain in $E[y]$:

$$h(\beta + y) = f(\alpha + ty) = ty\, F(ty), \qquad g(\beta + y) = y\, G(y).$$

Since $F(ty)$ and $G(y)$ are coprime and $y \nmid G(y)$, the gcd of these two polynomials in $E[y]$ is exactly y. Translating back, the gcd of $h(x)$ and $g(x)$ in $E[x]$ is $x - \beta$.

Let $m(x) \in L[x]$ be the minimal polynomial of β over L. Because $h(\beta) = f(\alpha) = 0$ and $g(\beta) = 0$, we have $m \mid h$ and $m \mid g$ in $L[x]$, hence also in $E[x]$. Therefore m divides $x - \beta$ in $E[x]$, so $\deg m = 1$ and $\beta \in L$. Then $\alpha = \theta + t\beta \in L$, hence $L = k(\alpha, \beta)$, i.e. $k(\alpha - t\beta) = k(\alpha, \beta)$. □

Lemma C.26. Let L be a field, and let $A, B \in L[y]$ be non-zero polynomials with $B(0) \neq 0$. Then for all but finitely many $t \in L$, $A(ty)$ and $B(y)$ are coprime.

Proof It suffices to show that for any irreducible factor $p(y)$ of $B(y)$, there are only finitely many $t \in L$ such that $p(y) \mid A(ty)$.

Let $K = L[y]/(p)$ and let β be the image of y in K. Because $B(0) \neq 0$, we know $p(0) \neq 0$, hence $\beta \neq 0$ in K. We have $p(y) \mid A(ty)$ in $L[y]$ if and only if $A(t\beta) = 0$ in K.

Consider $A(T\beta)$ in $K[T]$. This polynomial is nonzero because if $A(y) = a_d y^d + \cdots$, then $A(T\beta) = (a_d\beta^d)T^d + \cdots$ with $a_d\beta^d \neq 0$. Therefore, $A(y)$ has at most $\deg(A)$ zeroes in K. Consequently, there are at most $\deg(A)$ values of $t \in L$ that satisfy the above condition. As $B(y)$ has only finitely many irreducible factors, we see that the set of 'bad' t's is finite. □

C.8 Noether's Normalization Lemma

Noether's Normalization Lemma, due to Emmy Noether, is a fundamental result about algebras over a field. It is powerful because it allows us to reduce many questions about

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

general k-algebras to statements about the polynomial ring $k[x_1, \ldots, x_n]$. The lemma has many fundamental results as consequences. In particular, Hilbert's Nullstellensatz is a corollary.

Theorem C.27 (Noether's Normalization Lemma). Let k be a field and A be a finitely generated k-algebra. For some integer $0 \leqslant d \leqslant n$, there are elements $y_1, \ldots, y_d \in A$ such that A is an integral extension of the polynomial ring $k[y_1, \ldots, y_d]$.

If A is generated by $x_1, \ldots, x_n$ and k is infinite, the proof shows that $y_1, \ldots, y_d$ can be chosen as k-linear combinations of the $x_1, \ldots, x_n$. We begin with a basic lemma.

Lemma C.28. Let R be a ring and let $P \in R[t_1, \ldots, t_m]$ be a nonzero polynomial.

(i) For a sufficiently large integer s, we have

$$P(t_1 + t_m^s, \ldots, t_{m-1} + t_m^{s^{m-1}}, t_m) = g \cdot t_m^d + (\text{terms of order} < d \text{ in } t_m)$$

where $g \in R$ is nonzero coefficient of P.

(ii) If $R = k$ is an infinite field, then there are $a_1, \ldots, a_{m-1} \in k$ such that

$$P(t_1 + a_1 t_m, \ldots, t_{m-1} + a_{m-1} t_m, t_m) = g \cdot t_m^d + (\text{terms of order} < d \text{ in } t_m)$$

where $g \in k$.

Proof (i): Consider a monomial $t_1^{e_1} \ldots t_m^{e_m}$ in P. If we perform the substitution $t_i \mapsto t_i + t_m^{s^i}$ for $i = 1, \ldots, m-1$, we get a polynomial in t_m of the form $t_m^N +$(lower degree terms in t_m), where $N = e_m + e_1 s + e_2 s^2 + \cdots + e_{m-1} s^{m-1}$. Note that if $(e_1, \ldots, e_m)$ and $(e_1', \ldots, e_m')$ are two tuples which give the same N, then

$$(e_m - e_m') + (e_1 - e_1')s + \cdots + (e_{m-1} - e_{m-1}')s^{m-1} = 0$$

As this is a real polynomial in s, it has only finitely many roots. As P has only finitely many monomials, this means that we may choose s large enough so that all N's are distinct. This means that if we expand $P(t_1 + t_m^s, \ldots, t_{m-1} + t_m^{s^{m-1}}, t_m)$ into powers of t_m, the leading term will be of the desired form $g t_m^d$, where g is one of the nonzero coefficients of P.

(ii): Let $d = \deg(P)$ and write $P = P_d + Q$ where P_d is the homogeneous degree-d part of P. For $a = (a_1, \ldots, a_{m-1})$, the coefficient of t_m^d in $P(t_1 + a_1 t_m, \ldots, t_{m-1} + a_{m-1} t_m, t_m)$ is $P_d(a_1, \ldots, a_{m-1}, 1) \in k$. Since $P_d \neq 0$ and k is infinite, we may find some $a \in k^{m-1}$ such that $P_d(a_1, \ldots, a_{m-1}, 1) \neq 0$. For such a choice, the leading coefficient in t_m is $g = P_d(a, 1) \in k^\times$. □

Proof of Noether's Normalization Lemma We proceed by induction on n, the number of generators of A. If $n = 0$, then A is quotient of k, so we may take $d = 0$.

Assume $n \geqslant 1$ and fix a surjection $\varphi\colon k[t_1, \ldots, t_n] \to A$. Let $I = \operatorname{Ker} \varphi$. If $I = (0)$, then $A \simeq k[t_1, \ldots, t_n]$, and we may take $d = n$ with $y_i = x_i$.

If $I \neq (0)$, choose a nonzero polynomial $f \in I$. By Lemma C.28, we can perform a change of variables to new generators $t_1', \ldots, t_n'$ such that f is monic in t_n' (after multiplying with a suitable nonzero scalar in k).

Let $x_i' = \varphi(t_i')$ and let $B \subset A$ be the subalgebra generated by $x_1', \ldots, x_{n-1}'$. Since

$f(x_1', \ldots, x_n') = 0$ is a monic polynomial equation in x_n' with coefficients in B, the element x_n' is integral over B. Consequently, $A = B[x_n']$ is an integral extension of B.

Since B requires only $n-1$ generators, the induction hypothesis implies that B is an integral extension of a polynomial ring $k[y_1, \ldots, y_d]$. Because integrality is transitive, the same holds for A. □

Corollary C.29. Let A be a finitely generated algebra over a field k and let $\mathfrak{m}$ be a maximal ideal in A. Then $A/\mathfrak{m}$ is a finite field extension of k.

Proof The field $K = A/\mathfrak{m}$ is finitely generated as a k-algebra because A is. By Noether's Normalization Lemma, K is an integral extension over some polynomial ring $k[y_1, \ldots, y_r]$. If $r \geqslant 1$, then K has positive dimension, which is not possible. Therefore, $r = 0$, and K is a finite field extension of k. □

C.9 Graded rings

In this book, a *graded ring* will refer to a ring R which is graded by the non-negative integers, i.e. a ring which admits a decomposition

$$R = \bigoplus_{n \geqslant 0} R_n = R_0 \oplus R_1 \oplus \cdots$$

as an abelian group, such that $R_m \cdot R_n \subset R_{m+n}$ for each $m, n \geqslant 0$. Occasionally, we will also discuss *$\mathbb{Z}$-graded rings*, where we allow negative degrees as well.

A ring map $\phi\colon R \to S$ between two graded rings R and S is said to be a *map of graded rings* if it respects the grading, that is, if $\phi(R_n) \subset S_n$ for all n.

Note that R_0 is a subring of R and that R is an algebra over R_0. Moreover, each R_n is an R_0-module. The elements in R_n are said to be *homogeneous of degree* n, and one writes $\deg x = n$ when $x \in R_n$. (By convention, 0 is considered homogeneous of any degree.) Every nonzero element $x \in R$ can be expressed uniquely as a finite sum $x = \sum_n x_n$ with $x_n \in R_n$, and the nonzero terms in the sum are called the *homogeneous components* of x.

Example C.30. The simplest examples of graded rings are the polynomial rings $R = A[t_0, \ldots, t_r]$, with the *standard grading* where each variable t_i has degree 1 and the elements from A have degree 0. Each graded piece R_n is a free module over $R_0 = A$ with the monomials of degree n serving as a basis. △

An R-module M is *graded* if it has a similar decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ as an abelian group and $R_m M_n \subset M_{m+n}$ for all n and m. Note that we allow also elements of negative degrees. A *map of graded R-modules* is an R-linear map $\phi\colon M \to N$ satisfying $\phi(M_n) \subset N_n$ for all $n \in \mathbb{Z}$. With this notion of morphisms, the graded R-modules form a category, denoted GrMod_R.

As for graded rings, a nonzero element $x \in M$ is *homogeneous* of degree n if it lies in M_n. Any element $x \in M$ may be expressed in a unique way as a finite sum $x = \sum_n x_n$ with each x_n in M_n, and the nonzero terms are called the *homogeneous components* of x.

Most of the familiar definitions for modules carry over to the graded setting. For instance, the direct sum of a collection of graded modules $\bigoplus_i M_i$ is graded in a natural way such

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

that canoncal inclusions $M_j \hookrightarrow \bigoplus_i M_i$ preserve the grading. Likewise, the kernel and the cokernel of a map of graded modules are also graded in a natural way.

A sequence of graded modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

is exact if it is exact as sequence of ordinary modules. As maps of graded modules preserve the grading, this is equivalent to saying that each of the sequences

$$0 \longrightarrow M'_n \longrightarrow M_n \longrightarrow M''_n \longrightarrow 0.$$

are exact (as a sequence of R_0-modules).

An ideal $\mathfrak{a} \subset R$ is called *homogeneous* if the homogeneous components of each element in $\mathfrak{a}$ belongs to $\mathfrak{a}$. In other words, we may write $\mathfrak{a} = \bigoplus_n \mathfrak{a}_n$ with $\mathfrak{a}_n = \mathfrak{a} \cap R_n$. An ideal $\mathfrak{a}$ is homogeneous if and only if it is generated by homogeneous elements (see Exercise **??**). It is readily verified that radicals, intersections, sums and products of homogeneous ideals are homogeneous. If $\mathfrak{a}$ is an homogeneous ideal, the quotient $R/\mathfrak{a}$ inherits a grading from R and $R/\mathfrak{a} = \bigoplus_n R_n/\mathfrak{a}_n$.

A homogenous ideal $\mathfrak{p}$ is prime if and only if for each pair of homogeneous elements $f, g \in R$, $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

We will write R_+ for the direct sum $\bigoplus_{n>0} R_n$. This is naturally a homogeneous ideal of R, which we call the *irrelevant ideal*.

Example C.31. The irrelevant ideal of a polynomial ring $R = A[t_0, \ldots, t_r]$ is equal to $R_+ = (t_0, \ldots, t_r)$. △

Example C.32 (Veronese subrings)**.** For a graded ring R and $d \in \mathbb{N}$, we define $R^{(d)}$ to be the subring of R defined by $\bigoplus_{n \geqslant 0} R_{nd}$. As every element of $R^{(d)}$ has degree equal to a multiple of d, it is often useful to redefine the grading on $R^{(d)}$ by dividing all degrees by d.

For instance, when $R = k[x_0, x_1]$, the Veronese subring $R^{(2)}$ is given by $k[x_0^2, x_0x_1, x_1^2]$. We may adjust the grading by defining $\deg x_0 = \deg x_1 = \frac{1}{2}$, the elements of the ring $R^{(2)}$ will have integer degrees. A different viewpoint is to note that $R^{(2)}$ is isomorphic to the graded ring $k[u_0, u_1, u_2]/(u_1^2 - u_0u_2)$, where in the latter ring, the variables have degree 1. △

If $S \subset R$ is a multiplicative system consisting of homogeneous elements, and M is a graded module, the localization $S^{-1}M$ is naturally a graded R-module with degree n part equal to

$$(S^{-1}M)_n = \big\{m/s \in S^{-1}M \mid m \in M \text{ homogeneous}, s \in S \text{ and } \deg m - \deg s = n\big\}.$$

In particular, if f is a homogeneous element of positive degree, the localization R_f is a $\mathbb{Z}$-graded ring. The degree 0 part $(R_f)_0$ plays an important role in the Proj construction.

C.10 Discrete Valuation Rings

Let K be a field. A *discrete valuation* is a surjective map $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ satisfying, for all $f, g \in K$:

(1) $\nu(fg) = \nu(f) + \nu(g)$;
(2) $\nu(f+g) \geqslant \min\{\nu(f), \nu(g)\}$, with equality when $\nu(f) \neq \nu(g)$;
(3) $\nu(f) = \infty$ if and only if $f = 0$.

An integral domain A with fraction field K is a *discrete valuation ring* (DVR) if there exists a discrete valuation ν on K such that $A = \{\, x \in K \mid \nu(x) \geqslant 0 \,\}$.

Example C.33. Let $K = k(x)$ be the field of rational functions in one variable. Every nonzero $f \in K$ can be written uniquely as $f = x^n g(x)/h(x)$, where $n \in \mathbb{Z}$ and $g(x), h(x) \in k[x]$ satisfy $g(0), h(0) \neq 0$. Define the *order of vanishing at* 0 by $\nu_0(f) = n$ and $\nu_0(0) = \infty$. The elements of nonnegative valuation are exactly the rational functions that can be written as $g(x)/h(x)$ with $h(0) \neq 0$. Therefore the valuation ring is $A = k[x]_{(x)}$, with maximal ideal $(x) = \{\, f \in A \mid \nu_0(f) \geqslant 1 \,\}$. △

Example C.34. Let $K = k(x)$ and define $v_\infty \colon K^\times \to \mathbb{Z}$ by $v_\infty(f/g) = \deg g - \deg f$, where $f, g \in k[x]$ are nonzero. Set $v_\infty(0) = \infty$. This valuation measures order of vanishing at infinity; negative values correspond to poles at infinity. The corresponding valuation ring is $A = \{\, f/g \in k(x) \mid \deg f \leqslant \deg g \,\}$, with maximal ideal $\mathfrak{m} = \{\, f/g \in k(x) \mid \deg f < \deg g \,\}$. △

Example C.35. Let $K = \mathbb{Q}$ and let p be a prime number. Every nonzero rational number $y \in \mathbb{Q}$ can be expressed uniquely as $y = p^n a/b$, where $n \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ are not divisible by p. Define the *p-adic valuation* by $v_p(y) = n$ and $v_p(0) = \infty$. The valuation ring is $A = \mathbb{Z}_{(p)} = \{m/n \in \mathbb{Q} \mid p \nmid n\}$. △

Proposition C.36. Let A be a DVR with fraction field K, associated valuation ν, and maximal ideal $\mathfrak{m} = \{\, f \in K \mid \nu(f) \geqslant 1 \,\}$. Then:
(i) A is a local ring with maximal ideal $\mathfrak{m}$ and $A^\times = \{\, f \in K \mid \nu(f) = 0 \,\}$.
(ii) If $t \in A$ satisfies $\nu(t) = 1$, then $\mathfrak{m} = (t)$.
(iii) Every nonzero ideal of A is of the form (t^n) for some $n \geqslant 0$, where $\nu(t) = 1$.
(iv) A is Noetherian of Krull dimension 1.
(v) A is normal.

Proof (i): $f \in A$ is a unit if and only if $1/f \in A$, which holds if and only if $\nu(f) = 0$. So the non-units are exactly $\mathfrak{m} = \{\, f \mid \nu(f) \geqslant 1 \,\}$, which is an ideal (closed under addition by the ultrametric property and under multiplication by elements of A). Thus A is local with maximal ideal $\mathfrak{m}$.

(ii): If $f, g \in A$ with $\nu(g) \geqslant \nu(f)$, then $\nu(g/f) \geqslant 0$, so $g \in (f)$. For any $g \in \mathfrak{m}$, we have $\nu(g) \geqslant 1 = \nu(t)$, so $g \in (t)$. Thus $\mathfrak{m} \subset (t)$, and the reverse inclusion holds since $t \in \mathfrak{m}$.

(iii): Let $\mathfrak{a}$ be a nonzero ideal, and choose $f \in \mathfrak{a}$ with $\nu(f)$ minimal. For any $g \in \mathfrak{a}$, $\nu(g) \geqslant \nu(f)$, so $g \in (f)$ by the same argument. Hence $\mathfrak{a} = (f) = (t^{\nu(f)})$.

(iv): By (iii), the nonzero ideals of A are $(t^0) = A, (t), (t^2), \ldots$, so A is Noetherian. The only prime ideals are (0) and $(t) = \mathfrak{m}$ (if $\mathfrak{p} = (t^n)$ is prime and $n \geqslant 2$, then $t^n = t \cdot t^{n-1} \in \mathfrak{p}$ forces $t \in \mathfrak{p}$, so $\mathfrak{p} \supset \mathfrak{m}$). Thus $\dim A = 1$.

(v): Suppose $x \in K$ is integral over A, satisfying $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with $a_i \in A$. Let $d = \nu(x)$ and assume for contradiction that $d < 0$. Then $\nu(x^n) = nd$, while for each $i < n$ we have $\nu(a_i x^i) = \nu(a_i) + id \geqslant id > nd$ (since $d < 0$ reverses the

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

inequality $i < n$). By the ultrametric property, the term x^n has strictly smaller valuation than all others, so the sum $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has valuation $nd \neq \infty$ and cannot be zero—a contradiction. Thus $d \geqslant 0$, so $x \in A$. □

An element $t \in A$ with $\nu(t) = 1$ is called a *local parameter* (or sometimes *uniformizer*) for A. By Proposition C.36(ii), any element of $\mathfrak{m} - \mathfrak{m}^2$ is a local parameter. Every nonzero element of K can be written uniquely as ut^n with $u \in A^\times$ and $n \in \mathbb{Z}$: for $0 \neq f \in A$, n is the unique integer with $f \in \mathfrak{m}^n - \mathfrak{m}^{n+1}$; for general $f/g \in K^\times$, one has $\nu(f/g) = \nu(f) - \nu(g)$.

Proposition C.37. Let A be a Noetherian local domain of dimension 1, with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. The following are equivalent:

(i) A is a DVR.
(ii) $\mathfrak{m}$ is principal.
(iii) Every nonzero ideal of A is a power of $\mathfrak{m}$.
(iv) A is integrally closed in its fraction field.
(v) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.

Proof The implications (i)⇒(ii), (i)⇒(iii), (i)⇒(iv), and (i)⇒(v) all follow from Proposition C.36.

Assuming (ii), write $\mathfrak{m} = (t)$. We first show that

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0). \tag{C.7}$$

Suppose $a \in \bigcap_n (t^n)$. For each $n \geqslant 1$ write $a = a_n t^n$ with $a_n \in A$. Since A is a domain, $a_n = a_{n+1}t$, giving an ascending chain $(a_1) \subset (a_2) \subset \cdots$. By the Noetherian hypothesis, this chain stabilizes: $(a_N) = (a_{N+1}) = \cdots$ for some N, say $a_{N+1} = u_N a_N$ for a unit u_N. Then $a = a_{N+1}t^{N+1} = a_N t^N$ gives $(u_N t - 1)a_N t^N = 0$. Since $u_N t - 1$ is a unit ($t \in \mathfrak{m}$) and A is a domain, $a_N = 0$, so $a = 0$.

(ii)⇒(i): By (C.7), every nonzero $f \in A$ lies in some $\mathfrak{m}^n - \mathfrak{m}^{n+1}$ and factors as $f = ut^n$ with $u \in A^\times$. Define $\nu(f) = n$ for $f \in A - \{0\}$, and extend to $K^\times$ by $\nu(f/g) = \nu(f) - \nu(g)$. One checks that ν is a discrete valuation and $A = \{\, x \in K \mid \nu(x) \geqslant 0 \,\}$.

(ii)⇒(iii): Let $\mathfrak{a} \neq 0$. By (C.7), $\mathfrak{a} \not\subset \mathfrak{m}^n$ for some n; let n be the largest integer with $\mathfrak{a} \subset \mathfrak{m}^n$. Choose $a \in \mathfrak{a} - \mathfrak{m}^{n+1}$; then $a = bt^n$ with $b \notin \mathfrak{m}$, so b is a unit and $t^n \in \mathfrak{a}$, giving $\mathfrak{m}^n = (t^n) \subset \mathfrak{a}$. The reverse inclusion holds by definition of n, so $\mathfrak{a} = \mathfrak{m}^n$.

(iii)⇒(ii): Since $\dim A = 1$, Nakayama's lemma shows $\mathfrak{m} \neq \mathfrak{m}^2$ (otherwise $\mathfrak{m} = 0$, contradicting dimension 1). Pick $t \in \mathfrak{m} - \mathfrak{m}^2$. By (iii), $(t) = \mathfrak{m}^r$ for some r; since $t \notin \mathfrak{m}^2$ we have $r = 1$, so $\mathfrak{m} = (t)$.

(iv)⇒(ii): Suppose A is integrally closed. Let $x \in \mathfrak{m}$ be nonzero. Since $\dim A = 1$, the only nonzero prime of A is $\mathfrak{m}$, so $\sqrt{(x)} = \mathfrak{m}$ and $\mathfrak{m}^n \subset (x)$ for some n. Choose n minimal and pick $y \in \mathfrak{m}^{n-1} - (x)$. Let $z = x/y \in K$. Then

$$z^{-1}\mathfrak{m} \subset x^{-1}\mathfrak{m}^n \subset x^{-1}(x) = A,$$

so $z^{-1}\mathfrak{m}$ is an ideal of A. If $z^{-1}\mathfrak{m} \subset \mathfrak{m}$, then multiplication by z^{-1} is an A-module endomorphism of the finitely generated module $\mathfrak{m}$; by the Cayley–Hamilton theorem (applied with $M = \mathfrak{m}$, $\mathfrak{a} = A$, $\varphi = z^{-1}\cdot$), z^{-1} is integral over A, hence $z^{-1} \in A$. But then

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

$y = xz^{-1} \in (x)$, contradicting the choice of y. Therefore $z^{-1}\mathfrak{m} \not\subset \mathfrak{m}$, which forces $z^{-1}\mathfrak{m} = A$, so $\mathfrak{m} = (z)$.

(ii)$\Leftrightarrow$(v): If $\mathfrak{m} = (t)$, then $\mathfrak{m}/\mathfrak{m}^2$ is spanned by the class of t. Since $\mathfrak{m} \neq \mathfrak{m}^2$ (as above), $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$. Conversely, if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, choose $t \in \mathfrak{m}$ whose image generates $\mathfrak{m}/\mathfrak{m}^2$; then $\mathfrak{m} = (t) + \mathfrak{m}^2$, and Nakayama's lemma gives $\mathfrak{m} = (t)$. □

C.11 Unique Factorization Domains

In a Noetherian ring A, any nonzero nonunit $x \in A$ admits a factorization

$$x = x_1 \cdots x_n \tag{C.8}$$

where each x_i is irreducible (that is, an element which is nonzero, not a unit, and not a product of two nonunits). Indeed, if no such factorization existed, then we must be able to write $x = a_1 b_1$ with a_1, b_1 nonunits, then write $a_1 = a_2 b_2$ similarly and so on. Thus we obtain nonunits $a_1, a_2, \ldots$ in A such that a_{i+1} divides a_i for each $i \geqslant 1$ and a_{i+1}/a_i is not a unit. This means that the chain $(a_1) \subset (a_2) \subset (a_3) \subset \ldots$ is an infinite strictly increasing sequence of ideals, contradicting the fact that A is Noetherian.

A is a *unique factorization domain* (UFD) if the factorization (C.8) is unique up to order and multiplication by units.

Example C.38. Fields, $\mathbb{Z}$, $\mathbb{Z}[i]$, and all PID s are UFD s. △

Proposition C.39 (Gauss's lemma). Let A be a UFD with fraction field K. If $f \in A[x]$ is a polynomial that factors in $K[x]$ into a product of nonconstant polynomials, then it factors in $A[x]$ as well.

Comparing factorizations in $A[x]$ and $K(A)[x]$, one shows the following:

Example C.40. If A is a UFD, then so is $A[x]$. △

Proposition C.41. Let A be an integral domain and $f \in A$ a nonzero element.
(i) If (f) is prime, then f is irreducible.
(ii) If A is a UFD and f is irreducible, then (f) is prime.

Proof (i): Suppose (f) is prime and $f = gh$. Then $g \in (f)$ or $h \in (f)$; say $g = af$. Then $f = (af)h = f(ah)$, so $ah = 1$ and h is a unit. Therefore f is irreducible.

(ii): Suppose f is irreducible and $xy \in (f)$, say $xy = af$. Write $x = x_1 \cdots x_s$ and $y = y_1 \cdots y_t$ as products of irreducible elements. In the unique factorization of xy, f must appear as an associate of some x_i or y_j, so $x \in (f)$ or $y \in (f)$. □

Proposition C.42. A Noetherian integral domain A is a UFD if and only if every prime ideal of height 1 is principal.

Proof Suppose A is a UFD and $\mathfrak{p}$ has height 1. Choose $0 \neq x \in \mathfrak{p}$ and factor $x = x_1 \cdots x_n$ into irreducibles; since $\mathfrak{p}$ is prime, some $x_i \in \mathfrak{p}$. In a UFD, irreducible implies prime, so (x_i) is a prime ideal. The chain $(0) \subset (x_i) \subset \mathfrak{p}$ forces $\mathfrak{p} = (x_i)$ since $\operatorname{ht}(\mathfrak{p}) = 1$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Conversely, suppose every height-1 prime is principal. By (C.8) every element is a product of irreducibles, so it suffices to show irreducibles are prime. Let a be irreducible. Any prime $\mathfrak{p}$ minimal over (a) has height 1 by Krull's principal ideal theorem (since A is an integral domain). By hypothesis, $\mathfrak{p} = (\pi)$ is principal, and $(a) \subset (\pi)$ means $\pi \mid a$. Since a is irreducible, $a = u\pi$ for some unit u, so $(a) = (\pi) = \mathfrak{p}$ is prime. □

C.12 Regular sequences

Let A be a ring and let M be an A-module. A sequence of elements $x_1, \dots, x_r \in A$ is called an *M-regular sequence* if

(i) x_i is a nonzerodivisor on $M/(x_1, \dots, x_{i-1})M$ for each $i = 1, \dots, r$; and
(ii) $M/(x_1, \dots, x_r)M \neq 0$.

An A-regular sequence is simply called a *regular sequence*, and an ideal $\mathfrak{a} \subset A$ is said to be generated by a regular sequence if $\mathfrak{a} = (x_1, \dots, x_r)$ for some regular sequence $x_1, \dots, x_r$.

Condition (i) for $i = 1$ says that x_1 is a nonzerodivisor on M; condition (ii) rules out the degenerate case $(x_1, \dots, x_r)M = M$, which for A local and $x_i \in \mathfrak{m}$ is automatic by Nakayama's lemma.

The definition depends on the order of the elements (see Exercise C.12.1). However, if A is a *Noetherian local ring* and the x_i lie in the maximal ideal $\mathfrak{m}$, then any permutation of a regular sequence remains a regular sequence (Exercise C.12.2).

Example C.43. The variables $x_1, \dots, x_n$ form a regular sequence in the polynomial ring $A = k[x_1, \dots, x_n]$. Indeed, the quotient $A/(x_1, \dots, x_i)$ is the polynomial ring $k[x_{i+1}, \dots, x_n]$, an integral domain in which the image of x_{i+1} is nonzero, hence a nonzerodivisor; and $A/(x_1, \dots, x_n) = k \neq 0$. △

Example C.44. Let A be a ring and $f \in A$ a nonzerodivisor which is not a unit. Then f is a regular sequence of length 1, and $V(f) \subset \operatorname{Spec} A$ is the associated hypersurface. If A is a UFD and f, g are nonzero nonunits with no common irreducible factor such that $(f, g) \neq A$, then f, g is a regular sequence, and $V(f, g)$ has codimension 2. △

Example C.45. In $A = k[x, y]/(xy)$ the element x is a zerodivisor, since $xy = 0$ while $y \neq 0$; so x alone is not a regular sequence. This reflects the fact that $V(x)$ is one of the two coordinate axes making up $\operatorname{Spec} A$, hence contains an irreducible component of $\operatorname{Spec} A$ and does not cut the dimension down at all. The element $x - y$, on the other hand, is a nonzerodivisor, and $A/(x - y) \simeq k[x]/(x^2)$ is a fat point at the origin. △

The point of the definition is that a regular sequence cuts the codimension exactly: unlike in Example 8.55, each successive element increases the codimension by 1.

Exercise C.12.1 (Regular sequences and ordering)**.** Let $A = k[x, y, z]$. Show that x, $y(1-x)$, $z(1-x)$ form a regular sequence, but that $y(1-x)$, $z(1-x)$, x do not. Describe what happens in terms of planes in $\mathbb{A}^3_k$. What happens if we consider these elements in the localization at (x, y, z)?

Exercise C.12.2 (Permutability)**.** Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. Show that if $x_1, \dots, x_r \in \mathfrak{m}$ is an M-regular

sequence, then so is $x_{\sigma(1)}, \dots, x_{\sigma(r)}$ for every permutation σ of $\{1, \dots, r\}$. HINT: Reduce to the case of two elements $x, y \in \mathfrak{m}$.

C.13 Exercises

Exercise C.13.1 (Proof of Krull's Principal Ideal Theorem)**.** Let A be a Noetherian ring and let $f \in A$ be a nonunit. Let $\mathfrak{p}$ be a prime ideal minimal among those containing f. The aim of this exercise is to show that $\operatorname{ht} \mathfrak{p} \leqslant 1$, with equality when f is a nonzerodivisor.

a) Show that if $\mathfrak{p}$ has height 0, then f is a zerodivisor.
b) Show that $\operatorname{ht}(\mathfrak{p}A_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p})$ and reduce to proving the following: if A is a local ring with maximal ideal $\mathfrak{m}$, $f \in \mathfrak{m}$ is not contained in any other prime, and $\mathfrak{q} \subsetneq \mathfrak{m}$, then $A_{\mathfrak{q}}$ has dimension 0.
c) Show that $\sqrt{(f)} = \mathfrak{m}$ and that $A/(f)$ is an Artinian ring.
d) Define $\mathfrak{q}^{(n)} = \ell^{-1}(\mathfrak{q}^n A_{\mathfrak{q}})$ where $\ell \colon A \to A_{\mathfrak{q}}$ is the localization map. Show that $\mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \cdots$ and that if $xy \in \mathfrak{q}^{(n)}$ with $x \notin \mathfrak{q}$, then $y \in \mathfrak{q}^{(n)}$.
e) Show that there exists $N \in \mathbb{N}$ such that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + (f)$ for all $n \geqslant N$.
f) Prove that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)} + f\mathfrak{q}^{(n)}$ for all $n \geqslant N$.
g) Show that $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$. HINT: Consider $M = \mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}$.
h) Show that $\mathfrak{q}^n A_{\mathfrak{q}} = \mathfrak{q}^{n+1} A_{\mathfrak{q}}$.
i) Show that $\mathfrak{q}^n A_{\mathfrak{q}} = 0$. HINT: Consider $M = \mathfrak{q}^n A_{\mathfrak{q}}$.
j) Deduce that $A_{\mathfrak{q}}$ has dimension 0 and conclude the proof of the theorem.

C.14 Limits

C.14.0 Direct limits. A *preordered* set is a set equipped with a relation $i \leqslant j$ which is reflexive (that is, $i \leqslant i$ for all i), and transitive (that is, if $i \leqslant j$ and $j \leqslant k$, then $i \leqslant k$). A preordered set resembles a partially ordered set, except $i \leqslant j$ and $j \leqslant i$ need not imply that $i = j$. A preordered set I is *directed* if for any two elements i and j, there is a $k \in I$ such that $k \geqslant i$ and $k \geqslant j$.

A *directed system of A-modules* (M_i, ϕ_{ij}) consists of a collection $\{M_i\}_{i \in I}$ of A-modules, indexed by a directed set I, along with A-linear maps $\phi_{ij} \colon M_i \to M_j$ for each pair $i \leqslant j$, satisfying the two conditions

(i) $\phi_{ii} = \mathrm{id}_{M_i}$.
(ii) $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ whenever $i \leqslant j \leqslant k$.

The second condition may be illustrated by the diagram

$$M_k \xrightarrow[\phi_{jk}]{} M_j \xrightarrow[\phi_{ij}]{} M_i, \qquad \phi_{ik} \colon M_k \to M_i.$$

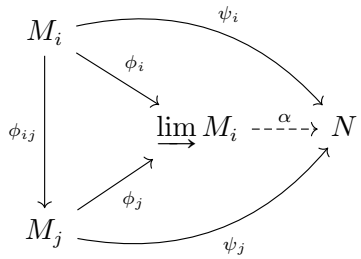
The *direct limit* of the directed system (M_i, ϕ_{ij}) of A-modules is an A-module $\varinjlim M_i$ together with a collection of A-linear maps

$$\phi_j \colon M_j \longrightarrow \varinjlim M_i$$

which satisfy $\phi_i \circ \phi_{ij} = \phi_j$. The module $\varinjlim M_i$ and the maps ϕ_i satisfy the following

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

universal property: for any A-module N and any system of A-linear maps $\psi_i\colon M_i \longrightarrow N$ such that $\psi_j \circ \phi_{ij} = \psi_i$, there is a *unique* map $\alpha\colon \varinjlim M_i \to N$ satisfying $\psi_i = \alpha \circ \phi_i$. This is pictured in the diagram below



The direct limit can be constructed explicitly as follows. Start with the disjoint union $\coprod_i M_i$, and define the equivalence relation $\sim$, where $x \in M_i$ and $y \in M_j$ are defined to be equivalent when there is an index k with $k \geqslant i$ and $k \geqslant j$ such that $\phi_{ki}(x) = \phi_{kj}(y)$. It is not hard to check that this is an equivalence relation. Then

$$\varinjlim M_i = \coprod_i M_i / \sim .$$

The A-module structure on $\varinjlim M_i$ is defined follows: for two equivalence classes $[x]$ and $[y]$, we can represent them by elements x and y from the same M_k. Then the module structure is defined by the formula $a[x] + b[y] = [ax + by]$ where $ax + by$ takes place in M_k.

The universal property follows from the fact that the ψ_i induce a map $\coprod_i M_i \to N$ which is compatible with the equivalence relation. Any such map descends uniquely to the quotient and gives the desired map $\alpha\colon \varinjlim M_i \to N$. As usual, this universal property ensures that the limit will be unique up to a unique isomorphism.

Apart from the universal property, there are two 'basic properties', one should bear in mind when computing with direct limits:

- Every element in $\varinjlim M_i$ is of the form $\phi_j(x)$ for some j and some $x \in M_j$.
- An element $x \in M_j$ maps to zero in $\varinjlim M_i$ if and only if $\phi_{ij}(x) = 0$ for some $i \geqslant j$.

C.14.0 Examples.

Example C.46 (Union as a direct limit)**.** If each M_i are submodules of some A-module M, and the maps $M_i \to M_j$ are given by inclusions $M_i \subset M_j$, then the direct limit is simply the union:

$$\varinjlim_i M_i = \bigcup_i M_i.$$

△

Example C.47 (Localization as a direct limit)**.** Let A be a ring and S a multiplicative subset. We define a preorder on S by $s \leqslant t$ if $t = us$ for some $u \in S$, and this makes S a directed set. Next, for $s \leqslant t$ with $t = us$, define $f_{st}\colon A_s \to A_t$ by $f_{st}(as^{-n}) = au^n t^{-n}$. Then the collection of rings $\{A_s\}_{s\in S}$ forms a directed system of rings.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

For each $s \in S$, there is a localization map $A_s \to S^{-1}A$, so from the universal property of the direct limit, we obtain a canonical A-linear map

$$\phi\colon \varinjlim_{s\in S} A_f \longrightarrow S^{-1}A.$$

We claim that this is an isomorphism. The map ϕ is surjective: any element in $S^{-1}A$ is of the form as^{-1} with $s \in S$; this element lies in A_s and hence in the image of ϕ. The map ϕ is injective: if $as^{-n} \in A_f$ is mapped to 0 in $S^{-1}A$, then for some $t \in S$ it holds that $ta = 0$, hence $as^{-n} = 0 \in A_{st}$, and ϕ is injective. △

Example C.48. As a special case of the above example, for an element $g \in A$, the localization A_g can be seen as the direct limit of the inverse system

$$A \to A \to A \to \cdots$$

where each map is multiplication by g. △

Example C.49 (Stalks as a direct limit)**.** Let X be a topological space, and consider the directed set I of open neighbourhoods U of a point $x \in X$ ordered by inclusion. If $\mathcal{F}$ is a presheaf on X, then setting $M_U = \mathcal{F}(U)$, the above construction of the direct limit $\varinjlim_U M_U$ is exactly the same as the definition of the stalk $\mathcal{F}_x$ as given in Chapter 3. △

Example C.50 (Cofinal index sets)**.** If $J \subset I$ is a directed subset, there is an induced A-linear map

$$\varinjlim_{J} M_j \longrightarrow \varinjlim_{I} M_i. \tag{C.9}$$

Then the inclusion of disjoint unions $\coprod_{i\in J} M_i \subset \coprod_{i\in I} M_i$ respects the equivalence relation, and therefore passes to the quotients and induces a map as in (C.9), which is A-linear.

If J is *cofinal* in I, meaning that for every $i \in I$ there is a j in J with $j \geqslant i$, then (C.9) is an isomorphism. Indeed, (C.9) is surjective, because any $x \in M_i$ is equivalent to an element $\phi_{ji}(x)$ contained in an M_j with $j \in J$ and $j \geqslant i$. If $x \in M_j$ is mapped to zero in $\varinjlim_{i\in I} M_i$, then it maps to zero in some $M_{i'}$ with $i' \geqslant i$. But then it maps to zero in some M_j with $j \in J$ and $j \geqslant i'$ as well, and so is zero in $\varinjlim_{j\in J} M_j$.

In particular, if the index set I has a largest element i_0, then $\varinjlim_{i\in I} M_i = M_{i_0}$. △

Suppose that (M_i, ϕ_{ij}) and (N_i, ψ_{ij}) are two directed systems of A-modules, indexed by the same directed set I. A map between them is a collection $\{\rho_i\}_{i\in I}$ of A-linear maps $\rho_i\colon M_i \to N_i$ such that $\rho_i \circ \psi_{ij} = \phi_{ij} \circ \rho_j$. This induces an A-linear map

$$\rho\colon \varinjlim_{i} M_i \longrightarrow \varinjlim_{i} N_i,$$

which is compatible with the natural maps of the two limits. Indeed, the collection $\{\rho_i\}$ yields a map already between the disjoint unions which respects the equivalence relations.

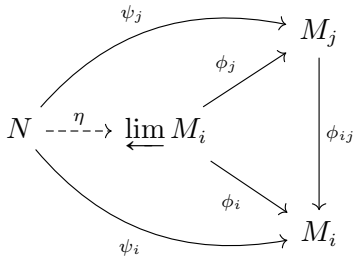
C.14.0 Inverse limits. The dual concept of a direct limit is the *inverse limit* (also called the *projective limit* or just the *limit*) of an *inverse system* $\{M_i\}_{i\in I}$. These systems and their limits are defined similarly to the direct systems, just with the arrows reversed.

An *inverse system* (M_i, ϕ_{ij}) of A-modules indexed by a directed set I is a set of A-modules $\{M_i\}_{i\in I}$, and for each pair $i, j \in I$ with $i \leqslant j$ an A-linear map $\phi_{ij}\colon M_j \to M_i$ such that

(i) $\phi_{ii} = \mathrm{id}_{M_i}$
(ii) $\phi_{ik} \circ \phi_{jk} = \phi_{ji}$ whenever $i \leqslant k \leqslant j$.

Note that the maps ϕ_{ij} go 'backwards' relatively to the order in I. The definition of the inverse limit is word for word the same as the definition of the direct limit except that all arrows are reversed.

The *inverse limit* of an inverse system (M_i, ϕ_{ij}) is an A-module $\varprojlim M_i$ together with a collection of A-linear maps $\phi_i\colon \varprojlim M_i \to M_i$ so that $\phi_i = \phi_{ji} \circ \phi_j$, which are universal in this respect. The universal property is illustrated with the diagram



The inverse limit can be constructed explicitly as follows. Define a submodule $L \subset \prod_{i\in I} M_i$ by

$$L = \{\, (x_i) \in \prod_{i\in I} M_i \mid x_i = \phi_{ij}(x_j) \text{ for all pairs } i, j \text{ with } i \leqslant j \,\} \tag{C.10}$$

The projection maps induce maps $\phi_i\colon L \to M_i$. The module L and the ϕ_i then satisfy the universal property of the inverse limit: a collection of maps $\psi_i\colon N \to M_i$ gives rise to a map $\eta\colon N \to \prod_i M_i$ by the assignment $x \mapsto (\psi_i(x))$, and it takes values in L when the ψ_i's satisfy the compatibility constraints $\psi_i = \phi_{ij} \circ \psi_j$.

If $J \subset I$ is a directed subset, there is a canonical projection

$$\prod_{i\in I} M_i \longrightarrow \prod_{i\in J} M_i$$

that simply remembers the components indexed by J. This respects the relations defining the inverse limit in (C.10), so there is an induced map

$$\varprojlim_{i\in I} M_i \longrightarrow \varprojlim_{i\in J} M_i \tag{C.11}$$

As for the direct limit, if J is cofinal in I, the map in (C.11) is an isomorphism.

C.14.0 Examples.

Example C.51. Let I be a set with the 'trivial ordering', that is, $i \leqslant j$ if and only if $i = j$. Then $\varprojlim_{i\in I} M_i = \prod_{i\in I} M_i$. △

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Example C.52 (Inverse limits and intersections)**.** If all the M_i's are submodules of some fixed module M, and the maps $M_j \to M_i$ are the inclusions, the inverse limit will simply be the intersection of the M_i's:

$$\varprojlim_{i\in I} M_i = \bigcap_{i\in I} M_i \subset M.$$

△

Example C.53 (The p-adic integers)**.** An important application of inverse limits is to form so-called 'completions of rings'. The primary example is the p-adic numbers. Let p be a prime number and consider the modules $\mathbb{Z}/p^i\mathbb{Z}$. They form an inverse system indexed by $\mathbb{N}$ with ϕ_{ij} being just the canonical reduction map $\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$, that for $j \geqslant i$ sends a class $[n]_{p^j}$ mod p^j to the class $[n]_{p^i}$ mod p^i. The system may be visualized by the sequence

$$\ldots \longrightarrow \mathbb{Z}/p^{i+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^i\mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

The inverse limit is called the *ring of p-adic integers*. △

Example C.54 (Inverse limits and sections)**.** In the context of sheaves, direct limits gives us stalks, and inverse limits give a way to compute sections.

Let $\mathcal{F}$ be a sheaf on a topological space X and let $U \subset X$ be an open subset. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of U which is directed under inclusion, that is, the intersection of two members from $\mathcal{U}$ contains a third. Then the groups $\mathcal{F}(U_i)$ form a directed system under the restriction maps, and there is an isomorphism $\mathcal{F}(U) \simeq \varprojlim_{i\in I} \mathcal{F}(U_i)$. Indeed, the restriction maps $\rho_{UU_i}\colon \mathcal{F}(U) \to \mathcal{F}(U_i)$ satisfy $\rho_{UU_i} = \rho_{U_jU_i} \circ \rho_{UU_j}$ for $U_i \subset U_j$, and they thus give a canonical map $\mathcal{F}(U) \to \varprojlim_{i\in I} \mathcal{F}(U_i)$.

This is isomorphism by the description in (C.10): that s maps to zero, means that $\rho_{UU_i}(s) = s|_{U_i} = 0$ for each i, which by the Locality axiom means that $s = 0$. Furthermore, sections $s_i \in \mathcal{F}(U_i)$ so that $s_j|_{U_i} = s_i$ for each inclusion $U_i \subset U_j$ may, by the Gluing axiom, be glued together to give a section of $\mathcal{F}$ over U, and hence the map is surjective.

△

C.14.0 Exercises.

Exercise C.14.1. Show that arbitrary direct and inverse limits exist in the categories Sets and Rings.

Exercise C.14.2. Find a directed system in the category FiniteSets of finite sets that does not have a direct limit.

Exercise C.14.3. Let $(M_i, \phi_{ij})_{i\in I}$ be a directed (respectively inverse) system of A-modules. Assume that I is *discrete*; that is, that no two elements are comparable (in other words, $i \leqslant j$ only when $i = j$). Show that $\varinjlim_{i\in I} M_i = \bigoplus_i M_i$, respectively $\varprojlim_{i\in I} M_i = \prod_i M_i$.

Exercise C.14.4. Assume that I is a directed set in which every element is dominated by a maximal element. Let $(M_i, \phi_{ij})_{i\in I}$ be a direct (respectively inverse) system of A-modules indexed by I. Show that $\varinjlim_{i\in I} M_i$ is isomorphic to the direct sum $\bigoplus M_j$, respectively $\varprojlim_{i\in I} M_i$ is isomorphic to the product $\prod M_j$, where the sum (respectively the product) extends over all maximal elements in I.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Exercise C.14.5 (Direct limits are exact)**.** Let (M_i, ϕ_{ij}), (M_i', ϕ_{ij}') and (M_i'', ϕ_{ij}'') be three directed systems of A-modules. Suppose given exact sequences

$$0 \longrightarrow M_i' \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} M_i'' \longrightarrow 0$$

with the collections $\{\alpha_i\}$ and $\{\beta_i\}$ being maps of direct systems (i.e. they are compatible with the transition maps). Show that the induces sequence of limits

$$0 \longrightarrow \varinjlim M_i' \xrightarrow{\alpha} \varinjlim M_i \xrightarrow{\beta} \varinjlim M_i'' \longrightarrow 0$$

is exact. In other words, taking direct limits is an exact functor.

Exercise C.14.6 (Inverse limits are left exact)**.** With setting as in the previous exercise except that the systems are inverse systems, show that the sequence of inverse limits

$$0 \longrightarrow \varprojlim M_i' \xrightarrow{\alpha} \varprojlim M_i \xrightarrow{\beta} \varprojlim M_i''$$

is exact. Show by finding an example that β is not always surjective. Hence the inverse limit is merely left exact.

Exercise C.14.7. Let A be a ring. Show that the inverse limit of the inverse system

$$\ldots \to A[x]/\mathfrak{m}^{i+1} \to A[x]/\mathfrak{m}^i \to \ldots \to A[x]/\mathfrak{m}^2 \to A[x]/\mathfrak{m}$$

where $\mathfrak{m} = (x)$, and the maps are the canonical reduction maps, is isomorphic to the ring of formal power series $A[[x]]$.

This material has been / will be published by Cambridge University Press as *Introduction to Schemes* by Geir Ellingsrud & John Christian Ottem
This pre-publication version is free to view and download for personal use only. Not for re-distribution, re-sale, or use in derivative works.
© Geir Ellingsrud and John Christian Ottem 2015–2026

Bibliography

Atiyah, Michael, and MacDonald, Ian Grant. 2018. *Introduction to Commutative Algebra*. CRC Press.

Cox, David A, Little, John B, and Schenck, Henry K. 2024. *Toric varieties*. Vol. 124. American Mathematical Society.

Dolgachev, I. 2003. Lectures on Invariant Theory. *London Mathematical Society Lecture Note Series/Cambridge University Press*, **296**.

Eisenbud, David. 2013. *Commutative Algebra: With a View Toward Algebraic Geometry*. Vol. 150. Springer Science & Business Media.

Fulton, William. 1993. *Introduction to toric varieties*. Princeton university press.

Fulton, William. 2013. *Intersection theory*. Vol. 2. Springer Science & Business Media.

Griffiths, Phillip, and Harris, Joseph. 1979. Algebraic geometry and local differential geometry. Pages 355–452 of: *Annales scientifiques de l'École Normale Supérieure*, vol. 12.

Hartshorne, Robin. 2013. *Algebraic Geometry*. Vol. 52. Springer Science & Business Media.

Kempf, George. 1993. *Algebraic varieties*. Vol. 172. Cambridge University Press.

Liu, Q. 2002. *Algebraic Geometry and Arithmetic Curves*. Vol. 6. Oxford Graduate Texts in Math.

Serre, Jean-Pierre. 1955. Un théorème de dualité. *Comment. Math. Helv*, **29**(9-26), 8.

Serre, Jean-Pierre. 2013. *Local fields*. Vol. 67. Springer Science & Business Media.

Stacks Project. 2018. *Stacks Project*.

Vakil, Ravi. 1999. Introduction to algebraic geometry. *Lecture notes for course*, **18.725**.

Vakil, Ravi. 2024. *Foundations of Algebraic Geometry. The Rising Sea.* Princeton University Press.