Introduction to Schemes

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Contents

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https://docs.google.com/document/d/1T7R9ROah2RyR6mXMesEHZgk2de4cSOC-GMXgR
edit?usp=sharing

Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of positive integers.

$\subset$ means 'is subset of', i.e., the same thing as $\subseteq$

All rings are commutative with 1.

Ring maps are required to send 1 to 1.

The zero ring is not an integral domain (and therefore not a field).

For a ring A, we write $A_\mathfrak{p}$ and A_x for the localizations in the multiplicative sets $S = A - \mathfrak{p}$ and $S = \{1, x, x^2, \dots\}$ respectively. Thus $\mathbb{Z}_p = \mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \nmid b\}$.

A 'map' is a morphism in the relevant category, e.g., a 'map of rings' is ring homomorphism.

We will occasionally write $A = B$ if there is a *canonical* isomorphism $A \simeq B$. So for instance, $\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z} = \mathbb{Z}$.

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Varieties

We begin by discussing varieties, which will serve as the main motivating example in the theory of schemes. We will contend ourselves to presenting the basic definitions and fundamental properties of the two most important classes of varieties, namely the affine varieties and the projective varieties. As we move forward in the book, we will develop the theory of varieties in greater depth.

Varieties are defined over a fixed ground field k, and in this chapter we shall assume that k is algebraically closed. It is useful to keep some specific fields in mind, e.g. the field of complex numbers $\mathbb{C}$, the field of algebraic numbers $\bar{\mathbb{Q}}$ or perhaps the algebraic closure $\bar{\mathbb{F}}_p$ of a finite field.

For reasons that will become clear when the notion of 'ring-valued points' is introduced, we shall write $\mathbb{A}^n(k)$ for the set k^n, and refer to it as the *affine n-space*. The change in notation from k^n to $\mathbb{A}^n(k)$ is meant to underline that there is more to $\mathbb{A}^n(k)$ than just the set of its elements; it will soon be equipped with a topology, and ultimately, it will be a scheme, denoted by $\mathbb{A}^n_k$.

1.1 Algebraic sets

We begin by introducing the *algebraic sets*. These are the subsets of the affine space $\mathbb{A}^n(k)$ whose points are the common solutions of a set of polynomial equations:

Definition 1.1. If S is a subset of polynomials in $k[x_1, \ldots, x_n]$, we define their *zero set* as

$$Z(S) = \{\, x \in \mathbb{A}^n(k) \mid f(x) = 0 \text{ for all } f \in S \,\}.$$

An *algebraic set* is a subset of $\mathbb{A}^n(k)$ of this form.

If $f_1, \ldots, f_r$ are elements of S, each expression $\sum_{i=1}^r b_i f_i$ with the b_i's being polynomials, also vanishes at the points of $Z(S)$. This means that the zero set of the ideal $\mathfrak{a}$ generated by the elements of S is the same as $Z(S)$; that is, $Z(S) = Z(\mathfrak{a})$. We will therefore almost exclusively work with ideals and tacitly replace a set of polynomials by the ideal they generate. Hilbert's basis theorem tells us that any ideal in $k[x_1, \ldots, x_n]$ is finitely generated, so that an algebraic subset is always described as the set of common zeros of *finitely* many polynomials. Note the two special cases $Z(1) = \varnothing$ and $Z(0) = \mathbb{A}^n(k)$.

The more constraints imposed, the smaller the solution set will be, so if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals with $\mathfrak{a} \subset \mathfrak{b}$, one has $Z(\mathfrak{b}) \subset Z(\mathfrak{a})$. Sending $\mathfrak{a}$ to $Z(\mathfrak{a})$ therefore gives an inclusion-reversing map from the partially ordered set of ideals in $k[x_1, \ldots, x_n]$ to the partially ordered set of subsets of $\mathbb{A}^n(k)$.

The map sending $\mathfrak{a}$ to $Z(\mathfrak{a})$ is not surjective. A polynomial can only have finitely many zeros, so any proper infinite subset of $\mathbb{A}^1(k)$ is not algebraic. To give an example in $\mathbb{A}^n(k)$ for any n, just take an infinite proper subset of one of the coordinate axes.

Neither is the map injective. Different ideals can define the same algebraic set. For instance, the ideals (t) and (t^2) in $k[t]$, both have the origin in the affine line $\mathbb{A}^1(k)$ as their zero set. More generally, any power $\mathfrak{a}^r$ of an ideal $\mathfrak{a}$ will have the same zeros as $\mathfrak{a}$; indeed, since $\mathfrak{a}^r \subset \mathfrak{a}$, it holds that $Z(\mathfrak{a}) \subset Z(\mathfrak{a}^r)$, and the other inclusion holds as well because a polynomial f vanishes at the same points as the power f^r. To deal with this ambiguity, we resort to the *radical* $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$, which we recall is defined as

$$\sqrt{\mathfrak{a}} = \{\, f \mid f^r \in \mathfrak{a} \text{ for some } r > 0 \,\}.$$

Then the argument above yields that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$. Indeed, the radical is finitely generated, so some power $(\sqrt{\mathfrak{a}})^m$ is contained in $\mathfrak{a}$. Two ideals with the same radical thus have coinciding zero sets, and Hilbert's Nullstellensatz, which we shortly shall see, tells us that the converse is true as well.

The product of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is generated by the products $f \cdot g$ with $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$, and hence $Z(\mathfrak{a} \cdot \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$. For the sum $\mathfrak{a} + \mathfrak{b}$ one checks that $Z(\mathfrak{a} + \mathfrak{b}) = Z(\mathfrak{a}) \cap Z(\mathfrak{b})$, and in fact, this holds for sums of any cardinality (For a proof, see Lemma 2.2).

Proposition 1.2 (Properties of algebraic sets). Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals and $\{\mathfrak{a}_i\}_{i \in I}$ a family of ideals in the polynomial ring $k[x_1, \ldots, x_n]$. Then:

(i) If $\mathfrak{a} \subset \mathfrak{b}$, then $Z(\mathfrak{b}) \subset Z(\mathfrak{a})$;
(ii) $Z(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} Z(\mathfrak{a}_i)$;
(iii) $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$;
(iv) $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

The identities (ii) and (iii) tell us that finite unions and arbitrary intersections of algebraic sets are again algebraic. Furthermore, as $\mathbb{A}^n(k) = Z(0)$ and $\varnothing = Z(1)$, the algebraic sets constitute the closed sets of a topology on the affine space $\mathbb{A}^n(k)$. It is called the *Zariski topology*.

If $X \subset \mathbb{A}^n(k)$ is any subset, we get an induced Zariski topology on X, by declaring that the open sets of X are of the form $X \cap U$, where U is an open set in $\mathbb{A}^n(k)$.

Example 1.3 (The Zariski topology on the affine line $\mathbb{A}^1(k)$)**.** Each non-zero and proper ideal $\mathfrak{a}$ in the polynomial ring $k[t]$ is generated by a single element, say $\mathfrak{a} = (f)$. As the ground field k is algebraically closed, f factors as a product of linear terms $f = (t - a_1)^{n_1} \cdots (t - a_r)^{n_r}$ with $a_i \in k$. Hence $Z(f) = \{a_1, \cdots, a_r\}$, and apart from the entire line $\mathbb{A}^1(k)$, the closed sets are just the finite sets.

In other words, the Zariski topology on $\mathbb{A}^1(k)$ is the finite complement topology, in which the proper open sets are those whose complement is finite. In particular, note that the Zariski topology on $\mathbb{A}^1(\mathbb{C})$ behaves very differently than the usual topology on $\mathbb{C}$; there are much fewer open sets.

There is a partial converse to the construction of the zero locus $Z(\mathfrak{a})$ of an ideal. One may consider the set of polynomials vanishing on a given subset of $\mathbb{A}^n(k)$, which actually is an ideal.

Definition 1.4. For a subset X of $\mathbb{A}^n(k)$, we let $I(X)$ denote the ideal consisting of polynomials in $k[x_1, \ldots, x_n]$ that vanish along X; that is,

$$I(X) = \{ f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X \}.$$

This gives an inclusion-reversing map $X \mapsto I(X)$ from the set of subsets of $\mathbb{A}^n(k)$ to the set of ideals in the polynomial ring $k[x_1, \ldots, x_n]$.

Examples

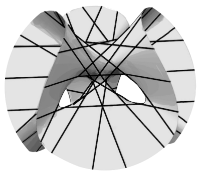
Example 1.5. The linear polynomial $x_1 + 2x_2 + 5x_3$ defines an algebraic set in $\mathbb{A}^3(k)$ which can be identified with a 2-dimensional plane. More generally, any linear subspace of $\mathbb{A}^n(k)$ is defined by linear equations and is therefore an algebraic set.

Example 1.6. Another classical examples are the conic sections. They are the closed algebraic sets in the affine plane $\mathbb{A}^2(k)$ given by quadratic equations. Three familiar examples include the circle $x^2 + y^2 = 1$, the parabola $y = x^2$ and the hyperbola $xy = 1$. If k is algebraically closed of characteristic $\neq 2$, any conic section can be reduced via a linear change of coordinates to one of these types.

Example 1.7. A more interesting example is the so-called *Clebsch cubic surface*; a surface in $\mathbb{A}^3(\mathbb{C})$ defined by the equation

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3.$$

The real points of the surface, i.e. the points in $\mathbb{A}^3(\mathbb{R})$ satisfying the equation, is depicted below. This surface contains 27 lines, all defined over the real numbers.

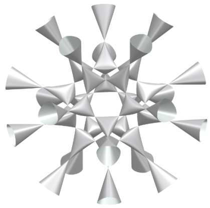


The Clebsch cubic surface

Example 1.8. Algebraic sets can show a high degree of complexity. The *Barth sextic* in $\mathbb{A}^3(\mathbb{C})$ is the zero locus of the degree 6 polynomial

$$4(\phi^2x^2 - y^2)(\phi^2y^2 - z^2)(\phi^2z^2 - x^2) - (1 + 2\phi)(x^2 + y^2 + z^2 - 1)^2$$

where $\phi = (1+\sqrt{5})/2$. This remarkable surface has 65 singular points, which is the maximal number for a degree 6 surface. A plot of the real points of the surface is depicted below.



The Barth Sextic with 65 singular points

Hilbert's Nullstellensatz

For an algebraic subset X it holds true that $Z(I(X)) = X$. Hilbert's Nullstellensatz is about the composition of I and Z the other way around, namely about $I(Z(\mathfrak{a}))$. Polynomials in the radical $\sqrt{\mathfrak{a}}$ vanish along $Z(\mathfrak{a})$ (if a power of f vanishes on a set, f vanishes there as well), and therefore $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$. The Nullstellensatz tells us that this inclusion is an equality.

Theorem 1.9 (Hilbert's Nullstellensatz). Assume that k is an algebraically closed field and that $\mathfrak{a}$ is an ideal in $k[x_1, \ldots, x_n]$. Then one has

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

A proof will be given in Section 12.2.

The Nullstellensatz has the following fundamental consequences.

Theorem 1.10 (Weak Nullstellensatz). Let k be an algebraically closed field and $\mathfrak{a}$ an ideal in the polynomial ring $k[x_1, \ldots, x_n]$.

(i) $Z(\mathfrak{a})$ is non-empty if and only if $\mathfrak{a}$ is not the unit ideal;

(ii) The maximal ideals in $k[x_1, \ldots, x_n]$ are precisely those of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $(a_1, \ldots, a_n) \in \mathbb{A}^n(k)$.

Proof It is clear that $Z(1) = \emptyset$. If $Z(\mathfrak{a}) = \emptyset$, requiring a polynomial to vanish along $Z(\mathfrak{a})$ imposes no constraint, so $1 \in I(Z(\mathfrak{a}))$, and the Nullstellensatz gives that $1 \in \mathfrak{a}$. This shows (i).

As to (ii), note that the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal being the kernel of the evaluation map $k[x_1, \ldots, x_n] \to k$ (which sends f to its value at $(a_1, \ldots, a_n)$). If $\mathfrak{m}$ is a maximal ideal, the Nullstellensatz yields that $Z(\mathfrak{m}) \neq \emptyset$. So take a point $(a_1, \ldots, a_n)$ in $Z(\mathfrak{m})$. Then $(x_1 - a_1, \ldots, x_n - a_n) \subset \mathfrak{m}$, and as the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal, we must have equality. □

Example 1.11. It is important to note that Hilbert's Nullstellensatz only holds when the ground field is algebraically closed. A simple example of a proper ideal with empty zero locus is the ideal $(x^2 + 1)$ in $\mathbb{R}[x]$.

Exercise 1.1.1. In any ring, the radical of an ideal $\mathfrak{a}$ equals the intersection of the prime ideals containing it. Using the Nullstellensatz, show that in the polynomial ring $k[x_1, \ldots, x_n]$, the radical $\sqrt{\mathfrak{a}}$ equals the intersection of all the *maximal* ideals containing $\mathfrak{a}$.

Exercise 1.1.2. Show that the Zariski topology on $\mathbb{A}^2(k)$ is not the product topology on $\mathbb{A}^2(k) = \mathbb{A}^1(k) \times \mathbb{A}^1(k)$.

Irreducible sets and varieties

Irreducibility is a notion from point set topology which plays a fundamental role in algebraic geometry.

Definition 1.12. A topological space X is said to be *irreducible* if it can not be written as the union of two proper closed subsets; that is, if $X = Z \cup Z'$ with Z and Z' closed, then either $Z = X$ or $Z' = X$.

Equivalently, the space X is irreducible if and only if the intersections of any two non-empty open subsets is non-empty. Indeed, to say that $U \cap V = \varnothing$ with U and V open, is to say that $U^c \cup V^c = X$. And so if X is irreducible, either $U^c = X$ or $V^c = X$; that is, either $U = \varnothing$ or $V = \varnothing$. A third way of expressing that X is irreducible, is to say that every non-empty open subset is dense.

For an algebraic set X, being irreducible means that the ideal $I(X)$ is prime:

Proposition 1.13. An algebraic set $X = Z(\mathfrak{a}) \subset \mathbb{A}^n(k)$ is irreducible if and only if the ideal $I(Z(X)) = \sqrt{\mathfrak{a}}$ is prime.

Proof Because $Z(\sqrt{\mathfrak{a}}) = Z(\mathfrak{a})$, it suffices to treat the case when $\mathfrak{a}$ is radical. Assume that $Z(\mathfrak{a}) = Z(\mathfrak{b}) \cup Z(\mathfrak{b}')$ with radical ideals $\mathfrak{b}$ and $\mathfrak{b}'$ both containing $\mathfrak{a}$. By (iii) of Proposition 1.2, it holds that $Z(\mathfrak{b}) \cup Z(\mathfrak{b}') = Z(\mathfrak{b} \cap \mathfrak{b}')$, and since the intersection of two radical ideals is radical, we get that $\mathfrak{b} \cap \mathfrak{b}' = \mathfrak{a}$ by the Nullstellensatz. So if $\mathfrak{a}$ is prime, then either $\mathfrak{b} \subset \mathfrak{a}$ or $\mathfrak{b}' \subset \mathfrak{a}$. That is, either $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b}' = \mathfrak{a}$.

The implication the other way is easier: if $\mathfrak{a}$ is not prime, it is the intersection of several different prime ideals. Dividing these into two groups and letting $\mathfrak{b}$ and $\mathfrak{b}'$ be the corresponding intersections, one obtains a decomposition $Z(\mathfrak{a}) = Z(\mathfrak{b}) \cup Z(b')$ of $Z(\mathfrak{a})$ into distinct closed subsets. □

Let us give the following preliminary definition of a variety:

Definition 1.14. An *affine variety* is an irreducible algebraic set in $\mathbb{A}^n(k)$.

The mappings $X \mapsto I(X)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$ give mutually inverse one-to-one inclusion reversing correspondences between the objects in columns of the following table, where $A = k[x_1, \ldots, x_n]$.

ALGEBRA	GEOMETRY
maximal ideals of A	points of $\mathbb{A}^n(k)$
prime ideals of A	irreducible algebraic sets in $\mathbb{A}^n(k)$
radical ideals of A	closed subsets of $\mathbb{A}^n(k)$
maximal ideals of $A/\mathfrak{a}$	points of $Z(\mathfrak{a})$

An *irreducible component* of a topological space X is a maximal closed irreducible subset. Every algebraic set can be written as a finite union of its irreducible components; this follows from the Lasker–Noether theorem, which implies that any radical ideal is the intersection of finitely many prime ideals. The affine varieties therefore constitute the building blocks of all algebraic sets in affine space. We will give a general treatment of decompositions into irreducibles in Chapter **??**. For now, let us give two examples illustrating how to find these components.

Example 1.15. Consider the algebraic set $X = Z(I)$ in $\mathbb{A}^3(k)$, where I is the ideal

$$I = (xz - y^2, x^2 - y).$$

Let us find the irreducible components of X. Let $p = (a, b, c) \in X$ be a point. Then the second equation implies that $b = a^2$. Plugging this into the first equation, we get $ac - a^4 = 0$, which implies that either $a = 0$ or $c = a^3$. Thus p lies in one of the irreducible subsets $X_1 = Z(x, y)$ or $X_2 = Z(y - x^2, z - x^3)$. Conversely, a point in X_1 clearly lies in X, and if $p = (a, b, c) \in X_2$, it holds that $b = a^2$ and $c = a^3$ so that $ac - b^2 = a^4 - a^4 = 0$, and p lies in X. Hence we find that

$$X = Z(x, y) \cup Z(y - x^2, z - x^3).$$

In geometric terms, X is the union of the z-axis and a curve called ‘the twisted cubic’ (which we shall meet at several later occasions).

Example 1.16. Consider the algebraic set $Z(I) \subset \mathbb{A}^2(k)$, where I is the ideal

$$I = (y - x^2, x^2 + (y - 1)^2 - 1).$$

Over the real numbers, we recognise the points of $Z(I)$ as the intersection points of the parabola $y = x^2$ and the circle of radius 1 with centre in $(0, 1)$. To find these intersection points, we compute a primary decomposition of the ideal:

$$\begin{aligned} I &= (y - x^2, x^2 + (x^2 - 1)^2 - 1) \\ &= (y - x^2, x^2(x - 1)(x + 1)) \\ &= (y - x^2, x^2) \cap (y - x^2, x - 1) \cap (y - x^2, x + 1) \\ &= (y, x^2) \cap (y - 1, x - 1) \cap (y - 1, x + 1). \end{aligned}$$

Thus $Z(I)$ consists of the three points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

The primary decomposition of an ideal I gives important information about the algebraic set $Z(I)$. In particular, it describes $Z(I)$ as the union of its irreducible components. Even in the present example, the decomposition gives more refined information than just the set of points of $Z(I)$; it reflects that $(0, 0)$ is different to the two others (the primary component being (y, x^2) shows that it has ‘multiplicity 2’).

Exercises

Exercise 1.1.3. Show that the algebraic set $Z(y^2 - x^3 - 1) \subset \mathbb{A}^2(k)$ is irreducible.

Exercise 1.1.4. For each of the following ideals $\mathfrak{a}$ find a decomposition of $Z(\mathfrak{a})$ into irreducible components.

a) (x^3, x^2y, xy^3);
b) (yz, xz, y^3, x^2y);
c) $(x^2 - y, xz - y^2, x^3 - xz)$.

Exercise 1.1.5. Identify $\mathbb{A}^{nm}(k)$ with the space of $m \times n$-matrices over the field k. Show that the set of matrices of rank less than a given number is an algebraic set.

Exercise 1.1.6. Let us continue the previous exercise with $m = n$.

a) Show that the set of symmetric matrices, i.e. matrices such that $A^T = A$, is an algebraic set in $\mathbb{A}^{n^2}(k)$;
b) Show that the set $GL_n(k)$ of invertible matrices is Zariski open in $\mathbb{A}^{n^2}(k)$;
c) Show that the set $SL_n(k)$ of matrices with determinant one is an algebraic set in $\mathbb{A}^{n^2}(k)$;
d) Show that the set X of matrices A such that $A^r = 0$ for a given natural number r, form an algebraic set in the affine space $\mathbb{A}^{n^2}(k)$. Compute the ideal $I(X)$ for $n = 2$ and $r = 2$.

1.2 Polynomial functions and polynomial maps

A *polynomial function* on algebraic subset $X \subset \mathbb{A}^n(k)$ is simply the restriction of a polynomial in $k[x_1, \ldots, x_n]$ to X. Two polynomials f and g restrict to the same function on X precisely when their difference $f - g$ vanishes on X, so the set of polynomial functions on X can be identified with the quotient ring

$$A(X) = k[x_1, \ldots, x_n]/I(X).$$

This ring is called the *affine coordinate ring* of X, and carries essentially all information about the set X. It has no nilpotent elements since $I(X)$ is a radical ideal and it is an integral domain if and only if X is irreducible.

The correspondence between prime ideals and irreducible closed subsets shows that the Krull dimension of $A(X)$ equals the dimension of the topological space X; that is, the length of the longest chain $X_0 \subset X_1 \subset \cdots \subset X_r = X$ of distinct irreducible closed sets in X.

Example 1.17. The square root $\sqrt{x}$ is not *per se* a polynomial function on $\mathbb{A}^1(\mathbb{C})$ (it is not even a well-defined function), but it defines a polynomial function on the parabola

$$X = Z(x - y^2)$$

in $\mathbb{A}^2(\mathbb{C})$. Indeed, there the sign ambiguity is resolved, and the square root is simply given by $(x, y) \mapsto y$. Note that the coordinate ring of X is

$$A(X) = \mathbb{C}[x, y]/(x - y^2) \simeq \mathbb{C}[y],$$

which is an integral domain of Krull dimension one, in accordance with the intuition that X is irreducible and of dimension one.

The notion of a 'polynomial function' can be extended to the notion of 'polynomial maps' between algebraic sets; these are maps that under composition carry polynomial functions to polynomial functions.

Definition 1.18 (Polynomial maps)**.** Let X and Y be two algebraic sets. A map $f\colon X \to Y$ is called *a polynomial map* if the composition $g \circ f$ is a polynomial function whenever g is a polynomial function on Y.

The composition of two polynomial maps is again a polynomial map, so the algebraic sets form a category $\mathsf{AlgSets}$ with the polynomial maps as morphisms. We say that a polynomial map is an *isomorphism* when it has an inverse map that is also a polynomial map.

When $f\colon X \to Y$ is a polynomial map and $g \in A(Y)$, the composition $g \circ f$ is again a polynomial map $X \to \mathbb{A}^1(k)$, which we denote by $f^\sharp(g) = g \circ f$. This gives us a map

$$\begin{aligned} f^\sharp\colon A(Y) &\longrightarrow A(X) \\ g &\mapsto g \circ f. \end{aligned} \tag{1.1}$$

The map $f^\sharp$ is a map of k-algebras since sums and products of polynomial functions on Y are computed pointwise, and constants clearly map to constants. It is a fundamental property of affine algebraic sets that all morphisms of k-algebras are realized in this way.

It also follows from the definitions that $Z(f^\sharp(g)) = f^{-1}Z(g)$. In particular, this shows that polynomial maps are continuous in the Zariski topology.

Theorem 1.19 (Main theorem of affine algebraic sets). Let X and Y be two algebraic sets. The map

$$\operatorname{Hom}_{\mathsf{AlgSets}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathsf{Alg/k}}(A(Y), A(X))$$

that sends f to $f^\sharp$, is a bijection from the set of polynomial maps to the set of maps of k-algebras.

Proof The map in the theorem is injective: assume that f_1 and f_2 are two different polynomial maps from X to Y. Then there is is a point $x \in X$ with $f_1(x) \neq f_2(x)$, and so there is a polynomial function g on Y with $g(f_1(x)) \neq g(f_2(x))$; that is, $f_1^\sharp(g) \neq f_2^\sharp(g)$.

To prove that the map is surjective, we begin with treating the case that $Y = \mathbb{A}^n(k)$. In this case, giving a map $f\colon X \to Y$ amounts to giving n functions $f_1, \ldots, f_n$ on X so that $f(x) = (f_1(x), \ldots, f_n(x))$, and f is a polynomial map precisely when the f_i's are polynomial functions. Indeed, if $u_1, \ldots, u_n$ are coordinates on $\mathbb{A}^n(k)$, a polynomial function g on Y is just a polynomial in the u_i. Hence the composition $g \circ f$ becomes a polynomial in the f_i's, which clearly is a polynomial function on X when the f_i's are. Now, if $\phi\colon A(Y) = k[u_1, \ldots, u_n] \to A(X)$ is a map of k-algebras, we may use the images $f_i = \phi(u_i)$ as components for a function f as above. Then $f^\sharp = \phi$ because the two maps agree on the generators u_i.

In the general case, we assume that $Y \subset \mathbb{A}^n(k)$. Note that a map $f\colon X \to \mathbb{A}^n(k)$ takes values in Y precisely when $f^\sharp(g) = g \circ f = 0$ for all $g \in I(Y)$. So if $\phi\colon A(Y) \to A(X)$ is given, the composition $k[u_1, \ldots, u_n] \to A(Y) \to A(X)$ of ϕ with the restriction map is a ring map that vanishes $I(Y)$. Hence by the first case, it yields a map $X \to \mathbb{A}^n(k)$ with components $f_i = \phi(u_i)$, and this map factors through Y because $g \circ f = \phi(g) = 0$ whenever $g \in I(Y)$. □

From a categorical angle, the theorem says that the category of algebraic sets is equivalent to the the category of finitely generated, reduced k-algebras (with arrows reversed). The subcategory of varieties; that is, the full subcategory with irreducible algebraic sets as objects, is then equivalent to the category of integral domains finitely generated over k (with arrows reversed).

Examples

Example 1.20. Any linear map $f\colon \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ is a polynomial map. Indeed, the components f_i of f are linear polynomials $f_i(x) = \sum_j a_{ij}x_j$.

Example 1.21. Consider the algebraic set $X = Z(y^2 - x^3)$ in $\mathbb{A}^2(k)$. The affine coordinate ring of X is given as

$$A(X) = k[x, y]/(y^2 - x^3),$$

which is an integral domain because the polynomial $y^2 - x^3$ is irreducible.

Consider the polynomial map $f\colon \mathbb{A}^1(k) \to \mathbb{A}^2(k)$ given by $t \mapsto (t^2, t^3)$. The image of f is contained in $X \subset \mathbb{A}^2(k)$, and, in fact, f is a bijection between $\mathbb{A}^1(k)$ and X. Indeed, observe that $f(t) = (0, 0)$ only for $t = 0$, and if $(x, y) \neq (0, 0)$ lies in X, the assignment $t = y/x$ defines the inverse. However, f is not an isomorphism. As $f^\sharp(x) = t^2$ and $f^\sharp(y) = t^3$, the induced map

$$f^\sharp\colon k[x, y]/(y^2 - x^3) \longrightarrow k[t]$$

has image $k[t^2, t^3]$ and so is not surjective.

Note that the 'same' X can be embedded into different $\mathbb{A}^n(k)$'s. For instance, the above X can be embedded in $\mathbb{A}^3(k)$ as the zero set $Z(y^2 - x^3, z)$ or as $Z(y^2 - x^3, z - xy)$.

Example 1.22 (The Frobenius map)**.** In this example, we assume that k is of positive characteristic p. The map $\phi\colon k[t] \to k[t]$ given by $t \mapsto t^p$, is a map of k-algebras, and the corresponding polynomial map $F\colon \mathbb{A}^1(k) \to \mathbb{A}^1(k)$ acts on points by sending a point a to a^p. The map F is bijective because every $a \in k$ has a unique p-th root. However it is not an isomorphism, because the ring map $F^\sharp = \phi\colon k[t] \to k[t]$ is not surjective.

Regular and rational functions

The coordinate ring $A(X)$ of an affine variety X being an integral domain has a fraction field $k(X)$, which is called the *function field* or *field of rational functions* on X. Elements of $k(X)$ can be interpreted as functions on open sets in X; indeed, a fraction $f = a/b$ yields a well defined function on the open set where b does not vanish.

One says that a rational function $f \in k(X)$ is *regular* at a point $x \in X$ if it can be expressed as a fraction $f = a/b$ with $b(x) \neq 0$. Such a function will automatically be regular in a neighbourhood of x: it is regular in the complement of the proper closed set $Z(b)$. Such complements are called *distinguished open sets*, and the standard notation is $D(b)$; that is, $D(b) = \{ x \in X \mid b(x) \neq 0 \}$.

It is easy to check that sums and products of rational functions regular at x, are again regular at x. Thus the functions regular at x form a subring of $k(X)$. This ring is called the *local ring* of X at the point x and is denoted by $\mathcal{O}_{X,x}$.

Proposition 1.23. The set $\mathcal{O}_{X,x}$ of rational functions which are regular at x is a local ring whose maximal ideal $\mathfrak{m}_x$ consists of the functions vanishing at x.

Proof Recall that a ring is local if it has just one maximal ideal $\mathfrak{m}$, or equivalently, it has a maximal ideal $\mathfrak{m}$ such that elements not in $\mathfrak{m}$ are invertible. In our case, the ideal $\mathfrak{m}_x$ is precisely the kernel of the evaluation map $\mathcal{O}_{X,x} \to k$, and so $\mathfrak{m}_x$ is maximal since k is a field. An element $f \in \mathcal{O}_{X,x}$ which does not vanish at x, can be expressed as $f = a/b$ with both $a(x) \neq 0$ and $b(x) \neq 0$. Hence the rational function $1/f = b/a$ is regular at x and belongs to $\mathcal{O}_{X,x}$. Thus elements not in $\mathfrak{m}_x$ are invertible in $\mathcal{O}_{X,x}$, and we conclude that $\mathcal{O}_{X,x}$ is a local ring. □

Note that a rational function a/b may be regular in a larger set than the distinguished open set $D(b)$. The standard example is as follows.

Example 1.24. Consider the variety $X \subset \mathbb{A}^4(k)$ whose equation is $xy - zw = 0$. In the function field $k(X)$ the equality $x/w = z/y$ holds, and the corresponding rational function is thus regular in the open set $U = D(w) \cup D(y)$. Now, the point is that U is strictly larger than both $D(w)$ and $D(y)$, just consider the points $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$.

Exercise 1.2.1. With notation as in the example above:

a) Verify that $xy - zw$ is an irreducible polynomial;
b) Verify that the rational function x/w is not regular in any open set containing the locus where $y = w = 0$.

Proposition 1.25. Let X be an affine variety. If a rational function $f \in k(X)$ is regular at all points of X, then f is a polynomial function. In other words,

$$A(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}.$$

Proof Consider the ideal $\mathfrak{a}_f = \{ b \in A(X) \mid bf \in A(X) \}$. It has the property that a rational function f is regular at x if and only if $x \notin Z(\mathfrak{a}_f)$; indeed, $x \notin Z(\mathfrak{a}_f)$ if and only if some $b \in \mathfrak{a}_f$ does not vanish at x, which in turn is equivalent to f being on the form $f = a/b$ for some b with $b(x) \neq 0$. So when f is regular everywhere, it follows that $Z(\mathfrak{a}_f) = \varnothing$, and the Nullstellensatz yields that $1 \in \mathfrak{a}_f$; that is, $f \in A(X)$. □

We shall need the following result later on.

Proposition 1.26. When $n \geqslant 2$, a rational function which is regular on the open set $\mathbb{A}^n(k) - \{0\}$ is the restriction of a polynomial function.

Proof Let f be regular in $\mathbb{A}^n(k) - \{0\}$. Viewing f as an element of the function field of $\mathbb{A}^n(k)$, we may express it as $f = a/b$ with a and b polynomials. As polynomial rings are UFD's, we may chose a and b without common factors, and then they are unique (up to units). Hence a/b is not regular along $Z(b)$. By Krull's Principal Ideal Theorem , $Z(b)$ is either empty (and b is constant), or it has dimension $n - 1$; that is, if not empty, $Z(b)$ will be a larger set than $\{0\}$ when $n \geqslant 2$. Thus b is either constant, in which case f is a polynomial, or f is not regular along $Z(b)$. □

Example 1.27. Let $X = Z(xy - 1) \subset \mathbb{A}^2(k)$ and consider the first projection

$$f\colon X \to \mathbb{A}^1(k) - \{0\}.$$

The map f is actually an isomorphism; an inverse is given by $g(x) = (x, x^{-1})$ (note that x^{-1} is indeed a regular function on $\mathbb{A}^1(k) - \{0\}$). This means that regular functions on $\mathbb{A}^1(k) - \{0\}$ are given by polynomials in x and x^{-1}.

Exercises

Exercise 1.2.2. Let $x_0, \ldots, x_n$ be coordintes on the affine $(n+1)$-space $\mathbb{A}^{n+1}(k)$ and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $x_1, \ldots, x_n$.

a) Show that the algebraic set $X = Z(x_0 - f)$ is isomorphic to $\mathbb{A}^n(k)$;
b) For which f's is the algebraic set $X = Z(x_0^2 - f)$ irreducible?
c) Find a bijection between the open set $\mathbb{A}^n(k) - Z(f)$ in $\mathbb{A}^n(k)$ and the algebraic set $Z(x_0 f - 1)$ in $\mathbb{A}^{n+1}(k)$.

Exercise 1.2.3.

a) Let $\phi\colon A \to B$ be a map of rings. Show that $\phi^{-1}\mathfrak{p}$ is a prime ideal if $\mathfrak{p} \subset B$ is one;
b) Assume further that A and B are finitely generated k-algebras. Show that $\phi^{-1}\mathfrak{m}$ is a maximal ideal if $\mathfrak{m} \subset B$ is one. HINT: Use the Nullstellensatz to see that $A/\phi^{-1}\mathfrak{m} = k$ (remember that k is assumed to be algebraically closed in this chapter).

Exercise 1.2.4. Let $X = Z(f)$ and $Y = Z(g)$ be two algebraic sets in $\mathbb{A}^2(k)$ with X irreducible. Show that either $X \cap Y$ is a finite set, or $X \subset Y$.

1.3 Projective varieties

Having defined affine varieties, we move on to introducing projective space and projective varieties, and we continue working over an algebraically closed ground field k.

Definition 1.28. *The projective n-space* $\mathbb{P}^n(k)$ is the quotient of $\mathbb{A}^{n+1}(k) - \{0\}$ by the equivalence relation

$$(a_0, \ldots, a_n) \sim (ta_0, \ldots, ta_n),$$

where $t \in k$ is non-zero.

Two points in $\mathbb{A}^{n+1}(k)$ are equivalent precisely when they lie on the same line through the origin, so one may think about $\mathbb{P}^n(k)$ as the set of lines in $\mathbb{A}^{n+1}(k)$ through the origin; or if you want, the set of one-dimensional linear subspaces of k^{n+1}.

The equivalence class of a point $a = (a_0, \ldots, a_n)$ in $\mathbb{A}^{n+1}(k) - \{0\}$ will be denoted by $(a_0 : \cdots : a_n)$. The a_i's are called the *homogeneous coordinates* of a. Note that they are not coordinates in the usual strict sense of the word; they are not even functions on $\mathbb{P}^n(k)$, only their ratios are well defined. Note also that no point in $\mathbb{P}^n(k)$ has all homogeneous coordinates equal to 0; the tuple $(0 : \cdots : 0)$ is forbidden.

The Zariski topology on $\mathbb{P}^n(k)$

Just like the affine spaces, the projective space $\mathbb{P}^n(k)$ comes equipped with a natural *Zariski topology*. It is best described by the quotient map

$$\pi \colon \mathbb{A}^{n+1}(k) - \{0\} \longrightarrow \mathbb{P}^n(k),$$

which sends $(a_0, \ldots, a_n)$ to $(a_0 : \cdots : a_n)$. This allows us to define the topology by declaring a subset $V \subset \mathbb{P}^n(k)$ to be closed if and only if the inverse image $\pi^{-1}(V)$ is closed.

There is a construction, similar to $Z(\mathfrak{a})$ in the affine case, that describes all closed sets in $\mathbb{P}^n(k)$ in terms of certain ideals in a polynomial ring. However, it is slightly more delicate as polynomials are not functions on the projective spaces. They are not invariant under scaling of the arguments and so not constant on equivalence classes. The solution is to use *homogeneous polynomials*; that is, polynomials such that for some natural number d one has

$$f(tx_0, \ldots, tx_n) = t^d f(x_0, \ldots, x_n)$$

for all t. The values of f still depend on t, but the point is that whether the value of f is zero or not, is independent of t. So we may define the zero set of f in $\mathbb{P}^n(k)$ as

$$Z_+(f) = \{\, x \in \mathbb{P}^n(k) \mid f(x) = 0 \,\}.$$

More generally, for each set S of homogeneous polynomials, one may put

$$Z_+(S) = \{\, x \in \mathbb{P}^n(k) \mid f(x) = 0 \text{ for all } f \in S \,\}.$$

These sets are Zariski closed; indeed, a homogeneous polynomial f vanishes at a point $x \in \mathbb{P}^n(k)$ precisely when it vanishes along the whole line $\pi^{-1}(x)$ in $\mathbb{A}^n(k)$; in other words, $\pi^{-1} Z_+(S) = Z(S) \cap \mathbb{A}^{n+1}(k) - \{0\}$.

Ideals $\mathfrak{a}$ whose zero-set equals a Zariski closed inverse image, are characterised by the property that if $a \in Z(\mathfrak{a})$, then the entire fibre $\pi^{-1}\pi(a)$ lies in $Z(\mathfrak{a})$, in other words, the inclusion $\pi^{-1}\pi(a) \subset Z(\mathfrak{a})$ holds. These ideals are precisely the *homogenous* ideals.

Recall that an ideal is said to be *homogeneous* if for each element $f \in \mathfrak{a}$ all homogeneous components of f lie in $\mathfrak{a}$. In other words, if $f = \sum_i f_i$ is the decomposition of f into a sum of homogeneous polynomials, then $f \in \mathfrak{a}$ if and only if $f_i \in \mathfrak{a}$ for all i. An ideal is homogeneous if and only if it is generated by homogeneous polynomials.

Being a homogeneous ideal is equivalent to $Z(\mathfrak{a})$ containing the line through the origin and each point $x \in Z(\mathfrak{a})$; indeed, if $\mathfrak{a}$ is homogeneous, $x \in Z(\mathfrak{a})$ implies that $t \cdot x \in Z(\mathfrak{a})$ for every $t \in k$. The following lemma shows that the converse holds as well.

Lemma 1.29. Let $a \in \mathbb{A}^{n+1}(k) - \{0\}$ be a point. A polynomial f vanishes at all points on the line through a and the origin if and only if all the homogeneous components of f do.

Proof Developing f in terms of the homogeneous components f_i, we find

$$f(tx) = t^d f_d(x) + \cdots + t f_1(x) + f_0(x).$$

For $x = a$ fixed, this is a polynomial in t. Since f is assumed to be zero on the entire line through a, it has infinitely many zeroes and hence must be the zero polynomial in t. It follows that $f_i(a) = 0$ for all i. □

We have thus established the desired description of the closed sets in projective space:

Proposition 1.30. The Zariski closed sets of $\mathbb{P}^n(k)$ are precisely those of the form $Z_+(\mathfrak{a})$ where $\mathfrak{a}$ is a homogeneous ideal.

Example 1.31 (The irrelevant ideal)**.** The ideal $\mathfrak{m}_+ = (x_0, \ldots, x_n)$ is called the *irrelevant* ideal. It is certainly homogeneous, but its zero locus is empty (no point has all homogeneous coordinates equal to zero). Similarly, any $\mathfrak{m}_+$-primary ideal $\mathfrak{q}$ has empty zero set because $Z(\mathfrak{q}) = Z(\mathfrak{m}_+)$, so that $Z(\mathfrak{q}) \cap (\mathbb{A}^{n+1}(k) - \{0\}) = \varnothing$.

Example 1.32 (The complex projective spaces)**.** The complex projective spaces $\mathbb{P}^n(\mathbb{C})$ (which topologists usually write as $\mathbb{CP}^n$) are also equipped with a Euclidean topology. It is just the quotient topology inherited from the standard Euclidean topology on $\mathbb{C}^{n+1}$. With this topology they are compact manifolds. Every one-dimensional subspace of $\mathbb{C}^{n+1}$ meets the unit sphere $\mathbb{S}^{2n+1}$ along a unit circle, so the restriction $\pi|_{\mathbb{S}^{2n+1}}$ is a continuous surjection $\pi|_{\mathbb{S}^{2n+1}} : \mathbb{S}^{2n+1} \to \mathbb{P}^n(\mathbb{C})$. Since the unit sphere $\mathbb{S}^{2n+1}$ is compact, if follows that $\mathbb{P}^n(\mathbb{C})$ is compact as well. It is noteworthy that $\pi|_{\mathbb{S}^{2n+1}}$ is a fibre bundle with unit circles as fibres.

The projective Nullstellensatz

The usual operations on ideals, like sums, products, intersections and the formation of radicals, yield homogeneous ideals when applied to homogeneous ideals. Moreover, the equalities between the associated closed sets, as stated in Proposition 1.2 in the affine cases, are still valid.

Proposition 1.33. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two homogeneous ideals and let $\{\mathfrak{a}_i\}_{i\in I}$ be a family of homogeneous ideals in the polynomial ring $k[x_0, \ldots, x_n]$.

(i) If $\mathfrak{a} \subset \mathfrak{b}$, then $Z_+(\mathfrak{b}) \subset Z_+(\mathfrak{a})$;
(ii) $Z_+(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} Z_+(\mathfrak{a}_i)$;
(iii) $Z_+(\mathfrak{a}\mathfrak{b}) = Z_+(\mathfrak{a} \cap \mathfrak{b}) = Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b})$;
(iv) $Z_+(\mathfrak{a}) = Z_+(\sqrt{\mathfrak{a}})$.

Proof The proposition follows directly from the affine case (Proposition 1.2) by intersecting with $\mathbb{A}^{n+1}(k) - \{0\}$ and pushing down by π. For instance, the last equality in (iii) follows by the following equalities:

$$Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b}) = \pi\big(\big(Z(\mathfrak{a}) \cup Z(\mathfrak{b})\big) \cap \mathbb{A}^{n+1}(k) - \{0\}\big) = \\ = \pi\big(Z(\mathfrak{a} \cap \mathfrak{b}) \cap \mathbb{A}^{n+1}(k) - \{0\}\big) = Z_+(\mathfrak{a} \cap \mathfrak{b}).$$

□

There is also a projective version of the Nullstellensatz. The statement is very similar to the one in the affine case, but there are two notable differences. First of all, the irrelevant ideal $\mathfrak{m}_+ = (x_0, \ldots, x_n)$ and all primary ideals with radical equal to $\mathfrak{m}_+$ have empty zero locus. Secondly, one must be careful when defining the vanishing ideal $I(S)$ for a subset $S \subset \mathbb{P}^n(k)$ and let it be the ideal generated by the homogeneous polynomials which vanish along S. Note that this ideal is only generated by homogeneous polynomials, not every element is homogeneous.

Theorem 1.34 (Projective Nullstellensatz). Let $\mathfrak{a}$ be a homogeneous ideal in the polynomial ring $k[x_0, \ldots, x_n]$.

(i) The zero locus $Z_+(\mathfrak{a})$ is empty if and only if either $1 \in \mathfrak{a}$ or $\sqrt{\mathfrak{a}} = \mathfrak{m}_+$;
(ii) If $Z_+(\mathfrak{a}) \neq \emptyset$, it holds true that $I(Z_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Proof To prove (i), note that the set $Z_+(\mathfrak{a})$ is non-empty if and only if $Z(\mathfrak{a}) \cap \mathbb{A}^{n+1}(k) - \{0\}$ is non-empty. There are two ways in which the inrtersection can be empty: either $Z(\mathfrak{a}) = \emptyset$, and $1 \in \mathfrak{a}$, or $Z(\mathfrak{a}) = \{0\}$, and $\sqrt{\mathfrak{a}} = \mathfrak{m}_+$.

To prove (ii), we observe as in Lemma 1.29 that $I(Z(\mathfrak{a}))$ equals the ideal generated by all homogeneous polynomials in $\mathfrak{a}$, which by definition is equal to $I(Z_+(\mathfrak{a}))$. By the affine Nullstellensatz, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, and we are done. □

As in the affine case, the maps I and Z_+ give a way to translate between algebra and geometry.

Proposition 1.35. The maps $\mathfrak{a} \mapsto Z_+(\mathfrak{a})$ and $S \mapsto I(S)$ are mutually inverse inclusion reversing bijections between the set of proper radical homogenous ideals in $k[x_0, \ldots, x_n]$ and the set of closed subsets of $\mathbb{P}^n(k)$.

Again one should note that the irrelevant ideal is special: a proper homogeneous ideal corresponds to the empty set if and only if its radical equals the irrelevant ideal.

Example 1.36 (The ideal of a point in $\mathbb{P}^n(k)$)**.** In the affine case the maximal ideals in $k[x_1, \ldots, x_n]$ correspond exactly to the points of $\mathbb{A}^n(k)$. In projective space the points correspond to lines in $\mathbb{A}^{n+1}(k)$, so their ideals are homogeneous, but they are not maximal. A convenient set of generators (certainly not minimal) for the ideal of a point $(a_0 : \cdots : a_n)$, are the 2×2-minors of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix} \tag{1.2}$$

Indeed, a variable point $(x_0 : \cdots : x_n)$ lies in in the same one-dimensional linear subspace as $(a_0 : \dots : a_n)$ precisely when the two corresponding vectors are dependent, i.e. precisely when the matrix in (1.2) has rank one.

There is also a projective analogue of Proposition 1.13, and the proof is essentially the same as in the affine case; it relies on (iii) in Proposition 1.33.

Proposition 1.37. A closed subset $Z_+(\mathfrak{a})$ is irreducible if and only if the radical $\sqrt{\mathfrak{a}}$ is prime.

This leads us to give the following definition.

Definition 1.38. A *projective variety* is a closed irreducible subset of a projective space $\mathbb{P}^n(k)$.

Exercise 1.3.1. Write out the details of the proof of Proposition 1.33.

Distinguished open sets

On the affine spaces $\mathbb{A}^n(k)$ one has coordinates $x_1, \dots, x_n$ so that any regular function is a polynomial in the x_i. A projective space $\mathbb{P}^n(k)$ do not have such global coordinates, but there is a class of standard open subsets where we have good coordinates. These are the so-called *distinguished open sets*. A point $a = (a_0 : \dots : a_n)$ in $\mathbb{P}^n(k)$ has at least one non-zero homogeneous coordinate, say $a_i \neq 0$, and then a belongs to the set

$$D_+(x_i) = \{\, (x_0 : \dots : x_n) \mid x_i \neq 0 \,\} \subset \mathbb{P}^n(k).$$

On this set the ratios x_j/x_i are well defined functions and can be used as coordinates.

There are two standard ways of transition between the homogenous coordinates and the coordinates in a distinguished open set $D_+(x_i)$, *homogenization* and *dehomogenization*. With any homogenous polynomial F one associates a dehomogenized polynomial F^d simply by setting $x_i = 1$; that is, $F^d = F(x_0, \dots, 1, \dots, x_n)$, and for any homogeneous ideal $\mathfrak{a}$ one lets $\mathfrak{a}^d$ be the ideal $\mathfrak{a}^d = \{\, F^d \mid F \in \mathfrak{a} \,\}$.

With any polynomial f of degree d in the n variables $t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_n$, one associates the homogeneous polynomial $f^h = x_i^d f(x_0/x_i, \dots, x_n/x_i)$. And for an ideal $\mathfrak{a}$, we let $\mathfrak{a}^h$ denote the homogeneous ideal generated by the f^h for $f \in \mathfrak{a}$.

Example 1.39. The practical recipe to find f^h is as follows. Fill up each term of f with a power of x_i whose exponent makes the degree become d. For instance, when homogenizing $f = x_3 + x_1x_2 + x_3^4$ with respect to x_0, one obtains $f^h = x_3x_0^3 + x_0^2x_1x_2 + x_3^4$.

One has a pair of maps $\Phi\colon D_+(x_i) \to \mathbb{A}^n(k)$ and $\Psi\colon \mathbb{A}^n(k) \to D_+(x_i)$ given by

$$\begin{aligned} \Phi &\colon (x_0 : \dots : x_n) \mapsto (x_0/x_i, \dots, 1, \dots, x_n/x_i) \\ \Psi &\colon (t_0, \dots, 1, \dots, t_n) \mapsto (t_0 : \dots : 1 : \dots : t_n), \end{aligned}$$

where the 1 appears in the i-th slot. (To avoid tortuous notation, we here consider $\mathbb{A}^n(k)$ as being the linear subspace $Z(x_i - 1)$ of $\mathbb{A}^{n+1}(k)$ where the i-th coordinate equals one.)

Lemma 1.40. The maps Φ and Ψ is a pair of mutually inverse homeomorphisms between $\mathbb{A}^n(k)$ and $D_+(x_i)$ equipped with the subspace topology.

Proof It is easy to check that the two maps are mutually inverse, so the main claim is that they are continuous, and this follows from the two identities

$$\begin{aligned}\Psi^{-1}(Z_+(\mathfrak{a}) \cap D_+(x_i)) &= Z(\mathfrak{a}^d) \\ \Phi^{-1}Z(\mathfrak{a}) &= Z_+(\mathfrak{a}^h) \cap D_+(x_i).\end{aligned} \tag{1.3}$$

□

Exercise 1.3.2. Verify the identities in (1.3) above.

Examples

Example 1.41 (The Quadratic surface)**.** Consider two copies of $\mathbb{P}^1(k)$, one with homogeneous coordinates $(u_0 : u_1)$ and the other with $(t_0 : t_1)$. There is a map $\mathbb{P}^1(k) \times \mathbb{P}^1(k) \to \mathbb{P}^3(k)$ defined by the assignment

$$(t_0 : t_1) \times (u_0 : u_1) \mapsto (t_0u_0 : t_0u_1 : t_1u_0 : t_1u_1).$$

This is well defined, because scaling $(t_0 : t_1)$ and $(u_0 : u_1)$ by respectively λ and μ, scales $(t_0u_0 : t_0u_1 : t_1u_0 : t_1u_1)$ by $\lambda\mu$, and since at least one of the t_i's and one of u_i's are non-zero, at least one of the products t_iu_j's is non-zero as well.

The image is closed in $\mathbb{P}^3(k)$, being equal to the zero locus of $w_0w_3 - w_1w_2$ with w_i's being homogeneous coordinates on $\mathbb{P}^3(k)$. For instance, in the open affine piece $D_+(w_0)$ it holds that $t_0u_0 = w_0 \neq 0$, so the inverse image equals $D_+(u_0) \times D_+(u_1)$. Normalizing, i.e. setting $w_0 = t_0 = u_0 = 1$, the map takes the form $(1 : t) \times (1 : u) \mapsto (1 : t : u : tu)$, and it becomes clear that the image equals $w_3 = w_1w_2$.

Example 1.42 (Rational normal curves)**.** Consider the map

$$\begin{aligned}\rho\colon \mathbb{P}^1(k) &\to \mathbb{P}^n(k) \\ (t_0 : t_1) &\mapsto (t_0^n : t_0^{n-1}t_1 : \cdots : t_0t_1^{n-1} : t_1^n).\end{aligned} \tag{1.4}$$

This is well defined because when t_0 and t_1 are scaled by λ, the products $t_0^{n-i}t_1^i$ are all scaled by λ^n, and of course, these products are never all zero. The image C_n is called a *rational normal curve of degree* n.

The map ρ is injective. Indeed, observe first that the image of ρ is contained in the union $D_+(x_0) \cup D_+(x_n)$. For points in the image lying in the distinguished open subset $D_+(x_n)$, one recovers the ratio t_0/t_1 as x_{n-1}/x_n, and for image points in $D_+(x_0)$ one finds $t_1/t_0 = x_1/x_0$.

The image C_n is a closed subset of $\mathbb{P}^n(k)$. It equals the common vanishing locus of the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} & x_n \end{pmatrix}. \tag{1.5}$$

When $n = 2$, we just get the conic section $x_0x_2 - x_1^2 = 0$ in the projective plane $\mathbb{P}^2(k)$. The curve C_3 is called the *twisted cubic curve*, 'twisted' because it does not lie in any plane in $\mathbb{P}^3(k)$.

Example 1.43 (The Veronese surface)**.** The projective plane $\mathbb{P}^2(k)$ can be embedded in a natural way in the projective space $\mathbb{P}^5(k)$ using all the quadratic monomials as coordinate functions:

$$\begin{gathered}\mathbb{P}^2(k) \to \mathbb{P}^5(k) \\ (t_0 : t_1 : t_2) \mapsto (t_0^2 : t_0t_1 : t_0t_2 : t_1^2 : t_1t_2 : t_2^2).\end{gathered} \tag{1.6}$$

The image is called the *Veronese surface*. Note that the definition makes sense, because a simultaneous scaling of the t_i's by λ simultaneously scales the monomials by λ^2, and they do not all vanish at the same time. In homogeneous coordinates $(x_0 : \cdots : x_5)$ on $\mathbb{P}^5(k)$, the homogeneous ideal of the the surface is given by the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}$$

Exercise 1.3.3. With reference to Example 1.41:

a) Show that closed sets in $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ induced by the Zariski closed sets in $\mathbb{P}^3(k)$ are given by the vanishing of *bihomogeneous* polynomial. Here a polynomial $F(x_0, x_1, y_0, y_1)$ is said to be bihomogeneous of bidegree (a, b) if $F(sx, ty) = s^a t^b F(x, y)$ for all $s, t \in k$.
b) Show that the subspace topology on $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$ is not the product topology.

Exercise 1.3.4. Show that the rational normal curve C_n equals the zero locus of the minors of the matrix (1.5). Hint: Consider distinguished open sets.

1.4 Regular functions on projective spaces

As we already observed, homogeneous polynomials do not define functions on the projective spaces. However, some quotients of homogeneous polynomials do give rational functions. These quotients must be invariant under scaling of the variables, and are consequently of the form g/h where g and h are homogeneous polynomials of the same degree, and then the quotient g/h is a well-defined function on the distinguished open set $D_+(h)$.

As in the affine case, a function f is said to be regular throughout an open set $U \subset \mathbb{P}^n(k)$ if each point x in U has a neighbourhood over which $f = g/h$ with $h(x) \neq 0$, and where g and h are homogeneous polynomials of the same degree. The functions regular at a point x form a ring $\mathcal{O}_{\mathbb{P}^n(k),x}$, which is a local ring whose maximal ideal consists of the regular functions vanishing at x.

However, contrary to the affine case, there are not many global regular functions on $\mathbb{P}^n(k)$. In fact, they are all constant. This statement is true for any projective variety, but for the sake of brevity we contend ourselves to proving it only for projective space itself.

Theorem 1.44. The only global regular functions on the projective space $\mathbb{P}^n(k)$ are the constants.

Proof Let f be a global regular function on $\mathbb{P}^n(k)$, and consider the composition $f \circ \pi$ of f with the canonical projection $\pi\colon \mathbb{A}^{n+1}(k) - \{0\} \to \mathbb{P}^n(k)$. It is a global regular function on $\mathbb{A}^{n+1} - \{0\}$, so by Proposition 1.26 on page 12, it is given by a polynomial. However, since this polynomial comes from a function on $\mathbb{P}^n(k)$, it must be constant on lines through the origin. This means that it must have degree 0, that is, it is constant everywhere. □

The fact that there are so few global regular functions basically forces us to instead work with regular functions $f\colon U \to k$ that are defined on open sets $U \subset X$. This is one of the reasons to introduce *sheaves* in the next few chapters.

2

The Prime Spectrum

In this chapter we make the first step towards the notion of a scheme, by defining the spectrum of a ring. The spectrum of a ring A is a topological space, denoted by $\operatorname{Spec} A$, with a Zariski-like topology whose closed sets are formed from the ideals of A.

To motivate the definition, assume for a moment that $A = A(X)$ is the coordinate ring of an affine variety X. By Hilbert's Nullstellensatz the points of X are in bijection with the set of maximal ideals in A: a point $x = (a_1, \ldots, a_n)$ corresponds to the maximal ideal $\mathfrak{m}_x = (x_1 - a_1, \ldots, x_n - a_n)$ of regular functions on X vanishing at x, and every maximal ideal is of this form. Thus there is no loss of information in replacing X with the set $\{ \mathfrak{m} \mid \mathfrak{m} \subset A \text{ is a maximal ideal} \}$. Note that a point $x \in X$ lies in $Z(\mathfrak{a})$ if and only if $\mathfrak{a} \subset \mathfrak{m}_x$. Therefore, under this identification, the closed sets $Z(\mathfrak{a})$ from Chapter 1 now take the form $Z(\mathfrak{a}) = \{ \mathfrak{m} \mid \mathfrak{m} \supset \mathfrak{a} \}$. Thus the ring A determines the topological space X. Moreover, maps from X to other affine varieties are determined by A as well: according to Theorem 1.19, polynomial maps $f : X \to Y$ correspond bijectively to maps of k-algebras $\phi : A(Y) \to A(X)$.

The rings that appear in the setting of varieties are rather special. They are integral domains and finitely generated k-algebras over an algebraically closed field k, and the assumption that k be algebraically closed, is essential in order to have the above correspondence between points and maximal ideals.

There is a natural way of generalizing this to all rings, which involves including all prime ideals, instead of just the maximal ideals. Given a ring A, the spectrum $\operatorname{Spec} A$ of A is simply the set of prime ideals of A. This set is then equipped with a topology, called the Zariski topology, whose closed sets are the sets of the form $V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subset \mathfrak{p} \}$ where $\mathfrak{a}$ is any ideal in A.

The idea of replacing maximal ideals by prime ideals is fundamental in scheme theory. From a categorical perspective this is a good choice, since inverse images of prime ideals under ring maps are prime ideals, and thus a ring map $A \to B$ induces a map $\operatorname{Spec} B \to \operatorname{Spec} A$. When X and Y were affine varieties, we were lucky that the induced map $A(Y) \to A(X)$, in fact, pulls maximal ideals back to maximal ideals, but this is no more true for general ring maps (a simple example is the inclusion $\mathbb{Z} \to \mathbb{Q}$).

2.1 The spectrum of a ring

Let A be a ring. As usual we assume that A is commutative with 1.

Definition 2.1. The *prime spectrum* of a ring A, or simply the *spectrum* is defined as the set of prime ideals in A.

$$\operatorname{Spec} A = \{\, \mathfrak{p} \mid \mathfrak{p} \subset A \text{ is a prime ideal} \,\}.$$

There is a topology on $\operatorname{Spec} A$ which generalizes the Zariski topology on a variety and which is also called the *Zariski topology*. The definitions are very similar; the closed sets in $\operatorname{Spec} A$ are those of the form

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a} \,\},$$

where $\mathfrak{a}$ is any ideal in A. Of course, one has to verify that the axioms for a topology are satisfied. These require that the union of two closed sets, and the intersection of any number (finite or infinite) of closed sets, is closed. And of course, both the whole space and the empty set must be closed. The following lemma tell us that the closed subsets $V(\mathfrak{a})$ indeed satisfy these axioms:

Lemma 2.2. Let A be a ring and assume that $\{\mathfrak{a}_i\}_{i\in I}$ is a family of ideals in A. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A. Then the following three statements hold true:

(i) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$;
(ii) $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$;
(iii) $V(A) = \varnothing$ and $V(0) = \operatorname{Spec} A$.

Proof Prime ideals are by definition proper ideals, so $V(A) = \varnothing$. Also, the zero ideal (0) is contained in every ideal, so $V(0) = \operatorname{Spec} A$. This proves (iii), and (ii) follows just as easily, because the sum of a family of ideals is contained in an ideal if and only if each of the ideals in the family is.

For statement (i): the inclusion $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b})$ is clear, so we need only to show that $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$. If $\mathfrak{b} \not\subseteq \mathfrak{p}$, there is an element $b \in \mathfrak{b}$ with $b \notin \mathfrak{p}$. If $a \in \mathfrak{a}$, then $ab \in \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$, and so $a \in \mathfrak{p}$ because $\mathfrak{p}$ is prime. Consequently, one has the inclusion $\mathfrak{a} \subset \mathfrak{p}$. □

Corollary 2.3. The collection of sets of the form $V(\mathfrak{a})$ constitute the closed sets a topology on $\operatorname{Spec} A$.

The next lemma is about inclusions between the closed sets of $\operatorname{Spec} A$, and we recognise them as analogues of some of the statements about subset of varieties in Proposition 1.2.

Lemma 2.4. For two ideals $\mathfrak{a}, \mathfrak{b} \subset A$ we have

(i) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$;
(ii) $V(\mathfrak{a}) \subset V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$;
(iii) $V(\mathfrak{a}) = \varnothing$ if and only if $\mathfrak{a} = A$;
(iv) $V(\mathfrak{a}) = \operatorname{Spec} A$ if and only if $\mathfrak{a} \subset \sqrt{(0)}$.

Proof Recall the following identity for the radical of an ideal:

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a}\subset\mathfrak{p}} \mathfrak{p}. \tag{2.1}$$

From this, we see that $\mathfrak{a}$ and $\sqrt{\mathfrak{a}}$ are contained in the same prime ideals, and we infer that $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$. To show (ii), let us assume that $V(\mathfrak{a}) \subset V(\mathfrak{b})$. From (2.1) we then obtain

$$\sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{p}\in V(\mathfrak{b})} \mathfrak{p} \subset \bigcap_{\mathfrak{p}\in V(\mathfrak{a})} \mathfrak{p} = \sqrt{\mathfrak{a}}.$$

Conversely, assume that $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$. If $\mathfrak{p} \in V(\mathfrak{a})$, then $\sqrt{\mathfrak{a}} \subset \mathfrak{p}$, and we deduce from the chain of inclusions $\mathfrak{b} \subset \sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}} \subset \mathfrak{p}$ that $\mathfrak{p} \in V(\mathfrak{b})$. This proves (ii).

Statement (iii) follows from Lemma 2.2 because $V(\mathfrak{a}) = V(1) = \operatorname{Spec} A$ if and only if $\sqrt{\mathfrak{a}} = (1)$, which happens if and only if $\mathfrak{a} = (1)$. Similarly, (iv) holds because $V(\mathfrak{a}) = V(0)$ if and only if $\mathfrak{a} \subset \sqrt{(0)}$. □

Corollary 2.5. The assignment $\mathfrak{a} \mapsto V(\mathfrak{a})$ gives a one-to-one correspondence between radical ideals of A and closed subsets of $\operatorname{Spec} A$.

Residue fields

Contrary to elements in the coordinate ring of a variety, elements in a general ring A cannot be interpreted as functions on $\operatorname{Spec} A$ into some fixed field. However, there still is an analogy between elements f of A and some sort of functions on $\operatorname{Spec} A$. If x is a point in $\operatorname{Spec} A$ which corresponds to $\mathfrak{p}$, the localization $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and one has the field $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, which will also be denoted by $k(x)$. It is canonically isomorphic to the fraction field of the domain $A/\mathfrak{p}A$. The residue class of an element f modulo $\mathfrak{p}$ gives an element $f(x) \in k(\mathfrak{p})$, which may be considered as the 'value' of f at x. It is important to note that these values lie in different fields which might vary with the point.

Definition 2.6. The field $k(\mathfrak{p})$ is called the *residue field* of $\operatorname{Spec} A$ at $\mathfrak{p}$.

For each $f \in A$, we may also speak of its 'zero set', i.e. the points $x \in X$ such that $f(x) = 0$ in $k(x)$. By definition $f(x) = 0$ if and only if $f \in \mathfrak{p}$, so the 'zero set' is exactly the closed set

$$V(f) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p} \,\}.$$

Furthermore, a closed set $V(\mathfrak{a})$ may be written as

$$V(\mathfrak{a}) = \{\, x \in \operatorname{Spec} A \mid f(x) = 0 \text{ for all } f \in \mathfrak{a} \,\}.$$

First examples

Example 2.7 (Fields)**.** If K is a field, the prime spectrum $\operatorname{Spec} K$ has only one point, which corresponds to the only prime ideal in K, the zero ideal.

Example 2.8 (Artinian rings)**.** The ring $A = \mathbb{C}[t]/(t^2)$ is not a field, but has only one prime ideal, namely the ideal (t). Note that the ideal (0) is not prime as $t^2 = 0$, but $t \notin (0)$.

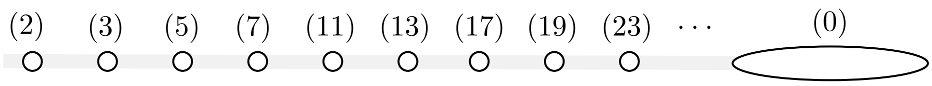
The ring $A = \mathbb{C}[t]/(t(t-1))$ has a spectrum which consists of two points. By the Chinese Remainder theorem,

$$A \simeq \mathbb{C}[t]/t \times \mathbb{C}[t]/(t-1) \simeq \mathbb{C} \times \mathbb{C},$$

which has exactly two prime ideals, namely $0 \times \mathbb{C}$ and $\mathbb{C} \times 0$.

More generally, an Artinian ring A has only finitely many prime ideals which are all maximal, so $\operatorname{Spec} A$ is a finite set, and the topology is the discrete topology.

Example 2.9 (The spectrum of the integers)**.** In the ring of integers $\mathbb{Z}$, there are two types of prime ideals: the zero-ideal (0) and the maximal ideals (p), one for each prime number p. The latter correspond to closed points in $\operatorname{Spec} \mathbb{Z}$, and one has $V(0) = \operatorname{Spec} \mathbb{Z}$.



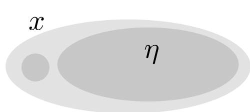
The spectrum of the integers

As $\mathbb{Z}$ is a principal ideal domain, any ideal is of the form (n) for some integer n. It follows that the closed subsets are of the form $V(n) = V(p_1) \cap \cdots \cap V(p_r)$ where the p_i are the prime factos of n. In other words, the closed sets are either finite sets of closed points or the whole space. Dually, the non-empty open sets are the complements of finite sets of closed points. In particular, this means that $\operatorname{Spec} \mathbb{Z}$ is not Hausdorff, as any open set must contain (0).

The residue field at a closed point (p) is equal to $k(p) = \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$, whereas the residue field at (0) is equal to $\mathbb{Z}_{(0)} = \mathbb{Q}$. Each element f of the ring $\mathbb{Z}$ gives rise to a function on $\operatorname{Spec} \mathbb{Z}$ with values in the various residue fields. For instance, the integer $f = 17$ takes the values $f((0)) = 17$, $f((2)) = \bar{1}$, $f((3)) = \bar{2}$, $f((5)) = \bar{2}$, $f((7)) = \bar{3}, \ldots$, in the fields $\mathbb{Q}, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \ldots$, respectively, where the bar indicates the residue class modulo the relevant prime.

Example 2.10 (Discrete valuation rings)**.** Consider a discrete valuation ring A, such as the series ring $\mathbb{C}[\![t]\!]$, or one of the localizations $k[t]_{(t)}$ or $\mathbb{Z}_{(p)}$. (See Appendix A for background on discrete valuation rings). The ring A has exactly two prime ideals, the maximal ideal $\mathfrak{m}$ and the zero ideal (0), and $\operatorname{Spec} A$ consists of just two points: $\operatorname{Spec} A = \{\eta, x\}$ with x corresponding to the maximal ideal $\mathfrak{m}$ and $\{\eta\}$ to (0). The closed sets are $\varnothing$, $\{x\}$ and $\{x, \eta\}$. Therefore $\{\eta\} = \operatorname{Spec} A - \{x\}$ is open; so η is an open point!

The open sets are $\varnothing$, $\operatorname{Spec} A$ and $\{\eta\}$. Again $\operatorname{Spec} A$ is not Hausdorff, as there is too few open sets to separate x and η.



The spectrum of a DVR

Exercises

Exercise 2.1.1. Describe $\operatorname{Spec}\mathbb{Z}_{(210)}$.

Exercise 2.1.2. Let $\mathfrak{p}$ be a prime ideal in a ring A. Show that there is a canonical inclusion $A/\mathfrak{p} \hookrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and that this yields an identification of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ with the fraction field of $A/\mathfrak{p}$.

Exercise 2.1.3. Let $\mathfrak{a} \subset A$ be an ideal. Show that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a}\subset\mathfrak{p}} \mathfrak{p}$. HINT: If $f \notin \sqrt{\mathfrak{a}}$ the ideal $\mathfrak{a}A_f$ is a proper ideal in the localization A_f, hence contained in a maximal ideal.

2.2 Generic points

The Zariski topology on $\operatorname{Spec} A$ is very different from the Euclidean topology on manifolds that we are used to. In fact, the topology can exhibit surprising behaviour, even compared to the usual Zariski topology on varieties. For instance, points can fail to be closed. In fact, the next proposition implies that a point $x \in \operatorname{Spec} A$ is closed if and only if the corresponding ideal is maximal.

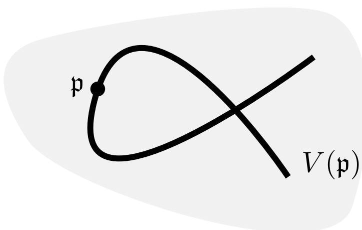
Proposition 2.11. The closure of a set $S \subset \operatorname{Spec} A$ is equal to $\overline{S} = V(\mathfrak{a})$ where $\mathfrak{a} = \bigcap_{\mathfrak{p}\in S} \mathfrak{p}$. In particular, the closure $\overline{\{\mathfrak{p}\}}$ of the one-point set $\{\mathfrak{p}\}$ equals the closed set $V(\mathfrak{p})$.

Proof Let $\mathfrak{b}$ be the radical ideal with $V(\mathfrak{b}) = \overline{S}$. Then every $\mathfrak{p} \in S$ contains $\mathfrak{b}$, and hence $\mathfrak{b} \subset \mathfrak{a}$. On the other hand, $V(\mathfrak{a})$ is closed, and $S \subset V(\mathfrak{a})$, so it follows that $\overline{S} \subset V(\mathfrak{a})$. Hence $V(\mathfrak{b}) \subset V(\mathfrak{a})$, and $\mathfrak{a} \subset \mathfrak{b}$ by Lemma 2.4. We conclude that $\mathfrak{a} = \mathfrak{b}$. □

This leads to the following definition:

Definition 2.12 (Genric points)**.** A point x in a closed subset Z of a topological space X is called a *generic point* for Z if $\overline{\{x\}} = Z$,

In our context, each point $\mathfrak{p} \in \operatorname{Spec} A$ is the generic point of the closed set $V(\mathfrak{p})$.



Example 2.13. When A is an integral domain A, the zero ideal (0) is prime, and as $V(0) = \operatorname{Spec} A$, it is the generic point of all of $\operatorname{Spec} A$. This explains the ‘fat’ points in the pictures in Examples 2.9 and 2.10, their closures are the whole space.

2.3 Affine spaces

The most important examples of prime spectra are the affine spaces.

Definition 2.14. For each natural number n, we define the *affine n-space* as

$$\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n].$$

More generally, for a ring A, we define the *affine n-space over A* by

$$\mathbb{A}^n_A = \operatorname{Spec} A[t_1, \ldots, t_n].$$

When k is an algebraically closed field, then $\mathbb{A}^n_k$ is the scheme analogue of the affine n-space $\mathbb{A}^n(k)$ (as defined in Chapter 1). In this setting, Hilbert's Nullstellensatz tells us that the points of $\mathbb{A}^n(k)$ are in one-to-one correspondence with the maximal ideals in $A = k[t_1, \ldots, t_n]$ (which all are of the form $(t_1 - a_1, \ldots, t_n - a_n)$ with $a_i \in k$). Thus $\mathbb{A}^n(k)$ occurs naturally as a subset of the spectrum $\mathbb{A}^n_k$, and moreover, the old Zariski topology on the variety $\mathbb{A}^n(k)$ is the one induced from the Zariski topology on $\mathbb{A}^n_k$. However, there are other prime ideals in A than just the maximal ideals; the zero ideal for instance. So $\mathbb{A}^n_k$ is strictly larger than $\mathbb{A}^n(k)$. The differences between $\mathbb{A}^n_k$ and $\mathbb{A}^n(k)$ become even more apparent if k is not algebraically closed.

Example 2.15 (The affine line)**.** The prime spectrum $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ is called the *affine line over k*. The polynomial ring $k[t]$ is a principal ideal domain, so the prime ideals are either of the form $(f(t))$ where $f(t)$ is an irreducible polynomial, or the zero ideal (0). In the first case the ideals are automatically maximal, so it follows that $\mathbb{A}^1_k$ has two types of points: the closed points and the generic point η.

If we assume that k is algebraically closed, then the maximal ideals are all of the form $(t - a)$ for $a \in k$. The residue fields at the corresponding points are of the form

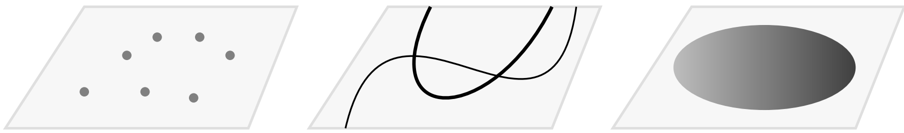
$$k(a) = k[t]_{(t-a)}/(t-a)k[t]_{(t-a)} \simeq k.$$

Thus for instance, $\mathbb{A}^1_{\mathbb{C}}$ consists of the generic point η and one point for each complex number.

When k is not algebraically closed, there can be other closed points than the ones of the form $(t - a)$. An interesting special case is when $k = \mathbb{R}$. Then $\mathbb{A}^1_{\mathbb{R}}$ is called the *real affine line*. By the Fundamental Theorem of Algebra, a non-zero prime ideal $\mathfrak{p}$ of $\mathbb{R}[t]$ is of the form $\mathfrak{p} = (f(t))$ where either $f(t)$ is linear; that is, $f(t) = t - a$ for an $a \in \mathbb{R}$, or f is quadratic with two conjugate complex non-real roots; that is, $f(t) = (t - a)(t - \bar{a})$ with $a \in \mathbb{C}$ but $a \notin \mathbb{R}$. This shows that the closed points in $\operatorname{Spec} \mathbb{R}[t]$ may be identified with the set of pairs $\{a, \bar{a}\}$ with $a \in \mathbb{C}$.

In the non-algebraically closed case, the residue fields of the 'extra' non-closed points can be more interesting. For instance, the maximal ideal $(t^2 + 1)$ defines a closed point in $\mathbb{A}^1_{\mathbb{R}}$ with residue field $\mathbb{C}$. In general, if a maximal ideal $\mathfrak{m}$ in $k[t]$ is generated by the irreducible polynomial $f(t)$, the residue field at the corresponding point in $\mathbb{A}^1_k$ is the extension of k obtained by adjoining a root of f.

Example 2.16 (The affine plane)**.** When k is algebraically closed, the maximal ideals of $k[t_1, t_2]$ are all of the form $(t_1 - a_1, t_2 - a_2)$, and these constitute all the closed points of $\mathbb{A}^2_k$. There are also the prime ideals of the form $\mathfrak{p} = (f)$ where f is an irreducible polynomial from $k[t_1, t_2]$. The prime ideal $\mathfrak{p}$ is the generic point of the closed subset $V(f)$. In addition to the point $\mathfrak{p}$, the points of $V(f)$ are the closed points corresponding to ideals $(t_1 - a_1, t_2 - a_2)$ containing f. This condition is equivalent to $f(a_1, a_2) = 0$, so the closed points of $V(f)$ correspond to what one in a variety setting would call *the curve* given by the equation $f(t_1, t_2) = 0$.



2.4 Irreducibility and connectedness

Recall from Chapter 1 that a topological space X is *irreducible* if it cannot be written as the union of two proper closed subsets. From Proposition 1.13, we know that the coordinate ring of an affine variety is an integral domain, and very simple examples indicate that reducibility of $\operatorname{Spec} A$ is closely linked to zero divisors in A (see Example 2.21 below). In general, we have the following:

Proposition 2.17. Let A be a ring. Then the following statements hold:
(i) If $\mathfrak{p} \subset A$ is a prime ideal, it holds that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$, and $\mathfrak{p}$ is the only generic point of $V(\mathfrak{p})$;
(ii) A closed subset $Z \subset \operatorname{Spec} A$ is irreducible if and only if Z is of the form $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$;
(iii) The space $\operatorname{Spec} A$ itself is irreducible if and only if A has just one minimal prime ideal; in other words, if and only if the nilradical $\sqrt{(0)}$ is prime.

Proof Statement (i) is just Proposition 2.11 on page 25. For the uniqueness part, when $V(\mathfrak{p}) = V(\mathfrak{q})$, it holds by Lemma 2.4 on page 22 that both $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$.

Proof of (ii): As the closure of any singleton is irreducible, and since we just showed that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, we conclude that $V(\mathfrak{p})$ irreducible. For the reverse implication, let $V(\mathfrak{a}) \subset \operatorname{Spec} A$ be a closed subset. Recall that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$. If $\sqrt{\mathfrak{a}}$ is not prime, there are more than one prime involved in the intersection. We may divide them into two different groups thus representing $\sqrt{\mathfrak{a}}$ as an intersection $\sqrt{\mathfrak{a}} = \mathfrak{b} \cap \mathfrak{b}'$ where $\mathfrak{b}$ and $\mathfrak{b}'$ are the intersections of the primes in the two groups, and hence are different radical ideals. One concludes that $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{b}')$, and $V(\mathfrak{a})$ is not irreducible.

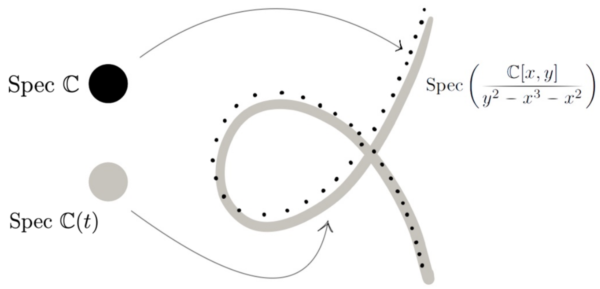
The third statement follows from the second, because $\operatorname{Spec} A = V\big(\sqrt{(0)}\big)$ (again by Lemma 2.4). □

A consequence of the proposition is that $\operatorname{Spec} A$ is irreducible whenever A is an integral domain, as in that case (0) is a minimal prime ideal. However, $\operatorname{Spec} A$ may well be irreducible for other rings as well. The ring $A = \mathbb{C}[t]/(t^2)$ is a simple example: it is not an integral domain, and has only one prime ideal, namely the principal ideal (t). Statement (iii) above tells us that this example is typical for such rings: every zerodivisor in the ring is nilpotent. Or, in the spirit of the analogy with functions, there are non-zero functions vanishing everywhere.

Example 2.18. Let $R = \mathbb{C}[x, y]/(y^2 - x^3 - x^2)$ and $X = \operatorname{Spec} R$. Then X is irreducible and is called the *nodal cubic curve* over $\mathbb{C}$. There are two types of points in X:

(i) Closed points $p \in X$. These correspond to maximal ideals $\mathfrak{m} = (x - a, y - b)$ where a, b satisfy $b^2 = a^3 + a^2$. The residue fields equal $k(p) = R/\mathfrak{m} \simeq \mathbb{C}$.

(ii) The generic point η. This corresponds to the zero ideal. The residue field equals the fraction field of R, which is isomorphic to $\mathbb{C}(t)$ (via the substitution $x = t^2 - 1, y = t^3 - t$).



Recall that a topological space is *connected* if it cannot be written as a disjoint union of two proper open subsets. All of the examples we have seen until now, with the exception of Example 2.8, are connected.

Example 2.19 (A disconnected spectrum)**.** Suppose that $A = A_1 \times A_2$ is the direct product of two non-trivial rings A_1 and A_2. In A we have the two *orthogonal idempotents* $e_1 = (1, 0)$ and $e_2 = (0, 1)$; they satisfy the relations $e_1^2 = e_1$, $e_1e_2 = 0$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$.

The spectrum $\operatorname{Spec} A$ decomposes as the disjoint union $\operatorname{Spec} A = V(e_1) \cup V(e_2)$ of the two closed sets $V(e_i)$; indeed, since $e_1 + e_2 = 1$, it holds that $V(e_1) \cap V(e_2) = V(e_1, e_2) = \varnothing$. And since $e_1e_2 = 0$, either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$ for each prime $\mathfrak{p} \in \operatorname{Spec} A$, so the $V(e_i)$'s cover $\operatorname{Spec} A$.

In fact, there is a converse to this example.

Proposition 2.20. A spectrum $\operatorname{Spec} A$ is disconnected if and only if A is isomorphic to a direct product $A = A_1 \times A_2$ of non-trivial rings A_1 and A_2.

While it would certainly be possible to give a direct proof of this proposition at the present stage, we wait until the next chapter; there is a much more conceptual proof using the structure sheaf (see Example 4.9 on page 57). For reduced rings however, the argument is straightforward (see Exercise 2.4.1)

Connectedness is a weaker topological condition than irreducibility in the sense that an irreducible space is also connected. However, it is possible to be connected yet reducible, as the following example shows:

Example 2.21. The prime spectrum $X = \operatorname{Spec} k[x,y]/(xy)$ is a good example of a space which is connected but not irreducible. The coordinate functions x and y are zero-divisors in the ring $k[x,y]/(xy)$, and their zero-sets $V(x)$ and $V(y)$ show that X has two components. Since these two components intersect at the origin, X is connected.

Exercise 2.4.1. Let A be a reduced ring. Show that $\operatorname{Spec} A$ is not connected if and only if $A = A_1 \times A_2$ for two non-trivial rings A_1 and A_2. HINT: If $\operatorname{Spec} A$ is the disjoint union $V(\mathfrak{a}) \cup V(\mathfrak{b})$, it holds true that $\mathfrak{a} + \mathfrak{b} = A$. Use this to find two non-trivial idempotents.

Exercise 2.4.2. Assume that X is a topological space that is not connected. Exhibit two non-constant orthogonal idempotents with sum unity in the ring of continuous functions on X. HINT: The characteristic functions of two disjoint open sets whose union equals X, will do.

2.5 Distinguished open sets

There is no way to describe the open sets in $\operatorname{Spec} A$ as simply and elegantly as the closed sets can be. However there is a natural basis for the topology on $\operatorname{Spec} A$ whose sets are easily defined, and which turns out to be very useful. For an element $f \in A$, we let $D(f)$ be the complement of the closed set $V(f)$; that is, we set

$$D(f) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \,\} = V(f)^c.$$

These are clearly open sets; we call them *distinguished open sets*.

Lemma 2.22. For a ring A and elements $f, g \in A$, we have
(i) $D(f) \cap D(g) = D(fg)$;
(ii) $D(g) \subset D(f)$ if and only if $g^n \in (f)$ for some natural number n. In particular, one has $D(f) = D(f^n)$ for all n.

Proof Statement (i): if $\mathfrak{p}$ is a prime ideal, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ hold true if and only if $fg \notin \mathfrak{p}$.

Proof of (ii): the inclusion $D(g) \subset D(f)$ holds if and only if $V(f) \subset V(g)$, and by Lemma 2.4 on page 22 this is true if and only if $(g) \subset \sqrt{(f)}$; in other words, if and only if $g^n \in (f)$ for a suitable n. □

Lemma 2.23.
(i) The collection of distinguished open sets form a basis for the topology of $\operatorname{Spec} A$;
(ii) A family $\{D(f_i)\}_{i \in I}$ forms an open covering of $\operatorname{Spec} A$ if and only if the f_i generate the unit ideal, i.e. if and only if there is a relation

$$1 = a_1 f_{i_1} + \cdots + a_n f_{i_n} \tag{2.2}$$

where $i_1, \ldots, i_n \in I$.

Proof Statement (i): we need to show that each open subset U of $\operatorname{Spec} A$ can be written as the union of distinguished open sets. Observe that, by definition, the complement U^c of U is of the form $U^c = V(\mathfrak{a})$ with $\mathfrak{a} \subset A$ an ideal, and choose a set $\{f_i\}$ of generators for $\mathfrak{a}$ (not necessarily a finite set). Then we have

$$U = V(\mathfrak{a})^c = V\big(\sum_i (f_i)\big)^c = \big(\bigcap_i V(f_i)\big)^c = \bigcup_i D(f_i). \tag{2.3}$$

Statement(ii): from the identity (2.3) with $U = \operatorname{Spec} A$, it follows that the open sets $D(f_i)$ constitute a covering of $\operatorname{Spec} A$ if and only if $V(\sum_i (f_i)) = \varnothing$, which happens if and only if $\sum_i (f_i) = (1)$. But this is the case if and only if 1 is a combination of finitely many of the f_i's. □

The lemma tells us that the $D(f)$'s form a basis for the topology: any open set $U \subset \operatorname{Spec} A$ can be written as a union of $D(f)$'s. Moreover, we deduce that any open cover may be refined to one whose members all are distinguished, and hence it can be reduced to a *finite* covering. A topological space with this property is said to be *quasi-compact*. [1].

Example 2.24. In the affine line $\mathbb{A}^1_k$ over a field, every closed set is of the form $V(f)$ for some polynomial f, so every open set is a distinguished open set $D(f)$. In $\mathbb{A}^2_k = \operatorname{Spec} k[u, v]$, the set $U = \mathbb{A}^2_k - V(u, v)$ is open, but not of the form $D(f)$. Still, we have $U = D(u) \cup D(v)$.

Example 2.25 (The circle)**.** Consider the unit circle $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. The maximal ideal $\mathfrak{m} = (x, y - 1)$ defines the point $(0, 1)$ on X. Note that this ideal is not a principal ideal. Nevertheless, the complement $X - \{(0, 1)\}$ is a distinguished open set. Indeed, it coincides with $D(y - 1)$ because modulo the ideal $(x^2 + y^2 - 1)$, it holds true that

$$(x, y - 1)^2 = (x^2, x(y - 1), (y - 1)^2) = (y - 1).$$

With the subspace topology inherited from $\operatorname{Spec} A$, the distinguished open sets are themselves prime spectra:

Lemma 2.26. A distinguished open subset $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$.

Proof Recall that for a multiplicative set $S \subset A$, the map $\mathfrak{p} \mapsto \mathfrak{p}S^{-1}A$ gives an inclusion-preserving bijection between the prime ideals of $S^{-1}A$ and the prime ideals $\mathfrak{p}$ of A satisfying $\mathfrak{p} \cap S = \varnothing$. Applying this to $S = \{1, f, f^2, \dots\}$, we get the lemma. □

For two open sets $D(f)$ and $D(g)$, we have the following implications:

$$\begin{aligned} D(f) \supset D(g) &\implies V(f) \subset V(g) \\ &\implies \sqrt{(f)} \supset \sqrt{(g)} \\ &\implies g \in \sqrt{(f)} \\ &\implies g^r = cf \text{ for some } r \in \mathbb{N}, c \in A \end{aligned}$$

[1] The terminology is a little bit unfortunate; spaces in which every open cover has a finite subcover are usually called 'compact'. However, some authors reserve the term 'compact' for quasi-compact and Hausdorff, and this jargon has caught on in the algebraic geometry literature.

If follows that there is a canonical localization map

$$\begin{aligned} \rho_{fg}\colon A_f &\longrightarrow A_g \\ \frac{a}{f^n} &\longmapsto \frac{ac^r}{g^{rn}}, \end{aligned} \tag{2.4}$$

which is a ring map factoring $A \to A_g$. From this we deduce that $D(g)$ may be identified with a distinguished open subset in $\operatorname{Spec}(A_f)$.

Exercises

Exercise 2.5.1. Show that the ideal $\mathfrak{m} = (x, y-1)$ in $A = \mathbb{R}[x,y]/(x^2+y^2-1)$ is not principal.

Exercise 2.5.2. Show that $D(f) = \varnothing$ if and only if f is nilpotent. HINT: Use that $\sqrt{(0)} = \bigcap_{\mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p}$.

Exercise 2.5.3. Show that $D(f) = D(g)$ if and only if there are integers m, n such that $g^m = u \cdot f^n$ for some unit $u \in A$.

Exercise 2.5.4. Check that for a nested inclusion $D(h) \subset D(g) \subset D(f)$, we have $\rho_{fh} = \rho_{gh} \circ \rho_{fg}$.

Exercise 2.5.5. Let A be a ring, let $\mathfrak{a}$ be an ideal in A and let $\{f_i\}_{i\in I}$ be elements from $\mathfrak{a}$. Show that the open distinguished sets $D(f_i)$ cover $\operatorname{Spec} A - V(\mathfrak{a})$ if and only if some power of each element $f \in \mathfrak{a}$ lies in the ideal generated by the f_i's.

Exercise 2.5.6. Let k be a field and let $A = k[t_0, t_1, \dots]$ be a polynomial ring in countably many variables. Let $\mathfrak{m}$ be the maximal ideal $\mathfrak{m} = (t_0, t_1, \dots)$. Show that $U = \operatorname{Spec} A - \mathfrak{m}$ is not quasi-compact. Conclude that U is not the spectrum of a ring. HINT: Consider the open covering $\{D(t_i)\}_{i\geqslant 0}$.

2.6 Maps between prime spectra

Let A and B be two rings and let $\phi\colon A \to B$ be a map of rings. The inverse image $\phi^{-1}\mathfrak{p}$ of a prime ideal $\mathfrak{p} \subset B$ is a prime ideal: that $ab \in \phi^{-1}\mathfrak{p}$ means that $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, so at least one of $\phi(a)$ or $\phi(b)$ has to lie in $\mathfrak{p}$. Hence sending $\mathfrak{p}$ to $\phi^{-1}\mathfrak{p}$ gives us a well-defined map

$$f = \operatorname{Spec}(\phi)\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A. \tag{2.5}$$

This map is continuous in the Zariski topology, because preimages of closed sets are closed by item (i) in the next proposition.

Proposition 2.27. Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be induced by the ring map $\phi : A \to B$.

(i) $f^{-1}V(\mathfrak{a}) = V(\phi(\mathfrak{a})B)$ for each ideal $\mathfrak{a} \subset A$.
(ii) $f^{-1}D(g) = D(\phi(g))$ for each $g \in A$;
(iii) $\overline{f(V(\mathfrak{b}))} = V(\phi^{-1}\mathfrak{b})$ for each ideal $\mathfrak{b}$ of B.

Proof To prove (i), let $\mathfrak{a} \subset A$ be an ideal. Then we have

$$f^{-1}V(\mathfrak{a}) = \{\, \mathfrak{p} \subset B \mid \mathfrak{a} \subset \phi^{-1}\mathfrak{p} \,\} = \{\, \mathfrak{p} \subset B \mid \phi(\mathfrak{a}) \subset \mathfrak{p} \,\} = V(\phi(\mathfrak{a})B).$$

Indeed, as $\mathfrak{a} \subset \phi^{-1}\phi(\mathfrak{a})$, the inclusion $\phi(\mathfrak{a}) \subset \mathfrak{p}$ holds if and only if $\mathfrak{a} \subset \phi^{-1}\mathfrak{p}$. In particular, the inverse image of any closed subset is again closed, so f is continuous.

For (ii), note that for each element $g \in A$ we have the following equalities:

$$f^{-1}D(g) = \{\, \mathfrak{p} \subset B \mid g \notin \phi^{-1}\mathfrak{p} \,\} = \{\, \mathfrak{p} \subset B \mid \phi(g) \notin \mathfrak{p} \,\} = D(\phi(g)).$$

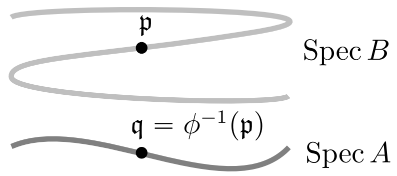
Finally we prove (iii): according to Corollary 2.11 on page 25, the closure $\overline{f(V(\mathfrak{b}))}$ equals $V(\mathfrak{a})$ with $\mathfrak{a}$ the ideal given by

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in f(V(\mathfrak{b}))} \mathfrak{p} = \bigcap_{\mathfrak{b} \subset \mathfrak{q}} \phi^{-1}\mathfrak{q}.$$

The equality holds because $\mathfrak{p} \in f(V(\mathfrak{b}))$ implies that $\mathfrak{p} = \phi^{-1}\mathfrak{q}$ for some $\mathfrak{q}$, with $\mathfrak{b} \subset \mathfrak{q}$. So we get that

$$\mathfrak{a} = \bigcap_{\mathfrak{b} \subset \mathfrak{q}} \phi^{-1}\mathfrak{q} = \phi^{-1}\Big(\bigcap_{\mathfrak{b} \subset \mathfrak{q}} \mathfrak{q}\Big) = \phi^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\phi^{-1}\mathfrak{b}}.$$

Hence $V(\mathfrak{a}) = V(\phi^{-1}\mathfrak{b})$, which gives the desired identity. □



Proposition 2.28. With notation as in Proposition 2.27, if ϕ is surjective, then f induces a homeomorphism from $\operatorname{Spec} B$ onto the closed subset $V(\operatorname{Ker}\phi) \subset \operatorname{Spec} A$. In particular, if $\mathfrak{a} \subset A$ is an ideal, the quotient map $A \to A/\mathfrak{a}$ induces a homeomorphism

$$f\colon\ \operatorname{Spec}(A/\mathfrak{a}A) \xrightarrow{\ \simeq\ } V(\mathfrak{a}) \subset \operatorname{Spec} A$$

Proof If $\phi\colon A \to B$ is surjective, we may assume $B = A/\mathfrak{a}$, where $\mathfrak{a} = \operatorname{Ker}\phi$. The map $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$ gives is an inclusion preserving one-to-one correspondence between prime ideals in $A/\mathfrak{a}$ and prime ideals in A containing $\mathfrak{a}$ (with inverse given by $\mathfrak{q} \mapsto \mathfrak{q}/\mathfrak{a}$). This shows that f is a continuous bijection onto the closed subset $V(\mathfrak{a})$. To show that f is a homeomorphism, it suffices to show that it is closed; this follows from the equalities

$$f(V(\mathfrak{b}/\mathfrak{a})) = \{\mathfrak{p} \in \operatorname{Spec} A | \mathfrak{b}/\mathfrak{a} \subset \mathfrak{p}/\mathfrak{a} \in \operatorname{Spec}(A/\mathfrak{a})\} = V(\mathfrak{b}). \qquad \square$$

The map $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$ is the standard example of a *closed embedding*. We will discuss these in more detail later.

Proposition 2.29. With notation as in Proposition 2.27, if ϕ is injective, then $f(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$. In fact, the image $f(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$ if and only if $\operatorname{Ker} \phi \subset \sqrt{(0)}$.

Proof Again, by (iii) of Proposition 2.27, the closure of $f(\operatorname{Spec} B) = f(V(0))$ equals $V(\phi^{-1}(0)) = V(\operatorname{Ker} \phi)$. So $f(\operatorname{Spec} B)$ is dense if and only if $V(\operatorname{Ker} \phi) = \operatorname{Spec} A$. But this happens exactly when $\operatorname{Ker} \phi \subset \mathfrak{p}$ for all $\mathfrak{p}$, or equivalently when $\operatorname{Ker} \phi \subset \sqrt{(0)}$. $\square$

Examples

Example 2.30 (Reduction modulo a prime p)**.** The reduction mod p-map $\mathbb{Z} \to \mathbb{F}_p$ induces a map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. The one and only point in $\operatorname{Spec} \mathbb{F}_p$ is sent to the point in $\operatorname{Spec} \mathbb{Z}$ corresponding to the maximal ideal (p). The inclusion $\mathbb{Z} \subset \mathbb{Q}$ of the integers in the field of rational numbers induces likewise a map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$, that sends the unique point in $\operatorname{Spec} \mathbb{Q}$ to the generic point η of $\operatorname{Spec} \mathbb{Z}$.

Example 2.31 (The circle)**.** Consider $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. The ring map

$$\phi\colon \mathbb{R}[x, y]/(u^2 + v^2 - 1) \to \mathbb{R}[x, y]/(x^2 + y^2 - 1) \tag{2.6}$$
$$u \mapsto x^2 - y^2 \tag{2.7}$$
$$v \mapsto 2xy$$

(originating from the 'squaring map' $z \mapsto z^2$) induces a map of spectra $f\colon X \to X$.

Example 2.32 (The twisted cubic)**.** Let k be a field. The ring map $\phi\colon k[x, y, z] \to k[t]$ given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ defines a map of prime spectra

$$f\colon \mathbb{A}^1_k \longrightarrow \mathbb{A}^3_k.$$

The image of f is the *twisted cubic curve* $V(\mathfrak{a}) \subset \mathbb{A}^3_k$ defined by the ideal $\mathfrak{a} = \operatorname{Ker} \phi = (y - x^2, z - x^3)$.

Example 2.33. Let k be a field. The ring map $\phi\colon k[x] \to k[x, y]/(xy - 1)$ induces a morphism

$$\operatorname{Spec} k[x, y]/(xy - 1) \longrightarrow \mathbb{A}^1_k.$$

On the level of closed points, when k is algebraically closed, this maps (a, a^{-1}) to a. Since $k[x, y]/(xy - 1)$ is an integral domain, it has a unique generic point η, and this is mapped to the generic point of $\mathbb{A}^1_k$. Note that $\operatorname{Spec} k[x, y]/(xy - 1) \simeq D(x) \subset \mathbb{A}^1_k$ via this morphism. In particular, the image is not closed in $\mathbb{A}^1_k$.

Exercise 2.6.1. Let $\phi : A \to B$ and $\psi : B \to C$ be two ring maps. Show that, with the notation of (2.5), $\operatorname{Spec}(\psi \circ \phi) = \operatorname{Spec}(\phi) \circ \operatorname{Spec}(\psi)$.

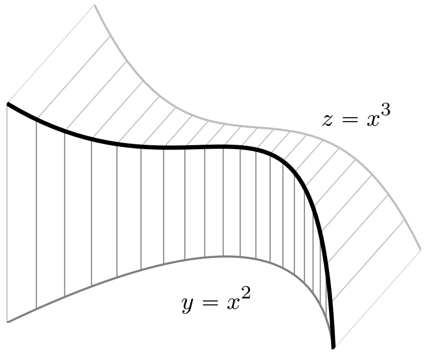


Figure 2.1 The twisted cubic curve

2.7 Scheme-theoretic fibres I

To understand a map of spectra $f : \operatorname{Spec} B \to \operatorname{Spec} A$, it is often useful to understand the inverse images of points, the fibres of f.

Let $\phi : A \to B$ be the ring map that induces f and let $y \in \operatorname{Spec} A$ be a point, corresponding to a prime ideal $\mathfrak{p}$ in A. We would like to understand the fibee $f^{-1}(y)$, i.e., the set of prime ideals $\mathfrak{q}$ in B such that $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.

If $y \in Y$ is a closed point, that is, $\mathfrak{p}$ is a maximal ideal, then we saw in Proposition 2.27 that the inverse image $f^{-1}V(\mathfrak{m})$ is equal to the closed set $V(\mathfrak{m}B)$. In other words, the fibre $f^{-1}(y)$ is homeomorphic to $\operatorname{Spec}(B/\mathfrak{m}B)$.

In general, the fibre $f^{-1}(y)$ may or may not be closed in $\operatorname{Spec} B$. The inverse image of the closure $V(\mathfrak{p})$ of y still equals $V(\mathfrak{p}B)$, but this set may contain other primes than the ones mapping to y. For instance, in the sitation when $\mathfrak{p} = (0)$ is prime, then $V(\mathfrak{p}B) = \operatorname{Spec} B$.

To remedy this, we consider the localization $(B/\mathfrak{p}B)_\mathfrak{p}$ of the A-module $B/\mathfrak{p}B$ in the multiplicative set $S = A - \mathfrak{p}$. Note the general equality $(B/\mathfrak{p}B)_\mathfrak{p} = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, and that we have canonical ring maps $B \to B/\mathfrak{p}B \to (B/\mathfrak{p}B)_\mathfrak{p}$. These induce maps of spectra:

$$\iota\colon\ \operatorname{Spec} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \longrightarrow \operatorname{Spec} B/\mathfrak{p}B \longrightarrow \operatorname{Spec} B. \tag{2.8}$$

Proposition 2.34. The map (2.8) induces a homeomorphism between $\operatorname{Spec} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ and the fibre $f^{-1}(\mathfrak{p}) \subset \operatorname{Spec} B$. In particular, if $\mathfrak{p} \in \operatorname{Spec} A$ is a closed point, then $f^{-1}(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$.

Proof Note the equalities

$$\{\, \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{p} \subset \phi^{-1}\mathfrak{q} \,\} = \{\, \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{p}B \subset \mathfrak{q} \,\} = V(\mathfrak{p}B).$$

In the particular case that $\mathfrak{p}$ is a maximal ideal, the inclusion $\mathfrak{p} \subset \phi^{-1}\mathfrak{q}$ must be an equality, and the sets above describe the fibre

$$f^{-1}(\mathfrak{p}) = \{\, \mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{p}B \subset \mathfrak{q} \,\} = V(\mathfrak{p}B).$$

The closed subset $V(\mathfrak{p}B)$ of $\operatorname{Spec} B$ with induced topology from $\operatorname{Spec} B$ is canonically homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$. Thus we have a homeomorphism between $\operatorname{Spec}(B/\mathfrak{p}B)$ and the fibre $f^{-1}(\mathfrak{p})$.

If $\mathfrak{p}$ is not a maximal ideal, the set $\operatorname{Spec}(B/\mathfrak{p}B)$ might, as we observed above, be bigger than the fibre. The extra prime ideals are those $\mathfrak{q}$ for which the inclusion $\mathfrak{p} \subset \phi^{-1}\mathfrak{q}$ is strict. That means that there exist elements $s \in S = A - \mathfrak{p}$ so that $\phi(s) \in \mathfrak{q}$. It follows that if we localize with respect to S, these extra prime ideals will cease being proper, because they contain invertible elements. It follows that the points in the fibre $f^{-1}(\mathfrak{p})$ correspond exactly to the primes in the localized ring $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$. Since this correspondence respects inclusions, the Zariski topology on the spectrum $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ coincides with the one induced from the Zariski topology of $\operatorname{Spec}(B/\mathfrak{p}B)$, and hence we get the statement in general. □

Example 2.35. Consider the map

$$f\colon \operatorname{Spec}\mathbb{C}[x,y,z]/(xy-z) \longrightarrow \operatorname{Spec}\mathbb{C}[z],$$

induced by the canonical map $\phi\colon \mathbb{C}[z] \to \mathbb{C}[x,y,z]/(xy-z) = B$. Let us compute the fibres $f^{-1}(\mathfrak{p})$ over the maximal ideals $\mathfrak{p} = (z-a)$. Note that

$$B/\mathfrak{p}B = \mathbb{C}[x,y,z]/(xy-z, z-a) \simeq \mathbb{C}[x,y]/(xy-a).$$

There are two cases. If $a \neq 0$, then $xy-a$ is an irreducible polynomial, and so $\operatorname{Spec} B/\mathfrak{p}B$ is irreducible. This is intuitive, as it corresponds to the hyperbola $V(xy-a)$ in $\mathbb{A}^2_\mathbb{C}$. If $a = 0$, we are left with $\operatorname{Spec}\mathbb{C}[x,y]/(xy)$, which is not irreducible; it has two components, the two coordinate axes $V(x)$ and $V(y)$.

Let us also consider the fibre over the generic point η of $\operatorname{Spec}\mathbb{C}[z]$, which corresponds to $\mathfrak{p} = (0)$. In this case, the ring $(B/\mathfrak{p}B)_\mathfrak{p}$ is the localization of B with respect to the multiplicative set $S = \mathbb{C}[z] - \{0\}$; that is, the ring

$$\mathbb{C}(z)[x,y]/(xy-z).$$

This is again an integral domain, so the fibre $f^{-1}(\eta)$ is irreducible. This fibre may be regarded as a hyperbola in the affine plane $\mathbb{A}^2_{\mathbb{C}(z)}$ over the field $\mathbb{C}(z)$.

Example 2.36. Let k be a field and consider the map

$$f\colon X = \operatorname{Spec} k[x,y]/(x-y^2) \longrightarrow \operatorname{Spec} k[x]$$

induced by the injection $k[x] \to k[x,y]/(x-y^2)$. Geometrically one would say this is just the projection of the 'horizontal' parabola onto the x-axis.

If $a \in k$, the fibre $f^{-1}(\mathfrak{p})$ over the maximal ideal $\mathfrak{p} = (x-a)$ is the spectrum of the ring

$$B/\mathfrak{p}B = k[x,y]/(x-y^2, x-a) \simeq k[y]/(y^2-a).$$

Let us first assume that k has characteristic different from 2. Several cases can occur:

(i) If $a \neq 0$ and has a square root in k, say $b^2 = a$, the polynomial $y^2 - a$ factors as $(y-b)(y+b)$, and by the Chinese Remainder theorem, the fibre becomes the product

$$\operatorname{Spec}\big(k[y]/(y-b) \times k[y]/(y+b)\big),$$

which is the disjoint union of two copies of $\operatorname{Spec} k$.

(ii) If $a \neq 0$, but does not have a square root in k, then the fibre equals $\operatorname{Spec} k(\sqrt{a})$, where $k(\sqrt{a})$ is a quadratic field extension of k. The fibre is a single point, but with 'multiplicity two' (in the sense that the degree of the field extension $k \subset k(\sqrt{a})$ is 2).

(iii) The final case is when $a = 0$. The fibre then equals $\operatorname{Spec} k[y]/(y^2)$, which is just a single point, but again there is a 'multiplicity two', accounted for by the presence of nilpotent elements in the ring (as vector space over k the ring $k[y]((y^2)$ has dimension two).

Example 2.37 (The Möbius Strip)**.** Consider the $\mathbb{R}$-algebra $A = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ and the circle $S = \operatorname{Spec} A$. There is a map

$$f\colon \operatorname{Spec} A[u, v]/(vx - uy) \longrightarrow S.$$

Let us compute the scheme theoretic fibres of f. Note that S is covered by the two affine subsets $D(x)$ and $D(y)$. If $\mathfrak{p} \in D(x)$, then x is invertible in $A_{\mathfrak{p}}$, and so, writing $B = A[u, v]/(vx - uy)$, we find

$$B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = (A_{\mathfrak{p}}/\mathfrak{p})[u, v]/(v - x^{-1}uy) \simeq k(\mathfrak{p})[u].$$

Hence the scheme theoretic fibre is isomorphic to $\mathbb{A}^1_{k(\mathfrak{p})}$. A similar argument works when $\mathfrak{p} \in D(y)$. Hence all fibres are isomorphic to affine lines.

Example 2.38. Consider the map

$$\pi : \operatorname{Spec} \mathbb{C}[t] \longrightarrow \operatorname{Spec} \mathbb{R}[t]$$

induced by the inclusion $\mathbb{R}[t] \subset \mathbb{C}[t]$. By Example 2.15 there are three cases to consider for the fiber $\pi^{-1}(y)$ of a point $y \in \mathbb{A}^1_{\mathbb{R}}$.

i) y corresponds to the maximal ideal $(t - a)$ where $a \in \mathbb{R}$. Then the fiber is given by

$$\pi^{-1}(y) = \operatorname{Spec}(\mathbb{C}[t]/(t - a)) \simeq \operatorname{Spec} \mathbb{C}.$$

Therefore the fibre is a single closed point with residue field $\mathbb{C}$.

ii) y corresponds a closed point corresponding to $\mathfrak{p} = (f(t))$ where $f \in \mathbb{R}[t]$ has two conjugate complex roots $a, \bar{a}$. Then

$$\pi^{-1}(y) = \operatorname{Spec}(\mathbb{C}[t]/(f(t))) \simeq \operatorname{Spec}(\mathbb{C}[t]/(t - a) \times \mathbb{C}[t]/(t + a))$$

Thus the fibre consists of two closed points, with residue fields $\mathbb{C}$.

iii) $y = \eta$ is the generic point. Then $f^{-1}(\eta)$ is given by the spectrum of the localization $S^{-1}\mathbb{C}[t] = \mathbb{C}(t)$ where $S = \mathbb{C}[t] - (0)$. In other words, $\pi^{-1}(\mathbb{C}(t)) = \mathbb{A}^1_{\mathbb{C}(t)}$ is the affine line over $\mathbb{C}(t)$.

The Galois group $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$ acts on the fibres of this example. More precisely, consider the conjugation map on the polynomial ring $\mathbb{C}[t]$ given by conjugating the coefficients of the polynomials; that is, sending a polynomial $\sum_i a_i t^i$ to $\sum_i \bar{a}_i t^i$. This defines an automorphism

$$\iota : \operatorname{Spec} \mathbb{C}[t] \longrightarrow \operatorname{Spec} \mathbb{C}[t]$$

Note that the sub-ring $\mathbb{R}[t] \subset \mathbb{C}[t]$ is unaltered by the conjugation map, so the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[t] & \xrightarrow{\quad \iota \quad} & \operatorname{Spec}\mathbb{C}[t] \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \operatorname{Spec}\mathbb{R}[t] & \end{array}$$

Thus $G = \langle \mathrm{id}, \iota \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts by automorphisms on the fibres of π. From this we see that $\operatorname{Spec}\mathbb{R}[t]$ can be viewed as the quotient space of $\operatorname{Spec}\mathbb{C}[t]$ by G, i.e., the space that parameterizes G-orbits. Indeed, by Example 2.15, the closed points of $\operatorname{Spec}\mathbb{R}[t]$ correspond exactly to the orbits of G and the generic point of $\operatorname{Spec}\mathbb{C}[t]$ is invariant and corresponds to the generic point of $\operatorname{Spec}\mathbb{R}[t]$.

Example 2.39 (The Gaussian integers)**.** The inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$ induces a morphism

$$f\colon \operatorname{Spec}\mathbb{Z}[i] \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

We will study $\operatorname{Spec}(\mathbb{Z}[i])$ by studying the fibres of this map. If $p \in \mathbb{Z}$ is a prime, the fibre over $(p)\mathbb{Z}$ consists of those primes that contain $(p)\mathbb{Z}[i]$. There are three cases:

(i) p stays prime in $\mathbb{Z}[i]$, and the fibre over $(p)\mathbb{Z}$ has one element, namely the prime ideal $(p)\mathbb{Z}[i]$. This happens if and only if $p \equiv 3 \mod 4$; [2]

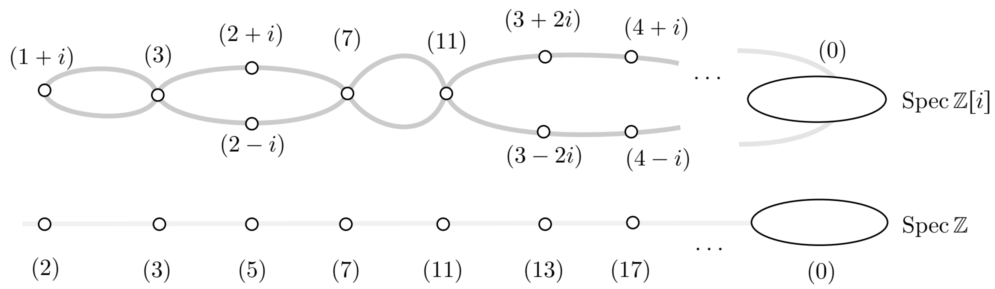
(ii) p splits into a product of two different primes, and the fibre consists of the corresponding two prime ideals. This happens if and only if $p \equiv 1 \mod 4$;

(iii) p factors into a product of repeated primes (such a prime is said to ‘ramify’). This happens only at the prime (2):

$$(2)\mathbb{Z}[i] = (2i)\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i].$$

This is not radical, and the fibre consists of the single prime $(1+i)\mathbb{Z}[i]$.

The following picture shows $\operatorname{Spec}\mathbb{Z}[i]$:



The spectrum $\operatorname{Spec}\mathbb{Z}[i]$

The Galois group $G = \operatorname{Gal}(\mathbb{Q}[i]/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $\operatorname{Spec}\mathbb{Z}[i]$. This group is generated by the complex conjugation map, which permutes the prime ideals in $\operatorname{Spec}(\mathbb{Z}[i])$ sitting over any (p) in $\operatorname{Spec}(\mathbb{Z})$. So for instance, if you look at the primes sitting over say (5), namely

[2] This is related to being able to write p as a sum of squares; if $p = x^2 + y^2$, then $p = (x+iy)(x-iy)$, so it is not prime in $\mathbb{Z}[i]$.

$(2+i)$ and $(2-i)$, you see that complex conjugation maps one into the other. Thus we picture $\operatorname{Spec}(\mathbb{Z}[i])$ as some curve lying above $\operatorname{Spec}(\mathbb{Z})$, with G permuting the points in each fibre (though some are fixed by G).

Example 2.40 (The affine line $\mathbb{A}^1_{\mathbb{Z}}$)**.** Consider the affine line $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$ and the morphism $f\colon \operatorname{Spec}\mathbb{Z}[t] \to \operatorname{Spec}\mathbb{Z}$ induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[t]$.

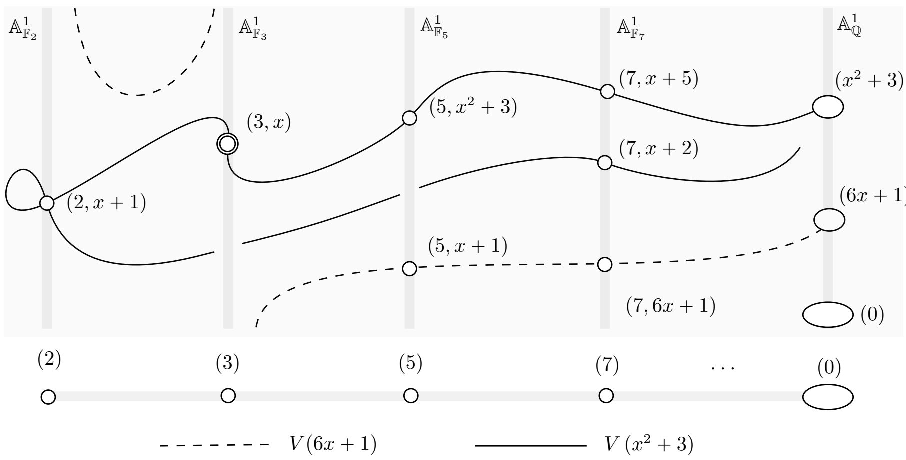
There are two cases for a fibre $f^{-1}(y)$ of a point $y \in \operatorname{Spec}\mathbb{Z}$. If y corresponds to the closed point $(p) \in \operatorname{Spec}\mathbb{Z}$, the fibre $f^{-1}(y)$ consists of all primes $\mathfrak{p} \subset \mathbb{Z}[t]$ such that $\mathfrak{p} \cap \mathbb{Z} = (p)$. According to Proposition 2.34, it is given by

$$V((p)\mathbb{Z}[t]) = \operatorname{Spec}(\mathbb{Z}[t]/p\mathbb{Z}[t]) = \mathbb{A}^1_{\mathbb{F}_p}.$$

Likewise, if $y = \eta$ is the generic point of $\operatorname{Spec}\mathbb{Z}$, corresponding to (0), Proposition 2.34 tells us that the fibre $f^{-1}(\eta)$ is the Spec of the localization $S^{-1}\mathbb{Z}[t] = \mathbb{Q}[t]$, where $S = \mathbb{Z} - 0$. In other words,

$$f^{-1}(\eta) = \operatorname{Spec}\mathbb{Q}[t] = \mathbb{A}^1_{\mathbb{Q}}.$$

The situation is shown in the figure below:



In the figure, we have depicted the two closed sets $V(6x+1)$ and $V(x^2+3)$. Note that $V(6x+1)$ is disjoint from the fibres above the primes 2 and 3 (why?). The closed subset $V(x^2+3)$ should be compared to Example 2.39).

Example 2.41 (The polynomial ring over a DVR)**.** Let A be a discrete valuation ring, such as the localization $\mathbb{Z}_{(p)}$. As in Example 2.10, we have $\operatorname{Spec} A = \{x, \eta\}$ with x a closed point (corresponding to a maximal ideal $\mathfrak{m}$) and η an open point (corresponding to (0)).

Consider the map $f\colon \operatorname{Spec} A[t] \to \operatorname{Spec} A$ corresponding to the inclusion $A \subset A[t]$. There are two fibres to consider, a closed fibre $f^{-1}(x)$ and an open fibre $f^{-1}(\eta)$.

The closed fibre consists of the primes $\mathfrak{p} \subset A[t]$ such that $\mathfrak{p} \cap A = \mathfrak{m}$. Writing $k = A/\mathfrak{m}$ for the residue field at x, we find using Proposition 2.34, that $f^{-1}(x)$ equals

$$V(\mathfrak{m}A[t]) = \operatorname{Spec}(A[t]/\mathfrak{m}A[t]) \simeq \operatorname{Spec} k[t].$$

Hence the fibre $f^{-1}(x)$ is homeomorphic to the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[t]$.

Using Proposition 2.34 again, we find that the open fibre is equal to $\operatorname{Spec} S^{-1}A[t]$ where

$S = A - (0)$. If we write $K = S^{-1}A$ for the fraction field of A, we have $S^{-1}A[t] = K[t]$ and so $f^{-1}(\eta)$ is isomorphic to the affine line $\mathbb{A}^1_K$.

Exercises

Exercise 2.7.1. In the same vein as Example 2.30, show that a ring A is a $\mathbb{Q}$-algebra (that is, it contains a copy of $\mathbb{Q}$) if and only if the canonical map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ factors through the generic point $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$.

Exercise 2.7.2. Show that the Zariski topology on $\operatorname{Spec} A$ is Hausdorff if and only if every prime ideal $\mathfrak{p}$ is maximal.

Exercise 2.7.3. Show that the closed points in $\operatorname{Spec} A$ form a dense set if and only if $\sqrt{0}$ equals the intersection $\bigcap_{\mathfrak{m} \subset A} \mathfrak{m}$ of all maximal ideals in A. HINT: Corollary 2.11 on page 25.

Exercise 2.7.4. Let A be an integral domain and $U \subset \operatorname{Spec} A$ an open non-empty subset. Show that there is no closed point in U if and only if there is an $f \in A$ such that A_f is a field. HINT: Consider distinguished open subsets $D(f) \subset U$.

Exercise 2.7.5. Let $\{A_i\}_{i \in I}$ be an infinite sequence of non-trivial rings, and let X be the disjoint union of the spectra $\operatorname{Spec} A_i$. Show that X is not homeomorphic to a spectrum of a ring.

Exercise 2.7.6 (Local rings)**.** Recall that a *local ring* is a ring A with only one maximal ideal.

a) Show that A is local if and only if $\operatorname{Spec} A$ has a unique closed point.
b) Give examples of local rings A so that $\operatorname{Spec} A$ consists of (i) one point; (ii) two points; (iii) infinitely many points.
c) A map of rings $\phi\colon A \to B$ is said to be local if $\phi(\mathfrak{m}_A) \subset \mathfrak{m}_B$. Show that ϕ is local if and only if the induced map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ maps the unique closed point of $\operatorname{Spec} B$ to that of $\operatorname{Spec} A$.
d) Give an example of a map of rings $f\colon A \to B$ which is not local. Describe your example in terms of the corresponding map on spectra.

Exercise 2.7.7. Show that $\operatorname{Spec} A$ has just one element if and only if A is a local ring all whose non-units are nilpotent, i.e. the radical $\sqrt{(0)}$ of the ring is a maximal ideal. For Noetherian rings this is equivalent to the ring being an Artinian local ring.

Exercise 2.7.8. Perform the analysis of the fibres of the map in Example 2.36 on page 35 when the field k has characteristic two.

Exercise 2.7.9.

a) Let A be a Noetherian ring such that $\operatorname{Spec} A$ is a finite set. Show that A is Artinian.
b) Show that the ring $A = \mathbb{C}[t_1, t_2, \ldots]/\mathfrak{m}^2$ where $\mathfrak{m} = (t_1, t_2, \ldots)$ is not Noetherian, but that $\operatorname{Spec} A$ is a single point.

Exercise 2.7.10. With reference to Example 2.39 on page 37:

a) Show that the fibre of ϕ over a prime ideal (p) is homeomorphic to

$$\operatorname{Spec} \mathbb{F}_p[x]/(x^2+1)$$

and that $\dim_{\mathbb{F}_p} \mathbb{F}_p[x]/(x^2+1)$ =2. HINT: Use that $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$.

b) Show that $\mathbb{F}_p[x]/(x^2+1)$ is a field if and only if x^2+1 does not have a root in $\mathbb{F}_p$.

c) Show that $\mathbb{F}_p[x]/(x^2+1)$ is a field if and only if $(p)\mathbb{Z}[i]$ is a prime ideal.

Exercise 2.7.11. Consider the ring map

$$\phi\colon \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y,z]/(xz-y)$$

which induces $f\colon \operatorname{Spec} \mathbb{C}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbb{C}}$. Show that the map f on the level of closed points sends (a, ab, b) to (a, ab), and the generic point to the generic point. Show that in this example, the image is neither open nor closed: it equals $D(x) \cup V(x,y)$.

Exercise 2.7.12. Let p and q be two different prime numbers and consider the morphism $\phi\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ induced from the map $k[x,y] \to k[t]$ which is defined by the assignments $x \mapsto t^p$ and $y \mapsto t^q$. Determine all scheme theoretic fibres of ϕ.

Exercise 2.7.13. Let k an algebraically closed field. Consider the k-algebra $A = k[x,y,z]/(xy,xz,yz)$ and let $X = \operatorname{Spec} A$. Consider the map $f\colon X \to \mathbb{A}^1$ dual to the k-algebra homomorphism $k[t] \to A$ that sends t to $x+y+z$. Determine all scheme theoretic fibres of f. HINT: Heuristics: $X(\mathbb{C})$ is the union of the three coordinate axes in $\mathbb{C}^3$, and the map sends points in $X(\mathbb{C})$ to the sum of their coordinates.

Exercise 2.7.14. Describe the scheme theoretic fibres in all points of the following morphisms.

a) $f\colon \operatorname{Spec} \mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec} \mathbb{C}[x]$;

b) $f\colon \operatorname{Spec} \mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec} \mathbb{C}[x]$;

c) $f\colon \operatorname{Spec} \mathbb{C}[x,y]/(xy) \to \operatorname{Spec} \mathbb{C}[x]$;

d) $f\colon \operatorname{Spec} \mathbb{Z}[x,y]/(xy^2-m) \to \operatorname{Spec} \mathbb{Z}$, where m is a non-zero integer.

Exercise 2.7.15. Determine all the scheme theoretic fibres of the morphism

$$\operatorname{Spec} \mathbb{Z}[(1+\sqrt{5})/2] \longrightarrow \operatorname{Spec} \mathbb{Z}[\sqrt{5}]$$

induced by the natural inclusion $\mathbb{Z}[\sqrt{5}] \subset \mathbb{Z}[(1+\sqrt{5})/2]$.

Exercise 2.7.16. Let $R = \mathbb{Z}[x,y]/(x^2-y^2-5)$ and consider the morphism $f\colon \operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$. Compute the fibres over (0), (2), (3) and (5). What happens if you replace R with the ring $\mathbb{Z}[x,y]/(3x^2-3y^2-15)$?

Exercise 2.7.17. For every ring A, there is a canonical map $\mathbb{Z} \to A$ which sends 1 to 1. Hence there is a canonical map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. Show that map factors through the canonical map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ if and only if A is of characteristic p,

Exercise 2.7.18. Describe the following prime spectra

a) $\operatorname{Spec} \mathbb{C}[x]/(x^3+x^2)$

b) $\operatorname{Spec} \mathbb{R}[x]/(x^3+x^2)$

Exercise 2.7.19. Study the fibres of the morphisms

a) $\operatorname{Spec} \mathbb{Z}[t]/(t^2 + t + 1) \to \operatorname{Spec} \mathbb{Z}$

b) $\operatorname{Spec} \mathbb{Q}(t)[x]/(x^3 + 3x + 1) \to \operatorname{Spec} \mathbb{Q}(t)$.

3

Sheaves

The concept of a sheaf was conceived in the German camp for prisoners of war called Oflag XVII, where French officers taken captive during the fighting in France in the spring 1940 were detained. Among them was the mathematician and lieutenant Jean Leray. In the camp he gave a series of lectures on algebraic topology(!!) during which he introduced some version of the theory of sheaves. In modern terms, Leray was aiming to compute the cohomology of a total space of a fibration in terms of invariants of the base and the fibres and the fibration itself. To achieve this, in addition to the concept of sheaves, he also invented 'spectral sequences'.

After the war, the theory of sheaves was developed further by Henri Cartan and Jean-Pierre Serre, and finally the theory was brought to the state as we know it today by Alexander Grothendieck.

3.1 Sheaves and presheaves

A common theme in mathematics is to study spaces by describing them in terms of their local properties. A manifold is a space which looks locally like Euclidean space; a complex manifold is a space which looks locally like open sets in $\mathbb{C}^n$; an algebraic variety is a space that looks locally like the zero set of a set of polynomials. Here it is clear that point set topology alone is not enough to fully capture the essence of these three notions. However, in each case, the spaces come equipped with a distinguished set of functions that adequately define them, respectively the C^∞-functions, the holomorphic functions, and the polynomials.

Sheaves provide a general framework for discussing such functions; they are objects that satisfy basic axioms valid in each of the examples above. To explain what these axioms are, let us consider the primary example of a sheaf: the sheaf of continuous maps on a topological space X. By definition, X comes with a collection of 'open sets', and these encode what it means for a map $f\colon X \to Y$ to another topological space Y to be continuous: for every open $U \subset Y$, the set $f^{-1}(U)$ should be open in X. For two topological spaces X and Y, we can define, for each open $U \subset X$, a set of continuous maps

$$C(U, Y) = \{\, f\colon U \to Y \mid f \text{ is continuous} \,\}.$$

Note that if $V \subset U$ is another open set, then the restriction $f|_V$ to V of a continuous function f is again continuous, so we obtain a map

$$\begin{aligned} \rho_{UV}\colon\ C(U, Y) &\longrightarrow C(V, Y) \\ f &\mapsto f|_V. \end{aligned} \tag{3.1}$$

Moreover, note that if $W \subset V \subset U$, we can restrict to W by first restricting to V, and so

$\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. The collection of the sets $C(U, Y)$ together with their restriction maps ρ_{UV} constitutes the *sheaf of continuous maps from X to Y*.

An essential feature of continuity is that it is a local property; f is continuous if and only if it is continuous in a neighbourhood of every point, and of course, two continuous maps that are equal in a neighbourhood of every point, are (tautologically) equal everywhere. A second property is that continuous functions can be glued together: given an open covering $\{U_i\}_{i \in I}$ of an open set U, and continuous functions $f_i \in C(U_i, Y)$ that agree on the intersections $U_i \cap U_j$ (formally: $f_i(x) = f_j(x)$ for all i and j and all $x \in U_i \cap U_j$), we can patch the maps f_i together to form a continuous map $f : U \to Y$, which satisfies $f|_{U_i} = f_i$ for each i; we simply define $f(x) = f_i(x)$ for any i such that $x \in U_i$.

Essentially, a *sheaf* on a topological space is a structure that encodes these properties. In each of the examples above, there is a corresponding sheaf of C^∞-functions, respectively holomorphic functions, and regular functions .

One may think of a sheaf as a collection of distinguished sets of functions, but they can also be much more general mathematical objects, which in a certain sense behave as sets of functions. The main aspect is that we want the distinguished properties to be preserved under restrictions to open sets, that the objects are determined from their local properties, and that 'gluing' is allowed.

Presheaves

The concept of a sheaf may be defined for any topological space, and the theory is best studied at this level of generality. We begin with the definition of a *presheaf*.

Definition 3.1 (Presheaf)**.** Let X be a topological space. A *presheaf of abelian groups* $\mathcal{F}$ on X consists of the following two sets of data:

(i) For each open $U \subset X$, an abelian group $\mathcal{F}(U)$;

(ii) For each pair of nested opens $V \subset U$, a map of groups

$$\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V);$$

These are called *restriction maps* and must satisfy the following two conditions:

(iii) For any open $U \subset X$, we have $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$;

(iv) For any three nested open subsets $W \subset V \subset U$, one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We will usually write $s|_V$ for $\rho_{UV}(s)$ when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called *sections* (or *sections over* U). The notation $\Gamma(U, \mathcal{F})$ for the group $\mathcal{F}(U)$ is also common usage; here Γ is the 'global sections'-functor (it is functorial in both U and $\mathcal{F}$).

The notion of a presheaf is not confined to presheaves of abelian groups. One may speak about presheaves of sets, rings, vector spaces etc. Indeed, for any category C one may define presheaves with values in C. The definition is essentially the same as for presheaves of abelian groups, the only difference being that one requires that the $\mathcal{F}(U)$ are objects from C, and of course, that restriction maps are all morphisms in C. We are certainly going to meet sheaves with more structure than the mere structure of abelian groups, e.g. sheaves of rings, but they will usually have an underlying structure of abelian group, so we start with these. We will also encounter sheaves of sets. Most of the results we establish for sheaves of abelian groups can be proved for sheaves of sets as well, as long as they can be formulated in terms of sets, and the proofs are essentially the same.

Sheaves

We are now ready to give the main definition of this chapter:

Definition 3.2 (Sheaf)**.** A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the two conditions:

(i) (Locality axiom) Suppose $U \subset X$ is an open set with an open covering $\mathcal{U} = \{U_i\}_{i \in I}$. If $s, t \in \mathcal{F}(U)$ are sections such that

$$s|_{U_i} = t|_{U_i}$$

for all i, then $s = t$;

(ii) (Gluing axiom) If U and $\mathcal{U}$ are as in (i), and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections that satisfy

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$ for all i.

These two axioms mirror the properties of continuous functions mentioned in the introduction. The Locality axiom says that sections are uniquely determined from their restrictions to smaller open sets. The Gluing axiom says that you are allowed to patch together local sections to a global one, provided they agree on overlaps.

A presheaf $\mathcal{G}$ is a *subpresheaf* of a presheaf $\mathcal{F}$ if $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open $U \subset X$, and such that the restriction maps of $\mathcal{G}$ are the restrictons of those of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $\mathcal{G}$ is naturally called a *subsheaf*.

There is a convenient way of formulating the two sheaf axioms at once. For each open cover $\mathcal{U} = \{U_i\}$ of an open set $U \subset X$, there is a sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{F}(U_i \cap U_j), \tag{3.2}$$

where the maps α and β are defined by the two assignments $\alpha(s) = (s|_{U_i})_i$, and $\beta(s_i) = (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i,j}$. Then $\mathcal{F}$ is a sheaf if and only if these sequences are exact. Indeed, exactness at $\mathcal{F}(U)$ means that α is injective, i.e. that $s|_{U_i} = 0$ for all i implies that $s = 0$ (this is equivalent to the Locality axiom). Exactness in the middle means that $\operatorname{Ker}\beta = \operatorname{Im}\alpha$; that is, elements s_i satisfying $s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$ come from an element $s \in \mathcal{F}(U)$ (the Gluing axiom).

This reformulation is sometimes handy when proving that a given presheaf is a sheaf. Moreover, since $\mathcal{F}(U) = \operatorname{Ker}\beta$, we can often use it to compute $\mathcal{F}(U)$ if the $\mathcal{F}(U_i)$'s and the $\mathcal{F}(U_i \cap U_j)$'s are known.

Example 3.3 (The empty set)**.** There is a subtle point about taking U to be the empty set in the definition of a sheaf. If $\mathcal{F}$ is a sheaf, we are forced to define $\mathcal{F}(\varnothing) = 0$. Indeed, the empty set is covered by the empty open covering, and since the empty product equals 0, the sheaf sequence (3.2) takes the form $0 \to \mathcal{F}(\varnothing) \to 0 \to 0$.

Morphisms between (pre)sheaves

A *morphism* (or simply a *map*) *of (pre)sheaves* $\phi\colon \mathcal{F} \to \mathcal{G}$ is a collection of maps of abelian groups $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open set in X, which are required to be compatible with the restriction maps. In other words, the following diagram commutes for each inclusion $V \subset U$ of open sets:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow[\phi_U]{} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V). \end{array} \tag{3.3}$$

In this way, the sheaves of abelian groups on X form a category, AbSh_X, whose objects are the sheaves and whose are morphisms the maps between them. The composition of two maps of sheaves is defined in the obvious way, as the composition of the maps on sections. Likewise, we have the category $\mathsf{AbPrSh}_\mathsf{X}$ with the presheaves of abelian groups as objects and morphisms the maps between them.

As usual, a map ϕ between two (pre)sheaves $\mathcal{F}$ and $\mathcal{G}$ is an *isomorphism* if it has a two-sided inverse, i.e. a map $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\phi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \phi = \mathrm{id}_{\mathcal{F}}$.

Examples

Example 3.4 (Continuous functions)**.** Take $X = \mathbb{R}^n$ and let $C(X, \mathbb{R})$ be the sheaf whose sections over an open set U is the ring of continuous real valued functions on U, and whose restriction maps ρ_{UV} are just the good old restriction of functions. Then $C(X, \mathbb{R})$ is a sheaf of rings (functions can be added and multiplied), and both sheaf axioms are satisfied. Indeed, any function $f\colon X \to \mathbb{R}$ which restricts to zero on an open covering of X is the zero function. Also, given continuous functions $f_i\colon U_i \to \mathbb{R}$ that agree on the overlaps $U_i \cap U_j$, we can form the continuous function $f\colon U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ for any i such that $x \in U_i$.

In fact, the argument from the beginning of this chapter shows that for any two topological spaces X and Y, the presheaf $\mathcal{F}(U) = C(U, Y)$ of continuous maps $f\colon U \to Y$ forms a sheaf (they are sheaves of sets, because we cannot in general add or multiply maps).

Example 3.5 (Differential operators)**.** Let $X = \mathbb{R}$ and let $C^r(X, \mathbb{R})$ be the sheaf of functions $f\colon U \to \mathbb{R}$ which are r times continuously differentiable (note that this is a subsheaf of $C(X, \mathbb{R})$). The differential operator $D = d/dx$ defines a morphism of sheaves $D\colon C^r(X, \mathbb{R}) \to C^{r-1}(X, \mathbb{R})$.

Example 3.6 (Holomorphic functions)**.** For a second familiar example, let $X \subset \mathbb{C}$ be an open set. On X one has the sheaf $\mathscr{A}_X$ of holomorphic functions. That is, for any open $U \subset X$, the sections $\mathscr{A}_X(U)$ is the ring of complex differentiable functions on U. Just like in the example above, one checks that $\mathscr{A}_X$ forms a sheaf. In fact, $\mathscr{A}_X$ is a *subsheaf* of the sheaf of continuous functions $U \to \mathbb{C}$.

One can relax the condition to get a larger sheaf $\mathscr{K}_X$ of meromorphic functions on X (these are functions holomorphic on all of U except for a set of isolated points, where they have poles). This sheaf contains $\mathscr{A}_X$ as a subsheaf, and the sections over an open U are the meromorphic functions on U.

In a similar way, one can get smaller sheaves contained in $\mathscr{A}_X$ by imposing vanishing conditions on the functions. For example if $x \in X$ is any point, one has the sheaf denoted $\mathfrak{m}_x$ of holomorphic functions vanishing at x. This is an example of an *ideal sheaf*: for each open $U \subset X$, $\mathfrak{m}_x(U)$ is an ideal of the ring $\mathscr{A}_X(U)$.

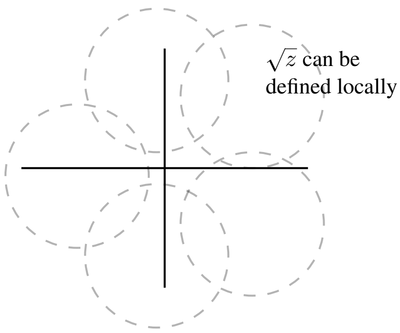
One of our main interests in this book will be the following:

Example 3.7 (Algebraic varieties)**.** Let X be an algebraic variety (e.g. an irreducible algebraic set in $\mathbb{A}^n(k)$ or $\mathbb{P}^n(k)$) with the Zariski topology. For each open $U \subset X$, define the presheaf

$$\mathcal{O}_X(U) = \{\, f\colon U \to k \mid f \text{ is regular} \,\}$$

where f is said to be regular if for each point $x \in U$ there is an affine neighbourhood in which f can be represented as a quotient of polynomials g/h with $h(x) \neq 0$.

This is a sheaf: locality holds, because if $f\colon U \to k$ restricts to the zero function on an open covering, it is the zero function. If we are given regular functions $f_i\colon U_i \to k$ on the members of an open overing $\{U_i\}$ of U that agree on the overlaps, they certainly glue to a continuous function $f\colon U \to k$; just define $f\colon U \to k$ by $f(x) = f_i(x)$ whenever $x \in U_i$. This function f is also regular because it restricts to f_i on U_i, and f_i is locally expressible as g/h there.



Example 3.8 (A presheaf which is not a sheaf)**.** Let us continue the set-up in Example 3.6 to exhibit an example of a presheaf which is not a sheaf. Let $X = \mathbb{C} - \{0\}$, and let $\mathscr{A}_X$ denote the sheaf of holomorphic functions. Inside $\mathscr{A}_X$ we find a subpresheaf given by

$$\mathcal{F}(U) = \{\, f \in \mathscr{A}_X(U) \mid f = g^2 \text{ for some } g \in \mathscr{A}_X(U) \,\}.$$

This is not a sheaf, because the Gluing axiom fails: the function $f(z) = z$ is holomorphic, and has a holomorphic square root near any point $x \in X$, but it is not possible to glue these together to a global square root function $\sqrt{z}$ on all of X. Note however, that the Locality axiom holds, because $\mathcal{F}$ is a subpresheaf of the sheaf $\mathscr{A}_X$ (which does satisfy Locality).

Example 3.9 (Constant presheaves)**.** For any space X and any abelian group A, one has the *constant presheaf* defined by $A(U) = A$ for any nonempty open set U (and $A(\varnothing) = 0$).

This is not a sheaf in general. For instance, if $X = U_1 \cup U_2$ is a disjoint union, and $A = \mathbb{Z}$, then any choice of integers $a_1, a_2 \in \mathbb{Z}$ will give sections of $A(U_1)$ and $A(U_2)$, and they automatically agree over the intersection, which is empty. But if $a_1 \neq a_2$, they cannot be glued to an element in $A(X) = \mathbb{Z}$. In fact, the constant presheaf is a sheaf if and only if any two non-empty open subsets of X have non-empty intersection. Algebraic varieties with the Zariski topology are examples of such spaces.

There is a quick fix for this. We can define the following sheaf A_X by letting

$$A_X(U) = \{\, f: U \to A \mid f \text{ is continuous} \,\}$$

where we give A the discrete topology. As before, we also must put $A_X(\varnothing) = 0$. For a connected open set U, we then have $A_X(U) = A$. More generally, since f must be constant on each connected component of U, it holds true that

$$A_X(U) \simeq \prod_{\pi_0(U)} A, \tag{3.4}$$

where $\pi_0(U)$ denotes the set of connected components of U.

The new presheaf A_X is called the *constant sheaf* on X with value A. It is a sheaf (e.g. by the final paragraph of Example 3.4). That being said, the sheaf A_X is not quite worthy of its name, as it is not quite constant.

Example 3.10 (Skyscraper sheaves)**.** Let A be a group. For $x \in X$, we can define a presheaf $A(x)$ by

$$A(x)(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is a sheaf. It is called the *skyscraper sheaf* of A at x.

Exercises

Exercise 3.1.1. Let X be the set with two elements with the discrete topology. Find a presheaf on X which is not a sheaf.

Exercise 3.1.2. In the notation of Example 3.6, the differential operator gives a map of sheaves $D: \mathscr{A}_X \to \mathscr{A}_X$, where as previously $X \subset \mathbb{C}$ is an open set. Show that the assignment

$$\mathscr{A}(U) = \{\, f \in \mathscr{A}_X(U) \mid Df = 0 \,\}$$

defines a subsheaf $\mathscr{A}$ of $\mathscr{A}_X$. Show that if U is a connected open subset of X, one has $\mathscr{A}(U) = \mathbb{C}$. In general for a not necessarily connected set U, show that $\mathscr{A}(U) = \prod_{\pi_0(U)} \mathbb{C}$ where the product is taken over the set $\pi_0(U)$ of connected components of U. So, in fact, $\mathscr{A}$ is the constant sheaf with value $\mathbb{C}$.

Exercise 3.1.3. Let $X \subset \mathbb{C}$ be an open set, and assume that $a_1, \ldots, a_r$ are distinct points in X and $n_1, \ldots, n_r$ natural numbers. Define $\mathcal{F}(U)$ to be the set of those functions meromorphic in U, holomorphic away from the a_i's and having a pole order bounded by n_i at a_i. Show that $\mathcal{F}$ is a sheaf of abelian groups. Is it a sheaf of rings?

Exercise 3.1.4 (The sheaf of homomorphisms)**.** Given two presheaves $\mathcal{F}$ and $\mathcal{G}$, we may form a presheaf $\mathscr{H}om_X(\mathcal{F}, \mathcal{G})$ by letting the sections over an open U be given by

$$\mathscr{H}om(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U), \tag{3.5}$$

and letting the restriction maps be the restrictions: if $V \subset U$ is another open set and $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is a map, the restriction of ϕ to V is simply the restriction $\phi|_V : \mathcal{F}|_V \to \mathcal{G}|_V$. Show that $\mathscr{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf whenever $\mathcal{G}$ is a sheaf.

3.2 Stalks

Suppose we are given a presheaf $\mathcal{F}$ of abelian groups on a topological space X. With every point $x \in X$ there is an associated abelian group $\mathcal{F}_x$ called the *stalk* of $\mathcal{F}$ at x. The stalk can be thought of as a way of keeping track of the behaviour of the sections of $\mathcal{F}$ in small neighbourhoods around x (regardless of how they may differ on different open sets of X.) The elements of $\mathcal{F}_x$ are called *germs of sections* or just *germs*, near x; they are essentially the sections of $\mathcal{F}$ defined in some sufficiently small neighbourhood of x. The group $\mathcal{F}_x$ is formally defined as the *direct limit* of the groups $\mathcal{F}(U)$ as U runs through the directed set of open neighbourhoods U of x (ordered by inclusion)[1]:

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

More concretely, the group $\mathcal{F}_x$ can be defined as follows. We begin with the *disjoint* union $\coprod_{x\in U} \mathcal{F}(U)$ whose elements we index as pairs (s, U) where U is an open neighbourhood of x and s is a section in $\mathcal{F}(U)$. We want to identify sections that coincide near x; that is, we declare (s, U) and (s', U') to be equivalent, and write $(s, U) \sim (s', U')$, if there is an open $V \subset U \cap U'$ with $x \in V$ such that s and s' coincide on V; that is, if one has

$$s|_V = s'|_V.$$

This is clearly a reflexive and symmetric relation, and it is transitive as well: if $(s, U) \sim (s', U')$ and $(s', U') \sim (s'', U'')$, one may find open neighbourhoods $V \subset U \cap U'$ and $V' \subset U' \cap U''$ of x over which s and s', respectively s' and s'', coincide. Clearly s and s'' then coincide over the intersection $V' \cap V$. The relation $\sim$ is therefore an equivalence relation.

Definition 3.11. The *stalk* $\mathcal{F}_x$ at $x \in X$ is defined as the set of equivalence classes

$$\mathcal{F}_x = \coprod_{x\in U} \mathcal{F}(U)/\sim .$$

In case $\mathcal{F}$ is a sheaf of abelian groups, the stalks $\mathcal{F}_x$ are all abelian groups. This is not

[1] For background on direct limits, see Appendix A

a priori obvious, because sections over different open sets can not be added. However, if (s, U) and (s', U') are given, the restrictions $s|_V$ and $s'|_V$ to any open $V \subset U \cap U'$ can be added, and this suffices to define an abelian group structure on the stalks.

The germ of a section

For any neighbourhood U of $x \in X$, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section s to the equivalence class where the pair (s, U) belongs. This class is called the *germ* of s at x, and a common notation for it is s_x. The map is a homomorphism of abelian groups (rings, modules, or whatever) as one easily verifies. One has $s_x = (s|_V)_x$ for any other open neighbourhood V of x contained in U, or in other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ {\scriptstyle \rho_{UV}} \downarrow & \nearrow & \\ \mathcal{F}(V). & & \end{array} \tag{3.6}$$

When working with sheaves and stalks, it is important to remember the three following working principles. The two first follow right away from the definition, and the third is easily deduced from the two first.

- The germ s_x of a section s vanishes if and only if s vanishes on some neighbourhood of x, i.e. there is an open neighbourhood U of x with $s|_U = 0$.
- All elements of the stalk $\mathcal{F}_x$ are germs, i.e. they are all of the form s_x for some section s over some open neighbourhood of x.
- The sheaf $\mathcal{F}$ is the zero sheaf if and only if all stalks are zero, i.e. $\mathcal{F}_x = 0$ for all $x \in X$.

Example 3.12. Let $X = \mathbb{C}$, and let $\mathscr{A}_X$ be the sheaf of holomorphic functions in X. What is the stalk $\mathscr{A}_{X,x}$ at a point x? Let f and g be two sections of $\mathscr{A}_X$ over a neighbourhood U of the point x having the same germ at x; that is, two functions holomorphic in neighbourhoods of x. The fact that f and g both admit Taylor series expansions around x, implies that $f = g$ in the connected component containing x of the set where they both are defined. The stalk $\mathscr{A}_{X,x}$ is therefore identified with the ring of power series that converge in a neighbourhood of x.

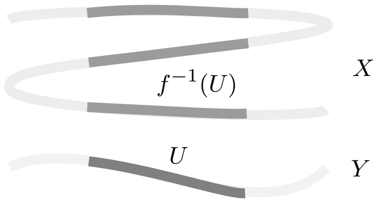
A map $\phi\colon \mathcal{F} \to \mathcal{G}$ of presheaves induces for every point $x \in X$ a map between the stalks

$$\phi_x \colon \mathcal{F}_x \longrightarrow \mathcal{G}_x.$$

Indeed, one may send a pair (s, U) to the pair $(\phi_U(s), U)$, and since ϕ behaves well with respect to restrictions, this assignment is compatible with the equivalence relations; if (s, U) and (s', U') are equivalent and s and s' coincide on an open set $V \subset U \cap U'$, the diagram (3.3) gives

$$\phi_U(s)|_V = \phi_V(s|_V) = \phi_V(s'|_V) = \phi_{U'}(s')|_V.$$

One checks that $(\phi \circ \psi)_x = \phi_x \circ \psi_x$ and $(\mathrm{id}_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}$, so the assignments $\mathcal{F} \mapsto \mathcal{F}_x$ and $\phi \mapsto \phi_x$ define a functor from the category of sheaves to the category of abelian groups.



Exercise 3.2.1. Let $\mathcal{F}$ be a sheaf and let $s, t \in \mathcal{F}(U)$ be two sections. Show that $s = t$ if and only if $s_x = t_x$ for every $x \in U$.

Exercise 3.2.2. Let $\phi, \psi \colon \mathcal{F} \to \mathcal{G}$ be maps of presheaves and assume that $\mathcal{G}$ is a sheaf. Prove that $\phi = \psi$ if and only if ϕ and ψ induce the same maps on all stalks, i.e. $\phi_x = \psi_x$ for every $x \in X$. HINT: : Use Exercise 3.2.2.

3.3 The pushforward of a sheaf

If $\mathcal{F}$ is a sheaf on a topological space X, and $f : X \to Y$ is a continuous map, we can define a sheaf $f_*\mathcal{F}$ on Y by defining

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U),$$

and the restriction maps $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ to be those coming from $\mathcal{F}$.

Definition 3.13. The sheaf $f_*\mathcal{F}$ is called the *pushforward* or *the direct image* of $\mathcal{F}$.

It is straightforward to verify that $f_*\mathcal{F}$ is a sheaf and not merely a presheaf. Indeed, if $\{U_i\}$ is an open covering of U, then $\{f^{-1}U_i\}$ is an open covering of $f^{-1}U$. A set of gluing data for $f_*\mathcal{F}$ and the given covering consists of sections $s_i \in \Gamma(U_i, f_*\mathcal{F}) = \Gamma(f^{-1}U_i, \mathcal{F})$ that agree on the intersections. This means that they coincide in $\Gamma(U_i \cap U_j, f_*\mathcal{F})$, which equals $\Gamma(f^{-1}U_i \cap f^{-1}U_j, \mathcal{F})$, and they may therefore be glued together to a section in $\Gamma(f^{-1}U, \mathcal{F}) = \Gamma(U, f_*\mathcal{F})$, as $\mathcal{F}$ is a sheaf. The Locality axiom follows for $f_*\mathcal{F}$, because it holds for $\mathcal{F}$.

Example 3.14. Let $\iota \colon \{x\} \to X$ be the inclusion of a closed point in X. If $\mathcal{A}$ is the constant sheaf of a group A on $\{x\}$, then $\iota_*\mathcal{A}$ is the skyscraper sheaf $A(x)$ from Example 3.10 on page 47.

The pushforward also depends functorially on the map f:

Lemma 3.15. If $g \colon X \to Y$ and $f \colon Y \to Z$ are continuous maps between topological spaces, and $\mathcal{F}$ is a sheaf on X, one has

$$(f \circ g)_*\mathcal{F} = f_*(g_*\mathcal{F}).$$

(This is indeed an equality, not merely an isomorphism.)

Exercise 3.3.1. Prove Lemma 3.15.

Exercise 3.3.2. Denote by $\{*\}$ a one point set. Let X be a topological space and $f\colon X \to \{*\}$ be the one and only map. Show that $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$ (where strictly speaking $\Gamma(X, \mathcal{F})$ stands for the constant sheaf on $\{*\}$ with value $\Gamma(X, \mathcal{F})$).

Exercise 3.3.3. Let X be a topological space and $x \in X$ a point that is not necessarily closed. Let $\iota : \{x\} \to X$ be the inclusion. Let $\mathcal{A}$ be the constant sheaf on $\{x\}$ with value the group A. Show that the stalks of $\iota_*\mathcal{A}$ are

$$(\iota_*\mathcal{A})_y = \begin{cases} A & \text{if } y \in \overline{\{x\}}; \\ 0 & \text{otherwise.} \end{cases}$$

3.4 Sheaves defined on a basis

Recall that a *basis* for the topology on X is a collection of open subsets $\mathscr{B}$ such that any open set of X can be written as a union of members of $\mathscr{B}$. In many situations, it turns out to be convenient to define a sheaf by saying what it should be over the open sets in a specific basis for the topology on X. The following definition makes this more precise.

Definition 3.16. A *$\mathscr{B}$-presheaf* $\mathcal{F}$ consists of the following data:

(i) For each $U \in \mathscr{B}$, an abelian group $\mathcal{F}(U)$;
(ii) For all pairs $U \supset V$ with U and V from $\mathscr{B}$, a restriction map $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$.

As before, these are required to satisfy the relations $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ and $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for each sequence $W \subset V \subset U$ of opens of $\mathscr{B}$. A *$\mathscr{B}$-sheaf* is a $\mathscr{B}$-presheaf satisfying the Locality and Gluing axioms for open sets in $\mathscr{B}$.

Since the intersections $V \cap V'$ of two sets $V, V' \in \mathscr{B}$ need not lie in $\mathscr{B}$, we need to clarify what we mean in the Gluing axiom. Given a cover of $U \in \mathscr{B}$ by subsets $U_i \in \mathscr{B}$. If $s_i \in \mathcal{F}(U_i)$ are sections such that $s_i|_V = s_j|_V$ for every i, j and every $V \subset U_i \cap U_j$ such that $V \in \mathscr{B}$, then the s_i should glue together to an element in $s \in \mathcal{F}(U)$.

The whole point with the notion of $\mathscr{B}$-sheaves is expressed in the following proposition. This construction will be used when we define the structure sheaf in Chapter 5.

Proposition 3.17. Let X be a topological space and let $\mathscr{B}$ be a basis for the topology on X. Then:

(i) Every $\mathscr{B}$-sheaf F extends to a sheaf $\mathcal{F}$ on X, which is unique up to unique isomorphism (which is the identity on $\mathcal{F}_0$);
(ii) If $\phi_0 : F \to G$ is a morphism of $\mathscr{B}$-sheaves, then ϕ_0 extends uniquely to a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ between the corresponding sheaves;
(iii) The stalk of the extended sheaf $\mathcal{F}$ at a point x can be computed as

$$\mathcal{F}_x = \varinjlim_{U \in \mathscr{B},\ x \in U} F(U).$$

Proof Let $U \subset X$ be an open set. The basic idea is to write U as a union of opens $U = \bigcup_{i \in I} V_i$, where $V_i \in \mathscr{B}$, and consider the set of 'compatible sections' $(s_i)_{i \in I}$, that is, sections $s_i \in F(V_i)$ such that $s_i|_W = s_j|_W$ whenever W is contained in $V_i \cap V_j$.

To define the group $\mathcal{F}(U)$ without reference to a choice of covering V_i, we consider all possible coverings at once, and define $\mathcal{F}(U)$ as the inverse limit of the $F(V)$, when V runs through the ordered set $\mathscr{B}_U$ of members of $\mathscr{B}$ contained in U,

$$\mathcal{F}(U) = \varprojlim_{\mathscr{B}_U} F(V),$$

Concretely, an element of $\mathcal{F}(U)$ is given by a collection $(s_V)_V$, one for each $V \in \mathscr{B}_U$, such that whenever $W \subset V$, we have $s_V|_W = s_W$.

The claimed properties follow from general functorial properties of the inverse limit. We begin with establishing (i). Observe that if the open set U is in $\mathscr{B}$, it will be a largest element in $\mathscr{B}_U$, and consequently we have the first equality in

$$F(U) = \varprojlim_{\mathscr{B}_U} F(V) = \mathcal{F}(U).$$

Hence $\mathcal{F}$ coincides with F on open sets in $\mathscr{B}$.

Secondly, if $U' \subset U$, clearly $\mathscr{B}_{U'} \subset \mathscr{B}_U$, and because inverse limits are functorial in the indexing set, we obtain maps $\mathcal{F}(U) \to \mathcal{F}(U')$, which serve as restriction maps.

Let us check that $\mathcal{F}$ is a sheaf. Locality: Suppose $s = (s_V) \in \varprojlim_{\mathscr{B}_U} F(V)$ is a collection of compatible elements and $\{U_i\}_{i \in I}$ is a covering of U such that $s|_{U_i} = 0$ for every i. Let $V \in \mathscr{B}_U$ be any subset. Since $\mathscr{B}$ is a basis, we can find a covering $\mathcal{V}$ of V consisting of open sets $B \in \mathscr{B}$ such that each B is contained in some U_j. Now, $s|_{U_i} = 0$ means that $s_B = 0$ for every $B \subset U_i$ and so $s_V|_B = s_B = 0$ for every $B \in \mathcal{V}$. In particular, by Locality for F, we get that $s_V = 0$. Since this happens for any V, we get that $s = 0$ as well.

Gluing: Let $\{U_i\}_{i \in I}$ be a cover of U and let $s^i \in \mathcal{F}(U_i)$ be a collection of compatible elements so that $s^i|_{U_i \cap U_j} = s^j|_{U_i \cap U_j}$ for all i, j. This means that $s^i_B = s^j_B$ for every $B \subset U_i \cap U_j$ in $\mathscr{B}$. Fix $V \in \mathscr{B}$ contained in U and let $\mathcal{V}$ be a cover of V by open sets $B \in \mathscr{B}$ so that $B \subset U_i$ for some i. First we claim that the elements $s^i_B \in F(B)$ for $B \in \mathcal{V}$ glue to an element $s_V \in F(V)$. So let $B \in \mathcal{V}$, and suppose B is contained in $U_i \cap B$, and let $s_V = s^i_V$. We note that this is independent of i, because if V is also contained in U_j, then $V \subset U_i \cap U_j$ and $s^j_V = s^i_V$, by the above. Now for the gluing: If $W \subset V \cap V'$ and $V \subset U_i$ $V' \subset U_j$, then

$$s_V|_W = s^i_W = s^j_W = s_{V'}|_W.$$

Hence, since F is a $\mathscr{B}$-sheaf, the elements s_V glue to an element $s_B \in F(B)$.

These elements are compatible, so it makes sense to define $s = (s_B) \in \varprojlim F(B) = \mathcal{F}(U)$. It is clear that $s|_{U_i} = s_i$ for every i. The presheaf $\mathcal{F}$ therefore satisfies the Gluing axiom.

The claim (iii) follow because we may use open sets from $\mathscr{B}$ when computing stalks.

Proof of (ii): saying $\phi_0 \colon F \to G$ is a map of $\mathscr{B}$-sheaves amounts to saying that the following diagram commutes for each pair $V' \subset V$ of opens in $\mathscr{B}$:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{(\phi_0)_V} & G(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(V') & \xrightarrow[(\phi_0)_{V'}]{} & G(V') \end{array}$$

Taking the inverse limit over all open subsets V from $\mathscr{B}$ contained in U, we obtain a natural map $\mathcal{F}(U) \to \mathcal{G}(U)$ which extends ϕ_0. These maps are moreover compatible with the restriction maps, so we get a map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$. Once again this must be unique, as it is completely determined by ϕ_0 on stalks. □

In the special case when $B \cap B' \in \mathscr{B}$ for every $B, B' \in \mathscr{B}$, a $\mathscr{B}$-presheaf F is a $\mathscr{B}$-sheaf if and only if the following sequence

$$0 \longrightarrow F(U) \longrightarrow \prod_{i \in I} F(U_i) \longrightarrow \prod_{i,j \in I} F(U_i \cap U_j) \tag{3.7}$$

is exact for every $U \in \mathscr{B}$ and covering $\{U_i\}_{i \in I}$ with $U_i \in \mathscr{B}$.

Exercise 3.4.1. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a space X and assume there is an open covering $\mathcal{U}$ of X and isomorphisms $\theta_U \colon \mathcal{F}|_U \simeq \mathcal{G}|_U$ that match on intersections; i.e. $\theta_U|_{U \cap U'} = \theta_{U'}|_{U \cap U'}$. Show that there is an isomorphism $\theta \colon \mathcal{F} \simeq \mathcal{G}$ extending the θ_U's. HINT: $\mathcal{F}$ and $\mathcal{G}$ define coinciding $\mathcal{B}$-sheaves.

4

Affine Schemes

In this chapter, we make one more step towards the general definition of a scheme, by defining *affine schemes*. Affine schemes serve as the building blocks for schemes in general, as every scheme has a open covering of affine schemes, and understanding the mechanics of affine schemes is essential for understanding schemes in general.

As any scheme, an affine scheme has two components: a topological space and a sheaf of rings. For the topological part, we use the spectrum of a ring $\operatorname{Spec} A$, and for the sheaf of rings, we use the *structure sheaf*, which define in Section 4.1. The definition of the structure sheaf is inspired by the sheaf of regular functions on an affine variety, so before giving the main definition, let us revisit the analogous concept in the setting of affine varieties.

Let $A = A(X)$ be the coordinate ring of an affine variety X, that is, A is the ring of globally defined regular functions on X. The fraction field K of A is the field of rational functions on X, i.e. the functions which are regular in some open subset $U \subset X$. For each open set U, the set of functions which are regular in each point of U forms subring $\mathcal{O}_X(U)$ of K. If $V \subset U$ is an open contained in U, the ring of regular functions $\mathcal{O}_X(V)$ in V, contains the ring $\mathcal{O}_X(U)$ of those regular in the bigger set U. The restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is nothing but the inclusion $\mathcal{O}_X(U) \subset \mathcal{O}_X(V)$; it simply considers functions in $\mathcal{O}_X(U)$ to lie in $\mathcal{O}_X(V)$.

Regular functions on the distinguished open set $D(f) = \{\, x \in X \mid f(x) \neq 0 \,\}$ are allowed to have powers of f in the denominator, and so they lie in the subring $A_f \subset K$ of elements of the form af^{-n} with $a \in A$ and n a non-negative integer. As explained in (2.4), if $D(g)$ is another distinguished open set with $D(g) \subset D(f)$, one may write $g^m = cf$ for some $c \in A$ and some suitable $m \in \mathbb{N}$, and hence there is an inclusion $A_f \subset A_g$ (since $f^{-1} = cg^{-m}$). Moreover, if $U \subset X$ is any subset, we have

$$\mathcal{O}_X(U) = \bigcap_{D(f) \subseteq U} \mathcal{O}_X(D(f)) \tag{4.1}$$

If one tries to carry out the above construction for a general ring A, one quickly runs into a few obstacles. For instance, there is no natural field K in which the rings $\mathcal{O}_X(U)$ lie as subrings. More critically, the localization maps $A_f \to A_g$ may fail to be injective. This happens already in the case $X = \operatorname{Spec} A$ with $A = k[x, y]/(xy)$, which corresponds to the union of x-axis and the y-axis in the affine plane. Since $xy = 0$, the element x maps to 0 via the localization map $A \to A_y$. Geometrically, this reflects the fact that the regular function x becomes zero over the open set $D(y)$ where $y \neq 0$, and similarly, the regular function y vanishes on $D(x)$. So this is by no means a big mystery; it naturally appears once we allow reducible spaces into the mix.

4.1 The structure sheaf on the spectrum of a ring

Motivated by the above discussion, it makes sense to define the sections of the structure sheaf over $D(f)$ to be the localized ring A_f. There is a small subtlety here, because different f's might give identical $D(f)$'s, and to avoid choices, we prefer to use a more canonical localization. Still, in the end, $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ will be isomorphic to A_f.

Let $\mathscr{B}$ be the collection of distinguished open sets $D(f)$. For $U \in \mathscr{B}$, we define the multiplicative system

$$\begin{aligned} S_U &= \{\, s \in A \mid s \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in U \,\} \\ &= \{\, s \in A \mid s(\mathfrak{p}) \neq 0 \text{ for all } \mathfrak{p} \in U \,\}. \end{aligned} \tag{4.2}$$

If $U \supset V$ are two distinguished opens, then $S_U \subset S_V$, so there is a canonical localization map

$$\rho_{UV} : S_U^{-1}A \longrightarrow S_V^{-1}A \tag{4.3}$$

With these ring maps, we have defined a $\mathscr{B}$-presheaf of rings on $\operatorname{Spec} A$. We will show below that it is in fact a $\mathscr{B}$-sheaf.

Lemma 4.1. For $U = D(f)$, there is a canonical isomorphism

$$S_U^{-1}A = A_f.$$

For $V = D(g) \subset D(f)$, the map ρ_{UV} is identified with the localization map $\rho_{fg} : A_f \to A_g$.

Proof Note that by assumption, $f \in S_U$, so there is a canonical localization map

$$\tau : A_f \longrightarrow S_U^{-1}A.$$

The main observation is that for an element $s \in S_U$, we have $s \notin \mathfrak{p}$ for every $\mathfrak{p} \in D(f)$, so $D(f) \subset D(s)$ This is equivalent to $\sqrt{(f)} \subset \sqrt{(s)}$, so one may write $f^n = cs$ for some $c \in A$ and $n \in \mathbb{N}$.

τ is injective: Suppose that $af^{-m} \in A_f$ maps to zero in $S_{D(f)}^{-1}A$. This means that $sa = 0$ for some $s \in S_{D(f)}$. But then $f^n a = csa = 0$, and therefore $a = 0$ in A_f.

τ is surjective: take any as^{-1} in $S_{D(f)}^{-1}A$ and write it as $as^{-1} = ca(f^n)^{-1} = caf^{-n}$. □

The notation $S_{D(f)}^{-1}A$ will only be present in the definition of $\mathcal{O}$. From now on, we will write $\mathcal{O}(D(f)) = A_f$, bearing in mind that it is defined in terms of a canonical localization.

Proposition 4.2. $\mathcal{O}$ is a $\mathscr{B}$-sheaf of rings.

Proof Let $D(f) \in \mathscr{B}$ and let $D(f) = \bigcup_{i \in I} D(f_i)$ be a covering with open sets in $\mathscr{B}$. We need to show that the $\mathscr{B}$-sheaf sequence (3.7)

$$0 \longrightarrow \mathcal{O}(D(f)) \longrightarrow \prod_i \mathcal{O}(D(f_i)) \longrightarrow \prod_{i,j} \mathcal{O}(D(f_i f_j))$$

is exact. By Lemma 2.23, we may reduce this covering to a finite one, so that $D(f)$ is covered by the sets $D(f_0), \ldots, D(f_s)$. It is then enough to show that the following sequence is exact

$$0 \longrightarrow A_f \xrightarrow{\alpha} \prod_{i=1}^s A_{f_i} \xrightarrow{\beta} \prod_{i,j=1}^s A_{f_i f_j} \tag{4.4}$$

where $\alpha(\frac{a}{f^n}) = \left(\frac{a}{f^n}, \ldots, \frac{a}{f^n}\right)$ and

$$\beta\left(\frac{a_1}{f_1^{n_1}}, \ldots, \frac{a_s}{f_s^{n_s}}\right)_{i,j} = \frac{a_i}{f_i^{n_i}} - \frac{a_j}{f_j^{n_j}} \tag{4.5}$$

We will show that this is exact by a series of reductions. As a sequence of A-modules, (4.4) is exact if and only if it is exact after being localized at every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Using the isomorphisms $(A_{f_i})_{\mathfrak{p}} = (A_{\mathfrak{p}})_{f_i}$ and $(A_{f_i f_j})_{\mathfrak{p}} = (A_{\mathfrak{p}})_{f_i f_j}$, the localized sequence takes the form

$$0 \longrightarrow A_{\mathfrak{p}} \xrightarrow{\alpha} \prod_{i=0}^n (A_{\mathfrak{p}})_{f_i} \xrightarrow{\beta} \prod_{i,j=0}^n (A_{\mathfrak{p}})_{f_i f_j}.$$

Up to reordering the indexes, we may assume that $\mathfrak{p} \in D(f_1)$, i.e. that f_1 is a unit in $A_{\mathfrak{p}}$. Replacing A by $A_{\mathfrak{p}}$, we reduce to showing that (4.6) below is exact when f_1 is a unit.

$$0 \longrightarrow A \xrightarrow{\alpha} \prod_{i=1}^n A_{f_i} \xrightarrow{\beta} \prod_{i,j=1}^n A_{f_i f_j} \tag{4.6}$$

Now the injectivity of α is clear, as the first component of α is the localization map $\rho_1 \colon A \to A_{f_1} = A$, which is an isomorphism, as f_1 is a unit. Moreover, given a sequence $(a_i) \in \operatorname{Ker} \beta$ with $\beta(a_i) = 0$, it holds that $a_i = a_1$ in $A_{f_1 f_i} \simeq A_{f_i}$ for $i \geqslant i$. We deduce that $a = \rho_1^{-1}(a_1) \in A$ is an element satisfying $\alpha(a) = (a_i)$. Therefore $\operatorname{Ker} \beta = \operatorname{Im} \alpha$, and the sequence is exact. □

Using Proposition 3.17 on page 51 we may now make the following definition:

Definition 4.3. The *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ is the unique sheaf extending the $\mathscr{B}$-sheaf $\mathcal{O}$.

The proof above tells us how to compute $\mathcal{O}_{\operatorname{Spec} A}(U)$ for any open set: cover U by finitely many distinguished opens $D(f_1), \ldots, D(f_s)$; then the sheaf sequence (3.2) shows that $\mathcal{O}_{\operatorname{Spec} A}(U)$ can be identified with the group

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \left\{ \left(\frac{a_i}{f_i^{n_i}}\right) \in \prod_{i=1}^s A_{f_i} \;\middle|\; \frac{a_i}{f_i^{n_i}} = \frac{a_j}{f_j^{n_j}} \text{ in } A_{f_i f_j} \text{ for all } i, j \right\}.$$

That being said, we will basically never need to know the group $\mathcal{O}_{\operatorname{Spec} A}(U)$ for U other than a distinguished open set $U = D(f)$. All that matters is that $\mathcal{O}_{\operatorname{Spec} A}$ is the unique sheaf that satisfies the two main properties we want:

Proposition 4.4 (Key properties of the structure sheaf). The sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ as defined above is a sheaf of rings satisfying the following two properties.

(i) Sections over distinguished opens:

$$\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$$

for every $f \in A$;

(ii) Stalks:

$$\mathcal{O}_{\operatorname{Spec} A, x} = A_{\mathfrak{p}},$$

where $\mathfrak{p} \subset A$ is the prime ideal corresponding to $x \in \operatorname{Spec} A$.

In particular, it holds that $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$.

Proof We defined $\mathcal{O}_{\operatorname{Spec} A}$ so that the first property would hold. For the second, we may compute the stalk using distinguished open sets:

$$\varinjlim_{x \in D(f)} \mathcal{O}(D(f)) = \varinjlim_{f \notin \mathfrak{p}} A_f = A_{\mathfrak{p}}.$$

(See also Example A.6 on page 414.) The last statement follows by taking $f = 1$ in (i). □

Examples

Example 4.5 (Spectrum of a field)**.** For a field K, the structure sheaf $\mathcal{O}_{\operatorname{Spec} K}$ is a constant sheaf with the value K at the single point of $\operatorname{Spec} K$.

Example 4.6. The structure sheaf of $\operatorname{Spec} \mathbb{Z}$ satisfies $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(n)) = \mathbb{Z}[\frac{1}{n}]$ for each natural number n. The stalks of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ at the closed point (p) is equal to $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}, p} = \mathbb{Z}_{(p)}$ and at the generic point the stalk equals $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}, (0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$.

Example 4.7. Let $X = \operatorname{Spec} \mathbb{C}[t]$. Then the stalk of $\mathcal{O}_X$ at the generic point $\eta = (0)$ is equal to $\mathcal{O}_{X,\eta} = \mathbb{C}(t)$. Each closed point $p \in X$ corresponds to a maximal ideal $(t - a)$, and the stalk of $\mathcal{O}_X$ at p is equal to $\mathcal{O}_{X,p} = \mathbb{C}[t]_{(t-a)}$.

Example 4.8. We continue Example 2.10 about spectra of DVR's. The spectrum $X = \operatorname{Spec} A = \{x, \eta\}$ of a DVR A has three open sets $\varnothing$, η, and X, and the structure sheaf takes the following values at these opens:

$$\mathcal{O}_X(\varnothing) = 0, \qquad \mathcal{O}_X(X) = A, \qquad \mathcal{O}_X(\eta) = A_x = K,$$

where K denotes the fraction field of A. The stalks are given by $\mathcal{O}_{X,x} = A_{(x)} = A$ and $\mathcal{O}_{X,\eta} = A_{(0)} = K$.

Example 4.9 (Disconnected spectra)**.** The structure sheaf may be used to prove that a ring A whose spectrum $\operatorname{Spec} A$ is not connected, decomposes as the direct product of two rings. Suppose $X = U_1 \cup U_2$, where U_1 and U_2 are open and closed subsets with $U_1 \cap U_2 = \varnothing$. The sheaf exact sequence takes the form

$$0 \longrightarrow \mathcal{O}_X(X) = A \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1 \cap U_2) = 0,$$

and we deduce that $A \simeq \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$.

Example 4.10. It is worthwhile to consider the special case when A is an integral domain. Then all the localizations A_f are subrings of the fraction field K of A, and the localization maps $A_f \to A_g$ for $D(g) \subset D(f)$ are simply inclusions of subrings of K. The intuitive picture from varieties is then correct: we may think of elements in $\mathcal{O}_{\operatorname{Spec} A}(U)$ simply as fractions a/b in K and

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \bigcap_{\mathfrak{p} \in U} A_{\mathfrak{p}} \subset K \tag{4.7}$$

In the general case, the intersection (4.7) is replaced by an inverse limit (Exercise 4.1.1).

Exercise 4.1.1. Show that the sections of $\mathcal{O}_{\operatorname{Spec} A}$ over an open set $U \subset X = \operatorname{Spec} A$, are given by the inverse limit of the localizations

$$\mathcal{O}_X(U) = \varprojlim_{D(f) \subset U} \mathcal{O}(D(f)) = \varprojlim_{D(f) \subset U} A_f. \tag{4.8}$$

Exercise 4.1.2 (A-module structure on $\mathcal{O}_{\operatorname{Spec} A}(U)$)**.** Let $a \in A$, show that there is a map of sheaves $[a] \colon \mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} A}$, inducing multiplication by a both on $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ and on the stalks $A_{\mathfrak{p}}$. HINT: For each distinguished open subset $D(f)$ of $\operatorname{Spec} A$ define $[a] \colon \mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f \to \mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ as the multiplication by a map; verify that this is a map of $\mathscr{B}$-sheaves.

4.2 Locally ringed spaces

We would like to define a scheme to be a space which is 'locally affine'; that is, one that looks like the spectrum of a ring near each point. To be able to make such a definition precise, we need a suitable category of spaces to work with. To this end, we use the two pieces of data we have in the affine case: the topological space $\operatorname{Spec} A$ together with its sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$.

Definition 4.11 (Locally ringed spaces)**.** A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that all the stalks $\mathcal{O}_{X,x}$ are local rings.

To make this into a category, we need to specify what a morphism between two locally ringed spaces is. Reflecting the above definition, a morphism betwen $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ should have two components, one map between the underlying topological spaces X and Y and one on the level of sheaves. Note that it does not make sense to talk about morphisms $\mathcal{O}_Y \to \mathcal{O}_X$, as these sheaves are defined on different spaces. Instead, once a continuous map $f \colon X \to Y$ is specified, the sheaf map should be a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y. This means that that for all open subsets $U \subset Y$, one has to specify ring maps

$$f_U^\sharp \colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}U),$$

compatible with the restriction maps; that is, such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f^\sharp_U} & \mathcal{O}_X(f^{-1}U) \\ \rho_{UV}\big\downarrow & & \big\downarrow \rho_{f^{-1}U\, f^{-1}V} \\ \mathcal{O}_Y(V) & \xrightarrow[f^\sharp_V]{} & \mathcal{O}_X(f^{-1}V). \end{array} \tag{4.9}$$

The intuition again comes from the theory of varieties, where we would like to think of $f^\sharp$ as a way of 'pulling back' functions on Y to X. If X and Y are affine varieties with sheaves of regular functions $\mathcal{O}_X$ and $\mathcal{O}_Y$ (as defined in Chapter 1), and $f : X \to Y$ is a polynomial map, there is an induced morphism $f^\sharp \colon A(Y) \to A(X)$ which sends a regular function $h \colon Y \to k$ to $h \circ f \colon X \to k$. If h is only regular on some open set $U \subset Y$, we may still define a pullback $f^\sharp(h) = h \circ f$, but this is only regular on $f^{-1}(U)$. In other words, $f^\sharp(h)$ defines a section in $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}U)$.

If $x \in X$ is a point, there is also an induced map between the localizations

$$f^\sharp_x \colon A(Y)_{\mathfrak{m}_y} \longrightarrow A(X)_{\mathfrak{m}_x}. \tag{4.10}$$

where $y = f(x)$. It sends a rational function g defined at $y = f(x)$, to $g \circ f$, which is regular at x. Moreover, if h vanishes at y, the corresponding pullback $f^\sharp(h) = h \circ f$ vanishes at x. This means that $f^\sharp_y$ maps the maximal ideal $\mathfrak{m}_y$ into $\mathfrak{m}_x$; or in other words, it is a *map of local rings*.

For a general locally ringed space, we do not have the luxury of speaking about functions into some fixed field k, so the ring maps $f^\sharp_U$ have to be specified as part of the data. We do not allow these to be completely arbitrary ring maps; there is a last condition saying that the induced map on stalks should have similar properties as the map (4.10).

First of all, for a point $x \in X$ and $y = f(x)$, the map in question is a map of rings

$$f^\sharp_x \colon \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}, \tag{4.11}$$

which is defined as follows: pick an element from $\mathcal{O}_{Y,y}$ and represent it as the germ s_y of a section $s \in \mathcal{O}_Y(U)$ over some open set $U \subset Y$. Then the section $t = f^\sharp(s)$ is a section of $\mathcal{O}_X(f^{-1}U)$. We define $f^\sharp_x(s)$ to be the germ of this section at x, i.e. $f^\sharp(s_x) = t_x \in \mathcal{O}_{X,x}$. This makes sense because $f^{-1}U$ contains x. Moreover, by the properties of direct limits, it is clear that this does not depend on the choice of U containing x.

The requirement we make on $f^\sharp$ is that the induced maps on stalks (4.11) is a map of local rings, i.e., $f^\sharp$ maps the maximal ideal $\mathfrak{m}_y$ into the maximal ideal $\mathfrak{m}_x$. Equivalently, $h \in \mathcal{O}_{Y,y}$ satisfies $g(y) = 0$ in $k(y)$ if and only if $f^\sharp_x(h)(x) = 0$ in $k(x)$. This is a natural choice in light of (4.10), but it is in no way automatic when starting from a general map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Definition 4.12 (Morphisms of locally ringed spaces)**.** A *morphism*, or simply *map*, of locally ringed spaces, is a pair

$$(f, f^\sharp)\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

where

(i) $f\colon X \to Y$ is a continuous map;

(ii) $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on Y, so that for each $x \in X$, with $y = f(x)$ the induced map on stalks

$$f_x^\sharp\colon \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

is a map of local rings; that is, $f_x^\sharp(\mathfrak{m}_y) \subset \mathfrak{m}_x$.

A second reason to include the requirement (ii) will appear in the proof of Proposition 4.17 below. Here is an example illustrating what can go wrong without it:

Example 4.13. Let $X = \operatorname{Spec}\mathbb{C}(t)$ and $Y = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t]$. There is a natural map $f\colon X \to Y$ induced by the inclusion $\mathbb{C}[t] \subset \mathbb{C}(t)$. Note that on the level of topological spaces, X consists of a single point ν, and f maps ν to the generic point η of Y. The corresponding stalk map $f_\eta^\sharp\colon \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\nu}$ is the identity map

$$\mathcal{O}_{Y,\eta} = \mathbb{C}[t]_{(0)} = \mathbb{C}(t) \to \mathbb{C}(t) = \mathcal{O}_{X,\nu},$$

which is certainly a map of local rings.

On the other hand, we could try to define a strange map $g\colon X \to Y$ by sending ν to some other point $y \in \mathbb{A}^1_{\mathbb{C}}$ corresponding to a maximal ideal $(t-a) \subset \mathbb{C}[t]$. The map g is clearly continuous, because X consists of a single point. However, the induced map

$$\mathcal{O}_{Y,y} = \mathbb{C}[t]_{(t-a)} \to \mathbb{C}(t) = \mathcal{O}_{X,\nu}$$

sends the maximal ideal to the unit ideal in $\mathbb{C}(t)$, so it is not a map of locally ringed spaces. This is as it should be, as the function $t-a$ vanishes at $y \in Y$, but its pullback, the image of $t-a$ in $\mathcal{O}_{X,\nu}$, does not vanish at $\nu \in X$. (In fact, it maps to itself via the evaluation map $\mathcal{O}_{X,\nu} \to k(\nu) = \mathbb{C}(t)$).

Maps between locally ringed spaces can be composed: given $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp)\colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, the map $X \to Z$ given by the composition $g \circ f$ on the level of topological spaces, and for the sheaf map we define $(g \circ f)^\sharp$ over an open set $U \subset Z$ as the composition

$$\mathcal{O}_Z(U) \xrightarrow{g^\sharp} \mathcal{O}_Y(g^{-1}U) \xrightarrow{f^\sharp} \mathcal{O}_X((g \circ f)^{-1}U) = (g \circ f)_*\mathcal{O}_X.$$

An *isomorphism of locally ringed spaces* is a morphism $f\colon X \to Y$ which admits an inverse morphism. In other words, there is a morphism $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In more concrete terms, this boils down to f being a homeomorphism such that $f_U^\sharp\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ is a ring isomorphism for every open $U \subset Y$.

Prime spectra are locally ringed spaces

For a ring A, the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is by design a locally ringed space. Indeed, the stalks of $\mathcal{O}_{\operatorname{Spec} A}$ are localizations of A at prime ideals so they are local rings. In this section, we show that ring maps induce morphisms of locally ringed spaces.

Recall that a map of rings $\phi\colon A \to B$ induces a continuous map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ that sends $\mathfrak{p}$ to $\phi^{-1}(\mathfrak{p})$. This will be the topological part of the induced map on locally ringed spaces

$$\operatorname{Spec}(\phi) = (f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \tag{4.12}$$

To specify the map $f^\sharp$ between the sheaves $\mathcal{O}_{\operatorname{Spec} A}$ and $f_*\mathcal{O}_{\operatorname{Spec} A}$, we use the $\mathscr{B}$-sheaf construction.

Consider a distinguished open set $D(g)$ in $\operatorname{Spec} A$. We have $\mathcal{O}_{\operatorname{Spec} A}(D(g)) = A_g$. By Proposition 2.27 (ii), the inverse image of $D(g)$ in $\operatorname{Spec} A$ equals $D(\phi(g))$. Therefore, we have

$$f_*\mathcal{O}_{\operatorname{Spec} B}(D(g)) = \mathcal{O}_{\operatorname{Spec} B}(f^{-1}D(g)) = \mathcal{O}_{\operatorname{Spec} B}(D(\phi(g))) = B_{\phi(g)}.$$

Now, there is a canonical localization map

$$\begin{array}{ccc} A_g & \longrightarrow & B_{\phi(g)} \\ a/g^n & \mapsto & \phi(a)/\phi(g)^n \end{array} \tag{4.13}$$

and this will be the desired ring map

$$f^\sharp_{D(g)}\colon \mathcal{O}_{\operatorname{Spec} A}(D(g)) \longrightarrow f_*\mathcal{O}_{\operatorname{Spec} B}(D(g)).$$

In this way, we obtain a map of $\mathscr{B}$-sheaves, which then extends to a map of sheaves by Proposition 3.17 (ii). To summarize, we have:

Proposition 4.14. Any map of rings $\phi\colon A \to B$ induces a map of locally ringed spaces

$$\operatorname{Spec}(\phi) = (f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}),$$

which satisfies the following properties:

(i) (Distinguished open sets) The map $f^\sharp_{D(g)}$ is the natural localization map

$$\mathcal{O}_{\operatorname{Spec} A}(D(g)) = A_g \longrightarrow B_{\phi(g)} = \mathcal{O}_{\operatorname{Spec} B}(D(\phi(g)))$$

given by the assignment (4.13).

(ii) (Stalks) The map induced by $f^\sharp$ between stalks at $\mathfrak{p} \in \operatorname{Spec} B$ and $\phi^{-1}(\mathfrak{p})$, is the localization map $A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ of ϕ.

Proof The first point is just a rephrasing of the definition of $f^\sharp$, and the second follows from Proposition 4.4 since we may use distinguished opens to compute stalks. Indeed, the map in (4.11) is a limit of localization maps of the form

$$A_g \longrightarrow B_{\phi(g)} \longrightarrow B_{\mathfrak{p}},$$

where $g \in A$ runs over elements such that $\phi^{-1}(\mathfrak{p}) \in D(g)$. These maps send $a/s \in A_g$ to $\phi(a)/\phi(s) \in B_\mathfrak{p}$, and in the limit we get the localization map $A_{\phi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$. □

The proposition reflects the statements in Proposition 4.4 on page 57 about sections and stalks of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$. To summarize, for affine schemes, both $\mathcal{O}_X$ and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ are determined by the localizations of the rings involved.

Example 4.15 (The cuspidal cubic)**.** Let k be an algebraically closed field, and let $A = k[u,v]/(u^2 - v^3)$. The assignments $u \mapsto t^2$, $v \mapsto t^3$ define a ring map

$$\phi\colon A \to k[t],$$

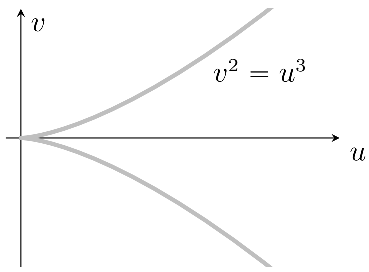
and hence a morphism of locally ringed spaces

$$f\colon \mathbb{A}^1_k \to \operatorname{Spec} A.$$

We claim that this is a homeomorphism, but not an isomorphism.

To see this, note that f sends a maximal ideal $(t-a) \in \operatorname{Spec} k[t]$ to the maximal ideal $(u - a^2, v - a^3) \in \operatorname{Spec} A$. So f is clearly injective on k-points. By the Nullstellensatz every maximal ideal of A is of the form $(u - \alpha, v - \beta)$, and since $v^2 - u^3$ vanishes at the corresponding point, setting $a = \beta\alpha^{-1}$, we find $\alpha = a^2$ and $\beta = a^3$. Therefore, it is also surjective.

Hence, since f maps the generic point of $\mathbb{A}^1_k$ to the generic point of $\operatorname{Spec} A$, the map f is a bijection. Each proper closed subset of $\mathbb{A}^1_k$ is finite, so f is closed, and hence a homeomorphism.



To see that f is not an isomorphism, let $x \in \mathbb{A}^1_k$ be the origin corresponding to the ideal (t). Then $f(x)$ is given by the ideal (u,v), and the induced stalk map $f^\sharp_x : \mathcal{O}_{X,f(x)} \to \mathcal{O}_{\mathbb{A}^1_k,x}$ is equal to the map of localizations

$$\bigl(k[u,v]/(u^2 - v^3)\bigr)_{(u,v)} \longrightarrow k[t]_{(t)}. \tag{4.14}$$

This map is not surjective (t is not in the image), and hence f is not an isomorphism. This moreover confirms our intuition that the cuspidal cubic is not even 'locally isomorphic' to $\mathbb{A}^1_k$ near the origin.

Exercise 4.2.1. Show that if $f\colon X \to Y$ is a morphism of locally ringed spaces, the stalk maps $f^\sharp_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induce maps between the residue fields $k(f(x))$ and $k(x)$. What happens when X and Y are affine varieties?

4.3 Affine schemes

We have now come to the definition of an affine scheme.

Definition 4.16. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Affine schemes form a category AffSch, a subcategory of the category of locally ringed spaces. This category is closely linked to the category of rings, as we will see next.

In (4.12) we defined, for each ring map $\phi : A \to B$, a map $\operatorname{Spec}(\phi)$ of locally ringed spaces between $\operatorname{Spec} B$ and $\operatorname{Spec} A$. Note that we have $\operatorname{Spec}\phi \circ \operatorname{Spec}\psi = \operatorname{Spec}\psi \circ \phi$, whenever ϕ and ψ are composable ring maps. This follows by the identity $\phi^{-1}\psi^{-1}\mathfrak{p} = (\psi\phi)^{-1}\mathfrak{p}$. And of course it holds that $\operatorname{Spec}\operatorname{id}_A = \operatorname{id}_{\operatorname{Spec} A}$. This shows that the assignment $A \mapsto \operatorname{Spec}(A)$ defines a contravariant functor from the category of rings Rings to the category of affine schemes AffSch.

There is also a contravariant functor Γ going the other way: taking global sections of the structure sheaf $\mathcal{O}_X$ gives us a ring $\mathcal{O}_X(X)$. Furthermore, a map of affine schemes $f\colon X \to Y$ comes equipped with a map of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, which on global sections yields a map

$$f^\sharp_Y\colon \mathcal{O}_Y(Y) \longrightarrow \Gamma(Y, f_*\mathcal{O}_X) = \mathcal{O}_X(X).$$

We therefore have a canonical 'global section map'

$$\Gamma\colon\ \operatorname{Hom}_{\mathsf{AffSch}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathsf{Rings}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \tag{4.15}$$

which sends a map $(f, f^\sharp)$ to the ring map $f^\sharp_Y\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. It is functorial in the sense that $(g\circ f)^\sharp_Z = f^\sharp_Y \circ g^\sharp_Z$ whenever $f\colon X \to Y$ and $g\colon Y \to Z$ are two scheme maps.

Proposition 4.17. If X and Y are affine schemes, the map Γ in (4.15) is bijective.

Proof We may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and then $A = \mathcal{O}_Y(Y)$ and $B = \mathcal{O}_X(X)$.

If $\phi\colon A \to B$ is a map of rings, it follows from Proposition 4.14 (i), that $\Gamma(\operatorname{Spec}\phi) = \phi$. To establish the bijection, we just need to show that $\operatorname{Spec}(\Gamma(f)) = f$ for a given a morphism $f\colon X \to Y$. We let $\phi = \Gamma(f)\colon A \to B$, that is, $\phi = f^\sharp_Y$.

Let $x \in X$ be a point which corresponds to the prime ideal $\mathfrak{q} \subset B$, and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to $f(x) \in Y$. The sheaf map $f^\sharp$ gives the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(Y) & \xrightarrow{f^\sharp_Y} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,f(x)} & \xrightarrow{f^\sharp_x} & \mathcal{O}_{X,x} \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_\mathfrak{p} & \xrightarrow[f^\sharp_x]{} & B_\mathfrak{q} \end{array}$$

We claim that $\phi(A - \mathfrak{p}) \subset B - \mathfrak{q}$. Indeed, $B - \mathfrak{q}$ is exactly the subset of elements in B that become invertible in $B_\mathfrak{q}$. By the diagram, this certainly happens to elements from $\phi(A - \mathfrak{p})$, because elements of $A - \mathfrak{p}$ become invertible in $A_\mathfrak{p}$. Hence we have $\phi(A - \mathfrak{p}) \subset B - \mathfrak{q}$, and consequently, $\phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$.

Next we use the assumption that the stalk map $f_x^\sharp$ is a map of local rings, so that $f_x^\sharp(\mathfrak{p}A_\mathfrak{p}) \subset \mathfrak{q}B_\mathfrak{q}$. By the diagram, this implies that $\phi(\mathfrak{p}) \subset \mathfrak{q}$. We conclude that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and so $\operatorname{Spec}\phi$ induces the same map as f on the underlying topological spaces.

Finally, we have two morphisms of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$, one induced by f and one induced by $\operatorname{Spec}\phi$. For each x, the induced stalk maps $f_x^\sharp$ and $(\operatorname{Spec}\phi)_x^\sharp$ are both equal to the map $A_\mathfrak{p} \to B_\mathfrak{q}$ above, and from this it follows as well that $f^\sharp = (\operatorname{Spec}\phi)^\sharp$, since maps of sheaves are detemined on stalks (see Exercise 3.2.2 on page 50). □

We have established the following important theorem, which is the scheme version of the Main Theorem of Algebraic Sets (Theorem 1.19 on page 10).

Theorem 4.18 (Main Theorem for Affine Schemes). The two functors Spec and Γ are up to equivalence mutually inverse and give an equivalence between the categories $\mathsf{Rings}^{\mathrm{op}}$ and AffSch.

Proof Note that there is an equality $\Gamma \circ \operatorname{Spec} = \mathrm{id}_{\mathsf{Rings}}$. Conversely, for each X, there is unique map $\operatorname{Spec}\mathcal{O}_X(X) \to X$ corresponding to the identity in $\operatorname{Hom}_{\mathsf{Rings}}(\mathcal{O}_X(X), \mathcal{O}_X(X))$. Therefore $\operatorname{Spec} \circ \Gamma$ is equivalent to $\mathrm{id}_{\mathsf{AffSch}}$. □

In summary, affine schemes X are completely characterized by their rings of global sections $\mathcal{O}_X(X)$, and morphisms between affine schemes $X \to Y$ are in bijective correspondence with ring homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. In particular, a map f between two affine schemes is an isomorphism if and only if the corresponding ring map is an isomorphism.

Example 4.19. Maps between affine schemes can very well have a homeomorphism as underlying topological map without being isomorphisms. The easiest examples are the spectra of fields: being sets with one element, they are all homeomorphic, but two are isomorphic as schemes only when the fields are isomorphic. For another example, closer to the world of varieties, see Example 4.15.

Example 4.20. There is one and only one morphism of schemes $\operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}$. Indeed, ring maps are required to send 1 to 1, so there is only one ring map $\mathbb{Z} \to \mathbb{Z}$.

Example 4.21 (Maps to $\mathbb{A}^1$ and $\mathcal{O}_X(X)$)**.** If A is a ring, then there is a bijection between ring maps $\phi : \mathbb{Z}[t] \to A$ and elements of A (ϕ is determined uniquely by the image of t). Therefore, by Theorem 4.18

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \mathbb{A}^1) = \mathcal{O}_X(X).$$

In clear text: an element of $\mathcal{O}_X(X)$ is the same thing as a map

$$f : X \longrightarrow \mathbb{A}^1.$$

Thus the global sections the structure sheaf $\mathcal{O}_X$ do indeed correspond to some sort of 'regular functions' on X – not into a field k – but into the affine line over $\mathbb{Z}$. We will see a generalization of this for general schemes in Example 6.8.

4.4 The sheaf associated to an A-module

The construction of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ works more generally. For each A-module M there is a parallel construction of a sheaf $\widetilde{M}$ on $\operatorname{Spec} A$. Over the distinguished open sets $D(f)$, the sections of $\widetilde{M}$ are given by

$$\widetilde{M}(D(f)) = M_f,$$

and the restriction maps are the canonical localization maps described as follows: when $D(g) \subseteq D(f)$, it holds true that $g^n = af$ for some $a \in A$ and some $n \in \mathbb{N}$, and the canonical localization map $M_f \to M_g$ sends bf^{-r} to $a^r bg^{-nr}$. The same proof as for the structure sheaf (Proposition 4.2 on page 55), but with obvious modifications, shows that this is actually a $\mathscr{B}$-sheaf. Hence it gives rise to a unique *sheaf* on $\operatorname{Spec} A$, which we will continue to denote by $\widetilde{M}$.

This tilde-construction is functorial in M. For any A-linear map $\phi\colon M \to N$, there is an induced map $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$. Indeed, according to Proposition 3.17 to define $\widetilde{\phi}$, it suffices to say what ut should be over each distinguished open set $D(f)$. Here we simply define $\widetilde{\phi}_{D(f)}\colon \widetilde{M}(D(f)) \to \widetilde{N}(D(f))$ to be the induced map between the localizations $\phi_f\colon M_f \to N_f$, given by $mf^{-r} \mapsto \phi(m)f^{-r}$. This is a map of $\mathscr{B}$-sheaves because the following diagram commutes for each f and g with $D(g) \subset D(f)$:

$$\begin{array}{ccc} M_f & \xrightarrow{\phi_f} & N_f \\ \downarrow & & \downarrow \\ M_g & \xrightarrow{\phi_g} & N_g \end{array}$$

Indeed, writing $g^n = af$ as above, and denoting the two localization maps respectively by ι_M and ι_N, we find:

$$\phi_g \iota_M(mf^{-r}) = \phi_g(a^r mg^{-nr}) = a^r\phi(m)g^{-nr} = \iota_N(\phi(m)f^{-r}) = \iota_N\phi_f(m).$$

Clearly one has $\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi}$, whenever ϕ and ψ are composable A-linear maps and consequently the 'tilde-operation' is a covariant functor from the category Mod_A of A-modules to the category $\mathsf{AbSh}_{\operatorname{Spec} A}$ of sheaves on $X = \operatorname{Spec} A$.

The sheaves $\widetilde{M}$ are rather special sheaves, and they play an important role in algebraic geometry. In particular, they are what one calls *$\mathcal{O}_X$-modules*. For each open set $U \subset X$, the group $\widetilde{M}(U)$ is an $\mathcal{O}_X(U)$-module in a natural way, and the restriction maps are module homomorphisms in the sense that if $V \subset U$, it holds that $as|_V = a|_V \cdot s|_V$, where $s \in \widetilde{M}(U)$ and $a \in \mathcal{O}_X(U)$. This is at least clear for the distinguished open sets $U = D(f)$: then $\widetilde{M}(D(f)) = M_f$ is a natural module over $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ and the restrictions are just localization maps. For a general U, it follows from the fundamental sequence for $\widetilde{M}(U)$ as described in part (iii) of Proposition 4.22 below.

The three main properties of the sheaf $\widetilde{M}$ are listed in the proposition that follows.

Proposition 4.22. Let A be a ring and M an A-module. The sheaf $\widetilde{M}$ on $\operatorname{Spec} A$ has the following properties.

(i) Stalks: let $x \in \operatorname{Spec} A$ be a point whose corresponding prime ideal is $\mathfrak{p}$. Then the stalk $\widetilde{M}_x$ of $\widetilde{M}$ at $x \in X$ is

$$\widetilde{M}_x = M_{\mathfrak{p}};$$

(ii) Sections over distinguished open sets: if $f \in A$, one has

$$\Gamma(D(f), \widetilde{M}) = M_f.$$

In particular, $\Gamma(\operatorname{Spec} A, \widetilde{M}) = M$;

(iii) Sections over arbitrary open sets: for any open subset U of $\operatorname{Spec} A$ covered by the distinguished open sets $\{D(f_i)\}_{i\in I}$, there is an exact sequence

$$0 \longrightarrow \Gamma(U, \widetilde{M}) \longrightarrow \textstyle\prod_i M_{f_i} \xrightarrow{\ \beta\ } \textstyle\prod_{i,j} M_{f_i f_j},$$

where β is given by

$$\beta\left(\frac{m_1}{f_1^{n_1}}, \ldots, \frac{m_s}{f_s^{n_s}}\right)_{i,j} = \frac{m_i}{f_i^{n_i}} - \frac{m_j}{f_j^{n_j}} \tag{4.16}$$

Proof The properties in the proposition are completely analogous to the ones in Proposition 4.4 on page 57 about the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$, and the proofs are very similar.

The first property follows because the stalks $\widetilde{M}_x$ and the localizations $M_{\mathfrak{p}}$ are direct limits of the same modules over the same directed system; the second follows from the way we defined $\widetilde{M}$, and the third follows from the sheaf exact sequence (3.2). □

Example 4.23. Let A be a ring and let $I \subset A$ be an ideal. Then $\widetilde{I}$ is an *ideal sheaf* in $\mathcal{O}_{\operatorname{Spec} A}$, i.e. for each $U \subset \operatorname{Spec} A$, the space of sections $\widetilde{I}(U)$ is an ideal of $\mathcal{O}_X(U)$. For $U = D(f)$, it holds that $\widetilde{I}(D(f))$ simply is the ideal IA_f in A_f.

Example 4.24. Let $A = k[u,v]/(u^2 + v^2 - 1)$ and $X = \operatorname{Spec} A$. Consider the A-module M given by the quotient

$$M = Ae_1 \oplus Ae_2/(ue_1 + ve_2).$$

Let us determine stalk $\widetilde{M}$ at the point $x \in X$ corresponding to the prime ideal $\mathfrak{p} = (u, v - 1)$.Since v is invertible in $A_{\mathfrak{p}}$, we can replace the relation $ue_1 + ve_2 = v(uv^{-1}e_1 + e_2)$ by $uv^{-1}e_1 + e_2$, which allows us to eliminate the factor $A_{\mathfrak{p}}e_2$. We find:

$$\widetilde{M}_x = M_{\mathfrak{p}} = A_{\mathfrak{p}}e_1 \oplus A_{\mathfrak{p}}e_2/(uv^{-1}e_1 + e_2) \simeq A_{\mathfrak{p}} = \mathcal{O}_{X,x}.$$

Similar arguments show that $(\widetilde{M})_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$ for every $\mathfrak{p}$, indeed, given $\mathfrak{p}$, either $u \notin \mathfrak{p}$ or $v \notin \mathfrak{p}$. Therefore $\widetilde{M}$ and $\mathcal{O}_X$ have the same stalks at every point.

5

Schemes in general

Finally, we can give the definition of a scheme:

Definition 5.1 (Schemes)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme. In other words, there is an open cover $\{U_i\}_{i\in I}$ of open subsets of X such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to some affine scheme $(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$.

Note that $(U_i, \mathcal{O}_X|_{U_i})$ is naturally a locally ringed space, as $\mathcal{O}_X|_{U_i}$ has the same local rings as $\mathcal{O}_X$ for points in U_i.

As for affine schemes, a scheme has two layers: a topological space X covered by open sets of the form $\operatorname{Spec} A_i$, and a structure sheaf $\mathcal{O}_X$ which restricts to the structure sheaves $\mathcal{O}_{\operatorname{Spec} A_i}$.

If $x \in X$ is a point, the stalk $\mathcal{O}_{X,x}$ is called the *the local ring* at x. Note that x is contained in some open subset $U = \operatorname{Spec} A$, and corresponds to some prime ideal $\mathfrak{p}$ in A, and then $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$. As before, we think of elements in $\mathcal{O}_{X,x}$ as 'rational functions defined at x', even if this is strictly true only for well-behaved schemes (see Proposition 5.27).

In the local ring $\mathcal{O}_{X,x}$ we also have the *maximal ideal* $\mathfrak{m}_x$, which in the setting above is equal to $\mathfrak{p}A_{\mathfrak{p}}$, and the corresponding *residue field* $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, which equals $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

A *morphism*, or *map* for short, between two schemes X and Y is simply a map f between X and Y regarded as locally ringed spaces. This also has two components: a continuous map, which we shall denote by f as well, and a map of sheaves of rings

$$f^\sharp \colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X,$$

with the additional requirement that the induced map on stalks $f^\sharp$ is a map of local rings, i.e., takes the maximal ideal $\mathfrak{m}_y$ into $\mathfrak{m}_x$.

In this way the schemes form a category, which we shall denote by Sch. It contains the category of affine schemes AffSch as a subcategory.

5.1 Relative schemes

There is also the notion of *relative schemes* where a base scheme S has been chosen. A *scheme over* S, or an S*-scheme*, is a scheme X together with a morphism $f \colon X \to S$, which we call the *structure map* or the *structure morphism*. A map between two schemes over S, say $X \to S$ and $Y \to S$, is a map $X \to Y$ of schemes compatible with the two structure maps; that is, a map such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

is commutative. The schemes over S form a category Sch/S, and the set of morphisms, as defined above, is denoted by $\mathrm{Hom}_S(X, Y)$.

When the base scheme S is affine, say $S = \operatorname{Spec} A$, we say that X is a *scheme over* A, and we write Sch/A for the category $\mathsf{Sch}/\mathrm{Spec\,A}$. To say that an affine scheme $\operatorname{Spec} B$ is a scheme over $\operatorname{Spec} A$ is the same thing as saying that B is an A-algebra: giving the structure map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is equivalent to giving the map of rings $f^\sharp\colon A \to B$. The Main Theorem of Affine Schemes (Theroerem 4.18 on page 64) has the following relative version.

Theorem 5.2. Let A be a ring. Then the category AffSch/A of affine schemes over A is equivalent to the category Alg/A of A-algebras (with the arrows reversed).

Note that each affine scheme $X = \operatorname{Spec} A$ has a canonical map $X \to \operatorname{Spec} \mathbb{Z}$, induced by the canonical ring map $\mathbb{Z} \to A$. In Example 6.7 we will show that the same holds for any scheme, so every scheme is a $\mathbb{Z}$-scheme.

The concept of relative schemes can be thought of as a vast generalisation of the concept 'varieties over k'. However, the extension to more general rings or even schemes turns out to be conceptually very fruitful, e.g. when discussing properties of morphisms (Chapter **??**) or fibre products (Chapter 10).

Example 5.3. The Möbius strip scheme

$$X = \operatorname{Spec} \mathbb{R}[x, y, u, v]/(vx - uy, x^2 + y^2 - 1)$$

from Example 2.37 can be viewed as a 2-dimensional scheme over $\mathbb{R}$, but one can also view it as a 1-dimensional scheme over $S = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. The latter perspective offers extra geometric insight, as all the fibres of $X \to S$ are affine lines.

5.2 Open embeddings and open subschemes

If X is a scheme and $U \subset X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on U making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is even a scheme, because if X is covered by affines $V_i = \operatorname{Spec} A_i$, then each $U \cap V_i$ is open in V_i, hence can be covered by distinguished open subsets, which are all affine schemes. Therefore there is a canonical scheme structure on U, and we call $(U, \mathcal{O}_X|_U)$ an *open subscheme* of X and say that U has the *induced scheme structure*. Moreover, a morphism of schemes $\iota\colon V \to X$ is an *open embedding* if it is an isomorphism onto an open subscheme of X.

When referring to 'an open affine' in X or 'an open affine covering' of X, we shall tacitly assume that the open sets involved are given the canonical scheme structure, and so are open *subschemes*. Thus $\operatorname{Spec} k \subset \operatorname{Spec} k[x]/(x^2)$ is not an open affine of $\operatorname{Spec} k[x]/(x^2)$ even though the subset is open and the scheme is affine.

Example 5.4. The open set $U = \mathbb{A}_k^1 - V(x)$ is an open subscheme of the affine line $\mathbb{A}_k^1 = \operatorname{Spec} k[x]$. Note the isomorphism $U \simeq \operatorname{Spec} k[x, x^{-1}] = \operatorname{Spec} k[x, y]/(xy - 1)$ of schemes.

Example 5.5 (Distinguished open subsets)**.** More generally, each distinguished open set $D(f)$ in an affine scheme $\operatorname{Spec} A$ is an open subscheme. It is affine, isomorphic to $\operatorname{Spec} A_f$. Indeed, by Lemma 2.26 the map $\iota\colon \operatorname{Spec} A_f \to \operatorname{Spec} A$ corresponding to the localization map $A \to A_f$ is a homeomorphism onto $D(f)$, and it follows readily from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_{D(f)}$ coincides with the structure sheaf on $\operatorname{Spec} A_f$.

A word of warning: an open subscheme of an affine scheme might not itself be affine.

Example 5.6. The open subset $U \subset \mathbb{A}_k^2 - V(u, v)$ of $\mathbb{A}_k^2 = \operatorname{Spec} k[u, v]$ is not an affine scheme. This is a consequence of the restriction map $\iota^\sharp\colon \mathcal{O}_{\mathbb{A}_k^2}(\mathbb{A}_k^2) \to \mathcal{O}_{\mathbb{A}_k^2}(U)$ being an isomorphism: if U were affine, the inclusion $\iota\colon U \to \mathbb{A}_k^2$ would be an isomorphism according to the Main Theorem of Affine Schemes (Theorem 4.18), but obviously it is not. To see that the restriction is an isomorphism, we resort to the sheaf sequence (**??**) on page **??** for the covering $\{D(u), D(v)\}$ of U. In view of the equalities $\mathcal{O}_{\mathbb{A}_k^2}(D(u)) = k[u, v]_u$ and $\mathcal{O}_{\mathbb{A}_k^2}(D(v)) = k[u, v]_v$, the sheaf sequence takes the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_U(U) & \xrightarrow{\alpha} & k[u,v]_u \oplus k[u,v]_v, & \xrightarrow{\beta} & k[u,v]_{uv} \\ & & \uparrow{\scriptstyle \iota^\sharp} & & & & \\ & & k[u,v] & & & & \end{array}$$

where also the restriction map $\iota^\sharp$ is indicated. Note that $\alpha(\iota^\sharp(c)) = (c, c)$. The map β sends an element $f = (au^{-n}, bv^{-m})$ to $au^{-n} - bv^{-m}$, and f lies in the kernel of β precisely when $au^{-n} = bv^{-m}$; or in other words, when $av^m = bu^m$. As the polynomial ring is a UFD, we conclude that $a = cu^m$ and $b = cv^m$ for some $c \in k[u, v]$, so that $f = (c, c)$. That is, $\iota^\sharp$ is surjective, and since it is clearly injective, it is an isomorphism.

Exercise 5.2.1. Consider the ring $R = \mathbb{Z}[t]$ and let $X = \operatorname{Spec} R$.

a) For a prime number p, show that $\mathfrak{m} = (t, p)$ is a maximal ideal of R.
b) Let $U = X - \{\mathfrak{m}\}$. Show that $U = D(p) \cup D(t)$ and that

$$\mathcal{O}_X(U) = \mathbb{Z}[t]$$

c) Deduce that U is not affine.

5.3 Closed embeddings and closed subschemes

In this section, we explain what it should mean to be a *closed subscheme* of a scheme. Intuitively, a closed subscheme is given by a scheme Z, which is embedded as a closed subset $Z \subset X$. Given that there are many possibilities for choosing the scheme structure on the same underlying closed set, and this makes the definition slightly more subtle than the one for open subscheme. The prototypical example to have in mind is $\operatorname{Spec}(A/\mathfrak{a})$, which as we have seen, embeds naturally as the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} A$ (Proposition 2.27 on page 32). In general, a closed subscheme is a scheme $(Z, \mathcal{O}_Z)$ with a morphism $\iota\colon Z \to X$, which locally looks like the map $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$. We formalize this in the next two definitions.

Definition 5.7 (Closed embeddings and closed subschemes)**.** A morphism $\iota : Z \to X$ is called a *closed embedding* if there is an affine cover $\{U_i\}_{i\in I}$ of X such that

(i) $\iota^{-1}(U_i)$ is affine for every $i \in I$;

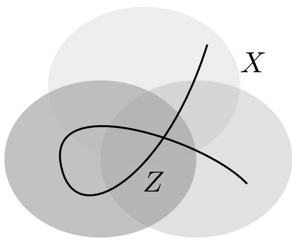
(ii) the ring map

$$\iota^\sharp : \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_Z(\iota^{-1}U_i)$$

is surjective for every i.

We say that Z is a *closed subscheme* of X. Two closed subschemes Z, Z' are said to be *equal* if there is an isomorphism $\phi : Z \to Z'$ such that $\iota = \iota' \circ \phi$.

In other words, X and Z are covered by affine schemes $U_i = \operatorname{Spec}(A_i)$, and $\iota^{-1}(U_i) = \operatorname{Spec} B_i$, so that for each i, the induced ring map $A_i \to B_i$ is surjective, which means that $B_i = A_i/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i$. Moreover, the morphism $\iota^{-1}U_i \to U_i$ is given by the canonical morphism $\operatorname{Spec}(A_i/\mathfrak{a}_i) \to \operatorname{Spec}(A_i)$.



Even if a closed subscheme Z is defined as an abstract scheme which maps into X, we usually think of it as a closed subset of X. This is reasonable because the image $V = \iota(Z)$ is a closed subset (the U_i's form an open cover of X, and each subset $\iota(Z) \cap U_i$ is closed being equal to $V(\mathfrak{a}_i)$). Moreover, we may put a structure sheaf on V by defining $\mathcal{O}_V$ to be $\iota_*\mathcal{O}_Z$.

Example 5.8. The schemes $\operatorname{Spec} k[x]/(x^n)$ with $n \in \mathbb{N}$ and k a field, give different subschemes of $\mathbb{A}^1_k$. Still, the underlying topological spaces are the same (a single point), and these spectra are homeomorphic. However, having non-isomorphic structure sheaves, they are not isomorphic as schemes.

Example 5.9. Consider the affine 4-space $\mathbb{A}^4_k = \operatorname{Spec} A$, with k a field and $A = k[x, y, z, w]$. Then the three ideals

$$I_1 = (x, y), \quad I_2 = (x^2, y) \text{ and } I_3 = (x^2, xy, y^2, xw - yz),$$

have the same radical (x, y), and thus give rise to the same closed subset $V(x, y) \subset \mathbb{A}^4_k$, but they give different closed subschemes of $\mathbb{A}^4_k$.

Classifying closed subschemes according to the above definition is not so easy, even for affine schemes. Of course, each ideal $\mathfrak{a} \subset A$, yields the closed subcheme $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$, but because the definition refers to a specific affine covering, it is a priori not obvious that all closed subschemes arise in this way, or even if every closed subscheme of $\operatorname{Spec} A$

is an affine scheme. This is nevertheless true, but we will need to postpone the proof until Chapter **??**, where we give a more systematic treatment of closed subschemes in terms of ideal sheaves.

Proposition 5.10. Let $X = \operatorname{Spec} A$ be an affine scheme. The map $\mathfrak{a} \mapsto \operatorname{Spec}(A/\mathfrak{a})$ is a one-to-one correspondence between the set of ideals of A and the set of closed subschemes of X. In particular, each closed subscheme of an affine scheme is also affine.

For later use, we include the following definition, which combines the two types of embeddings we have seen:

Definition 5.11 (Locally closed embeddings)**.** A morphism $f\colon Z \to X$ is said to be a *locally closed embedding* if it is the composition of an open and a closed imbedding. That is, if $f = g \circ h$ with $g\colon U \to X$ an open embedding and $h\colon Z \to X$ a closed embedding.

Exercises

Exercise 5.3.1. Show that being a closed embedding is a property which is 'local on the target'. In clear text: given a morphism $f\colon Z \to X$ and an open cover $\{U_i\}$ of X. Let $V_i = f^{-1}U_i$ and assume that each restriction $f|_{V_i}\colon V_i \to U_i$ is a closed embedding. Prove that then also f is a closed embedding.

Exercise 5.3.2. Show that being a locally closed embedding is 'local on the image'. Assume that $f\colon Z \to X$ is a morphism and that $\{U_i\}$ is a family of open subsets of X covering the image $f(Z)$. Assume further that each restriction $f|_{f^{-1}U_i}\colon f^{-1}U_i \to U_i$ is a closed embedding, then f is a locally closed embedding.

Exercise 5.3.3. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. Prove that if both f and g are closed embeddings, then $g \circ f$ is one as well.

5.4 R-valued points

A point in an affine variety X over an algebraically closed field k can be viewed as a solution in k to a finite set of polynomial equations. This perspective is lost when transitioning to spectra of general rings. Still, given a scheme such as

$$X = \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n]/(f_1, \ldots, f_r), \tag{5.1}$$

we can still talk about solutions to the defining equations, but there are many choices of fields where to consider solutions. In fact, since the polynomials have integer coefficients, the equations $f_1(t) = \cdots = f_r(r) = 0$ are meaningful over any ring R. This leads to the notion of an 'R-valued point' of a scheme.

Formally, an *R-valued point*, or an *R-point*, of a scheme X is simply a morphism

$\operatorname{Spec} R \to X$. The set of all such morphisms will be denoted by $X(R)$; that is, we define

$$X(R) = \operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} R, X).$$

Note that if $f\colon X \to Y$ is a map of schemes, composition gives an induced map of sets $X(R) \to Y(R)$. The sets $X(R)$ also depend functorially on R. To every ring map $R \to S$ there is a corresponding map of schemes $\operatorname{Spec} S \to \operatorname{Spec} R$, which induces a map of sets $X(R) \to X(S)$. Therefore the scheme X determines a *functor* $X\colon \mathsf{Rings} \to \mathsf{Sets}$. We will explore the link between a scheme and its associated functor in Chapter 10.7.

Example 5.12. The R-points of the affine space over $\mathbb{Z}$, $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n]$, is just R^n. Indeed, elements $f \in \mathbb{A}^n(R)$ are by definition maps of schemes

$$\operatorname{Spec} R \longrightarrow \mathbb{A}^n,$$

which according to Theorem 4.18 on page 64 correspond bijectively to ring maps

$$\phi\colon \mathbb{Z}[t_1, \ldots, t_n] \longrightarrow R. \tag{5.2}$$

These in turn, are in bijection with the n-tuple $(\phi(t_1), \ldots, \phi(t_n))$ in R^n.

In particular, for fields k it holds that $\mathbb{A}^n(k) = k^n$, which explains the notation $\mathbb{A}^n(k)$ used in Chapter 1.

Example 5.13. Going one step further, given an ideal $\mathfrak{a} = (g_1, \ldots, g_r) \subset \mathbb{Z}[t_1, \ldots, t_n]$, consider the corresponding affine scheme $X = \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n]/\mathfrak{a}$. Scheme maps $\operatorname{Spec} R \to X$ are in a one-to-one correspondence with ring maps

$$\phi\colon \mathbb{Z}[t_1, \ldots, t_n]/\mathfrak{a} \longrightarrow R,$$

again according to Theorem 4.18. Such maps are in turn in bijection with ring maps ϕ as in (5.2) that vanishes on the ideal $\mathfrak{a}$; that is, they are in bijection with n-tuples $(a_1, \ldots, a_n) \in R^n = \mathbb{A}^n(R)$ such that $g_i(a_j) = 0$.

Example 5.14. For a specific example, consider the scheme

$$X = \operatorname{Spec} \mathbb{Z}[u, v]/(u^2 + v^2 - 1).$$

Then the set $X(\mathbb{R})$ of $\mathbb{R}$-points consists of the points of the unit circle in $\mathbb{R}^2$; the $\mathbb{Z}$-points $X(\mathbb{Z})$ consists of the four points $(\pm 1, 0)$ and $(0, \pm 1)$, while one may verify that the rational points; that is, the $\mathbb{Q}$-points, are given by

$$X(\mathbb{Q}) = \left\{ \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right) \bigm| t \in \mathbb{Q} \right\} \cup \{(0, -1)\}.$$

In Exercise **??** you are asked to verify this.

Example 5.15 (A conic with no real points)**.** Let $X = \operatorname{Spec} A$, where A is the algebra $A = \mathbb{R}[u, v]/(u^2 + v^2 + 1)$. The equation $u^2 + v^2 + 1 = 0$ has no real solutions, so $X(\mathbb{R}) = \varnothing$. However, the set $X(\mathbb{C})$ is infinite because the equation has infinitely many complex solutions (two for each choice of $v \in \mathbb{C}$). Note also that A has infinitely many maximal ideals, so that the underlying topological space of X is infinite.

The sets $X(R)$ of R-points are clearly important in number theory. A rather extreme example of this is Fermat's Last Theorem, which asks about the set $X(\mathbb{Q})$ where $X = \operatorname{Spec}\mathbb{Z}[x,y,z]/(x^n + y^n - z^n)$. This example shows that even when R is a field, it can be very difficult to describe the set $X(R)$ of R-valued points, or even determining whether $X(R) \neq \varnothing$. However, sometimes scheme theory can shed light on this problem, e.g. showing that $X(K) \neq \varnothing$ provided, say $X(L) \neq \varnothing$ for some suitable field extension $K \subset L$.

Example 5.16 (Non-existence of $\mathbb{Z}$-points)**.** The equation $3x^2 - 7y^2 = 1$ has no solution in integers x and y. Indeed, modulo 3, the equation reduces to $2y^2 = 1 \mod 3$, but $2y^2$ must be 0 or 2 modulo 3. In geometric terms, the scheme

$$X = \operatorname{Spec}\mathbb{Z}[x,y]/(3x^2 - 7y^2 + 1)$$

has no $\mathbb{Z}$-points; any $\mathbb{Z}$-point of X would survive via the map $X(\mathbb{Z}) \to X(\mathbb{F}_3)$ induced by the reduction mod 3 map $\mathbb{Z} \to \mathbb{F}_3$.

Likewise, $X(\mathbb{R}) \neq \varnothing$ is a necessary condition for the existence of $\mathbb{Z}$-points.

One says that a scheme X satisfies the *Hasse principle* if these conditions are also sufficent, that is, if $X(\mathbb{R}) \neq 0$ and $X(\mathbb{F}_p) \neq 0$ for all primes p implies $X(\mathbb{Z}) \neq 0$. The Hasse principle holds in some cases, e.g., when X is defined by a quadratic polynomial, but it fails in general. The *Selmer curve*

$$X = \operatorname{Spec}\mathbb{Z}[x,y]/(3x^3 + 4y^3 + 5)$$

has points over $\mathbb{R}$ and every $\mathbb{F}_p$, but none over $\mathbb{Z}$.

In the examples above, the sets $X(R)$ are rather manageable. However, the sets $X(R)$ can in fact be enormous even when K is a field. For instance, the next example shows that the set $X(\mathbb{C})$ is uncountable, even for $X = \operatorname{Spec}\mathbb{C}$.

For this reason it is important to consider the relative situation. When X is a scheme over some base ring A, and R is an A-algebra, one has the sets

$$X_A(R) = \operatorname{Hom}_{\mathsf{Sch}/A}(\operatorname{Spec} R, X)$$

of relative morphisms over A. These satisfy the same functorial properties as the sets $X(R)$ above, but in many cases they will be more manageable.

Example 5.17. We have

(i) $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{C}}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}) = \{\mathrm{id}_{\operatorname{Spec}\mathbb{C}}\}$;

(ii) $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{R}}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C}) = \{\mathrm{id}_{\operatorname{Spec}\mathbb{C}}, \iota\}$, where $\iota\colon \mathbb{C} \to \mathbb{C}$ is complex conjugation map;

(iii) $\operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec}\mathbb{C}, \operatorname{Spec}\mathbb{C})$ is the set of all field automorphisms of $\mathbb{C}$, or in other words, the Galois group of $\mathbb{C}$ over $\mathbb{Q}$. This is an uncountable group.

Points in schemes

A single scheme X gives rise to many sets of R-valued points $X(R)$. For instance, $\mathbb{A}^n$ simultaneously gives rise to all the possible $\mathbb{A}^n(k)$'s from Chapter 1, by varying the field k.

For a scheme X, looking at maps from spectra of fields into X help us understand the points of X. Every point of X is a K-point for some field K.

The residue fields play an important role here. If $x \in X$ is a point, there is a canonical map

$$\iota_x : \operatorname{Spec} k(x) \to X$$

which maps the only point of $\operatorname{Spec} k(x)$ to x. To see this, suppose x is contained in an open affine subset $U = \operatorname{Spec} A$ and corresponds to a prime ideal $\mathfrak{p} \subset A$. Then the residue field is given by $k(x) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, and there is a morphism $\iota_x : \operatorname{Spec} k(x) \to X$ defined by the composition

$$\operatorname{Spec} k(x) \to \operatorname{Spec}(A_\mathfrak{p}) \to U \to X.$$

It is not hard to see that this is independent of the choice of U (see Exercise 5.4.6).

Thus this is a way to organize the points of X according to their residue fields.

The residue field $k(x)$ and the morphism $\iota_x : \operatorname{Spec} k(x) \to X$ satisfy a certain universal property for K-points in general:

Lemma 5.18. Let X be a scheme and let $x \in X$ be a point. For a field K, there are natural bijections between:

(i) K-valued points $f : \operatorname{Spec} K \to X$ with image x;
(ii) Maps of local rings $\mathcal{O}_{X,x} \to K$;
(iii) Maps of fields $k(x) \to K$;

Proof (i)⇒ (ii). If $f : \operatorname{Spec} K \to X$ is a morphism which maps the point $y \in \operatorname{Spec} K$ to x, the sheaf part of the morphism gives a map of local rings $f_y^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} K,y} = K$.

(ii)⇒ (iii). If $\mathcal{O}_{X,x} \to K$ is a map of local rings, it maps the maximal ideal of $\mathcal{O}_{X,x}$ to the maximal ideal of K, namely (0), and hence it induces a map between the fields $k(x) \to K$.

(iii)⇒ (i) Let $k(x) \to K$ be a map of fields. Let $U = \operatorname{Spec} A$ be an affine open set containing x, so that x corresponds to a prime ideal $\mathfrak{p}$ in A. Then composing $k(x) \to K$ with the map ι_x, we get a map of schemes $\operatorname{Spec} K \to X$, i.e., a K-point with image x. □

Corollary 5.19. Let X be a scheme and let K be a field. Then there is a bijection

$$X(K) = \left\{ (x, \alpha) \;\middle|\; \begin{array}{l} x \in X \text{ is a point;} \\ \alpha\colon k(x) \to K \text{ is a field embedding} \end{array} \right\}.$$

Exercises

Exercise 5.4.1. Let $X = \operatorname{Spec} \mathbb{Z}$. Compute $X(\mathbb{F}_p)$, $X(\mathbb{Q})$ and $X(\mathbb{C})$.

Exercise 5.4.2. Verify the claim about $X(\mathbb{Q})$ in Example 5.14. HINT: Compute the second intersection point a general line trough $(0, 1)$ has with the unit circle.

Exercise 5.4.3. With reference to Example 5.14, show that one may interpret $X(\mathbb{Q})$ as the set of Pythagorean triples:

$$X(\mathbb{Q}) = \{\, (a, b, c) \in \mathbb{Z}^3 \mid a^2 + b^2 = c^2 \text{ and } a, b, c \text{ relatively prime} \,\}.$$

Exercise 5.4.4. With reference to Example 5.14, let p be a prime such that $p \not\equiv 1 \mod 4$. Show that the description in Example 5.14 also is valid for $X(\mathbb{F}_p)$.

Exercise 5.4.5. With reference to Example 5.15, consider the natural inclusion

$$A = \mathbb{R}[u, v]((u^2 + v^2 + 1) \subset \mathbb{C}[u, v]/(u^2 + v^2 + 1) = A_{\mathbb{C}}.$$

For each point $z = (a, b) \in X(\mathbb{C})$ consider the ideal $\mathfrak{n}_z = \mathfrak{m}_z \cap A$. Show that $\mathfrak{n}_x$ is maximal and that $\mathfrak{n}_z = \mathfrak{n}_w$ if and only if $w = (\bar{a}, \bar{b})$ with $z = (a, b)$. Conclude that A has infinitely many maximal ideals.

Exercise 5.4.6. Let X be a scheme and let $x \in X$ be a point.

a) Show that there is a canonical morphism

$$f : \operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow X$$

b) Show that the map $\iota_x : \operatorname{Spec} k(x) \to X$ definied in the text factors via f.
c) Show that on the level of topological spaces, the image of f is the intersection of all open neighbourhoods containing x.
d) Compute the image of f when:
   (i) x is the generic point of an irreducible scheme;
   (ii) x is a closed point of $\mathbb{A}^2_{\mathbb{C}}$.

5.5 Basic geometric properties of schemes

There are a few basic properties of schemes that only concern the underlying topological space. We have seen some of these already:

- X is *irreducible* if it cannot be decomposed as $X = Y \cup Z$ where Y, Z are proper closed subsets.
- X is *connected* if it cannot be decomposed as $X = U \cup V$ where U, V are disjoint open sets.
- X is *quasi-compact* if any open cover has a finite subcover.

We have already studied these notions for affine schemes. Here $\operatorname{Spec} A$ is irreducible if and only if A has a unique minimal prime, i.e., if $\sqrt{(0)}$ is prime (Proposition 2.17). $\operatorname{Spec} A$ is connected if and only if $A \neq B \times C$ for two non-trivial rings B, C (Proposition 2.20). $\operatorname{Spec} A$ is always quasi-compact (see page 30). (See Exercise 2.5.6 for a scheme which is not quasi-compact.)

Exercise 5.5.1. Find an example of a connected scheme X with a disconnected open subset $U \subset X$.

Reduced schemes and integral schemes

Recall that a ring A is said to be *reduced* if it has no non-zero nilpotent elements. We define a scheme $(X, \mathcal{O}_X)$ to be *reduced* if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

Lemma 5.20. A scheme X is reduced if and only if for every open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no non-zero nilpotents.

Proof Assume first that X is reduced. Any non-zero nilpotent element in one of the rings $\mathcal{O}_X(U)$ would have a non-zero germ in at least one local ring $\mathcal{O}_{X,x}$, which would then not be reduced. For the reverse implication, let $x \in X$ be a point and let $s \in \mathcal{O}_{X,x}$ be any element. We may write s as the germ of some section $t \in \mathcal{O}_X(U)$, which can not be nilpotent; hence s is not nilpotent either. □

Example 5.21. An affine scheme $X = \operatorname{Spec} A$ is reduced precisely when A is a reduced ring. Thus $\mathbb{A}^n_k$ is reduced, but $\operatorname{Spec} k[x]/(x^n)$ for $n \geqslant 2$, is not.

One says that a scheme is *integral* if it is both irreducible and reduced. An affine scheme $\operatorname{Spec} A$ is integral if and only if A is an integral domain. Indeed, $\operatorname{Spec} A$ is reduced if and only if A has no nilpotents; that is, if and only if the nilradical vanishes, and $\operatorname{Spec} A$ is irreducible if and only if the nilradical is prime. These two statements imply that the zero-ideal is prime, and so A is an integral domain.

Moreover, it is not hard to prove the following:

Proposition 5.22. A scheme X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open $U \subset X$.

One important fact about integral schemes is that they have a *function field*, $k(X)$, which contains all the rings $\mathcal{O}_X(U)$ as subrings.

To define $k(X)$, recall that any integral scheme has a unique generic point η. The generic point is the only point which is dense, i.e., belongs to every open non-empty subset of X. If $U = \operatorname{Spec} A$ is an open affine, η corresponds to the zero ideal (0) of A, and the local ring $\mathcal{O}_{X,\eta}$ is equal to the field of fractions $K(A)$ of A. We define the *function field* $k(X)$ of X to be $K(A)$. That is, $k(X)$ is the local ring $\mathcal{O}_{X,\eta}$ at the generic point.

Example 5.23. The function field of $\operatorname{Spec} \mathbb{Z}$ equals $\mathcal{O}_{\operatorname{Spec} \mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$.

Example 5.24. The function field of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ equals the field $k(x_1, \ldots, x_n)$ of rational functions in $x_1, \ldots, x_n$.

Example 5.25 (The quadratic cone)**.** The quadratic cone $Q = \operatorname{Spec} k[x, y, z]/(x^2 - yz)$ is integral being the spectrum of an integral domain ($x^2 - yz$ is irreducible), and the function field of Q is equal to

$$K(k[x, y, z]/(x^2 - yz)) \simeq k(x, y)$$

since we can eliminate z using that $z = y^{-1}x^2$ (note that y is invertible in $k(Q)$).

We showed in Example 4.10 that each $\mathcal{O}_X(U)$ is a subring of $k(X)$ when X was an integral affine scheme. The same argument as in the example works more generally. For any non-empty open U the ring $\mathcal{O}_X(U)$ is an integral domain with fraction field $(\mathcal{O}_X(U))_0 = \mathcal{O}_{X,\eta} = k(X)$, and the canonical germ map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ is identified with the inclusion $\mathcal{O}_X(U) \subset k(X)$. These inclusions are moreover compatible with restrictions, i.e., all diagrams

$$\begin{array}{ccc} \mathcal{O}_X(U) & \hookrightarrow & k(X) \\ {\scriptstyle \rho_{UV}}\downarrow & \nearrow & \\ \mathcal{O}_X(V) & & \end{array}$$

where $V \subset U$ are two open subsets, commute. This shows that we may view $\mathcal{O}_X(U)$ as a subsheaf of the constant sheaf $k(X)$ on X.

Taking direct limits, we see that also all the local rings $\mathcal{O}_{X,x}$ lie as subrings of $k(X)$. We say that an element $f \in k(X)$ is *defined* at the point x if $f \in \mathcal{O}_{X,x}$.

Lemma 5.26. Let X be an integral scheme and let $f \in k(X)$. The set

$$U_f = \{\, x \in X \mid f \in \mathcal{O}_{X,x} \,\}$$

is open.

Proof Let $x \in U_f$ and let $\operatorname{Spec} A$ be an affine neighbourhood of x. Consider the ideal $\mathfrak{a}_f = \{\, b \in A \mid bf \in A \,\}$. If $\mathfrak{p}$ is a prime in A, then $f \in A_\mathfrak{p}$ if and only if $\mathfrak{a}_f \nsubseteq \mathfrak{p}$; that is, $V(\mathfrak{a}_f)$ is the complement of $U_f \cap \operatorname{Spec} A$ in $\operatorname{Spec} A$. □

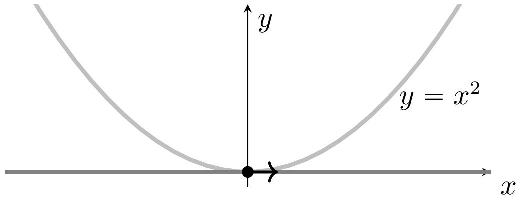
Proposition 5.27. Let X be an integral scheme with function field $k(X)$ and let $U \subset X$ be open. Then

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} = \left\{ f \in k(X) \;\middle|\; \begin{array}{l} \text{for each point } x \in U,\ f \text{ can be} \\ \text{represented as } g/h \text{ where } h(x) \neq 0 \end{array} \right\} \subset k(X).$$

Proof There are two equalities to prove here. To prove the first, assume first that U is affine, say $U = \operatorname{Spec} A$. Then the first equality amounts to the equality $A = \bigcap A_\mathfrak{p}$ where the intersection extends over all prime ideals in A. The inclusion $A \subset \bigcap A_\mathfrak{p}$ is trivial. To verify the other, assume that the ideal $\mathfrak{a}_f$ is proper. It will then be contained in a maximal ideal $\mathfrak{m}$, and consequently $f \notin A_\mathfrak{m}$. If U is a general open subset, the equality follows from the equality $\mathcal{O}_X(U) = \bigcap \mathcal{O}_X(V)$, where the intersection extends over all non-empty open affine subsets $V \subset U$. This holds because $\mathcal{O}_X(U)$ equals the inverse limit $\mathcal{O}_X(U) = \varprojlim \mathcal{O}_X(V)$, and this inverse limit becomes the intersection when all the rings are identified with subrings of $k(X)$.

To prove the second equality, let $x \in X$ be a point, and let $\operatorname{Spec} A$ be an open affine subset containing x. Then $k(X)$ equals the fraction field K of A. An element $f \in K$ lies in $\mathcal{O}_{X,x} = A_\mathfrak{p} \subset K$ if and only if it can be expressed as a quotient $f = a/s$ where $s \notin \mathfrak{p}$. □

Example 5.28. Non-reduced schemes appear frequently when two schemes X and Y intersect. For instance, consider the parabola $X = \operatorname{Spec} k[x,y]/(y - x^2)$ and the line $Y = \operatorname{Spec} k[x,y]/(y)$. The intersection of these is given by the ideal $I = (y - x^2, y) = (x^2, y)$, which is not a radical ideal. The nilpotent elements of $k[x,y]/(x^2,y) = k[x]/(x^2)$ in some sense account for the 'tangency' of the intersection $X \cap Y$.



Example 5.29. Here is a similar example in $\mathbb{A}^3_k$. Consider

$$X = \operatorname{Spec} k[x, y, z]/(z - xy^2)$$

which is a closed subscheme of $\mathbb{A}^3$ (a cubic surface). The intersection of X with the plane of equation $x = 0$ is given by the ideal $I = (z, xy^2)$, whose primary decomposition is

$$(z, xy^2) = (z, y^2) \cap (x, z).$$

The intersection $\operatorname{Spec} k[x, y, z]/I$ therefore is the union of the lines $y = z = 0$ and $x = z = 0$. Being defined by the non-radical ideal (z, y^2), the component along the former has 'multiplicity 2', which reflects the fact that the plane is tangent to X along that line. So the intersection is neither irreducible nor reduced.

Example 5.30 (Schemes of matrices)**.** Consider the scheme

$$\mathbb{M}_{n\times n} = \mathbb{A}^{n^2} = \operatorname{Spec} \mathbb{Z}[x_{ij} | 1 \leqslant i, j \leqslant n]$$

As the notation suggests, the k-points of this scheme parameterize $n \times n$-matrices with entries in k. The scheme $\mathbb{M}_n$ contains several interesting subschemes:

The *general linear group* GL_n is the subset of $\mathbb{M}_n$ consisting of invertible matrices. It is an open subscheme, in fact, it equals the the distinguished open set $D(\det M)$, where $\det M$ is the determinant of the matrix of variables $M = (x_{ij})_{ij}$.

There is also the the *special linear group* SL_n consisting of matrices of determinant one, is the set of closed points in $V(\det M - 1) \subset \mathbb{M}_{n\times n}$.

The *orthogonal group* $\mathrm{O}(n)$ corresponds to the matrices such that $M^t M$ is the identity matrix. It a closed subscheme, defined by the ideal I generated by the entries in $n \times n$ matrix $M^t M - I$ (which are polynomials in the x_{ij}'s). If we further impose the condition $\det M = 1$, we obtain the *special orthogonal group* $\mathrm{SO}(n)$.

Example 5.31 (Nilpotent matrices)**.** Particularly interesting examples of subschemes of $\mathbb{M}_n$ are the set of nilpotent matrices, i.e. matrices A such that $A^k = 0$ for some $k > 0$.

We continue working with the matrix M of variables from the prevous example. The equation $M^n = 0$ gives n^2 degree n polynomial relations in the variables x_{ij}, and the ideal J they generate define a closed subscheme $N = \operatorname{Spec}(\mathbb{Z}[x_{ij}]/J)$ of $\mathbb{M}_n$. Bearing in mind that an $n \times n$-matrix A is nilpotent if and only if $A^n = 0$, the k-points of N is the set of nilpotent matrices in k.

Interestingly, the subscheme N is typically non-reduced. Indeed, recall that the characteristic polynomial $\det(tI_n - A)$ of a matrix A equals t^n if and only if A is nilpotent (equivalent to all eigenvalues being zero), so in particular the trace $\operatorname{Tr} A$ (the sum of the eigenvalues or the subleading coefficient) of A vanishes. This means that $\operatorname{Tr} M = \sum x_{ii}$ vanishes in all closed points of N. So $\operatorname{Tr} M$ induces a nilpotent element in $k[x_{ij}]/J$, but being linear, $\operatorname{Tr} M$ does not lie in J.

One may put a different scheme structure on the set of nilpotent matrices, using the fact that a matrix A is nilpotent if and only if it has characteristic polynomial equal to t^n. Note that the coefficients of the characteristic polynomial

$$\det(tI - M) = t^n - c_1(M)t^{n-1} + \cdots + (-1)^n c_n(M)$$

are polynomials in the entries of M, so we see that we get n equations $c_1(M) = \cdots = c_n(M) = 0$, that define a subscheme in $\mathbb{M}_n$ with the same underlying topological space as N. In fact, it is not too hard to check that the ideal I generated by the $c_i(M)$'s is radical, so that $\operatorname{Spec}(k[x_{ij}]/I)$ is reduced.

Exercise 5.5.2. Describe $X = \operatorname{Spec} \mathbb{Z}[x]/(5x - 15)$. Is X irreducible? Reduced? What are the fibres of the canonical map $X \to \operatorname{Spec} \mathbb{Z}$?

Exercise 5.5.3. Let X be an integral scheme and $U \subset X$ an open subset. Show that $x \in U$ if and only if $\mathcal{O}_X(U) \subset \mathcal{O}_{X,x}$ inside $k(X)$.

5.6 Affine varieties and integral schemes

We have mentioned a few times that schemes are generalizations of algebraic varieties. On the other hand, we have also seen that even the simplest schemes, e.g. $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, behave differently than varieties in the sense that they have many non-closed points. Thus for this statement to make sense, we should expect there to be a natural way to ‘add non-closed points’ to an algebraic variety so that the resulting topological space has the structure of a scheme. Let us explain what this means more precisely.

Let k be an algebraically closed field and let X be an affine variety over k. Let $A = A(X)$ denote its affine coordinate ring; it is canonically attached to X, being the ring of regular functions on X. From A, we can build the affine scheme $X^s = \operatorname{Spec} A$. Note that the closed points of X^s are in bijection with the points of X (that is, $X^s(k) = X$) by the Nullstellensatz. In particular, there is a natural injection $X \subset X^s$. Thus as a set, X^s is obtained by adding to X the non-maximal prime ideals $\mathfrak{p}$ in A; there is one for each subvariety of X of positive dimension. Note that $V(I) \cap X = Z(I)$, so the classical Zariski topology on X is simply the induced topology from $X^s = \operatorname{Spec} A$.

The ring A is a finitely generated k-algebra with no zerodivisors. This means that X^s is an integral scheme over k.

The structure sheaf $\mathcal{O}_{X^s}$ on X^s is also constructed via the ring A via the various localizations. Proposition 5.27 tells us that the elements of $\mathcal{O}_{X^s}(U)$ over an open set $U \subset X^s$ can be identified with the ring of regular functions $f : U(k) \to k$

Associating X with X^s also behaves well with regard to morphisms. The fundamental theorem of affine varieties tells us that maps $\phi\colon X \to Y$ between two affine varieties are in one-one-correspondence with k-algebra homomorphisms $\phi^\sharp\colon A(Y) \to A(X)$. This exactly parallels our Theorem **??** for schemes. Hence putting $\phi^s = \operatorname{Spec} \phi^\sharp$, we obtain a morphism $\phi^s\colon X^s \to Y^s$ which extends ϕ. As $\phi^\sharp$ is a map of k-algebras, the morphism ϕ^s is a morphism of schemes over $\operatorname{Spec} k$. Moreover, any morphism of schemes $X^s \to Y^s$ arises in this way. This means that there is a functorial bijection

$$\operatorname{Hom}_{\mathsf{AlgSets}/k}(X, Y) = \operatorname{Hom}_{\mathsf{Sch}/k}(X^s, Y^s).$$

In particular, the assignment $X \mapsto Y^s$ gives a fully faithful functor from affine varieties to affine schemes over k. In particular, two varieties give rise to isomorphic schemes over k if and only if they are isomorphic as varieties, and each scheme isomorphism is uniquely determined by the variety isomorphism. In particular, this tells us that the category of varieties Var/k is equivalent to a full subcategory of Sch/k. We have already seen that this is a strict subcategory, e.g. $\operatorname{Spec} k[x]/(x^2)$ does not come from a variety.

5.7 Exercises

Exercise 5.7.1. Which of the topologies on a set with three points is the underlying topology of a scheme?

Exercise 5.7.2. Let X be a scheme.

a) Show that any irreducible and closed subset $Z \subset X$ has a unique generic point. HINT: Reduce to the affine case.
b) Show that in general schemes are not Hausdorff. What are the possible underlying topologies of affine schemes that are Hausdorff?
c) Show that X satisfies the zeroth separation axiom (they are T_0); that is, given two points x and y in X, there is an open subset of X containing one of them but not the other.

Exercise 5.7.3 (The sheaf of units)**.** Let X be a scheme with structure sheaf $\mathcal{O}_X$. We say that $s \in \mathcal{O}_X(U)$ is a *unit* if there exists a multiplicative inverse $s^{-1} \in \mathcal{O}_X(U)$.

a) Show that $s \in \mathcal{O}_X(U)$ is a unit if and only if for all $x \in U$, the germ s_x is a unit in the ring $\mathcal{O}_{X,x}$; that is, if and only if s_x does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.
b) We let $\mathcal{O}_X^\times(U)$ denote the subgroup of units in $\mathcal{O}_X(U)$. Show that $\mathcal{O}_X^\times(U)$ is a sub*sheaf* of $\mathcal{O}_X$.

Exercise 5.7.4 (The Frobenius morphism)**.** Let p be a prime number and let A be a ring of characteristic p. The ring map $F_A : A \to A$ given by $a \mapsto a^p$ is called the *Frobenius map* on A.

a) Show that F_A induces the identity map on $\operatorname{Spec} A$;
b) Show that if A is local, then F_A is a map of local rings;

c) For a scheme X over $\mathbb{F}_p$, define the *Frobenius morphism* $F_X : X \to X$ by the identity on the underlying topological space and with $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$. Show that F_X is a morphism of schemes;
d) Show that F_X is natural in the sense that if $f : X \to Y$ is a morphism of schemes over $\mathbb{F}_p$, we have $f \circ F_X = F_Y \circ f$.

In particular, this exercise shows that for a morphism of schemes $f : X \to Y$, in order to check that f is an isomorphism, is not enough to check that f is a homeomorphism; also the map $f^\sharp$ must be an isomorphism.

Exercise 5.7.5. Let X an integral scheme over a ring A, and let $f \in k(X)$. Show that there is a morphism $\phi\colon U_f \to \mathbb{A}^1_A$ such that $\phi^\sharp\colon A[t] \to \Gamma(U_f, \mathcal{O}_X)$ is given by $t \mapsto f$.

Exercise 5.7.6. Prove Proposition **??**. That is, prove that a scheme X is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open $U \subseteq X$.

Exercise 5.7.7. Let X be a scheme and let $x \in X$ be a point. Show that x is a closed point if and only if the corresponding morphism $\operatorname{Spec} k(x) \to X$ is finite.

Exercise 5.7.8. Let $X = \operatorname{Spec} k[x, y, z, w]/(xw - yz)$ and consider the open set $U = X - V(x, y)$. Use the above strategy as in Example 5.6 to compute $\mathcal{O}_X(U)$. Conclude that U is not affine.

Exercise 5.7.9. Prove Corollary 5.22.

Exercise 5.7.10. Prove that a composition of two closed embeddings is a closed embedding.

6

Gluing

It is sometimes said that 'algebraic geometry is the study of the geometry of zero sets of polynomials'. After Grothendieck, perhaps a more precise slogan would be that 'algebraic geometry is the geometry of rings'.

While this certainly has an amount of truth to it, the theory of schemes is much richer than just the spectra of rings. This is essentially due to the enormous flexibility gluing gives: we are allowed to glue together new schemes out of old ones, as well as sheaves on them, and also morphisms between these. The aim of this chapter is to explain the conditions under which this can be done. We begin with gluing together sheaves and maps between them (which is the easiest case and which works for any topological space), and then move on to schemes and morphisms. In the final part of the chapter we outline some applications of these constructions to the study of schemes.

6.1 Gluing of sheaves

In this section, X will be a topological space and $\{U_i\}_{i\in I}$ will be an open cover of X. We will write U_{ij} and U_{ijk}, respectively, for the intersections $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$, where $i, j, k \in I$.

Gluing maps of sheaves

Gluing maps of sheaves is the simplest gluing situation we will encounter. The following proposition gives the precise conditions under which this can be done:

Proposition 6.1 (Gluing conditions for maps for sheaves). Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. Let $\{U_i\}_{i\in I}$ be an open cover of X and assume that we are given a map of sheaves $\phi_i\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$, so that for all $i, j \in I$

$$\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}} \tag{6.1}$$

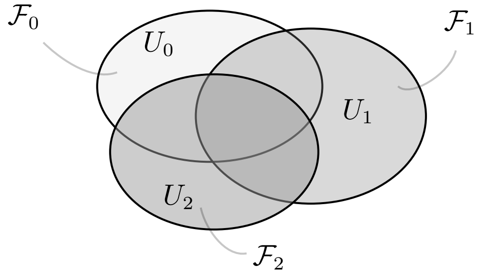
Then there exists a unique map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$.

Proof Take a section $s \in \mathcal{F}(V)$ where $V \subset X$ is open. Then over $V_i = U_i \cap V$, the section $\phi_i(s|_{V_i})$ is a well defined element in $\mathcal{G}(V_i)$, and we have $\phi_i(s|_{V_{ij}}) = \phi_j(s|_{V_{ij}})$ by the compatability assumption (6.1). Hence the sections $\phi_i(s|_{V_i})$'s of the $\mathcal{G}|_{V_i}$'s glue together to a section of $\mathcal{G}$ over V, which we define to be $\phi(s)$. It is clear that this association is additive, and compatible with restrictions, so we have the desired map of sheaves.

The uniqueness also follows: if ϕ and ψ are two morphisms of sheaves so that $\phi(s)|_{U_i} = \psi(s)|_{U_i}$ for all $i \in I$, then $\phi(s) = \psi(s)$ by the Locality axiom for $\mathcal{G}$, and consequently $\phi = \psi$. □

Gluing Sheaves

For gluing sheaves, the setting is as follows: for each open set U_i in the covering, we have a sheaf $\mathcal{F}_i$ on U_i, and our goal is to construct a global sheaf $\mathcal{F}$ on X that restricts to $\mathcal{F}_i$ for every U_i. A necessary condition for such an $\mathcal{F}$ to exist is that the $\mathcal{F}_i$'s should be isomorphic over the intersections U_{ij}. In fact, by specifying the precise conditions that these isomorphisms must satisfy (the 'gluing data'), we get not just a necessary but also a sufficient condition.



Proposition 6.2 (Gluing conditions for sheaves). Let $\{U_i\}_{i\in I}$ be a covering of X and suppose we have, for each i, a sheaf $\mathcal{F}_i$ on U_i. Suppose we are given isomorphisms

$$\tau_{ji} \colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}},$$

satisfying the three conditions
(i) $\tau_{ii} = \mathrm{id}_{\mathcal{F}_i}$
(ii) $\tau_{ji} = \tau_{ij}^{-1}$
(iii) $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$
(where (iii) takes place over the triple intersection U_{ijk}). Then there exists a sheaf $\mathcal{F}$ on X, unique up to isomorphism, such that there are isomorphisms $\nu_i \colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ satisfying $\nu_j = \tau_{ji} \circ \nu_i$ over each intersection U_{ij}.

Observe that the three conditions (i)–(iii) parallel the three requirements for a relation to be an equivalence relation; the first reflects reflectivity, the second symmetry and the third transitivity.

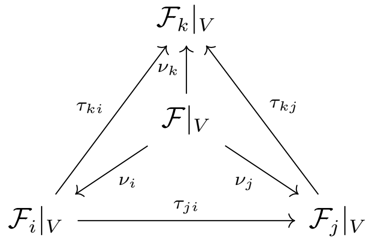
To motivate these a bit further, note that if we have managed to construct $\mathcal{F}$ and ν_i, the isomorphisms $\tau_{ji} = \nu_j \circ \nu_i^{-1}$ appear as the composition

$$\mathcal{F}_j|_{U_{ij}} \simeq \mathcal{F}|_{U_{ij}} \simeq \mathcal{F}_i|_{U_{ij}}$$

But isomorphisms of this form naturally satisfy (i)–(iii). For instance, to verify (iii):

$$\tau_{kj} \circ \tau_{ji} = (\nu_k \circ \nu_j^{-1}) \circ (\nu_j \circ \nu_i^{-1}) = \nu_k \circ \nu_i^{-1} = \tau_{ki}.$$

In terms of diagrams, the requirement $\nu_j = \tau_{ji} \circ \nu_i$ means that each of the small triangles in the figure below commute, and therefore the outer triangle commutes as well, which is exactly the condition (iii).



Proof If $W \subset X$ is an open set, we will write $W_i = U_i \cap W$ and $W_{ij} = U_{ij} \cap W$.

The construction of $\mathcal{F}$ is conceptually straightforward: the sections over an open set V is given by the collection of sections $s_i \in \mathcal{F}_i(V_i)$, chosen so that for each i and j, s_i and s_j agree over V_{ij}, meaning that τ_{ji} maps $s_i|_{V_{ij}}$ to $s_j|_{V_{ij}}$. In other words, we define

$$\mathcal{F}(V) = \left\{ (s_i)_{i \in I} \;\middle|\; \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \right\} \subset \prod_{i \in I} \mathcal{F}_i(V_i). \tag{6.2}$$

The τ_{ji} are maps of sheaves and are therefore compatible with all restriction maps. Therefore, if $W \subset V$ is another open set, we have $\tau_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}$ if $\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}$. The defining condition (6.2) is compatible with componentwise restrictions, and these can therefore be used as the restriction maps $\mathcal{F}(V) \to \mathcal{F}(W)$. We have thus defined a presheaf on X and proceed to check the two sheaf axioms.

Locality: let $s = (s_i) \in \mathcal{F}(V)$ be a section, and let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open cover of V. If $s|_{V_\alpha} = 0$ in $\mathcal{F}(V_\alpha)$ for every α, we must have that $s_i|_{U_i \cap V_\alpha} = 0$ in $\mathcal{F}_i(V_\alpha \cap U_i)$ for all α and i. But as $V_\alpha \cap U_i$ forms a cover of $V \cap U_i$, and $\mathcal{F}_i$ is a sheaf on U_i, this means that $s_i = 0$ in $\mathcal{F}(V \cap U_i)$. And since this holds for every i, we get $s = 0$.

Gluing: Let $s^\alpha \in \mathcal{F}(V_\alpha)$ be compatible sections over the opens of a covering $\{V_\alpha\}_{\alpha \in \Lambda}$ of V. This means that s^α and s^β are equal when restricted to $V_{\alpha\beta} = W_\alpha \cap W_\beta$. For $i \in I$ fixed, we then have a compatible family of sections $s_i^\alpha \in \mathcal{F}(U_i \cap V_\alpha)$, which, since $\mathcal{F}_i$ is a sheaf, glue to an element $s_i \in \mathcal{F}(U_i)$. We have $\tau_{ij}(s_j) = s_i$ in $\mathcal{F}(V \cap U_i \cap U_j)$ because this holds when restricted to $V_\alpha \cap U_{ij}$, since $s^\alpha \in \mathcal{F}(V_\alpha)$. The section $s = (s_i)$ therefore defines an element of $\mathcal{F}(V)$, which by construction restricts to s^j on each W_j.

Note that we haven't used the third condition yet. It will be neeed in order to construct the isomorphisms $\nu_i \colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$. To avoid getting confused by the names of the indices, we shall work with a fixed index $\alpha \in I$. Suppose $V \subset U_\alpha$ is an open set. Then naturally one has $V = V_\alpha$, and projecting from the product $\prod_i \mathcal{F}_i(V_i)$ onto the component $\mathcal{F}_\alpha(V) = \mathcal{F}_\alpha(V_\alpha)$ gives us a map

$$\nu_\alpha \colon \mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha.$$

We proceed to show that the ν_α's give the desired isomorphisms.

To begin with, we note that on the intersections $V_{\alpha\beta}$ the requirement in the proposition, that $\nu_\beta = \tau_{\beta\alpha} \circ \nu_\alpha$, is fulfilled. This follows directly from the definition in (6.2) that $s_\beta|_{V_{\alpha\beta}} = \tau_{\beta\alpha}(s_\alpha|_{V_{\alpha\beta}})$.

ν_α is injective: this is clear, since if $s = (s_i) \in \mathcal{F}(V)$ is a section such that $s_\alpha = 0 \in \mathcal{F}_\alpha(V)$, it follows that $s_i = s_i|_{V_{i\alpha}} = \tau_{i\alpha}(s_\alpha) = 0$ for all $i \in I$, and hence $s = 0$.

ν_α is surjective: take any section $\sigma \in \mathcal{F}_\alpha(V)$ over some $V \subset U_\alpha$ and define $s = (\tau_{i\alpha}(\sigma|_{V_{i\alpha}}))_{i \in I}$. Note that

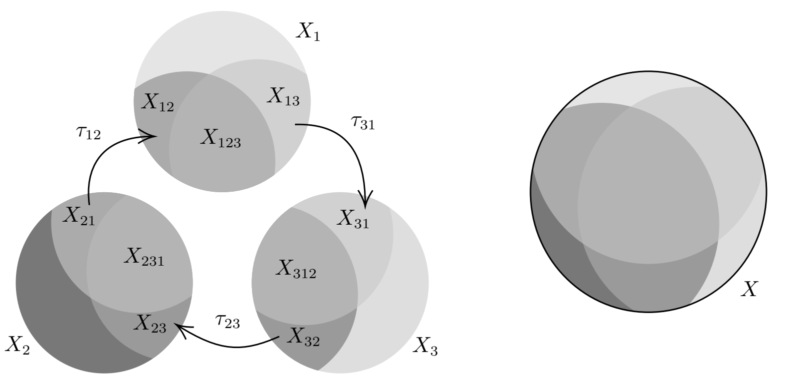
$$\tau_{ji}(\tau_{i\alpha}(\sigma|_{V_{ji\alpha}})) = \tau_{j\alpha}(\sigma|_{V_{ji\alpha}})$$

for every $i, j \in I$. Therefore, we see that s satisfies the condition in (6.2), and defines an element of $\mathcal{F}(V)$. As $\tau_{\alpha\alpha}(\sigma|_{V_{\alpha\alpha}}) = \sigma$ by the first gluing condition, the element s projects to the section σ of $\mathcal{F}_\alpha$. □

6.2 Gluing schemes

The ability to glue different schemes together along open subschemes is a fundamental property in the theory of schemes. As we will see in Chapter 7, this gives a plethora of new examples of schemes. The gluing of schemes is also an important part in many general existence proofs, such as the construction of the fibre product.

When we talk about gluing schemes, we are given a family $\{X_i\}_{i \in I}$ of schemes indexed by a set I. In each of the schemes X_i we are given a collection of open subschemes X_{ij}, one for each $j \in I$. The goal is to produce a new scheme X by gluing together all the X_i's along these open subschemes. This is done by identifying the open sets $X_{ij} \subset X_i$ and $X_{ji} \subset X_j$ using scheme isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$. If we let $X_{ijk} = X_{ik} \cap X_{ij}$ (these are the various triple intersections before the gluing has been done), we require that $\tau_{ji}(X_{ijk}) = X_{jik}$. Notice that X_{ijk} is an open subscheme of X_i.



There are three gluing conditions, similar to the ones we saw for sheaves, which must be satisfied for the gluing to be possible.

Proposition 6.3 (Gluing conditions for schemes). Suppose that we are given: a collection of schemes $\{X_i\}_{i\in I}$; for each i, j an open subschemes $X_{ij} \subset X_i$ and scheme isomorphisms $\tau_{ji} : X_{ij} \to X_{ji}$ satisfying

(i) $\tau_{ii} = \mathrm{id}_{X_i}$

(ii) $\tau_{ij} = \tau_{ji}^{-1}$

(iii) τ_{ij} takes X_{ijk} into X_{jik} and $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ over X_{ijk}.

Then there exists a scheme X with open embeddings $g_i \colon X_i \to X$ onto an open subscheme $U_i = g_i(X_i) \subset X$ such that

- $\{U_i\}_{i\in I}$ forms an open cover of X.
- For each $i, j \in I$, $g_i(X_{ij}) = U_i \cap U_j$ and the following diagram commutes:

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\quad\tau_{ij}\quad} & X_{ji} \\ & {\scriptstyle g_i}\searrow \qquad \swarrow{\scriptstyle g_j} & \\ & U_i \cap U_j & \end{array}$$

The scheme X is uniquely characterized by these properties up to a unique isomorphism.

Proof To construct the scheme X, we first build the underlying topological space X and then equip it with a sheaf of rings. For the latter, we rely on the gluing technique for sheaves explained in Proposition **??**. The fact that X is locally affine will follow immediately once the embeddings g_i are in place, because the X_i's are schemes and therefore locally affine.

To constructing the underlying topological space, we introduce an equivalence relation on the disjoint union $\coprod_i X_i$ by declaring two points $x \in X_{ij}$ and $x' \in X_{ji}$ to be equivalent when $x' = \tau_{ji}(x)$. Note that if the point x does not lie in any X_{ij} with $i \neq j$, we leave it alone; it will not be declared equivalent to any other point.

The three gluing conditions imply readily that this is an equivalence relation. The first requirement means that the relation is reflexive, the second that it is symmetric, and the third ensures it is transitive. The topological space X is then defined to be the quotient of $\coprod_i X_i$ by this relation equipped with the quotient topology. That is, if $\pi \colon \coprod_i X_i \to X$ denotes the quotient map, a subset U of X is open if and only if $\pi^{-1}(U)$ is open.

Topologically, the maps $g_i \colon X_i \to X$ are just the maps induced by the open inclusions $X_i \xrightarrow{\coprod}_i X_i$. They are clearly injective, because a point $x \in X_i$ is never equivalent to another point in X_i. Now, with the quotient topology on X, a subset U of X is open if and only if $g_i^{-1}(U) = X_i \cap \pi^{-1}(U)$ is open for all i. In view of the formula

$$\pi^{-1}(g_i(U)) = \bigcup_j \tau_{ji}(U \cap X_{ij}),$$

we conclude that each g_i is an open map, hence a homeomorphism onto its image.

We write U_i for $g_i(X_i)$ so that $U_{ij} = g(X_i) \cap g(X_j)$ and $U_{ijk} = U_i \cap U_j \cap U_k$. On X_{ij}, we have the isomorphisms $\tau_{ji}^{\sharp} \colon \mathcal{O}_{X_j}|_{X_{ij}} \to \mathcal{O}_{X_i}|_{X_{ij}}$; the sheaf maps of the scheme isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$. In view of the third gluing condition $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$, valid on X_{ijk}, we obviously have $\tau^\sharp_{ki} = \tau^\sharp_{ji} \circ \tau^\sharp_{kj}$. The two first gluing conditions translate into $\tau^\sharp_{ii} = \text{id}$ and $\tau^\sharp_{ji} = (\tau^\sharp_{ij})^{-1}$. Consequently, the gluing properties required to apply Proposition **??** are satisfied, and we are allowed to glue the different $\mathcal{O}_{X_i}$'s together and thus to equip X with a sheaf of rings. This sheaf of rings restricts to $\mathcal{O}_{X_i}$ on each of the open subsets X_i, and therefore its stalks are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed space which is locally affine, hence a scheme.

We leave it to the reader to prove the uniqueness statement in the proposition. □

Exercise 6.2.1. Prove the uniqueness part in the above proposition.

Gluing morphisms of schemes

Finally, we consider conditions under which we can glue morphisms of schemes

Proposition 6.4 (Gluing conditions for morphisms of schemes). Let X and Y be a schemes and let $\{U_i\}_{i\in I}$ be an open cover of X. Suppose we are given scheme morphisms

$$f_i : U_i \longrightarrow Y$$

satisfying $f_i|_{U_i\cap j} = f_j|_{U_i\cap U_j}$ for each i and j. Then there is a unique map of schemes

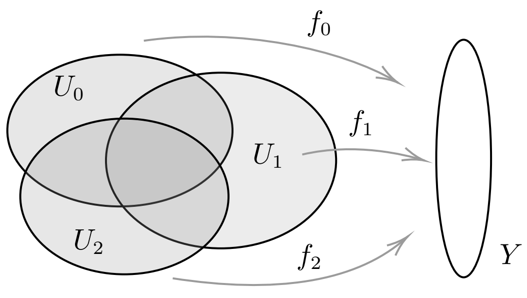
$$f : X \longrightarrow Y$$

such that $f|_{U_i} = f_i$ for every i.

Proof On the level of topolofical spaces, we define $f(x) = f_i(x)$ if $x \in U_i$. This is well-defined because $f_i(x) = f_j(x)$ for $x \in U_i \cap U_j$, and it is clear that it is continuous.

Next, we define the sheaf map $f^\sharp$. If $V \subset Y$ is an open set, we need to define a ring map $f^\sharp \colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. To do this, take any section $s \in \mathcal{O}_Y(V)$. Using the sheaf maps $f^\sharp_i$ over U_i, we get sections $t_i = f^\sharp_i(s)$ in $\mathcal{O}_X(f^{-1}V \cap U_i)$. But since $f^\sharp_i$ and $f^\sharp_j$ restrict to the same map on U_{ij}, it holds that $t_i|_{f^{-1}V\cap U_{ij}} = t_j|_{f^{-1}V\cap U_{ij}}$ in $\mathcal{O}_X(f^{-1}V \cap U_{ij})$. The t_i therefore patch together to a section $t \in \mathcal{O}_X(f^{-1}V)$, and we can define $f^\sharp(s)$ to be t. It is clear that $f^\sharp$ is a ring map, and that $f^\sharp = f^\sharp_i(s)$ when $V \subset U_i$. The pair $(f, f^\sharp)$ is therefore a map of locally ringed spaces because it is locally given by the f_i.

Proving the uniqueness statement is left to the reader. □



Exercise 6.2.2. Let X and Y be schemes and let $\mathscr{B}$ be a basis for the topology on X.

Suppose that there is a collection of morphisms $f_U : U \to Y$, one for each $U \in \mathscr{B}$, such that if $V \in \mathscr{B}$ satisfies $V \subset U$, we have

$$f_U|_V = f_V.$$

Show that there exists a unique morphism of schemes $f : X \to Y$ such that $f|_U = f_U$.

6.3 Maps into affine schemes

As a first application of the gluing theorems in this chapter, we prove the following important theorem about morphisms of schemes into affine schemes, which generalizes The Main Theorem for Affine Schemes (Theorem 4.18).

Theorem 6.5 (Maps into affine schemes). For any scheme X, the canonical map

$$\Phi_X : \operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_X(X))$$

given by $(f, f^\sharp) \mapsto f^\sharp_X$ is bijective.

Proof Let $\{U_i\}_{i \in I}$ be an open affine cover of X. By the affine case, (Theorem 4.18), we know that each Φ_{U_i} is bijective. We first claim that Φ_X is injective. Given two morphisms $f, g : X \to \operatorname{Spec} A$ that induce the same ring map $\beta : A \to \mathcal{O}_X(X)$, their restrictions are morphisms $f_i : U_i \to \operatorname{Spec} A$ and $g_i : U_i \to \operatorname{Spec} A$. For each i these both correspond to the ring map $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U_i)$ obtained by composing β with the restriction; thus $g_i = f_i$, because Φ_{U_i} is bijective. It follows that $f = g$ by the uniqueness part of Proposition 6.4, so Φ_X is injective.

Next we show that Φ_X is surjective. Let $\beta : A \to \mathcal{O}_X(X)$ be a ring map. Composing β with the appropriate restriction maps, one obtains ring maps

$$\beta_i : A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U_i),$$

and these induce morphisms $f_i : U_i \to \operatorname{Spec} A$. We claim that the f_i's may be glued together to a map $f : X \to \operatorname{Spec} A$. For this, we need to show that they agree over the overlaps $U_i \cap U_j$. The latter intersection might not be affine, however, it is enough to show that $f_i|_V = f_j|_V$ for every affine $V \subset U_i \cap U_j$. For this, consider the diagram

$$\begin{array}{ccccccc}
 & & \mathcal{O}_X(U_i) & & & & \\
 & \overset{\beta_i}{\nearrow} & \uparrow & \searrow & & & \\
A & \xrightarrow{\beta} & \mathcal{O}_X(X) & \longrightarrow & \mathcal{O}_X(U_i \cap U_j) & \longrightarrow & \mathcal{O}_X(V). \\
 & \underset{\beta_j}{\searrow} & \downarrow & \nearrow & & & \\
 & & \mathcal{O}_X(U_j) & & & &
\end{array}$$

The diagram tells us that the restrictions $f_i|_V$ and $f_j|_V$ induce the same ring map $A \to \mathcal{O}_X(V)$ and we conclude that they are equal by The Main Theorem for Affine Schemes (Theorem 4.18). As this is true for any V, the f_i's are equal on all of $U_i \cap U_j$. Hence the f_i can be glued together to a morphism $f : X \to \operatorname{Spec} A$. It must hold that $\Phi_X(f) = \beta$, because Φ_X is injective (which we just proved) and since $f|_{U_i}$ maps to β_i via Φ_{U_i} for each i. This completes the proof. □

For a general scheme X, it is natural to consider the affine scheme $\operatorname{Spec}(\mathcal{O}_X(X))$. This is in general very different from X, as the examples of Chapter 7 will show. There is however always a canonically defined morphism $X \to \operatorname{Spec}(\mathcal{O}_X(X))$, which satisfies a universal property with respect to morphisms into affine schemes:

Corollary 6.6. Let X be any scheme. Then there is a canonical map of schemes

$$f\colon X \longrightarrow \operatorname{Spec}(\mathcal{O}_X(X))$$

so that $f^\sharp$ induces identity on global sections. It is universal among morphism from X to affine schemes, that is, given a morphism $g\colon X \to \operatorname{Spec} A$, it holds that $g = \operatorname{Spec}(g^\sharp) \circ f$.

Proof The first part follows by applying the theorem to $A = \mathcal{O}_X(X)$, and the second follows, again from the theorem, in view of the equality

$$(\operatorname{Spec}(g^\sharp) \circ f)^\sharp = f^\sharp \circ g^\sharp = \mathrm{id}_{\mathcal{O}_X(X)} \circ g^\sharp = g^\sharp.$$

□

Example 6.7. As a special case, we note that there is a canonical bijection

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec}\mathbb{Z}) = \operatorname{Hom}_{\mathsf{Rings}}(\mathbb{Z}, \mathcal{O}_X(X)).$$

Since ring maps always preserve the unit element, the set on the right is clearly a one-point set. This means that there exist one and only one morphism of schemes $X \to \operatorname{Spec}\mathbb{Z}$. In categorical terms this means that $\operatorname{Spec}\mathbb{Z}$ is a *final object* in the category of schemes Sch.

The category Sch also has an *initial object*, the empty scheme; it equals the spectrum of the zero ring, $\operatorname{Spec} 0$, which has the empty set as underlying topological space. Given any scheme X there is clearly a unique morphism $\operatorname{Spec} 0 \to X$, which on the level of sheaves sends every section of $\mathcal{O}_X$ to zero.

Example 6.8 (Maps to $\mathbb{A}^1$ and $\mathcal{O}_X(X)$)**.** In the special case when $A = \mathbb{Z}[t]$, any morphism of rings $\mathbb{Z}[t] \to \mathcal{O}_X(X)$ is determined uniquely by the image of t. Thus by Theorem 6.5, we have

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \mathbb{A}^1) = \mathcal{O}_X(X)$$

Hence for any scheme, there is a bijection between the elements $f \in \mathcal{O}_X(X)$ and scheme maps

$$f : X \longrightarrow \mathbb{A}^1$$

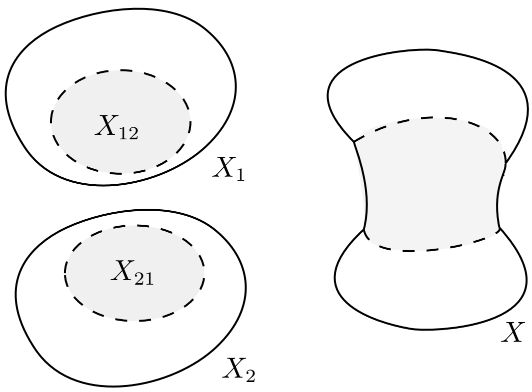
7

Examples constructed by gluing

7.1 Gluing two schemes together

To make the gluing techniques introduced in Chapter 6 a bit more concrete, we will study in detail the simple case of schemes obtained by gluing together just two schemes.

We start out with two schemes X_1 and X_2 with respective open subsets $X_{12} \subset X_1$ and $X_{21} \subset X_2$; these are open subschemes equipped with their canonical induced scheme structures obtained by restricting the structure sheaves. Furthermore, we assume we are given an isomorphism $\tau\colon X_{21} \to X_{12}$. For just two schemes, the gluing conditions are automatically fulfilled, and these data allow us to glue together X_1 and X_2 along X_{12} and X_{21} to construct a new scheme X.



On the level of topological spaces, X is obtained from the disjoint union $X_1 \coprod X_2$ by forming the quotient modulo the equivalence relation with $x \sim \tau(x)$ for $x \in X_{21} \subseteq X_2$ and giving X the quotient topology.

Each of the open embeddings $g_i\colon X_i \to U_i \subset X$ (where $i = 1$ or 2) allows us to view each X_i as an open subset of X, providing an open cover of X. For an open subset $V \subset X$, we may identify the sheaf sequence

$$0 \to \mathcal{O}_X(V) \to \mathcal{O}_X(V \cap U_1) \times \mathcal{O}_X(V \cap U_2) \to \mathcal{O}_X(V \cap U_1 \cap U_2)$$

with the following sequence

$$0 \to \mathcal{O}_X(g^{-1}V) \xrightarrow{\alpha} \mathcal{O}_{X_1}(g_1^{-1}V) \times \mathcal{O}_{X_2}(g_2^{-1}V) \xrightarrow{\beta} \mathcal{O}_{X_{12}}(g_1^{-1}V \cap X_{12})$$

where $\alpha(s) = (g_1^\sharp(s|_{V\cap U_1}), g_2^\sharp(s|_{V\cap U_1}))$ and $\beta(s_1, s_2) = s_1|_{X_{12}} - \tau^\sharp(s_2|_{X_{21}})$.

The main example to keep in mind is when X_1 and X_2 are both affine, say $X_1 = \operatorname{Spec} R$ and $X_2 = \operatorname{Spec} S$, and they are glued together along two distinguished open subsets $D(u)$ and $D(v)$ for some $u \in R$ and $v \in S$. The gluing map τ is induced from a ring isomorphism between the localizations

$$\phi : R_u \longrightarrow S_v.$$

We picture this by the following diagram of schemes

$$\operatorname{Spec} R \supset \operatorname{Spec} R_u = D(u) \xleftarrow[\simeq]{\tau} D(v) = \operatorname{Spec} S_v \subset \operatorname{Spec} S$$

To compute $\mathcal{O}_X(X)$, the sheaf exact sequence takes the form

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow R \times S \xrightarrow{\rho} S_v. \tag{7.1}$$

Here $\rho(r, s) = s/1 - \phi(r/1)$ with $s/1$ and $r/1$ denoting the images of s and r respectively in S_v and R_u. In other words, elements in $\mathcal{O}_X(X)$ correspond to pairs $(r, s) \in R \times S$ such that $s/1 = \phi(r/1)$ in the localized ring S_v.

We can also study sheaves on the glued scheme X. Proposition **??** tells us that giving a sheaf $\mathcal{F}$ on X is equivalent to specifying (i) a sheaf $\mathcal{F}_1$ on X_1; (ii) a sheaf $\mathcal{F}_2$ on X_2; (iii) a sheaf isomorphism

$$\nu_{12} \colon \mathcal{F}_2|_{D(v)} \longrightarrow \mathcal{F}_1|_{D(u)},$$

where we use the isomorphism τ to identify $D(u)$ and $D(v)$. In the special case that $\mathcal{F}_1 = \widetilde{M}$ and $\mathcal{F}_2 = \widetilde{N}$ for modules M and N over S and R respectively, it is equivalent to specify an isomorphism of R_u-modules

$$v_{12} \colon N_u \longrightarrow M_v.$$

(See Section 4.4 for the construction of $\widetilde{M}$). Many important examples arise from this basic construction. We will now survey a few of these.

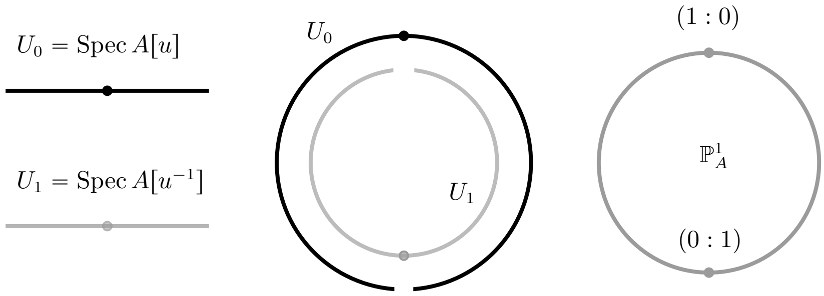
7.2 The projective line

The Riemann sphere $\mathbb{CP}^1$ is the complex plane $\mathbb{C}$ with one point added, the point at infinity. As a complex manifold, it is covered by two charts, both isomorphic to $\mathbb{C}$. There is a complex coordinate z centred at the origin, and its inverse $w = z^{-1}$ serves as the coordinate centred at infinity.

The construction of $\mathbb{CP}^1$ can be vastly generalized to work over any ring A. Let u be a variable ('the coordinate at the origin') and let $U_0 = \operatorname{Spec} A[u]$. The inverse u^{-1} is a variable as good as u ('the coordinate at infinity'), and we let $U_1 = \operatorname{Spec} A[u^{-1}]$. Both are copies of the affine line $\mathbb{A}^1_A$ over A.

Inside U_0, we have the distinguished open set $U_{01} = D(u)$, which is canonically isomorphic to $\operatorname{Spec} A[u, u^{-1}]$, and the open embedding $U_{01} \to U_0$ comes from the inclusion $A[u] \subset A[u, u^{-1}]$. Similarly, inside U_1 there is the distinguished open set $U_{10} = D(u^{-1})$, which is also identified with $\operatorname{Spec} A[u^{-1}, u]$ by the inclusion $A[u^{-1}] \subset A[u^{-1}, u]$. Hence U_{01} and U_{10} are isomorphic schemes, and we may glue U_0 to U_1 along U_{01}. The result is called the *projective line over* A and is denoted by $\mathbb{P}^1_A$.

The projective line over A indeed is a scheme over A. Indeed, there are canonical maps $\pi_0 \colon U_0 \to \operatorname{Spec} A$ and $\pi_1 \colon U_1 \to \operatorname{Spec} A$, which are induced by the inclusions $A \subset A[u]$ and $A \subseteq A[u^{-1}]$, respectively. Over the intersection $U_0 \cap U_1$, these morphisms agree, since both are induced by the inclusion $A \subset A[u, u^{-1}]$. Therefore, they can be glued to define a morphism $\pi \colon \mathbb{P}^1_A \to \operatorname{Spec} A$.



Gluing two affine lines to get $\mathbb{P}^1_A$

If $\operatorname{Spec} A$ is irreducible, then so is $\mathbb{P}^1_A$. This is because $\mathbb{P}^1_A$ contains $U_0 \simeq \mathbb{A}^1_A$ as a dense open subset and $A[u]$ is an integral domain if A is. Likewise, $\mathbb{P}^1_A$ is reduced if A is, because it has the same local rings as U_0 and U_1, which are reduced. Hence $\mathbb{P}^1_A$ is integral if $\operatorname{Spec} A$ is.

Note that the complement of U_1 equals $V(u) \subset U_0 = \operatorname{Spec} A[u]$, which is isomorphic to $\operatorname{Spec} A$. So when $A = k$ is a field, $\mathbb{P}^1_k$ is $\mathbb{A}^1_k$ with a single point added. In particular, when k is algebraically closed, the set of k-points $\mathbb{P}^1(k)$ coicides with the projective line defined in Chapter 1.

The following computation is very important.

Proposition 7.1. We have $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}) = A$.

Proof The projective line $\mathbb{P}^1_A$ is covered by the two open affines U_0 and U_1, and the standard exact sequence (7.1) above takes the form

$$\begin{array}{ccccccc} 0 \longrightarrow \Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}) & \longrightarrow & \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1_A}) \times \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1_A}) & \longrightarrow & \Gamma(U_{01}, \mathcal{O}_{\mathbb{P}^1_A}) \\ & & \downarrow \wr & & \downarrow \wr \\ & & A[u] \times A[u^{-1}] & \xrightarrow{\rho} & A[u, u^{-1}], \end{array}$$

where the map ρ sends a pair $(f(u), g(u^{-1}))$ of polynomials with coefficients in A, one in the variable u and one in u^{-1}, to the difference $g(u^{-1}) - f(u)$.

The group $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A})$ is therefore identified with the kernel of ρ. But this kernel consists of elements (a, a) where $a \in A$: if $f(u) - g(u^{-1}) = 0$ in $A[u, u^{-1}]$, then both f and g must have degree 0 as polynomials in u. □

In particular, for a field k, the group of global sections of $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k)$ is just the 'constants', k, as in Theorem 1.44 on page 19. Over the complex numbers, this can be seen as a special case of Liouville's theorem, that the only global holomorphic functions are the constants.

We note that we also have got yet another example of a scheme which is not affine: if $\mathbb{P}^1_{\mathbb{C}}$ were affine, it would have to be isomorphic to $\operatorname{Spec}\mathbb{C}$ according to Theorem 4.18 on page 64. But this is clearly not the case, as $\mathbb{P}^1_{\mathbb{C}}$ contains infinitely many closed points. Another morale to extract is that the group $\mathcal{O}_X(X)$ does not give much information about X for general schemes.

The projective line $\mathbb{P}^1_A$ as a quotient

$\mathbb{P}^1_A$ is in fact related to the first example of a non-affine scheme of Example **??** on page ??, namely the affine plane $\mathbb{A}^2_A = \operatorname{Spec} A[u,v]$ with the 'origin' $V(u,v)$ removed. In fact, there is a natural morphism between them:

$$\pi\colon \mathbb{A}^2_A - V(u,v) \longrightarrow \mathbb{P}^1_A.$$

On the level of closed points, when $A = k$ is an algebraically closed field, the morphism π is exactly the morphism used in the construction of the projective line as a quotient space in Chapter 1. $V(u,v)$ is the origin, and π collapses each line through the origin to its corresponding point in $\mathbb{P}^1(k)$.

The map π is constructed by gluing together the two morphisms

$$f_1\colon D(u) = \operatorname{Spec} A[u,u^{-1},v] \longrightarrow \operatorname{Spec} A[vu^{-1}]$$
$$f_2\colon D(v) = \operatorname{Spec} A[u,v,v^{-1}] \longrightarrow \operatorname{Spec} A[uv^{-1}]$$

which are induced from the inclusions $A[vu^{-1}] \subset A[u,u^{-1},v]$ and $A[uv^{-1}] \subset A[u,v,v^{-1}]$. Note that the two targets, $\operatorname{Spec} A[vu^{-1}]$ and $\operatorname{Spec} A[uv^{-1}]$, are two copies of $\mathbb{A}^1_A$ which glue to the projective line $\mathbb{P}^1_A$ (using uv^{-1} as the variable).

The union of the sources equals $D(u) \cup D(v) = \mathbb{A}^2_A - V(u,v)$ and $D(u) \cap D(v) = D(uv) = \operatorname{Spec} A[u,u^{-1},v,v^{-1}]$. Applying Spec to the following commutative diagram then shows that f_1 and f_2 satisfy the gluing condition:

$$\begin{array}{ccccc} A[u,u^{-1},v] & \hookleftarrow & A[u^{-1}v] & & \\ & \searrow & & \searrow & \\ & & A[u,u^{-1},v,v^{-1}] & \hookleftarrow & A[u^{-1}v,uv^{-1}] \\ & \nearrow & & \nearrow & \\ A[u,v,v^{-1}] & \hookleftarrow & A[uv^{-1}]. & & \end{array}$$

Exercise 7.2.1. Let K be a field. Show that the K-points of the projective line $\mathbb{P}^1$ are in bijection with the set of lines in K^2 passing through the origin $(0,0)$ HINT: Any map $\operatorname{Spec} K \to \mathbb{P}^1$ must factor via either U_0 or U_1.

Exercise 7.2.2. Let $X = \operatorname{Spec} A$ be an affine scheme over a field k. Show that every morphism $\mathbb{P}^1_k \to X$ is constant, i.e. it factors through some k-valued point of X.

Exercise 7.2.3. Show that $\mathbb{P}^1_A$ is not affine for any ring A. HINT: The canonical map $\mathbb{P}^1_A \to \operatorname{Spec} A$ is never an isomorphism (restrict to U_1).

A family of sheaves on $\mathbb{P}^1_A$

The projective spaces, in particular the projective line $\mathbb{P}^1_A$, carry a family of sheaves, which play an important role in algebraic geometry. There is one for each integer m, and the sheaves will be denoted by $\mathcal{O}_{\mathbb{P}^1_A}(m)$. We shall construct these sheaves using the gluing theorems for sheaves.

Let $U_0 = \operatorname{Spec} A[u]$ and $U_1 = \operatorname{Spec} A[u^{-1}]$ be the usual cover of $\mathbb{P}^1_A$, and consider the intersection $U_0 \cap U_1 = \operatorname{Spec} A[u, u^{-1}]$. Multiplication by u^m gives an isomorphism

$$A[u, u^{-1}] \xrightarrow{u^m} A[u, u^{-1}.$$

and by Exercise 4.1.2 on page 58, this induces an isomorphism of sheaves

$$\tau\colon \mathcal{O}_{U_1}|_{U_0\cap U_1} \longrightarrow \mathcal{O}_{U_0}|_{U_0\cap U_1}.$$

Now we define a sheaf $\mathcal{O}_{\mathbb{P}^1_A}(m)$ by gluing $\mathcal{O}_{U_1}$ to $\mathcal{O}_{U_0}$ along $U_0 \cap U_1$ using this isomorphism. Note that the direction of τ matters; we could of course have used the multiplication map in the reverse direction, but this would have yielded a different sheaf, namely $\mathcal{O}_{\mathbb{P}^1_A}(-m)$.

By construction, the sheaf $\mathcal{O}_{\mathbb{P}^1_A}(m)$ restricts to the structure sheaf on both open subsets U_0 and U_1; that is, $\mathcal{O}_{\mathbb{P}^1_A}(m)|_{U_0} \simeq \mathcal{O}_{U_0}$ and $\mathcal{O}(m)_{\mathbb{P}^1_A}|_{U_1} \simeq \mathcal{O}_{U_1}$ (in the jargon of Chapter **??** it is a *locally free sheaf*). However, when $m \neq 0$, the sheaf $\mathcal{O}_{\mathbb{P}^1_A}(m)$ is not isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^1_A}$. As we shall see, their global sections are different. In particular, this gives another illustration that a sheaf is not determined by its stalks alone.

To compute the global sections of $\mathcal{O}_{\mathbb{P}^1_A}(m)$, we use the the standard sheaf sequence applied to U_0, U_1. With $\mathcal{O}_X(U_0) = A[u]$, $\mathcal{O}_X(U_1) = A[u^{-1}]$, and $\mathcal{O}_X(U_{01}) = A[u, u^{-1}]$, the sequence takes the form

$$0 \longrightarrow \Gamma(\mathcal{O}_{\mathbb{P}^1_A}, \mathcal{O}_{\mathbb{P}^1_A}(m)) \longrightarrow A[u] \oplus A[u^{-1}] \xrightarrow{\rho} A[u, u^{-1}],$$

where $\rho(p(u), q(u^{-1})) = u^m q(u^{-1}) - p(u)$. If $m < 0$, then there are no non-trivial polynomials p and q such that $u^m q(u^{-1}) = p(u)$, and we infer that $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(m)) = \operatorname{Ker} \rho = 0$. For $m \geqslant 0$ however there are solutions. Indeed, every polynomial $p(u)$ of degree at most m is of the form $u^m q(u^{-1})$, and where q is uniquely determined by p. We have shown the following:

Proposition 7.2. The global sections of $\mathcal{O}_{\mathbb{P}^1_A}(m)$ are given by

$$\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(m)) = \begin{cases} A \oplus Au \oplus \cdots \oplus Au^m & \text{when } m \geqslant 0; \\ 0 & \text{when } m < 0. \end{cases}$$

Closed subschemes of $\mathbb{P}^1_A$

Let us have a closer look at the sheaf $\mathcal{O}_{\mathbb{P}^1_A}(-1)$ on $\mathbb{P}^1_A$. We claim that there is a sheaf map

$$\phi\colon \mathcal{O}_{\mathbb{P}^1_A}(-1) \to \mathcal{O}_{\mathbb{P}^1_A},$$

which makes $\mathcal{O}_{\mathbb{P}^1_A}(-1)$ into a subsheaf of $\mathcal{O}_{\mathbb{P}^1}$. We construct ϕ by defining it on each of the open sets U_0 and U_1 and then make sure that the maps glue using Lemma **??**. On the open set

$U_0 = \operatorname{Spec} A[u]$ we define $\phi_0\colon \mathcal{O}_{U_0} \to \mathcal{O}_{U_0}$ by multiplying sections by u. On U_1, we let $\phi_1\colon \mathcal{O}_{U_1} \to \mathcal{O}_{U_1}$ be the identity map. To see that the two maps glue, we need to check that they agree on the intersection $U_0 \cap U_1 = \operatorname{Spec} A[u, u^{-1}]$. But this follows directly from the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{O}_{U_0}|_{U_0\cap U_1} & \xrightarrow{\ u\ } & \mathcal{O}_{U_0}|_{U_0\cap U_1} \\ {\scriptstyle u^{-1}}\uparrow & & \uparrow\| \\ \mathcal{O}_{U_1}|_{U_0\cap U_1} & \xrightarrow{\ =\ } & \mathcal{O}_{U_1}|_{U_0\cap U_1}. \end{array}$$

The four sheaves are all equal to $\mathcal{O}_{U_0\cap U_1}$. The right vertical map is the gluing map for the sheaf $\mathcal{O}_{\mathbb{P}^1_A}$ and the left one that for $\mathcal{O}_{\mathbb{P}^1_A}(-1)$. The horizontal maps are the restrictions of the local maps to $\phi_0|_{U_0\cap U_1}$ and $\phi_1|_{U_0\cap U_1}$. Thus we have the desired map $\phi\colon \mathcal{O}_{\mathbb{P}^1_A}(-1) \to \mathcal{O}_{\mathbb{P}^1}$.

More generally, any non-zero section of $\phi\colon \mathcal{O}_{\mathbb{P}^1_A}(m)$ gives rise to a map $\mathcal{O}_{\mathbb{P}^1_A}(-m) \to \mathcal{O}_{\mathbb{P}^1_A}$. In the previous section, we showed that such a section is determined by a pair of polynomials $p(u)$ and $q(u^{-1})$ satisfying $p(u) = u^m q(u^{-1})$.The map is obtained by gluing together maps defined over U_0 and U_1, and the key point is the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U_0}|_{U_0\cap U_1} & \xrightarrow{\ p(u)\ } & \mathcal{O}_{U_0}|_{U_0\cap U_1} \\ {\scriptstyle u^{-m}}\uparrow & & \uparrow\| \\ \mathcal{O}_{U_1}|_{U_0\cap U_1} & \xrightarrow{\ q(u^{-1})\ } & \mathcal{O}_{U_1}|_{U_0\cap U_1}, \end{array}$$

where the left vertical arrow is gluing map for $\mathcal{O}_{\mathbb{P}^1_A}(-m)$ and the horizontal ones are restrictions of the multiplication maps to $U_0 \cap U_1$. As above, the right vertical map is just the gluing map for $\mathcal{O}_{\mathbb{P}^1_A}$.

Exercise 7.2.4. This exercise indicates how a non-zero section σ gives rise to a closed subscheme $V(\sigma)$ of $\mathbb{P}^1_A$. (This is part of a more general story, explored in Chapter 18.)

a) Show that the image of the map $\phi\colon \mathcal{O}_{\mathbb{P}^1_A}(-m) \longrightarrow \mathcal{O}_{\mathbb{P}^1_A}$ associated with a section σ of $\mathcal{O}_{\mathbb{P}^1_k}(m)$ is a principal ideal in each ring $\mathcal{O}_{\mathbb{P}^1_A}(U)$, and thus defines closed subscheme $Z_i \subset U_i$.
b) Show that the two ideals become equal in the ring $\mathcal{O}_{\mathbb{P}^1_A}(U_0 \cap U_1)$, and that the Z_i's can be glued together to a closed subscheme $V(\sigma) \subset \mathbb{P}^1_A$.

Exercise 7.2.5. Suppose $A = k$ is a field and let σ be a non-zero section of $\mathcal{O}_{\mathbb{P}^1_A}(m)$. Show that $V(\sigma)$, as defined in the previous exercise, consists of m points counted with multiplicity, that is, $m = \sum_{x\in V(\sigma)} \dim_k \mathcal{O}_{V(s),x}$. HINT: Show that $\deg p(u) = m - \deg q(u^{-1})$, where $\deg q(u^{-1})$ is the degree of $q(u^{-1})$ as a polynomial in u^{-1}.

7.3 The affine line with a doubled origin

The next example is obtained by gluing together two copies X_1 and X_2 of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[u]$ over a field k along their common open subset $X_{12} = \operatorname{Spec} k[u, u^{-1}]$ with the identity morphism $\phi : k[u, u^{-1}] \to k[u, u^{-1}]$ on the open set. The resulting scheme X is covered by two $\mathbb{A}^1_k$'s which overlap outside the origin. However, as the gluing process does nothing over the origins of each $\mathbb{A}^1_k$, there are now *two* points in X that replace the origin. X is called the *affine line with two origins*.

This scheme is not affine. Indeed, the sheaf sequence from before takes the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) \oplus \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) & \longrightarrow & \Gamma(X_{12}, \mathcal{O}_{X_{12}}) \\ & & & & \downarrow \Vert & & \downarrow \Vert \\ & & & & k[u] \oplus k[u] & \xrightarrow{\quad\rho\quad} & k[u, u^{-1}] \end{array}$$

where now $\rho(p(u), q(u)) = p(u) - q(u)$, and it follows that either open inclusion $\iota\colon \mathbb{A}^1_k \to X$ induces an isomorphism $\Gamma(X, \mathcal{O}_X) \simeq \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) = k[u]$. However, the open inclusion $\iota\colon \mathbb{A}^1_k = \operatorname{Spec} k[u] \to X$ is not an isomorphism (it is not surjective, since the image misses one of the two origins).

Note that the scheme X is both irreducible and reduced, with function field equal to $K = k(u)$. The two local rings $\mathcal{O}_{X,0_1}$ and $\mathcal{O}_{X,0_2}$ both lie as subrings of K; they are both equal to $k[u]_{(u)}$. This is somewhat unsetteling: any rational function which is regular at 0_1 is automatically regular at 0_2 and it takes the same value there. This is related to the property of 'separatedness', which we will discuss in Chapter 11.

Exercise 7.3.1. Let X be the affine line with two origins, as defined above.

a) Imitate the construction of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$ to form a family of sheaves $\mathcal{O}_X(m)$ on X, one for each integer m.
b) Show that $\mathcal{O}_X(m)$ and $\mathcal{O}_X(n)$ are not isomorphic unless $m = n$. HINT: Consider the behaviour of sections at the two origins.

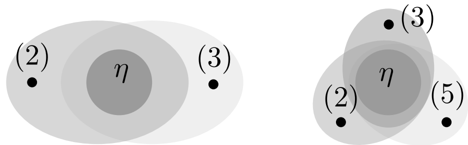
7.4 Semi-local rings

Semi-local rings are rings with finitely many maximal ideals. In the next two examples we give a few examples of such rings and how they can be described as local rings glued together.

Example 7.3 (Semi-local rings)**.** Consider the two rings $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$. These are both discrete valuation rings with with a common fraction field $\mathbb{Q}$ and maximal ideals (2) and (3) respectively. Their prime spectra $X_1 = \operatorname{Spec} \mathbb{Z}_{(2)}$ and $X_2 = \operatorname{Spec} \mathbb{Z}_{(3)}$ consist each of two points; the maximal ideal and a generic point $\eta_i = (0)$ which are open. (as described in Example 2.10 on page 24). The generic points given the open embeddings $\operatorname{Spec} \mathbb{Q} \to X_i$ for $i = 1, 2$. Hence we can glue together X_1 and X_2 along the two generic points and thus obtain a scheme X with one open point η and two closed points. Let us compute the global sections of $\mathcal{O}_X$ using the sheaf sequence for the open cover $\{X_1, X_2\}$:

$$\begin{array}{ccccc} 0 \to \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(X_1, \mathcal{O}_X) \times \Gamma(X_2, \mathcal{O}_X) & \longrightarrow & \Gamma(X_1 \cap X_2, \mathcal{O}_X) \\ & & \downarrow \wr\wr & & \downarrow \wr\wr \\ & & \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} & \xrightarrow[\rho]{\qquad\qquad} & \mathbb{Q}. \end{array}$$

The map ρ sends a pair (an^{-1}, bm^{-1}) to the difference $an^{-1} - bm^{-1}$, hence the kernel equals the the set (a, a) with $a \in \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$. This is a semi-local ring with the two maximalideals (2) and (3). By the Main Theorem of Maps into Affine schemes (Theorem 4.18) there is a map $X \to \operatorname{Spec} \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$, and it is left as an exercise to show that this is an isomorphism.



Example 7.4 (More semi-local rings)**.** More generally, if $P = \{p_1, \ldots, p_r\}$ is a finite set of distinct prime numbers, one may let $X_p = \operatorname{Spec} \mathbb{Z}_{(p)}$ for $p \in P$. There is, as in the previous case, a canonical open embedding $\operatorname{Spec} \mathbb{Q} \to X_p$ for each p. Let the images be $\{\eta_p\}$. Obviously, the conditions for gluing the η_p's together are all satisfied (the transition maps are all equal to $\mathrm{id}_{\operatorname{Spec} \mathbb{Q}}$, and $X_{pq} = \{\eta_p\}$ for all p), and we may glue the X_i together to a scheme X. Again, the global sections of the structure sheaf are found using the sheaf sequence

$$\begin{array}{ccccc} 0 \to \Gamma(X, \mathcal{O}_X) & \longrightarrow & \prod_{p \in P} \Gamma(X_p, \mathcal{O}_X) & \longrightarrow & \prod_{p,q \in P} \Gamma(X_p \cap X_q, \mathcal{O}_X) \\ & & \downarrow \wr\wr & & \downarrow \wr\wr \\ & & \prod_{p \in P} \mathbb{Z}_{(p)} & \xrightarrow[\rho]{\qquad\qquad} & \prod_{p,q \in P} \mathbb{Q}. \end{array}$$

The map ρ sends a sequence $(a_p)_{p \in P}$ to the sequence $(a_p - a_q)_{p,q \in P}$, and it follows that the kernel of ρ equals the intersection $A_P = \bigcap_{p \in P} \mathbb{Z}_{(p)}$. This is a semi-local ring whose maximal ideals are the $(p)A_P$'s for $p \in P$. There is a canonical morphism $X \to \operatorname{Spec} A_P$, and again we leave it to the reader to verify that this is an isomorphism.

Exercise 7.4.1. Verify the claims in Examples 7.3 and 7.4 above that X is isomorphic respectively to $\operatorname{Spec} \mathbb{Z}_2 \cap \mathbb{Z}_3$ and to $\operatorname{Spec} A_P$. HINT: Use the uniqueness statement in Proposition 6.3 on page 86.

Exercise 7.4.2. Glue $\operatorname{Spec} \mathbb{Z}_{(2)}$ to itself along the generic point to obtain a scheme X. Show that X is not affine. HINT: Show that $\mathcal{O}_X(X) = \mathbb{Z}_{(2)}$.

7.5 The blow-up of the affine plane

In this section we will construct the *blow-up of* $\mathbb{A}^2_k$ *at the origin*, by gluing together two affine schemes. We begin by recalling the classical construction for varieties. As in Chapter 1, we write $\mathbb{A}^2(k)$ for the variety, and $\mathbb{A}^2_k$ for the scheme, etc.

The blow-up as a variety

Let k be an algebraically closed field, and consider the affine plane $\mathbb{A}^2(k)$. There is a morphism $f\colon \mathbb{A}^2(k) - \{(0,0)\} \longrightarrow \mathbb{P}^1(k)$ that sends a point (x, y) to the point $(x : y)$ (in homogeneous coordinates on $\mathbb{P}^1(k)$). This map is not defined at the origin $(0, 0)$, but we can still consider the closure of the graph $\{(x, f(x))\}$ which is a subset of $\mathbb{A}^2(k) \times \mathbb{P}^1(k)$.

To describe the graph in more detail, we write $(s : t)$ for homogenous coordinates on $\mathbb{P}^1(k)$. Points in the product are then of the form $(x, y) \times (s : t)$, and those in the graph satisfy $(x : y) = (s : t)$. This means that $x = \alpha s$ and $y = \alpha t$ for some non-zero scalar α, and by eliminating α, we find the relation $xt-ys = 0$. Hence X is defined in $\mathbb{A}^2(k)\times\mathbb{P}^1(k)$ by that single equation, and we have:

$$X = Z(xt - ys) \subset \mathbb{A}^2(k) \times \mathbb{P}^1(k).$$

We also have two projection maps $p\colon X \to \mathbb{A}^2(k)$ and $q\colon X \to \mathbb{P}^1(k)$. Let us analyze the

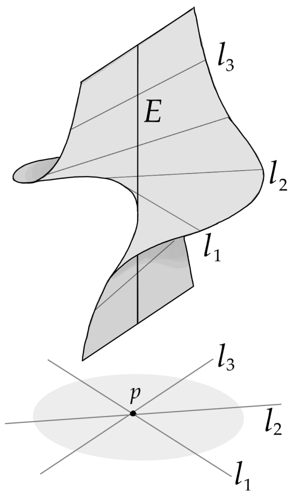


Figure 7.1 *The blow-up of the plane at a point*

fibres of the two maps. The fibres of p are easy to describe. If $(x, y) \in \mathbb{A}^2(k)$ is not the origin, then $p^{-1}(x, y)$ consists of a single point: the equation $xt = ys$ allows us to determine the point $(s : t)$ uniquely since either $x \neq 0$ or $y \neq 0$. However, when $(x, y) = (0, 0)$, any choices of s and t satisfy the equation, so $p^{-1}(0, 0) = (0, 0) \times \mathbb{P}^1(k)$. In particular, this inverse image is one-dimensional; it is called the *exceptional divisor* of X, and is frequently denoted by E.

Similarly, if $(s : t) \in \mathbb{P}^1(k)$ is a point, the fibre

$$q^{-1}(s : t) = \{(x, y) \times (s : t) \,|\, xt = ys\} \subset \mathbb{A}(k)^2 \times (s : t)$$

is the line in $\mathbb{A}^2(k)$ with $sx - ty = 0$ as equation, s and t being the coefficients. The map q is an example of a *line bundle*; all of its fibres are affine lines; that is, $\mathbb{A}^1(k)$'s. We will see these again later on in the chapter.

Using the standard cover of $\mathbb{P}^1(k)$ as a union of two $\mathbb{A}^1(k)$, we can give a a cover of X

consisting of two affine planes. For points in the open set $U \subset \mathbb{P}^1(k)$ where $s \neq 0$, we can normalize the coordinates by setting $s = 1$, and the equation $xt = sy$ then becomes $y = tx$. Hence x and t may serve as affine coordinates on $q^{-1}(U)$, and it follows that $q^{-1}(U) \simeq \mathbb{A}^2(k)$. In these normalized coordinates, the morphism $p\colon X \to \mathbb{A}^2_k$ restricts to the map $\mathbb{A}^2(k) \to \mathbb{A}^2(k)$ given by $(x,t) \mapsto (x, xt)$. Similarly, if V denotes the open set where $t \neq 0$, it holds that $q^{-1}(V) = \mathbb{A}^2(k)$ with affine coordinates y and s, and the map p is given there as $(y,s) \mapsto (sy, y)$.

The blow-up as a scheme

Inspired by the above discussion, we proceed to define the scheme-analogue of the blow-up of $\mathbb{A}^2_k$ at a point. It will be defined as a scheme over $\mathbb{Z}$ rather than over a field k (we get a blow-up of $\mathbb{A}^2_A$ for any ring A by replacing $\mathbb{Z}$ in everything below by A). Also, in addition to the scheme X, we want two morphisms of schemes $p\colon X \to \mathbb{A}^2$ and $q\colon X \to \mathbb{P}^1$ having similar properties to the morphisms in the example above.

Consider the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x,y]$. The prime ideal $\mathfrak{p} = (x,y) \subset \mathbb{Z}[x,y]$ corresponds to the origin of $\mathbb{A}^2(k)$ in the analogy with varitities. Consider the diagram

$$\begin{array}{ccccc} & & \mathbb{Z}[x,y] & & \\ & \swarrow & & \searrow & \\ \mathbb{Z}[x,t] & & & & \mathbb{Z}[y,s] \\ & \searrow & & \swarrow & \\ & & R = \mathbb{Z}[x,y,s,t]/(xt-ys, st-1), & & \end{array}$$

where the two skew maps in the upper part are given by $x \mapsto x$, $y \mapsto xt$ and $y \mapsto y$, $x \mapsto ys$ respectively, and the two others are induced by obvious inclusions.

Note that the ring R is isomorphic to $\mathbb{Z}[x,s,t]/(st-1) = \mathbb{Z}[x,t,t^{-1}]$ as well as to $\mathbb{Z}[y,s,t]/(st-1) = \mathbb{Z}[y,s,s^{-1}]$. Since this ring is a localization of both $\mathbb{Z}[x,t]$ and $\mathbb{Z}[y,s]$, we can identify its spectrum both as an open subscheme of $\operatorname{Spec} \mathbb{Z}[x,t]$ and as an open subscheme of $\operatorname{Spec} \mathbb{Z}[y,s]$. From this we get a diagram

$$\begin{array}{ccccc} & & \operatorname{Spec} \mathbb{Z}[x,y] & & \\ & \nearrow & & \nwarrow & \\ U = \operatorname{Spec} \mathbb{Z}[x,t] & & & & \operatorname{Spec} \mathbb{Z}[y,s] = V \\ & \nwarrow & & \nearrow & \\ & & \operatorname{Spec} R, & & \end{array}$$

where the bottom skew maps are open embeddings. Hence we can glue the two affine schemes U and V together along $\operatorname{Spec} R$ to obtain a new scheme X. By construction, the restrictions of the maps $\operatorname{Spec} \mathbb{Z}[x,t] \to \operatorname{Spec} \mathbb{Z}[x,y]$ and $\operatorname{Spec} \mathbb{Z}[y,s] \to \operatorname{Spec} \mathbb{Z}[x,y]$ to

$\operatorname{Spec} R$ coincide with the map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}[x, y]$, which is induced by $\mathbb{Z}[x, y] \to R$. Therefore they may be glued together to a morphism (the 'blow-up morphism')

$$p\colon X \to \mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x, y].$$

To complete the discussion, we should define the corresponding morphism $q\colon X \to \mathbb{P}^1$. Again we work locally. On the affine open $U = \operatorname{Spec} \mathbb{Z}[x, t]$ we have a map $U \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$ induced by the inclusion $\mathbb{Z}[t] \subset \mathbb{Z}[x, t]$. Similarly, on $V = \operatorname{Spec} \mathbb{Z}[y, s]$ we have a map $V \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[s]$. Checking if they can be glued together, amounts to seeing what happens on the overlap $U \cap V = \operatorname{Spec} R$. However, on $\operatorname{Spec} R$ it holds that $t = s^{-1}$, so using the standard description of $\mathbb{P}^1$ as being glued together of two affine lines, we see that the maps $\mathbb{Z}[t] \to R$ and $\mathbb{Z}[s] \to R$ induce the desired morphism $q\colon X \to \mathbb{P}^1$.

Exercise 7.5.1. Compute the space $\mathcal{O}_X(X)$ of global sections of the blow-up X and describe the canonical map $X \to \operatorname{Spec} \mathcal{O}_X(X)$.

Exercise 7.5.2. Imitate the construction above to define the blow-up of $\mathbb{A}^n$ along a codimension 2 linear space $V(x, y)$.

7.6 Projective spaces

In Chapter 1, we defined the projective spaces $\mathbb{P}^n(k)$ as varieties. In this section, we will construct the projective spaces as schemes over any ring A. In contrast to what we did before, when $\mathbb{P}^n(k)$ was constructed as a quotient space, we will construct $\mathbb{P}^n_A$ by gluing together $n + 1$ copies of the affine space $\mathbb{A}^n_A$. The gluing process resembles the one used for the projective line in Section 7.2.

Fix a ground ring A and variables $x_0, \dots, x_n$. For each $i = 0, \dots, n$, define the ring

$$R_i = A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right].$$

This is a polynomial ring over A in n variables, so each $U_i = \operatorname{Spec} R_i$ is isomorphic to $\mathbb{A}^n_A$. Note that each R_i is a subring of the ring

$$A[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}].$$

For each pair of indices i and j, there are equalities of subrings

$$R_i\left[\frac{x_i}{x_j}\right] = R_j\left[\frac{x_j}{x_i}\right] \tag{7.2}$$

which follows from the identities $x_l/x_i = x_l/x_j \cdot x_j/x_i$, valid for all i, j and l. The ring $R_i[x_i/x_j]$ is a localization of R_i in x_j/x_i, so we may identify $U_{ij} = \operatorname{Spec} R_i[x_i/x_j]$ with distinguished open subscheme $D(x_j/x_i) \subset U_i$. Then, using the equality (7.2), and the identity maps $\tau_{ij} : U_{ij} \to U_{ji}$ as gluing maps, the gluing conditions are clearly satisfied. The resulting scheme, is called the *projective n-space over A*, and is denoted by $\mathbb{P}^n_A$.

Note that all rings R_i are A-algebras, so each U_i is a scheme over A and comes with a structure map $U_i \to \operatorname{Spec} A$. These structure maps agree on U_{ij} and glue together to a map $\mathbb{P}^n_A \to \operatorname{Spec} A$, making $\mathbb{P}^n_A$ an A-scheme.

In analogy with Theorem 1.44 for $\mathbb{P}^n(k)$, and Proposition 7.1 for $\mathbb{P}^1_A$, we have the following result about global sections of the structure sheaf.

Proposition 7.5. Let A be a ring. Then

$$\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A.$$

Hypersurfaces in $\mathbb{P}^n_A$

Let $G \in A[x_0, \ldots, x_n]$ be a nonzero homogeneous polynomial of degree d. G determines a closed subscheme of $\mathbb{P}^n_A$ as follows. In the affine space $U_i = \operatorname{Spec} R_i$, we can consider the 'dehomogenization' of G with respect to the variable x_i, given by

$$g_i = G\left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\right) \in R_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

We can consider the affine subscheme of U_i given by

$$X_i = \operatorname{Spec}\left(R_i/(g_i)\right).$$

Note that if g_i and g_j denote the dehomogenizations of a polynomial g with respect to x_i and x_j, it holds that $g_i = (x_j/x_i)^d g_j$. The ideals (g_i) and (g_j) therefore become equal in the localization $R_{ij} = R_i[x_i/x_j] = R_j[x_j/x_i]$. This means that the subschemes X_i coincide in the intersections U_{ij}, and consequently they may be glued together to a closed subscheme $X \subset \mathbb{P}^n_A$. We call this the *projective hypersurface* defined by G.

More generally, any homogeneous ideal $I \subset A[x_0, \ldots, x_n]$ determines a closed subscheme of $\mathbb{P}^n_A$. We will explore this in greater detail in Chapter 9.

The projective plane

The *projective plane* $\mathbb{P}^2_A$ deserves special attention. It is constructed by gluing together the three affine planes $U_i = D_+(x_i) = \operatorname{Spec} R_i$ for $i = 0, 1, 2$.

It is sometimes helpful to rewrite these charts using the 'U_0-coordinates', i.e., writing $x = x_1/x_0$ and $y = x_2/x_0$. We can then express the other ratios in terms of x and y. For instance $x_2/x_1 = x^{-1}$. With this convention, the three affine opens become

$$U_0 = \operatorname{Spec} k[x, y], \quad U_1 = \operatorname{Spec} k[x^{-1}, yx^{-1}], \quad U_2 = \operatorname{Spec} k[y^{-1}, xy^{-1}].$$

Consider the hypersurface given by the homogeneous polynomial $G = x_2$. In the open set $U_0 = \operatorname{Spec} R_0$ the the ideal (x_2) becomes the ideal $(y) \subset A[x, y]$, and so

$$\operatorname{Spec} R_0/(g_0) \simeq \operatorname{Spec} A[x] \simeq \mathbb{A}^1_A.$$

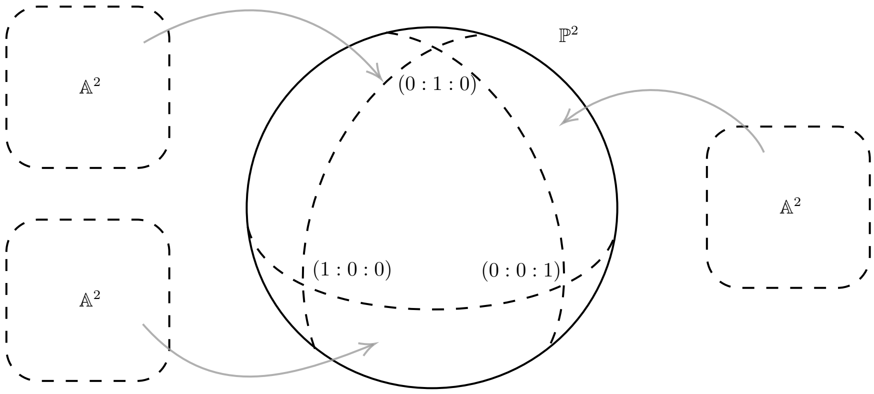
In U_1, the ideal (x_2) dehomogenizes to $(x_2/x_1^{-1} = x^{-1}y$, so that

$$\operatorname{Spec} R_1/(g_1) = \operatorname{Spec} A[x^{-1}] \simeq \mathbb{A}^1_A.$$

In $R_2 = A[y^{-1}, xy^{-1}]$, x_2 dehomogenizes to $x_2/x_2 = 1$, and so it defines the empty subscheme of U_2.

The hypersurface given by x_2 is therefore obtained by gluing two copies of $\mathbb{A}^1_A$ using the gluing map $x \mapsto x^{-1}$ over the overlaps, and so it is isomorphic to the projective line $\mathbb{P}^1_A$. As the subscheme is locally defined by a linear equation, and is isomorphic to $\mathbb{P}^1_A$ it deserves the name ‘a line’. In a completely symmetric way we find two other lines in $\mathbb{P}^2_A$ given by x_0 and x_1. Thus we picture $\mathbb{P}^2_A$ as U_0 with the ‘line at infinity’ given by the subscheme x_2.

If we choose a polynomial of higher degree, say $G = x_0^d + x_1^d + x_2^d$, we obtain more complicated subschemes X of $\mathbb{P}^2_A$. We think of them as ‘plane curves of degree d’, although the geometry as a scheme might be quite intricate if A is a general ring.



Exercises

Exercise 7.6.1. Prove Proposition 7.5. (A more general result will be proved in Chapter **??**).

7.7 Line bundles on $\mathbb{P}^1$

The sheaf $\mathcal{O}_{\mathbb{P}^1_k}(m)$ on the projective line $\mathbb{P}^1_k$, which we constructed in Example 7.2, has a geometric alter ego, the so-called line bundle L_m, which is a scheme with a morphism

$$\pi\colon L_m \longrightarrow \mathbb{P}^1_k,$$

Each fibre of π is an affine line; hence the name ‘line bundle’. In this section we shall construct these schemes explicitly and study some of them in detail.

For simplicity, we will work over a field k and will keep the convention that $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$ denote the standard affine cover of $\mathbb{P}^1_k$. Their intersection is equal to $U_0 \cap U_1 = \operatorname{Spec} k[u, u^{-1}]$.

Recall that the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(m)$ are obtained by gluing $\mathcal{O}_{U_0}$ and $\mathcal{O}_{U_1}$ together by means of the multiplication by u^m map on $\mathcal{O}_{U_0 \cap U_1}$. The new schemes L_m will be constructed essentially by the same gluing process, but schemes and not sheaves, will be glued together. Two copies of $\mathbb{A}^2_k$, $V_0 = \operatorname{Spec} k[u, s]$ and $V_1 = \operatorname{Spec} k[u^{-1}, t]$, will be glued together using the isomorphism

$$D(u) = \operatorname{Spec} k[u, u^{-1}, t] \xrightarrow{\simeq} \operatorname{Spec} k[u, u^{-1}, s] = D(u^{-1}),$$

which is induced by the isomorphism of k-algebras $\rho\colon k[u, u^{-1}, s] \to k[u, u^{-1}, t]$ that sends s to $u^m t$ and u to u. (The attentive reader will observe a change of sign in the exponent compared to the sheaf case.)

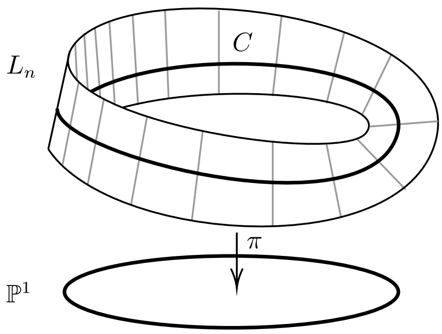
The situation is described with the following commutative diagram of ring maps:

$$\begin{array}{ccccccc} k[u,s] & \hookrightarrow & k[u,u^{-1},s] & \xrightarrow[\simeq]{\rho} & k[u,u^{-1},t] & \hookleftarrow & k[u^{-1},t] \\ \uparrow & & \nwarrow & & \nearrow & & \uparrow \\ k[u] & \hookrightarrow & & k[u,u^{-1}] & & \hookleftarrow & k[u^{-1}], \end{array}$$

where the maps other than ρ are the inclusions. Applying Spec, we get the following diagram of affine schemes:

$$\begin{array}{ccccc} \mathbb{A}^2 = V_0 & \hookleftarrow & D(u) \simeq D(u^{-1}) & \hookrightarrow & V_1 = \mathbb{A}^2_k \\ \downarrow & & \downarrow & & \downarrow \\ U_0 & \hookleftarrow & U_0 \cap U_1 & \hookrightarrow & U_1. \end{array}$$

The gluing conditions are trivially fulfilled (only a single morphism is involved), and hence we obtain a scheme L_m. It admits a morphism $\pi\colon L_m \to \mathbb{P}^1$ since the lower row gives the gluing data for $\mathbb{P}^1_k$. Note that if $x \in \mathbb{P}^1$ is a closed point, say $x \in U_0$, then the fibre $\pi^{-1}(x)$ is isomorphic to the affine line $\mathbb{A}^1_{k(x)}$. As noted at the top, this is the reason for the term 'line bundle': intuitively L_m is a family of affine lines parameterized by the base space $\mathbb{P}^1_k$.



There is a copy of $\mathbb{P}^1_k$ embedded in L_m which is called the *zero section* of L_m; that is, there is a closed embedding $\iota\colon \mathbb{P}^1_k \to L_m$ whose image is a closed subscheme $C \subset L_m$ that meets each fibre $\pi^{-1}(x) = \mathbb{A}^1_{k(x)}$ in the origin. Intuitively, this subscheme is defined by one of the equations $s = 0$ or $t = 0$ in each fibre. More precisely, C is given by $C \cap V_0 = V(s)$ and $C \cap V_1 = V(t)$. In the ring $\Gamma(V_0 \cap V_1, \mathcal{O}_{L_m})$, the relation $s = u^m t$ holds, and as u is invertible in $\Gamma(V_0 \cap V_1, \mathcal{O}_{L_m})$, the principal ideals (s) and (t) are equal. The two closed subschemes $V(s) \cap V_0 \cap V_1$ and $V(t) \cap V_0 \cap V_1$ coincide, and $V(s)$ and $V(t)$ can be patched together to a subscheme C.

We claim that C is a section of the morphism π; that is, it holds that $\pi \circ \iota = \mathrm{id}_{\mathbb{P}^1_k}$. As $V(s) = \operatorname{Spec} k[u,s]/(s) = \operatorname{Spec} k[u]$ as a subscheme of V_0, and $V(t) = \operatorname{Spec} k[u^{-1},t]/(t) = \operatorname{Spec} k[u^{-1}]$ inside V_1, we infer that $C \simeq \mathbb{P}^1_k$. Consider the composition of the maps

$$k[u] \longrightarrow k[u,s] \longrightarrow k[u,s]/(s) = k[u],$$

where the first map is the canonical inclusion andcorresponds geometrically to $\pi|_{V_0}$, and the second is the canonical quotient map and corresponds to the inclusion $\iota_0\colon V(s) = C \cap V_0 \to V_0$. Clearly, it holds that $\pi \circ \iota_0 = \mathrm{id}_{U_0}$. In a similar manner, it follows that $\pi|_{V_1} \circ \iota_1 = \mathrm{id}_{U_1}$, hence $\pi \circ \iota = \mathrm{id}_{\mathbb{P}^1_k}$ and C is a section.

A few particular cases

The schemes L_m give a rich source of examples in algebraic geometry, and we will come back to them several times in the book. For now let us study some of them in more detail.

Example 7.6 (The line-bundle L_0)**.** The scheme L_0 is glued together of two copies of $\mathbb{A}^2_k$ with the help of the inclusions

$$k[u,t] \longrightarrow k[u,u^{-1},t] \longleftarrow k[u^{-1},t].$$

In addition to π, the bundle L_0 admits a morphism $L_0 \to \mathbb{A}^1_k$ obtained by gluing together the two maps $\operatorname{Spec} k[u,t] \to \operatorname{Spec} k[t]$ and $\operatorname{Spec} k[u^{-1},t] \to \operatorname{Spec} k[t]$. The scheme L_0 is identified with the 'fibre product' $\mathbb{P}^1 \times_k \mathbb{A}^1_k$ (fibre products will be study in detail in Chapter 10), and is the scheme associated with the product variety $\mathbb{P}^1(k) \times \mathbb{A}^1(k)$.

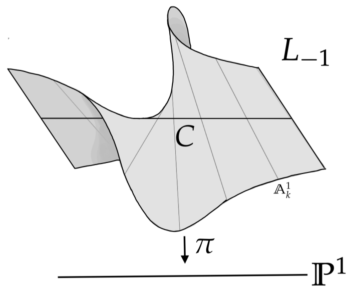
Example 7.7 (The line-bundle L_1)**.** The scheme L_1 is isomorphic to the complement of a closed point P in the projective plane, i.e. $Y = \mathbb{P}^2_k - \{P\}$. Indeed, choose coordinates x_0, x_1 and x_2 in the projective plane and consider the two distinguished open subschemes $V_0 = \operatorname{Spec} k[x_1/x_0, x_2/x_0]$ and $V_1 = \operatorname{Spec} k[x_0/x_1, x_2/x_1]$. Their union in $\mathbb{P}^2_k$ equals the complement of the closed point $P = (0:0:1)$. Renaming the variables $u = x_0/x_1$, $s = x_2/x_1$ and $t = x_2/x_0$, we find that $V_0 = \operatorname{Spec} k[u,s]$ and $V_1 = k[u^{-1},t]$, and the identity $x_2/x_1 = x_0/x_1 \cdot x_2/x_0$ turns into the equality $s = ut$, which is precisely the gluing data for L_1.

Geometrically the morphism $\mathbb{P}^2_k - \{P\} \to \mathbb{P}^1_k$ is given by 'projection from the point P'. The fibres are the lines in $\mathbb{P}^2_k$ through P (with the point P removed), and the zero section equals the line 'at infinity'; i.e. the line $V(x_2)$.

Example 7.8 (The line-bundle L_{-1})**.** We have in fact seen the scheme L_{-1} before: it is isomorphic to the blow-up of $\mathbb{A}^2_k$ at the origin. Recall that the blow-up X comes equipped with a map $q\colon X \to \mathbb{P}^1_k$, which is described in detail at the end of Section 7.5. One checks without much difficulties that the gluing maps used for forming q are the same as for making L_{-1}. The zero-section C corresponds to the exceptional divisor E in the blow-up. See also Exercise 7.7.1 below.

Example 7.9 (The line-bundle L_{-2})**.** The scheme L_{-2} is quite interesting. It is the so-called *desingularization of a quadratic cone*. The quadratic cone is the subscheme $Q = V(y^2 - xz)$ of $\mathbb{A}^3_k$, which is equal to $\operatorname{Spec} R$ with $R = k[x,y,z]/(y^2 - xz)$. We claim that there is a surjective morphism $\sigma\colon L_{-2} \to \operatorname{Spec} R$, which is an isomorphism outside the curve C. (The morphism σ is helpful for understanding the quadratic cone. In the terminology of Chapter 13, Q has a 'singularity' at the origin, whereas L_{-2} is 'non-singular'.)

We shall construct σ by giving the restrictions σ_i to each of the two opens V_0 and V_1 that make up L_{-2}. Recall that $V_0 = \operatorname{Spec} k[u,s]$ and $V_1 = \operatorname{Spec} k[u^{-1},t]$ with gluing map $\operatorname{Spec} k[u,u^{-1},s] \simeq \operatorname{Spec} k[u,u^{-1},t]$ given by the assignment $s \mapsto u^{-2}t$. The maps σ_i



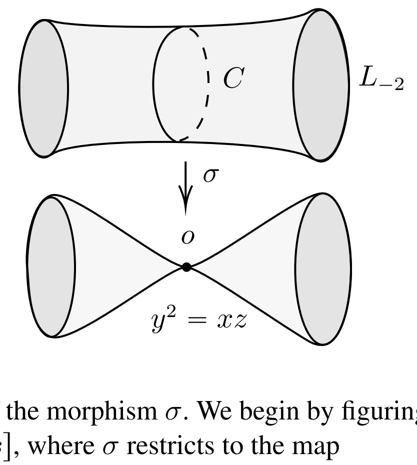
are Spec's of the ring maps $\phi_0\colon R \to k[u,s]$ and $\phi_1\colon R \to k[u^{-1},t]$ coming from the assignments

$$\phi_0\colon x \mapsto s,\ y \mapsto us,\ z \mapsto u^2 s;$$
$$\phi_1\colon x \mapsto u^{-2}t,\ y \mapsto u^{-1}t,\ z \mapsto t.$$

It holds that $\phi_0(y^2 - xz) = (us)^2 - u(us) = 0$ and $\phi_1(y^2 - xz) = (u^{-1}t)^2 - u^{-1}(u^{-1}t) = 0$, so the ϕ_i's are well defined. The σ_i's are compatible with the transitions function and can be glued together to the desired map $\sigma\colon L_{-2} \to \mathbb{P}^1_k$; indeed, one easily verifies that the diagram

$$\begin{array}{ccc} & R & \\ {\scriptstyle \phi_0}\swarrow & & \searrow{\scriptstyle \phi_1} \\ k[u,u^{-1},s] & \xrightarrow[\rho]{} & k[u,u^{-1},t] \end{array}$$

commutes; for instance, $\rho(\phi_0(x)) = \rho(s) = u^{-2}t = \phi_1(x)$.



Let us analyse the fibres of the morphism σ. We begin by figuring out what happens over the open set $V_0 = \operatorname{Spec} k[u,s]$, where σ restricts to the map

$$\sigma_0\colon \operatorname{Spec} k[u,s] \to Q$$

corresponding to ϕ_0. Consider the maximal ideal $\mathfrak{m} = (x,y,z) \subset R$ of the origin. The fibre over $\mathfrak{m}$ corresponds to prime ideals in $\mathfrak{p} \subset k[u,s]$ containing $\mathfrak{m}k[u,s] = (s, su, su^2) = (s)$; that is, the fibre equals the closed set $\sigma^{-1}(V(\mathfrak{m})) = V(s)$. This means that the whole

'u-axis' $V(s)$ in $\mathbb{A}^2 = \operatorname{Spec} k[u,s]$ is collapsed onto the origin in Q. Likewise, the 'u^{-1}-axis' in $\mathbb{A}^2 = \operatorname{Spec} k[u^{-1},t]$ is collapsed to the origin; in oher words, the whole zero-section C in L_{-2} is mapped to the origin. In fact, the C is the only subscheme of L_{-2} which is contracted; σ is an isomorphism outside C:

Proposition 7.10. The map σ restricts to an isomorphism $L_{-2} - C \xrightarrow{\simeq} Q - \{p\}$, where p is the origin in Q.

Proof The complement $Q - \{p\}$ of the origin is covered by the two distinguished open sets $D(x)$ and $D(z)$ (note that $D(y) = D(y^2) = D(xz)$ by the quadratic relation defining R).Likewise, the complement $L_{-2} - C$ of the zero-section is covered by the distinguished open subsets $D(s) \subset V_0 = \operatorname{Spec} k[u,s]$ and $D(t) \subset V_1 = \operatorname{Spec} k[u^{-1},t]$. It holds that $\sigma_0^{-1}(V(x)) = V(s) \subset \operatorname{Spec} k[u,s]$, and this means that the restriction $\sigma|_{V_0} = \sigma_0$ maps $D(s)$ onto $D(x)$. In fact, using the identification $D(x) = \operatorname{Spec} R_x$, and the identity $R_x = (k[x,y,z]/(y^2 - xz))_x \simeq k[x,y]_x$,we see that σ_0 is the map

$$\operatorname{Spec} k[u,s]_s \to \operatorname{Spec} k[x,y]_x$$

induced by the ring map such that $x \mapsto s$ and $y \mapsto us$. This is an isomorphism because we have inverted s. Hence $\sigma|_{V_0}$ is an isomorphism over $D(x)$. A symmetric argument shows that $\sigma|_{V_1}$ is an isomorphism over $D(z)$; all together, σ is an isomorphism outside C. □

Exercises

Exercise 7.7.1. Check that L_{-1} is indeed the blow-up constructed in Section 7.5.

Exercise 7.7.2. Show that for $m \geqslant 0$, the scheme L_{-m} admits a morphism $\sigma\colon L_{-m} \to Y$ contracting the zero-section C to a point.

Exercise 7.7.3. For the canonical morphism $\pi\colon L_m \to \mathbb{P}^1_k$, show that

$$\pi_* \mathcal{O}_{L_m} = \bigoplus_{i \geqslant 0} \mathcal{O}_{\mathbb{P}^1_k}(-im).$$

Exercise 7.7.4 (A variety perspective)**.** When k is an algebraically closed field the k-points of L_m are described by expressions resembling homogeneous coordinates.

a) Show that the k-points of L_m are precisely the equivalence classes of triples

$$(x_0 : x_1 \mid t),$$

where $x_0, x_1, t \in k$, with $(x_0, x_1) \neq (0,0)$ under the relation

$$(x_0 : x_1 \mid t) = (\alpha x_0 : \alpha x_1 \mid \alpha^m t),$$

for $\alpha \in k$ a non-zero scalar.

b) Show that the zero section is the the set of points of the form $(x_0 : x_1 \mid 0)$, and that if $m \geqslant 0$ and $p(x_0, x_1)$ is a homogeneous polynomial of degree m, then the map $\mathbb{P}^1(k) \to L_m(k)$ given by the assignment

$$(x_0 : x_1) \mapsto (x_0 : x_1 \mid q(x_0, x_1) t)$$

is a well defined section of $L_m(k) \to \mathbb{P}^1(k)$ (at least in a set-theoretic sense).

Exercise 7.7.5 (A variety perspective)**.** Define $f\colon L_{-m}(k) \to \mathbb{A}^{m+1}(k)$ by

$$(x_0 : x_1 \mid t) \mapsto (tx_0^m, tx_0^{m-1}x_1, \ldots, tx_0x_1^{m-1}t, x_1^m)$$

Show that this map is well defined and collapses the zero-section to the origin. Define and describe a scheme version of this map.

7.8 Double covers of projective space

Consider a polynomial in n variables over a base ring A, $f \in R = A[x_1, \ldots, x_n]$. This defines a closed subscheme X of the affine space $\mathbb{A}_A^{n+1} = \operatorname{Spec} R[y]$ given by

$$X = \operatorname{Spec} R[y]/(y^2 - f)$$

There is a morphism $\sigma : X \to \mathbb{A}_A^n$ induced by the ring map $R \to R[y]/(y^2 - f)$. We call X, with the map σ, the *double cover* of $\mathbb{A}_A^n = \operatorname{Spec} R$ associated to f. The name comes from the following example:

Example 7.11. Let $A = k$ be an algebraically closed field, and let $p \in \mathbb{A}_k^n$ be the closed point corresponding to the maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ in R. By Proposition 2.34, the fibre $\sigma^{-1}(p)$ is given by

$$\operatorname{Spec}(R[y]/((y^2 - f) + \mathfrak{m})) \simeq \operatorname{Spec} \mathbb{C}[y]/(y^2 - f(a_1, \ldots, a_n)).$$

If $f(a_1, \ldots, a_n) \neq 0$, the fibre consists of two points, and if $f(a_1, \ldots, a_n) = 0$, the fibre has one (it is given by $\operatorname{Spec} \mathbb{C}[y]/y^2$).

We will also consider double covers of projective spaces by gluing together the double coverings we just constructed. We begin with the case of $\mathbb{P}^1$.

Hyperelliptic curves

Let k be a field and consider a polynomial

$$p(x) = a_{2g+1}x^{2g+1} + \cdots + a_1x$$

of degree $2g + 1$ in $k[x]$.

Consider the two affine schemes $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, where

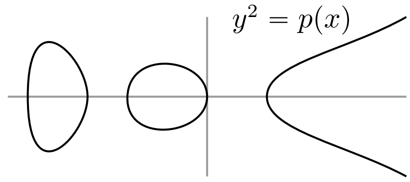
$$A = k[x, y]/(y^2 - p(x)) \text{ and } B = k[u, v]/(v^2 - u^{2g+2}p(u^{-1})).$$

Note that $u^{2g+2}p(u^{-1}) = a_{2g+1}u + \cdots + a_1u^{2g+1}$ indeed is a polynomial in u. The two distinguished open sets $D(x) = \operatorname{Spec} A_x$ and $D(u) = \operatorname{Spec} B_u$ are isomorphic: the assignments $\phi(u) = x^{-1}$ and $\phi(v) = x^{-g-1}y$ define an isomorphism $\phi\colon B_u \to A_x$. The map is well defined, as the little calculation

$$\phi(v^2 - u^{2g+2}p(u^{-1})) = x^{-2(g+1)}y^2 - x^{-(2g+2)}p(x) = x^{-(2g+2)}(y^2 - p(x))$$

shows that the defining ideal for B_u maps into the one defining A_x, and one verifies easily that the maps $x \mapsto u^{-1}$ and $y \mapsto vu^{-g-1}$ define the inverse. We can therefore glue X_1 and X_2 together along the open subsets $D(x)$ and $D(u)$.

The resulting scheme X is called a *hyperelliptic curve*, and it is a *double covering of* $\mathbb{P}^1_k$. In the case $g = 1$, the curve X is an example of an *elliptic curve*. Below is an illustration of the real points of one ofthe distinguished opensfor $g = 2$:



The scheme X admits a morphism π to $\mathbb{P}^1_k$: consider the two inclusions $k[x] \subset A$ and $k[u] \subset B$. Under the identification map $\phi\colon B_u \to A_x$ above, $k[u]$ is mapped into $k[x]$ and u maps to x^{-1}, as in the commutative diagram:

$$\begin{array}{ccc} k[u] & \xrightarrow{u \mapsto x^{-1}} & k[x] \\ \downarrow & & \downarrow \\ B_u & \xrightarrow{\phi} & A_x. \end{array}$$

The two inclusions yield maps $X_1 \to U_0 = \operatorname{Spec} k[x] \subset \mathbb{P}^1_k$ and $X_2 \to U_1 = \operatorname{Spec} k[u] \subset \mathbb{P}^1_k$, where U_0 and U_1 are joined together to a $\mathbb{P}^1_k$ according to the rule $x \leftrightarrow u^{-1}$. By the observation above, this is compatible with the way X_1 and X_2 are joined together, and so we get the desired morphism.

Observe that π is a double cover. Consider for instance the open set X_1. If $z \in U_0 \subset \operatorname{Spec} k[x]$ is a closed point with maximal ideal $\mathfrak{m} \subset k[x]$, the fibre over z is equal to

$$\pi^{-1}(p) = \operatorname{Spec} A/\mathfrak{m}A = \operatorname{Spec} K[y]/(y^2 - a),$$

where $K = k[x]/\mathfrak{m}$ and a is the residue of p in $K = k[x]/\mathfrak{m}$. If the characteristic of k is not two and $a \in k$ with a square root $b \in k$, then $K[y]/(y^2 - a) = k[y]/(y - b) \times k[y]/(y + b) \simeq k \times k$, and the fibre has two points. In the other cases, there is just one point in the fibre, but the vector space dimension of $K[y]/(y^2 - a)$ over K is still two, and the moniker 'double cover' persists being meaningful.

Notice that the construction of X is very similar to how the schemes L_m from Section 7.7 were made. In fact, X is a closed subscheme of L_{-g-1} in a natural way. Indeed, L_{-g-1} is the union of of $U = \operatorname{Spec} k[x, y]$ and $V = \operatorname{Spec}[u, v]$ and they are glued together with the maps defined by the same assignments as ϕ and ψ, and as these pass to the quotients A_x and B_u, we infer that X_1 and X_2 patch together to a closed subscheme of L_{-g-1}.

Higher-dimensional double coverings

The above construction generalizes in a straightforward manner to higher-dimensional projective spaces. We will even consider projective spaces over any ring A.

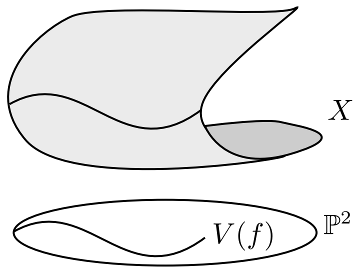
Let A be a ring and let $R = A[x_0, \ldots, x_n]$ with the usual grading. Let $f \in R$ be a homogeneous polynomial of degree $2d$, and for each $0 \leqslant i \leqslant n$ let

$$S_i = A[\tfrac{x_0}{x_i}, \ldots, \tfrac{x_n}{x_i}, \tfrac{y}{x_i^d}]/\big((\tfrac{y}{x_i^d})^2 - f(\tfrac{x_0}{x_i}, \ldots, \tfrac{x_n}{x_i})\big)$$

For each pair i, j letting $S_{ij} = S_i[x_i x_j^{-1}]$, one checks that $S_{ij} = S_{ji}$; indeed, this reduces to the identity

$$\left(\frac{x_i}{x_j}\right)^{2d}\left(\left(\frac{y}{x_i^d}\right)^2 - f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right) = \left(\frac{y}{x_j^d}\right)^2 - f\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

It is then straightforward to verify that the $\operatorname{Spec} S_i$'s glue together along the open subschemes $\operatorname{Spec} S_{ij}$'s to a scheme X. Moreover, keeping the notation R_i from the previous section, the morphisms $\operatorname{Spec} S_i \to \operatorname{Spec} R_i$, induced by the inclusions $R_i \to S_i$, glue together to a morphism $\pi\colon X \to \mathbb{P}^n_A$.



Exercise 7.8.1. Assume that k is algebraically closed. Let $a_{2g+1} = 1$ and $a_1 = -1$ and $a_i = 0$ for the other indices. Determine the image of $D(x)$ and $D(u)$ in $\mathbb{P}^1_k$. Find all points in $\mathbb{P}^1_k$ where the fibre of the double covering f does not consist of exactly two points. How many are there?

7.9 Hirzebruch surfaces

The Hirzebruch surfaces form a family of schemes showing several similarities with the line bundles in Section 7.7. There is one Hirzebruch surface $\mathbb{F}_m$ for each natural number m, and like the L_m's, they are fibrations over a $\mathbb{P}^1$; that is, they come equipped with morphisms

$$\pi\colon \mathbb{F}_m \longrightarrow \mathbb{P}^1.$$

The fibres, however, are not affine lines $\mathbb{A}^1$'s, but projective lines $\mathbb{P}^1$.

The construction of the Hirzebruch surfaces is very similar to how the L_m's were made, the difference being that the fibres of π are projective lines instead of affine lines. It works over $\mathbb{Z}$ as base ring, but to aid the intuition and ease notation, we shall work over a field k.

We begin with describing what kind of isomorphisms will constitute the gluing data. Consider two copies of the projective line $\mathbb{P}^1_A$ where A is a ring. The standard cover of one of the copies will be $U_0 = \operatorname{Spec} A[s]$ and $U_1 = \operatorname{Spec} A[s^{-1}]$, and the other has standard cover formed by $V_0 = \operatorname{Spec} A[t]$ and $V_1 = \operatorname{Spec} A[t^{-1}]$. Now, each invertible element $a \in A$ gives rise to two ring maps

$$\begin{aligned} A[s] \to A[t] &\quad s \mapsto at; \\ A[s^{-1}] \to A[t^{-1}] &\quad s^{-1} \mapsto a^{-1}t, \end{aligned}$$

which clearly agree on the overlap (that is, they induce the same map on $A[s, s^{-1}]$), and so we can patch them together to get a morphism $f_a\colon \mathbb{P}^1_A \to \mathbb{P}^1_A$. The map f_a is an isomorphism because a is invertible. Since the maps above do not affect elements from A, the map f_a is compatible with the structure maps of the $\mathbb{P}^1_A$'s:

$$\begin{array}{ccc} \mathbb{P}^1_A & \xrightarrow{f_a} & \mathbb{P}^1_A \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A. & \end{array}$$

We are now ready to construct the Hirzebruch surfaces. We view the base $\mathbb{P}^1_k$ as the union of the two affine pieces $U = \operatorname{Spec} k[u]$ and $V = \operatorname{Spec} k[u^{-1}]$, whose intersection equals $\operatorname{Spec} A$ with $A = k[u, u^{-1}]$. We work with two copies $\mathbb{P}^1_U$ and $\mathbb{P}^1_V$ of projective lines, the first one over $U = \operatorname{Spec} k[u]$ and the second over $V = \operatorname{Spec} k[u^{-1}]$. The structure maps are p and q respectively. Coordinates will be u, s and v, t on $\mathbb{P}^1_U$ and $\mathbb{P}^1_V$ respectively, so they come with covers $\{U_0, U_1\}$ and $\{V_0, V_1\}$ as described above.

Inside the base $\mathbb{P}^1_k$ we have the intersection $U \cap V = \operatorname{Spec} A$, and the idea is to glue together $\mathbb{P}^1_U|_{U\cap V} = p^{-1}(U \cap V)$ and $\mathbb{P}^1_V|_{U\cap V} = q^{-1}(U \cap V)$. Note that both of these are copies of $\mathbb{P}^1_A$ as above, and the gluing map ρ will be of the form f_a given there with $a = u^m$. The following diagram gives the situation map:

$$\begin{array}{ccccccc} \mathbb{P}^1_U & \hookleftarrow & \mathbb{P}^1_U|_{U\cap V} & \xrightarrow[\simeq]{\rho} & \mathbb{P}^1_V|_{U\cap V} & \hookrightarrow & \mathbb{P}^1_V \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} \\ U & \hookleftarrow & U \cap V & \xleftrightarrow{=} & U \cap V & \hookrightarrow & V \end{array}$$

The gluing conditions are trivially fulfilled, and the result is the Hirzebruch surface $\mathbb{F}^m$. The maps p and q are joined together to yield $\pi\colon \mathbb{F}^m \to \mathbb{P}^1$.

The gluing data is given by the maps:

$$\begin{aligned} k[u, u^{-1}, s] \to k[u, u^{-1}, t] &\quad s \mapsto u^n t \\ k[u, u^{-1}, s^{-1}] \to k[u, u^{-1}, t^{-1}] &\quad s^{-1} \mapsto u^{-n} t^{-1} \end{aligned}$$

Relation with the line bundles L_m

The similarities between the construction of the line bundles L_m and the Hirzebruch surface suggest a close relationship, and indeed there is one. The map $\rho\colon \mathbb{P}^1_A \to \mathbb{P}^1_A$ respects the standard covers; it takes $\operatorname{Spec} k[u, s]$ into $\operatorname{Spec} k[u, t]$ where it acts like $u \mapsto u$ and $s \mapsto u^n t$. This is exactly the gluing for the line bundle L_m, and we recognize L_m an open subset of $\mathbb{F}_m$ respecting the morphisms to $\mathbb{P}^1_l$:

$$\begin{array}{ccc} L_m & \hookrightarrow & \mathbb{F} \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1_k & \end{array}$$

In the same vein, ρ maps $\operatorname{Spec} k[u, u^{-1}, s^{-1}]$ into $\operatorname{Spec} k[u, u^{-1}, t^{-1}]$ by the assignments $u \mapsto u$ and $s^{-1} \mapsto u^{-m}t^{-1}$ which is the transition function for the line bundle L_{-m} (note that here the 'fibre coordinates' are s^{-1} and t^{-1}).

Note further that the complement of L_m is the zero-section C_{-m} in L_{-m} and the complement of L_{-m} is the zero-section C_m in L_m.

Example 7.12. In Example 7.7, we explained that L_1 is isomorphic to $\mathbb{P}^2_k - P$ with $P = (0 : 0 : 1)$. In this continuation we shall see that the isomorphism in fact extends to a morphism $\mathbb{F}_1 \to \mathbb{P}^2$ and that this map collapses the zero section C_{-m} to the point P; it is the blow-up of $\mathbb{P}^2_k$ in P.

We choose homogeneous coordinates x_0, x_1 and x_2 on $\mathbb{P}^2_k$ and write

$$V_0 = \operatorname{Spec} k\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \simeq \operatorname{Spec} k[u, s]$$
$$V_1 = \operatorname{Spec} k\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}\right] \simeq \operatorname{Spec} k[u, t]$$

by setting $u = x_0/x_1$, $s = x_2/x_1$ and $t = x_2/x_0$. Together they give an open embedding $L_1 \to \mathbb{P}^2_k$ with image $V_0 \cup V_1$.

We want to extend this over the open subscheme $W = \operatorname{Spec} k[u^{-1}, t^{-1}]$ in $\mathbb{F}_1$. This is done using the map ring map $k[x_0/x_2, x_1/x_2] \to k[u^{-1}, t^{-1}]$ given by the assignments $x_0/x_2 \mapsto t^{-1}$ and $x_1/x_2 \mapsto u^{-1}t^{-1}$. These are compatible with the above settings, and so the corresponding map $\operatorname{Spec} k[u^{-1}, t^{-1}] \to \operatorname{Spec} k[x_0/x_2, x_1/x_2] \subset \mathbb{P}^2_k$ can be patched to the one above, to yield a map $L_m \cup W \to \mathbb{P}^2_k$

Exercises

Exercise 7.9.1 ($\mathbb{F}_m$ as a variety)**.** Let k be an algebraically closed field. Show that the k-points of $\mathbb{F}_m(k)$ are in a one-to-one correspondence with the equivalence classes of quadruples

$$(x_0 : x_1 \mid y_0 : y_1)$$

where equivalence means

$$(x_0 : x_1 \mid y_0 : y_1) \sim (\alpha x_0 : \alpha x_1 \mid \alpha^m \beta y_0 : \beta y_1)$$

for non-zero scalars α and β.

Exercise 7.9.2. The different Hirzebruch surfaces are closely related, as this exercise shows.

a) Show that for some point P there is a map $\mathbb{F}_m - P \to \mathbb{F}_{m-1}$ that induces an isomorphism on the complement of two fibres.
b) Show that for some point P there is a map $\mathbb{F}_{m-1} - P \to \mathbb{F}_m$ that induces an isomorphism on the complement of two fibres.

HINT: The 'variety versions' are $(x_0 : x_1 \mid y_0 : y_1) \mapsto (x_0 : x_1 \mid y_0 : x_1y_1)$ with $P = (1 : 0 \mid 0 : 1)$ and $(x_0 : x_1 \mid y_0 : y_1) \mapsto (x_0 : x_1 \mid x_1y_0 : y_1)$ with $P = (1 : 0 \mid 1 : 0)$.

Exercise 7.9.3. Show that the open subschemes $L_m - C_{-m}$ and $L_m - C_m$ of respectively L_m and L_{-m} are isomorphic over $\mathbb{P}^1_k$. Show that gluing them together gives $\mathbb{F}_m$.

Exercise 7.9.4. The construction of $\mathbb{F}_m$ does not require that m is positive. Show that using $-m$ would yield a scheme isomorphic to $\mathbb{F}_m$.

Exercise 7.9.5. Let $X = \mathbb{P}^1_k$. Show that any element $\mathcal{O}_X(X)$ corresponding to a map $X \to \mathbb{A}^1$ factors via a "constant map" $\operatorname{Spec} k \to \mathbb{A}^1$.

8

Finiteness coditions and dimension

8.1 Noetherian schemes

Recall that a ring A is Noetherian if every ideal is finitely generated. This is a strong requirement for the ring, which has many important consequences. An equivalent condition is that any ascending chain of ideals eventually stabilizes. Note that an ascending chain of ideals $\{\mathfrak{a}_i\}$ in A corresponds to a descending chain $\{V(\mathfrak{a}_i)\}$ of closed subsets of $\operatorname{Spec} A$, which will be eventually constant when A is Noetherian.

This inspires the notion of a *Noetherian topological space*. These are spaces that satisfy the descending chain condition on closed subsets: every descending chain

$$\ldots \subset X_{i+1} \subset X_i \subset \ldots\ldots \subset X_2 \subset X_1 \tag{8.1}$$

of closed subsets stabilizes. Or in other words, $X_{i+1} = X_i$ for sufficiently large i.

Another aspect of Noetherian rings is that any non-empty collection of ideals has a maximal element (ordered by inclusion). Noetherian spaces have the analogous property that every non-empty collection of closed subsets has a minimal element.

Lemma 8.1.

(i) A topological space X is Noetherian if and only if every non-empty collection of closed subsets has a minimal element.

(ii) A Noetherian space is quasi-compact;

(iii) Every subspace Y of a Noetherian space X is Noetherian.

Proof Proof of (i): assume we have a non-empty set Σ of closed subsets that does not have a minimal element. Selecting any element $V_1 \in \Sigma$, we find that it must strictly contain an another element V_2 from Σ. This second element V_2 must in turn contain yet another element V_3 from Σ as a strict subset, and the process continues indefinitely. In this way, we construct an infinite strictly descending chain $V_1 \supset V_2 \supset V_3 \supset \ldots$ closed subsets of X, contradicting the assumption that X is Noetherian. The opposite implication follows by the definition of Noetherian space.

Proof of (ii): Let $\{U_i\}_{i\in I}$ be an open cover for X. Start with any U_{i_1}, and pick U_{i_2} so that $U_{i_1} \subsetneq U_{i_1} \cup U_{i_2}$. Then pick U_{i_3} so that $U_{i_1} \cup U_{i_2} \subsetneq U_{i_1} \cup U_{i_2} \cup U_{i_3}$ and so on. This produces a strictly increasing chain of open subsets of X, which must stabilize because X is Noetherian. Since every point of X is contained in some U_i, we find that X is expressed as a finite union of the subsets $U_{i_1}, \ldots, U_{i_n}$, and hence it is quasi-compact.

Proof of (iii): let $Y \subset X$ be a subspace of X and consider a non-empty collection Σ of closed subsets of Y. We define a corresponding collection Σ' in X as follows:

$$\Sigma' = \{W \mid W \text{ is closed in } X \text{ and } W \cap Y \in \Sigma\}.$$

This is also non-empty because in the subspace topology the closed sets in Y are exactly the intersections with Y of closed sets in X. Since X is Noetherian, there exists a minimal element W_0 in Σ'. We claim that $W_0 \cap Y$ is minimal in Σ. For any closed subset Z' in Y that is contained in $W_0 \cap Y$, there exists some closed set W' in X such that $W' \cap Y = Z'$. Now $W' \cap W_0$ is also in Σ' and contains $W' \cap W_0 \cap Y = Z'$. Minimality of W_0 in Σ' implies $W' \cap W_0 = W_0$, and thus $Z' = W_0 \cap Y$. Therefore, $W_0 \cap Y$ is indeed minimal in Σ. Therefore Y is Noetherian by (i).

□

The prototype example of a Noetherian space is $\operatorname{Spec} A$ where A is a Noetherian ring. However, $\operatorname{Spec} A$ can be Noetherian even without A being Noetherian; the condition is equivalent to the weaker condition that ascending chains of *radical ideals* eventually stabilize, and there are many rings which satisfy this without being Noetherian. Here is a simple example:

Example 8.2. Consider the polynomial ring $k[t_1, t_2, t_3, \dots]$ and the maximal ideal $\mathfrak{m} = (t_1, t_2, \dots)$. The ring

$$A = k[t_1, t_2, t_3, \dots]/\mathfrak{m}^2$$

has only one prime ideal, the maximal ideal $\mathfrak{m}$. Therefore, $\operatorname{Spec} A$ consists of a single point, and is therefore Noetherian as a topological space. The ring A however is not Noetherian, as $\mathfrak{m}$ requires infinitely many generators, namely all the t_i's.

In light of this example, we take a different route to define Noetherian schemes (we want $\operatorname{Spec} A$ to be a Noetherian scheme precisely when A is a Noetherian ring):

Definition 8.3.
(i) A scheme is *locally Noetherian* if it can be covered by open affine subschemes $\operatorname{Spec} A_i$ with each A_i being a Noetherian ring;
(ii) A scheme is *Noetherian* if it is both locally Noetherian and quasi-compact.

Note that a scheme is Noetherian if and only if it can be covered by finitely many open affines $\operatorname{Spec} A_i$ where each A_i is Noetherian.

Proposition 8.4. The spectrum $\operatorname{Spec} A$ is a Noetherian scheme if and only if A is a Noetherian ring.

Proof The 'if'-direction is clear, so assume that $\operatorname{Spec} A$ is Noetherian, which means that it may be covered by finitely many open affine subschemes $\operatorname{Spec} A_i$ with A_i Noetherian. Refining the cover using distinguished open sets, we may assume that each A_i is of the form A_{g_i}.

We want to show that each ideal $\mathfrak{a}$ in A is finitely generated. By assumption, the ideals

$\mathfrak{a}A_{g_i}$ are all finitely generated, and since g_i is a unit in A_{g_i}, we can find generators which are images of elements a_{ij} from A.

Consider then the A-linear map $\phi\colon \bigoplus_{i,j} A \to \mathfrak{a}$ that sends the standard basis vector e_{ij} to a_{ij}. Since the $D(g_i)$'s cover $\operatorname{Spec} A$, the localization $\phi_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in \operatorname{Spec} A$, and from this follows that ϕ is surjective as well. Consequently, $\mathfrak{a}$ is finitely generated. □

Proposition 8.5. If X is a Noetherian scheme, its underlying topological space is Noetherian.

Proof Since X is quasi-compact, it may be covered by a finite number of open affine subsets. A descending chain stabilizes if the intersection with each of those open sets stabilizes, so we reduce the proof to showing the proposition for $X = \operatorname{Spec} A$ with A a Noetherian ring. But that case is clear by the previous proposition. □

Proposition 8.6. Let X be a (locally) Noetherian scheme. Then any open or closed subscheme of X is also (locally) Noetherian.

Proof It will suffice to treat the case that X is Noetherian. Let $\{\operatorname{Spec} A_i\}_{i\in I}$ be a finite affine cover with each A_i Noetherian. It suffices to prove that if $Y \subset X$ is a closed or open subscheme, then $Y \cap \operatorname{Spec} A_i$ is Noetherian. In particular, since $Y \cap \operatorname{Spec} A_i$ is closed or open subscheme of an affine scheme, it suffices to consider the case where $X = \operatorname{Spec} A$ and A is Noetherian.

When Y is an open open subscheme: then there are elements $g_1, \ldots, g_n \in A$ such that we have $Y = \bigcup_{i=1}^n \operatorname{Spec} A_{g_i}$. If A is Noetherian, then so is each of the localizations A_{g_i}, and consequently Y is Noetherian.

When Y is a closed subscheme. $Y = \operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. If A is Noetherian, then so is $A/\mathfrak{a}$, and again $Y = \operatorname{Spec} A/\mathfrak{a}$ is Noetherian. □

Examples

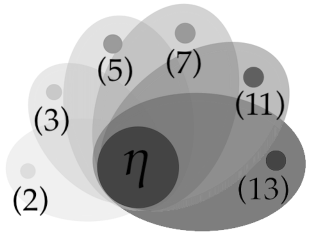
Example 8.7. All of the examples from Chapter 7 are Noetherian. They are all glued together by finitely many schemes of the form $\operatorname{Spec} A$ where A is a Noetherian ring.

Example 8.8. For a field k, the disjoint union $X = \coprod_{i=1}^\infty \operatorname{Spec} k$ is not Noetherian. In fact, it is not even quasi-compact.

Example 8.9. The scheme $X = \operatorname{Spec}\left(\prod_{i=1}^\infty k\right)$ is affine, hence quasi-compact. However it is not Noetherian, because the ring is not Noetherian.

In fact, the set of prime ideals in infinite products of fields is remarkably complicated: it is described by the set of so-called ‘ultrafilters’ on $\mathbb{N}$. (See also Exercise 8.1.9 for a related example).

Example 8.10. In Example 7.4 on page 97 we glued together schemes $X_p = \operatorname{Spec} \mathbb{Z}_{(p)}$ with p from a finite set of primes $\mathcal{P}$. However, in the gluing conditions for schemes, there are no restrictions on the number of schemes to be glued together, and we are free to take $\mathcal{P}$ infinite; for example, we can use the set $\mathcal{P}$ of all primes.



The resulting scheme $X_{\mathcal{P}}$ is rather peculiar: it is neither affine nor Noetherian, but it is locally Noetherian. As a scheme over $\mathbb{Z}$, the canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ is bijective and continuous, but it is not a homeomorphism. Moreover, for all open subsets $U \subset \operatorname{Spec}\mathbb{Z}$ the map induced on sections $\pi^\sharp\colon \Gamma(U, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}) \to \Gamma(\pi^{-1}U, \mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism; in other words, $\pi^\sharp\colon \mathcal{O}_{\operatorname{Spec}\mathbb{Z}} \to \pi_*(\mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism of sheaves!

As in Example 7.4 the scheme $X_{\mathcal{P}}$ is constructed by gluing the different $\operatorname{Spec}\mathbb{Z}_{(p)}$'s together along the generic points. However, when computing the global sections, we see things changing. As in Example 7.4 the global sections are computed with the help of the sheaf sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \Gamma(X, \mathcal{O}_X) & \longrightarrow & \prod_{p\in\mathcal{P}} \Gamma(X_p, \mathcal{O}_X) & \longrightarrow & \prod_{p,q\in\mathcal{P}} \Gamma(X_p \cap X_q, \mathcal{O}_X) \\ & & \| \downarrow & & \downarrow \| \\ & & \prod_{p\in\mathcal{P}} \mathbb{Z}_{(p)} & \xrightarrow{\quad\rho\quad} & \prod_{p,q\in\mathcal{P}} \mathbb{Q}, \end{array}$$

and the kernel of ρ is still $\bigcap_{p\in\mathcal{P}} \mathbb{Z}_{(p)}$, but now this intersection equals $\mathbb{Z}$; indeed, a rational number $\alpha = a/b$ lies in $\mathbb{Z}_{(p)}$ precisely when the denominator b does not have p as factor, so lying in all $\mathbb{Z}_{(p)}$, means that b has no non-trivial prime-factor. That is, $b = \pm 1$, and hence $\alpha \in \mathbb{Z}$.

One can understand the canonical map $\pi : X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ as follows. Each of the schemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ maps in a natural way into $\operatorname{Spec}\mathbb{Z}$, by the map induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}_{(p)}$. Here the generic point of $\operatorname{Spec}\mathbb{Z}_p$ map to generic point of $\operatorname{Spec}\mathbb{Z}$, and the closed point maps to $(p) \in \operatorname{Spec}\mathbb{Z}$. As the maps agree on the generic points, they glue to the canonical map $\pi : X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$. This is a continuous bijection by construction, but it is not a homeomorphism. Indeed, the subsets $\operatorname{Spec}\mathbb{Z}_{(p)}$ are open in $X_{\mathcal{P}}$ by the gluing construction, but they are not open in $\operatorname{Spec}\mathbb{Z}$, as their complements are infinite.

The underlying topological space of $X_{\mathcal{P}}$ is not Noetherian, as the subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ form an open cover that obviously cannot be reduced to a finite cover. However, it is locally Noetherian as the open subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ are Noetherian. The sets $U_p = X_{\mathcal{P}} - \{(p)\}$ map bijectively to $D(p) \subset \operatorname{Spec}\mathbb{Z}$ and $\Gamma(U_p, \mathcal{O}_{X_{\mathcal{P}}}) = \mathbb{Z}_p$, but U_p and $D(p)$ are not isomorphic.

Decomposition into irreducibles

A fundamental result about Noetherian rings is the Lasker–Noether theorem stating that every ideal $\mathfrak{a}$ in a Noetherian ring A admits an irredudant primary decomposition; in other words, $\mathfrak{a}$

can be expressed as an intersection

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r, \tag{8.2}$$

where the $\mathfrak{q}_i$'s are primary ideals with no inclusion relation among them. Such a decomposition is not always unique, but there are partial uniqueness results. The associated prime ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ are unique, as are the primary components $\mathfrak{q}_i$ whose associated prime ideals $\mathfrak{p}_i$ are minimal among the associated primes.

Geometrically, the decomposition (8.2) means that the closed subset $V(\mathfrak{a}) \subset \operatorname{Spec} A$ can be written as a finite union of irreducible closed subsets:

$$V(\mathfrak{a}) = V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2) \cup \cdots \cup V(\mathfrak{p}_r). \tag{8.3}$$

Of course, only minimal primes matter. If $\mathfrak{p}_i \subset \mathfrak{p}_j$, then $V(\mathfrak{p}_j)$ is contained in $V(\mathfrak{p}_i)$, and we can disregard it. Since these embedded components do not show up for radical ideals and since $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$, we get a clear and clean uniqueness statement. The closed sets appearing in (8.3) are unique up to ordering.

In general, if $Y \subset X$ is a closed subset of a topological space X, a decomposition

$$Y = Y_1 \cup \cdots \cup Y_r \tag{8.4}$$

into irreducible closed subsets is said to be *irredundant* if $Y_i \not\subset Y_j$ for every $i \neq j$. Or equivalently, no Y_i can be dropped without changing the union. The Lasker–Noether theorem has the following analogue:

Theorem 8.11. Every closed subset Y of a Noetherian topological space X has an irredundant decomposition into closed and irreducible subsets. Furthermore, the subsets appearing in the decomposition are unique up to order.

The irreducible closed subsets in the decomposition are the *irreducible components* of Y.

Proof Consider the family Σ of those closed subsets of X that cannot be decomposed into a finite union of irreducible closed subsets; or phrased in a different way, the set of counterexamples to the assertion. If the theorem does not hold, $\Sigma \neq \emptyset$. By assumption X is Noetherian, so Σ has a minimal element Y, which can not be irreducible. Hence $Y = Y_1 \cup Y_2$ where both Y_1 and Y_2 are proper subsets of Y and therefore do not belong to Σ. Either is thus a finite union of closed irreducible subsets, and the same is then true for their union Y. We have a contradiction, and Σ must be empty, and the theorem holds.

As to uniqueness, assume there are two irredundant decompositions such that

$$Y_1 \cup \cdots \cup Y_r = Z_1 \cup \cdots \cup Z_s$$

and such that one of the Y_i's, say Y_1, does not equal any of the Z_k's. Since Y_1 is irreducible and $Y_1 = \bigcup_j \big(Z_j \cap Y_1\big)$, it follows that $Y_1 \subset Z_j$ for some index j. A similar argument gives $Z_j = \bigcup_i \big(Z_j \cap Y_i\big)$ and Z_j being irreducible, it holds that $Z_j \subset Y_i$ for some i. Therefore $Y_1 \subset Z_j \subset Y_i$. Since the union of the Y_i's is irredundant, we infer that $Y_1 = Y_i$, and hence that $Y_1 = Z_j$. Contradiction. □

Example 8.12. Consider the closed set $Y = V(\mathfrak{a}) \subset \mathbb{A}^3_k$ given by the ideal

$$\mathfrak{a} = (x^2 - y, xz - y^2, x^3 - xz).$$

A primary decomposition of $\mathfrak{a}$ is given by $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3$, where

$$\mathfrak{q}_1 = (x, y), \quad \mathfrak{q}_2 = (x - 1, y - 1, z - 1), \quad \mathfrak{q}_3 = (x^2 - y, xy, y^2, z).$$

Taking radicals, we find that the primes associated to $\mathfrak{a}$ are the following:

$$\mathfrak{p}_1 = (x, y), \quad \mathfrak{p}_2 = (x - 1, y - 1, z - 1), \quad \mathfrak{p}_3 = (x, y, z).$$

Note that $\mathfrak{p}_1 \subset \mathfrak{p}_3$, and $\mathfrak{p}_3$ is thus an embedded component, which does not show up in the decomposition above. We therefore have $V(\mathfrak{a}) = V(x, y) \cup V(x - 1, y - 1, z - 1)$.

Exercises

Exercise 8.1.1. Show that in a topological space the closure of a singleton is irreducible.

Exercise 8.1.2 (Properties of irreducible subsets)**.** Let X be a topological space.

a) Show that if a subset $Z \subset X$ is irreducible, then so is the closure $\overline{Z}$;
b) Show that X is irreducible if and only if every non-empty open subset is dense;
c) If $f\colon X \to Y$ is a continuous map, show that $f(X)$ is irreducible if X is.

Exercise 8.1.3 (Irreducible components)**.** The maximal closed irreducible subsets of a topological space X are called the *irreducible components* of X.

a) Prove that any irreducible subset of a topological space X is contained in an irreducible component. HINT: Zorn's lemma;
b) Prove that X is the union of its irreducible components;
c) If X is Noetherian, prove that the irreducible components are precisely the sets appearing in the Lasker–Noether decomposition of X.

Exercise 8.1.4. Let X be a topological space and let $Z \subset X$ be an irreducible component of X. Let U be an open subset of X and assume that $U \cap Z$ is nonempty. Show that $Z \cap U$ is an irreducible component of U.

Exercise 8.1.5. Let X be a topological space. Show that the following two conditions are equivalent.

(i) X is Noetherian;
(ii) Every open subset of X is quasi-compact.

Exercise 8.1.6. Compute a primary decomposition for the following ideals and describe their corresponding closed subsets.

a) $I = (x^2y^2, x^2z, y^2z)$ in $k[x, y, z]$;
b) $I = (x^2y, y^2x)$ in $k[x, y]$;
c) $I = (x^3y, y^4x)$ in $k[x, y]$;
d) $I = (x, y, x - yz)$ in $k[x, y, z]$;
e) $I = (x^2 + (y - 1)^2 - 1, y - x^2)$ in $k[x, y]$.

Exercise 8.1.7 (A one dimensional non-Noetherian domain)**.** The ring A in this exercise was originally constructed by Krull as an example of a non-Noetherian domain with just one non-zero prime ideal. The spectrum $\operatorname{Spec} A$ has two points and is a Noetherian topological space, while A is not a Noetherian ring.

The example is no more exotic than the ring of rational functions $f(x, y)$ in two variables over $\mathbb{C}$ that are defined and constant on the y-axis. The elements of A, when written in lowest terms, have a denominator not divisible by x, and $f(0, y)$, which is then meaningful, is constant.

a) Show that the ideals $\mathfrak{a}_r = (x, xy^{-1}, \dots, xy^{-r})$ with $r \in \mathbb{N}$ form an ascending chain that does not stabilize. Conclude that R is not Noetherian.
b) Show that R is local with the set $\mathfrak{m}$ of elements $f \in R$ that vanish along the y-axis as the maximal ideal.
c) Prove that there are no other primes than $\mathfrak{m}$ and (0) in R. HINT: Show first that any element in R is of the form $x^i y^j \alpha$ where $i \geqslant 0$, $j \in \mathbb{Z}$ and α is a unit in R.

Exercise 8.1.8 (Perfect rings)**.** This exercise provides an abundance of non-Noetherian domains with Noetherian spectrum. Let A be a Noetherian reduced ring of characteristic p which is not a field, and let $F\colon A \to A$ denote the Frobenius homomorphism $a \mapsto a^p$. Consider the direct system $\{A_i\}_{i\in\mathbb{N}}$ with $A_i = A$ for all i and maps given by the sequence

$$A \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} \dots\dots$$

Let A^∞ denote the direct limit $\varinjlim A_i$ and let $\phi_i\colon A = A_i \to A^\infty$ denote the canonical maps.

a) Show that F is not surjective;
b) Let $a \in A$ be a non-unit. Show that the principal ideals $(\phi_i(a))$ in A^∞ form an ascending chain which is not stationary. Conclude that A^∞ not Noetherian;
c) Show that each $\operatorname{Spec} \phi_i$ is a homeomorphism $\operatorname{Spec} A \simeq \operatorname{Spec} A^\infty$, and conclude that $\operatorname{Spec} A^\infty$ is a Noetherian topological space.

Exercise 8.1.9 (The ring of eventually constant sequences)**.** Consider the subring A of $\prod_{i=1}^\infty \mathbb{Z}/2\mathbb{Z}$ consisting of sequences $(e_i)_{i\geqslant 1}$ which are eventually constant, that is, sequences with $e_i = e_{i+1}$ for $i \gg 0$.

a) Show that all elements of A are idempotents and conclude that every prime ideal is maximal. HINT: the only idempotents in a domain are 0 and 1.
b) Let $\mathfrak{m}_n$ denote the ideal generated by $1 - a_n$ where $a_n = (0, \dots, 0, 1, 0, \dots)$ with a '1' in the n-th factor. Show that $\mathfrak{m}_n$ is a maximal ideal.
c) Show that $D(a_n) = \{m_n\}$ and conclude that the one-point set $\{\mathfrak{m}_n\}$ is both open and closed in $\operatorname{Spec} A$.
d) Let $\mathfrak{m}_\infty$ denote the ideal consisting of sequences which are eventually zero, i.e., $e_i = 0$ for all $i \gg 0$. Show that $\mathfrak{m}_\infty$ is a maximal ideal. HINT: Consider the 'limit map' $A \to \mathbb{Z}/2$.
e) Show that A is not Noetherian. HINT: Show that $\mathfrak{m}_\infty$ is not finitely generated.
f) Show that these are all the prime ideals of A, i.e., that $\operatorname{Spec} A = \{\, \mathfrak{m}_i \mid i \in$

$\mathbb{N}\,\} \cup \{\mathfrak{m}_\infty\}$. HINT: Consider the cases $a_i \notin \mathfrak{m}$ for some i and $a_i \in \mathfrak{m}$ for all i separately; use the identity $a_i(1-a_i) = 0$.

g) Show that $\operatorname{Spec} A$ is homeomorphic to the set $\{\, \frac{1}{n} \mid n \in \mathbb{N}\,\} \cup \{0\}$ (with the standard topology).

8.2 Finite morphisms and morphisms of finite type

Let A be a ring and let B be an A-algebra. Recall that one says that B is *finitely generated* or *of finite type* over A if there is a finite set $b_1, \ldots, b_r$ of elements from B such that each $b \in B$ can be expressed as a polynomial with coefficient from A in the b_i's. One says that B is a *finite over* A if it is finitely generated as an A-module. In other words, there is a finite set of elements $b_1, \ldots, b_r$ so that each b is a linear combination $b = \sum a_i b_i$ with $a_i \in A$.

Even though the names are similar, the two notions are quite different. To say that B is of finite type, is to say that B is a ring quotient of a polynomial ring $A[t_1, \ldots, t_r]$, where as B being finite means that B is a quotient module of a free module A^r of finite rank. Thus $\mathbb{Z}[t]$ is of finite type, but not finite over $\mathbb{Z}$.

Morphisms of finite type

The scheme-theoretic analogue of the notion 'finitely generated algebra' is as follows:

Definition 8.13 (Morphisms of finite type)**.** Let X be a scheme over S with structure morphism $f\colon X \to S$.

(i) One says that f or X/S is of *locally finite type* if S has a cover consisting of affine open subschemes $V_i = \operatorname{Spec} A_i$ such that each $f^{-1}V_i$ can be covered by affine open subschemes $\operatorname{Spec} B_{ij}$, where each B_{ij} is finitely generated as an A_i-algebra;

(ii) One says that f or X/S is of *finite type* if in (i) one can do with a finite number of subschemes $\operatorname{Spec} B_{ij}$ for each i.

In case $S = \operatorname{Spec} A$, a scheme over A is said to be of *finite type* (respectively of *locally finite type*) over A if the morphism $X \to \operatorname{Spec} A$ is of finite type (respectively of finite type). Note that being (locally) of finite type is local on the target; if S can be covered by opens U_i so that all restrictions $f|_{f^{-1}U_i}$ are (locally) of finite type, then clearly f is (locally) of finite type as well.

The prototype example of a morphism of finite type is $f : \operatorname{Spec} B \to \operatorname{Spec} A$, where B is a finitely generated A-algebra, and f is induced by the natural map $A \to B$. The converse, that B is of finite type when f is, holds true as well, though this is slightly tricky to prove (see Corollary 8.19 below).

Proposition 8.14. A morphism $f : X \to S$ is of locally finite type if and only if for *any* affine cover $S_i = \operatorname{Spec} A_i$ of S, $f^{-1}(S_i)$ can be covered by affine subschemes $\operatorname{Spec} B_{ij}$ with each B_{ij} a finitely generated A_i-algebra.

Example 8.15. Both the affine spaces $\mathbb{A}^n_A$ and the projective spaces $\mathbb{P}^n_A$ are of finite type over $\operatorname{Spec} A$. The morphism $\coprod_{i=1}^{\infty} \mathbb{A}^1_k \to \mathbb{A}^1_k$ which is the identity on each component, is locally of finite type, but not of finite type.

Example 8.16. A closed embedding $\iota\colon X \to Y$ is of finite type. Indeed, by definition there is an open affine cover $\{\operatorname{Spec} A_i\}$ of Y so that $\iota^{-1}U_i \simeq \operatorname{Spec} A_i/\mathfrak{a}_i$, and $A_i/\mathfrak{a}_i$ is of finite type.

Example 8.17 (Open embeddings)**.** An open embedding $\iota\colon U \to X$ is locally of finite type, but is not of finite type in general. For instance, the open immersion

$$\bigcup_{t\in\mathbb{N}} D(t_i) \longrightarrow \operatorname{Spec} k[t_1, t_2, \dots]$$

is not of finite type, because the scheme on the left is not quasi-compact (and thus cannot be covered by finitely many affine subschemes).

However, if U is quasi-compact, then ι is of finite type. In that case, for any open affine $\operatorname{Spec} A$ in X, $U \cap \operatorname{Spec} A$ is open in $\operatorname{Spec} A$, and can be covered by *finitely many* distinguished open sets $D(g_i) = \operatorname{Spec}(A_{g_i})$, and each A_{g_i} is finitely generated over A (being generated by g_i^{-1}). In particular, if X is Noetherian, then any open embedding $\iota : U \to X$ is of finite type.

Note that Definition 8.13 refers to a specific affine cover $\{V_i\}$ of the base S and $\{\operatorname{Spec} B_{ij}\}_j$ of the inverse images $f^{-1}V_i$. It is an important fact that the conditions will in fact hold for any open affine cover.

Proposition 8.18. Let $f\colon X \to S$ be a morphism of finite type. Then for any open affine subscheme $\operatorname{Spec} A \subset S$ and each open affine $\operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$, the algebra B is finitely generated over A.

In particular, when both X and S are both affine, we have the following corollary.

Corollary 8.19. A morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is of finite type if and only if B is an A-algebra of finite type.

We will prove this result in Section 8.5.

Affine and finite morphisms

The other finiteness condition of this is section is that of a *finite morphism*. In addition to satisfy a rather strong finiteness requirement, finite morphism are required to be *affine*.

Definition 8.20 (Affine and finite morphisms)**.** Let $f\colon X \to Y$ be a morphism. One says that

(i) f is *affine* if there is an open covering $\{V_i\}$ of Y such that each inverse image $f^{-1}V_i$ is affine;
(ii) f is *finite* if there is a cover of Y by open affines $V_i = \operatorname{Spec} A_i$ such that each $f^{-1}V_i = \operatorname{Spec} B_i$, with B_i an A_i-algebra finitely generated as A_i-module.

One also says that X is finite over Y, and if $Y = \operatorname{Spec} A$, that X is finite over A.

As in the definition of finite type morphisms, the definitions of affine and finite morphisms make reference to a specific affine cover of the base. Therefore, it is not a priori clear whether a scheme which is affine over another affine scheme, is necessarily an affine scheme itself. This is nevertheless true, and is a particular case of the following more general result.

Proposition 8.21.

(i) If $f\colon X \to Y$ is affine morphism, the inverse image $f^{-1}U$ of each open affine subset $U \subset Y$ is affine.
(ii) If $f\colon X \to Y$ is a finite morphism and $U = \operatorname{Spec} A \subset Y$ is an open affine subscheme, then $f^{-1}U = \operatorname{Spec} B$ where B is a finite A-module.

The proof will be postponed until Section 8.5.

To underline the huge difference between the two finiteness conditions of this section, we observe the following: X is of finite type over a field k simply means it can be covered by open affine subschemes of the form $\operatorname{Spec} k[t_1, \ldots, t_r]/\mathfrak{a}$.

On the other hand, for X is to be finite over a field k means that $X = \operatorname{Spec} A$ is affine, and A is a k-algebra of finite dimension over k. Such a ring A is Artinian and has only finitely many prime ideals all being maximal. Hence the spectrum $\operatorname{Spec} A$ is a finite set, and the underlying topology is discrete.

Example 8.22. For $n \geqslant 1$, the structure morphisms $\mathbb{A}^n_k \to \operatorname{Spec} k$ and $\mathbb{P}^1_k \to \operatorname{Spec} k$ are of finite type, but not finite.

Example 8.23. The embedding $\operatorname{Spec} A_g \hookrightarrow \operatorname{Spec} A$ of a distinguished open subscheme is of finite type, but generally not finite.

An important fact about finite morphisms is that they have finite fibres.

Proposition 8.24. If $f : X \to Y$ is a finite morphism, then each scheme-theoretic fiber X_y has an underlying topological space which is finite and discrete.

Proof If $y \in Y$ is a point, choose an affine $U = \operatorname{Spec} A$ containing it. As f is finite, $f^{-1}(U) = \operatorname{Spec} B$ is also affine, so we reduce to the case where X and Y are affine, and f is induced by a ring map $A \to B$, making B into a finite A-module.

In this situation, y corresponds to a prime ideal $\mathfrak{p} \subset A$, and it follows that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a finite vector space over $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ (images of generators persist being generators). In other words, $B_{\mathfrak{b}}/\mathfrak{p}B_{\mathfrak{p}}$ is an Artinian ring, and hence its spectrum $X_y = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ is finite and discrete. □

Example 8.25. The converse of Proposition 8.24 does not hold. The open embedding $\mathbb{A}^1_k - \{0\} \hookrightarrow \mathbb{A}^1_k$ has at most one point in the fibre, but it is not a finite morphism.

There is a collection of results, the Cohen–Seidenberg Theorems, about prime ideals in integral extension with important applications to finite morphisms. We summarize them here without proofs. They are formulated with the more general hypothesis that the extension is integral, but finite ring extensions are integral.

Theorem 8.26. Let $A \subset B$ be an integral extension of rings.

(i) (Lying–Over) If $\mathfrak{p}$ prime ideal in A, there is prime ideal $\mathfrak{q}$ in B so that $\mathfrak{q} \cap A = B$;
(ii) If $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals in B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then $\mathfrak{q} = \mathfrak{q}'$;
(iii) (Going–Up) If $\mathfrak{p} \subset \mathfrak{p}'$ are two prime ideals in A and $\mathfrak{q} \in \operatorname{Spec} B$ with $\mathfrak{q} \cap A = \mathfrak{p}$, there is a $\mathfrak{q}' \in \operatorname{Spec} B$ with $\mathfrak{q}' \cap A = \mathfrak{q}$;
(iv) (Going–Down) Assume that A is integrally closed and that $\mathfrak{p}' \subset \mathfrak{p}$ are two prime ideals. If $\mathfrak{q} \in \operatorname{Spec} B$ is such that $\mathfrak{q} \cap A = \mathfrak{p}$, then there is a $\mathfrak{q}' \in \operatorname{Spec} B$ such that $\mathfrak{q}' \cap A = \mathfrak{p}'$.

For the moment, we shall only apply the two first parts. Translated into geometric language they give the following result about finite morphisms. One says that a morphism $f\colon X \to Y$ is *dominant*, if the image is a dense subset of Y.

Proposition 8.27 (Lying-Over). Let $f\colon X \to Y$ be a finite morphism between two schemes.

(i) The fibres of f are finite and discrete;
(ii) If f is dominant, it is surjective;
(iii) f is a closed map.

Proof We may assume that X and Y are affine, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$.

Statement (i) was discussed above. To prove (ii), note that according to Proposition 2.29 on page 33, the mophism f is dominant precisely when the kernel of the correspondig map $f^\sharp\colon A \to B$ between algebras is contained in the nilradical $\sqrt{0}$ of A. Hence when f is dominant, the map $\operatorname{Spec}(A/\operatorname{Ker} f^\sharp) \to \operatorname{Spec} A$ is a homeomorphism. We may thua assume that $A \subset B$, and Lying-Over applies.

Statement (iii) follows from (ii): by (iii) of Proposition 2.27 on page 32, the closure $\overline{f(V(\mathfrak{b}))}$ of the image of a closed subset $V(\mathfrak{b}) \subset \operatorname{Spec} B$ equals $V(\mathfrak{b} \cap A)$. Applying Lying-Over to the inclusion $A/\mathfrak{b} \cap A \subset B/\mathfrak{b}$, we see that $f(V(\mathfrak{b})) = V(\mathfrak{b} \cap A)$. □

Example 8.28. Let k be an algebraically closed field and consider the closed subset $X = V(y^2 + P(x)) \subset \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ where $P(x)$ is a polynomial in $k[x]$. Let $\pi\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ denote the projection onto the x-axis; that is, the map induced by the inclusion $k[x] \subset k[x,y]$. Then the restriction $\pi|_X$ will be finite. Indeed, its algebraic counterpart is the inclusion

$k[x] \subset k[x,y]/(y^2 + P(x))$, and the latter ring has a basis as module over $k[x]$ consisting of 1 and y.

On the contrary, if $Y = V((x-a)y^2 + P(x))$ where $a \in k$ is not root of $P(x)$, then $\pi|_Y$ is not a finite morphism. Indeed, the point $a \in \mathbb{A}^1(k)$ does not belong to its image, and so $\pi|_Y$ is not a closed map.

Exercises

Exercise 8.2.1. For each of the following rings A, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type.

$$\mathbb{Z}[i],\ \mathbb{Z}[1/p],\ \mathbb{Z}_{(p)},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}[x].$$

Exercise 8.2.2. Show that the composition of two morphisms (locally) of finite type is (locally) of finite type. Show that if S is quasi-compact and $f\colon X \to S$ is of finite type, then X will be quasi-compact.

Exercise 8.2.3. Assume that $\iota\colon \operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ is an open embedding. Show that B is of finite type over A.

Exercise 8.2.4. Assume that S is a Noetherian scheme and that $f\colon X \to S$ is of finite type. Prove that X is Noetherian. HINT: Hilbert's Basis Theorem.

Exercise 8.2.5. Let $A \subset B$ an integral extension of domains. Show that A is a field if and only if B is a field. If $\mathfrak{p}$ is a prime in A, show that $\mathfrak{p}$ lies in the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ if and only if $\mathfrak{p}A_{\mathfrak{p}}$ lies in the image of $\operatorname{Spec} B_{\mathfrak{p}} \to \operatorname{Spec} A_{\mathfrak{p}}$. Conclude that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Exercise 8.2.6. Let $f : X \to Y$ be an affine morphism and let $V \subset Y$ be an open set. Show that $f^{-1}(V) \to V$ is affine.

More generally, if $V \to Y$ is any morphism, show that the base change morphism $X \times_Y V \to V$ is affine. Thus affine morphisms are stable under base change.

Exercise 8.2.7. Show that the composition of two finite morphisms is finite.

8.3 The dimension of a scheme

Recall that the *Krull dimension* of a ring A is the supremum of the length of strictly ascending chains of prime ideals in A. For schemes, we make the following similar definition, which in fact works for any topological space.

Definition 8.29 (Dimension of topological spaces)**.** Let X be a topological space. The dimension of X is the supremum of all integers n such that there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct irreducible closed subsets of X.

This supremum might not be finite, in which case we declare that $\dim X = \infty$. A chain is said to be *saturated* if no new term can be inserted, and it is *maximal* if it is saturated and cannot be extended. If X is a scheme, we define the dimension of X as the dimension of the underlying topological space. In particular, it holds true that $\dim X = \dim X_{\text{red}}$.

Lemma 8.30.
(i) If $Y \subset X$ is any subset, then $\dim Y \leqslant \dim X$;
(ii) Assume that $\dim X < \infty$. If $Y \subset X$ is a closed and irreducible and $\dim Y = \dim X$, then Y is an irreduible component of X;
(iii) If $\{U_i\}_{i \in I}$ is an open cover of X, then $\dim X = \sup_{i \in I} \dim U_i$.

Proof Statement (i): the closure of irreducible subsets are irreducible, and since any closed $Z \subset Y$ satisfies $\overline{Z} \cap Y = Z$, a chain $\{Z_i\}$ of distinct irreducible closed subsets of Y will yield a chain $\{\overline{Z_i}\}$ of distinct irreducible closed subsets of X.

Statement (ii): were Y not a maximal closed irreducible subset of X, any maximal chain $Z_0 \subset \ldots \subset Z_r \subset Y$ in Y could be augmented to a longer chain in X.

Statement (iii): observe that if $Z \subset X$ is closed an irreducible and $Z \cap U \neq \varnothing$, then $\overline{Z \cap U} = Z$; indeed, were $\overline{Z \cap U}$ a proper subset, Z would be the union $Z = \overline{Z \cap U} \cup (Z - U)$ of two proper closed subsets. Hence if $Z_0 \subset \ldots \subset Z_n$ is a chain in X and U an element of the cover such that $U \cap Z_0 \neq \varnothing$, then $\{Z_i \cap U\}$ is a chain in U; and consequently $n \leqslant \dim U$. This shows that when $\dim X = \infty$, the supremum $\sup_{i \in I} \dim U_i$ will be infinite as well, and when $\dim X$ is finite, taking the chain to be maximal, we see that $\dim X = n \leqslant \dim U$. □

In the case where $X = \operatorname{Spec} A$ is affine, the closed irreducible subsets of X are of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal. Using this observation we find

Proposition 8.31. The dimension of $X = \operatorname{Spec} A$ equals the Krull dimension of A.

Having finite dimension does not guarantee that a scheme is Noetherian; see Example 8.2 for a 'trivial counterexample'. More seriously, there are even Noetherian rings whose Krull dimension is infinite. The first example was constructed by Masayoshi Nagata. Although each maximal chain of prime ideals in a Noetherian ring will be of finite length (prime ideals satisfy the descending chain condition) there can be arbitrary long ones.

The following is a consequence of the Going-Up part of the Cohen–Seidenberg theorems:

Proposition 8.32. If $f \colon X \to Y$ is a finite surjective morphism, then $\dim X = \dim Y$.

Proof Assume first that $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. Since f is dominant, we may further assume that $A \subset B$. By (ii) of Theorem A.17, any chain of distinct prime ideals in B remains a chain of distinct prime ideals when intersected with A. Hence $\dim X \leqslant \dim Y$. On the other hand, by succsessive application of Going-Up, any chain of distinct primes may be extended to a chain of distinct prime ideals in B. Hence $\dim Y \leqslant \dim X$.

In general, if $\{U_i\}$ is any affine cover of Y the inverse images $f^{-1}U_i$ form an affine cover of X, and we are through by the affine case and (iii) of Lemma 8.30. □

Example 8.33.

(i) The dimension of $\operatorname{Spec}\mathbb{Z}$ equals one. The maximal chains of prime ideals have the form $V(p) \subset V(0) = \operatorname{Spec}\mathbb{Z}$ for prime numbers p;

(ii) $\dim \operatorname{Spec} k[\epsilon]/(\epsilon^2) = 0$;

(iii) The dimension of $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$ is equal to $n + \dim A$ when A is a Noetherian ring (for general rings $\dim \mathbb{A}^n_A$ takes values between $\dim A + n$ and $\dim A + 2n$, and all values are possible). In particular, when $A = k$ is a field, $\mathbb{A}^n_k$ has dimension n. An instance of a maximal chain of irreducible closed subsets is

$$V(x_1, \ldots, x_n) \subset \ldots \subset V(x_1, x_2) \subset V(x_1) \subset \mathbb{A}^n_k.$$

(iv) The dimension of $\mathbb{A}^1_{\mathbb{Z}}$ equals two; maximal chains of prime ideals in $\mathbb{Z}[x]$ are shaped like $(0) \subset (p) \subset (f(x), p)$, where p is a prime number and $f(x)$ a polynomial which is irreducible mod p.

Example 8.34 (Zero-dimensional schemes)**.** The schemes

$$\operatorname{Spec}\mathbb{Z}/p\mathbb{Z}, \qquad \operatorname{Spec}\mathbb{C}[x]/(x^n), \qquad \operatorname{Spec}\mathbb{C}[x,y]/(x^2, xy, y^3),$$

have dimension zero. More generally, the spectrum of an Artinian ring has dimension zero (and when A is Noetherian, $\operatorname{Spec} A$ has dimension zero if and only if A is Artinian). However, there are non-Noetherian rings, e.g. the ring $\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, which have dimension zero and even infinitely many points (see Exercise **??**). The ring $A = \prod_{n=1}^{\infty} \mathbb{Z}/2^n\mathbb{Z}$ has infinite Krull dimension, yet $\operatorname{Spec} A$ is still Noetherian as a topological space.

Codimension

For a closed subset $Y \subset X$ the dimensions $\dim Y$ and $\dim X$ are defined in terms of closed irreducible subsets contained in Y and X respectively. When Y is irreducible, there is also a relative notion, the *codimension* of Y in X, denoted by $\operatorname{codim}(Y, X)$, which is defined in terms of closed irreducible subsets of X containing Y. These three numbers will in some important cases be related by the equality $\dim Y + \operatorname{codim}(Y, X) = \dim X$ (which justifies the name ‘codimension’), although this formula does not hold in general, it is not even true for all spectra of Noetherian integral domains (see Example 8.37 below).

Definition 8.35 (Codimension)**.** Let $Y \subset X$ be an irreducible closed subset of X. The *codimension* of Y is the supremum of all integers n such that there exists a chain

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of X.

In an affine case there is a bijective correspondence between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals in A, and the codimension of a closed subset $V(\mathfrak{p})$ will be equal to the height of the prime ideal $\mathfrak{p}$; that is, the maximal length of a chain of distinct prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r = \mathfrak{p}$, or equivalently, the Krull dimension $\dim A_{\mathfrak{p}}$ of the localized ring $A_{\mathfrak{p}}$. In a general scheme one has an analogue result:

Proposition 8.36. Let X be a scheme and $x \in X$ be a point. Set $Y = \overline{\{x\}}$. Then $\dim \mathcal{O}_{X,x} = \operatorname{codim}(Y, X)$.

Proof Given a chain $Y \subset Y_1 \subset \ldots Y_n$ of distinct irreducible closed subsets, the generic points $\eta_1, \ldots, \eta_n$ of the Y_i's are contained in each open neighbourhood U of x. In particular, if $U = \operatorname{Spec} A$ is an affine, these generic points correspond to prime ideals $\mathfrak{p}_n \subset \ldots \subset \mathfrak{p}_1 \subset \mathfrak{p}_x$ in A. Taking the supremum gives the claim. □

A chain of distinct irreducible closed subsets of Y may be spliced with one between Y and X, to yield a chain in X. Hence, taking suprema we find the inequality

$$\dim Y + \operatorname{codim}(Y, X) \leqslant \dim X.$$

As mentioned above, equality does not hold in general. And in fact, there are quite simple example with the inequality being strict. For integral schemes of finite type over fields however, the theory is much simpler, and in Chapter xxx we shall study the dimension in terms of the function field.

Example 8.37. Let A be a DVR with maximal ideal $\mathfrak{m} = (p)$ and fraction field K; for instance, the local ring $\mathbb{Z}_{(p)}$ with p a prime number. Consider the principal ideal $\mathfrak{n} = (tp - 1)$ in the polynomial ring $A[t]$. It is a maximal ideal as it equals the kernel of the map $A[t] \to K$ that sends $P(t)$ to $P(1/p)$, and one easily checks that it does not properly contain any non-zero prime ideal, so it is of height one. Letting $Y = V(\mathfrak{n})$ and $X = \operatorname{Spec} A[t]$, we find $\dim Y = 0$ and $\operatorname{codim} Y = 1$, but it holds that $\dim X = 2$.

8.4 Distinguished properties

In this section we will describe a small lemma which is very convenient when working with properties of schemes.

A property $\mathcal{P}$ of open affine subschemes of a scheme X is said to be *distinguished* if the following two conditions are satisfied:

(D1) If U has $\mathcal{P}$ and $g \in \mathcal{O}_X(U)$, then $D(g)$ has $\mathcal{P}$;
(D2) If $\{D(g_i)\}$ is a finite cover of U, and each $D(g_i)$ has $\mathcal{P}$, then U has $\mathcal{P}$.

Lemma 8.38. Let $\mathcal{P}$ be a distinguished property of open affine subschemes of X. If there is one open affine cover $\{U_i\}_{i \in I}$ of X so that each U_i has $\mathcal{P}$, then every open affines in X have $\mathcal{P}$. Moreover, it suffices that **(D2)** is satisfied for all covers by two distinguished opens.

Proof The set of distinguished open sets contained in some U_i form a basis $\mathcal{B}$ for the topology on X, and by property **(D1)**, they all have property $\mathcal{P}$. If V is an open affine in X, then being quasi-compact, it may be covered by finitely many opens of the basis $\mathcal{B}$, and so requirement **(D2)** ensures that V also has $\mathcal{P}$.

For the second statement in the lemma, assume **(D2)** is fulfilled for covers with two elements. We shall apply induction of the number r of opens in a given cover $\{D(g_i)\}$ of V.

Because the $D(g_i)$'s cover V, there is a relation $a_1g_1 + \cdots + a_rg_r = 1$ in $\mathcal{O}_V(V)$. Let $g = a_2g_2 + \cdots + a_rg_r$. Each $D(g_ig)$ with $i \geqslant 2$ is distinguished in $D(g_i)$, and hence has property $\mathcal{P}$ by **(D1)**. On the other hand, they are also distinguished in $D(g)$ and cover $D(g)$, hence $D(g)$ has $\mathcal{P}$ by induction. Now, V is the union of $D(g_1)$ and $D(g)$ and thus has $\mathcal{P}$ by the $r = 2$ case. □

8.5 Independence of the affine cover

Finite type

Proof of Proposition 8.18 The proof has two parts: a separate treatment of the affine case (i.e. a proof of the Corollary) followed by a reduction to that case (which relies on the notion of distinguished properties).

We begin with the affine case. Suppose that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of finite type, so that there is an affine cover $\{\operatorname{Spec} B_i\}$ of $\operatorname{Spec} B$ with each B_i finitely generated over A. We need to show that B is finitely generated over A. In the course of the proof we shall use the following elementary lemma:

Lemma 8.39. Assume there is a relation $\sum_{1 \leqslant i \leqslant r} a_ig_i = 1$ between elements from a ring R. Then for each natural number n, one may write $\sum_i c_ig_i^n = 1$ where the c_i's are polynomials with integer coefficients in the a_i's and the g_i's.

Proof Expand $(\sum_i a_ig_i)^{2nr}$ and observe that each term contains some power g_i^m with $m \geqslant n$. Then collect appropriate terms. □

Shrinking the $\operatorname{Spec} B_i$'s we may assume that each $\operatorname{Spec} B_i$ is a distinguished open subset $D(g_i)$ in $\operatorname{Spec} B$. As $\operatorname{Spec} B$ is quasi-compact, we may further assume that the $D(g_i)$'s are finite in number. Since the $D(g_i)$'s cover $\operatorname{Spec} B$, there is a relation $\sum_{1 \leqslant i \leqslant r} a_ig_i = 1$ with $a_i \in B$.

Let $t_{ij} \in B_{g_i}$ be generators for B_{g_i} as an A-algebra, and for each i write $g_i^{n_i}t_{ij} = b_{ij}$ with $b_{ij} \in B$ and $n_i \in \mathbb{N}$. We contend that the b_{ij}'s together with the a_i's and the g_i's generate B as an algebra over A. Indeed, pick an element $b \in B$. In each B_{g_i} there is an equality $b = P_i(t_{ij})$ with P_i a polynomial with coefficients in A, and multiplying up denominators, one finds relations $g_{ij}^n b = Q_i(b_{ij})$ in B, where the Q_i's also are polynomials with coefficients from A.

Now, by the lemma, there is a relation $1 = \sum_i c_ig_i^n$ with the c_i's being integral polynomials in a_i's and the g_i's. This yields

$$b = \sum_i bc_ig_i^n = \sum_i c_iQ_i(b_{ij}),$$

and since the c_i's are polynomials in a_i's and the g_i's, the right hand side is a polynomial in the b_{ij}'s, the a_i's and the g_i's with coefficients from A, and we are done.

Next we reduce to the affine case. Let $\mathcal{P}$ be the property of an open affine subscheme $\operatorname{Spec} A \subset S$ that for each open affine $\operatorname{Spec} B \subset f^{-1} \operatorname{Spec} A$, the algebra B is finitely generated over A. Since f is assumed to be finite, there is one affine cover of S, all of whose opens have $\mathcal{P}$. We proceed to check that $\mathcal{P}$ is a distinguished property; this will imply the Proposition.

The first requirement is straightforward, since if B is finitely generated over A, then B_g will be finitely generated over A_g.

For the second requirement, assume that a family $\{D(g_i)\}$ covers $\operatorname{Spec} A$ and that property $\mathcal{P}$ holds for each $\operatorname{Spec} A_{g_i}$. If $\operatorname{Spec} B$ is a given open affine subscheme of $f^{-1}\operatorname{Spec} A$, then the $\operatorname{Spec} B_{g_i}$ is open in $f^{-1}\operatorname{Spec} A_{g_i}$, and hence each B_{g_i} is finitely generated over A_{g_i}. But then it will be finitely generated over A as well, and we may apply Corollary 8.19 to conclude that B is finitely generated over A. □

Affine morphisms

Proof of Proposition 8.21 (i) We show that the property that $f^{-1}(U)$ is affine, is a distinguished property of open affine subsets U. Then the proposition follows from Lemma 8.38.

The first requirement, **(D1)**, comes for free since it holds true that $f^{-1}D(g) = D(f^\sharp(g))$ (see Proposition 2.27 on page 32).

For **(D2)**, let $V = f^{-1}U$, and assume that the distinguished open subsets $D(g_1)$ and $D(g_2)$ form a cover of U with each inverse image $V_i = f^{-1}D(g_i)$ being affine, say $f^{-1}D(g_i) = \operatorname{Spec} B_i$.

We begin with establishing that $B_i \simeq \mathcal{O}_V(V)_{g_i}$. To this end, consider the sheaf exact sequence

$$0 \longrightarrow \mathcal{O}_V(V) \xrightarrow{\alpha} B_1 \times B_2 \xrightarrow{\beta} B_{12}. \tag{8.5}$$

Here $B_{12} = \mathcal{O}_V(f^{-1}(U_1 \cap U_2))$, which equals both $(B_1)_{g_2}$ and $(B_2)_{g_1}$. As usual, the components of the map α are the restriction maps, and the map β sends (a, b) to the difference $a/1 - b/1$ in B_{12}.

Now we localize (8.5) in g_1. Note that both B_1, and B_{12} already are A_{g_1}-modules and so do not change when localized. Thus we obtain the sequence

$$0 \longrightarrow \mathcal{O}_V(V)_{g_1} \longrightarrow B_1 \times (B_2)_{g_1} \xrightarrow{\beta} (B_2)_{g_1}$$

where now $\beta(0, b) = b$. This sequence is actually split exact; the map $B_1 \to$

Either by the Snake Lemma or by a direct reasoning, one infers that the restriction map induces an isomorphism $\mathcal{O}_V(V)_{g_1} \simeq B_1$, and of course, by symmetry, $\mathcal{O}_V(V)_{g_2} \simeq B_2$.

Next, consider the canonical morphism $\theta\colon V \to \operatorname{Spec} \mathcal{O}_V(V)$ from Proposition **??** on page ??. It lives in commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\theta} & \operatorname{Spec} \mathcal{O}_V(V) \\ \uparrow & & \uparrow \\ f^{-1}D(g_i) & \xrightarrow{\simeq} & \operatorname{Spec} B_i, \end{array}$$

and since the $f^{-1}D(g_i)$'s cover V (by hypothesis) and the $\operatorname{Spec} B_i$ cover $\operatorname{Spec} \mathcal{O}_V(V)$ (by what we just did), the morphism θ is an isomorphism. □

Finite morphisms

Proof of Proposition 8.21 From Proposition 8.21(i) above we know that $f^{-1}V = \operatorname{Spec} B$ for some B, and it only remains to prove that B is a finite A-module. Now, the point is that having the spectrum of a finite algebra as inverse image, is a distinguished property of affine open subschemes of Y, and when this is established, we will be through.

Clearly $f^{-1}\operatorname{Spec} A_g = \operatorname{Spec} B_g$ so the first requirement is fulfilled. As to the second, assume that a finitely many $D(g_i)$'s cover V and that $f^{-1}D(g_i) = \operatorname{Spec} B_{g_i}$ with each B_i a finite modules over A_{g_i}. Let t_{ij} be generators of B_{g_i} over A_{g_i}, which we may chose to be images of elements b_{ij} in B. We contend that the b_{ij}'s generate B over A.

Given an element $b \in B$, it holds that $g_i^n b = \sum_j a_{ij} b_{ij}$ for some $n \in \mathbb{N}$ independent of i and with $a_{ij} \in A$. Since the $D(g_i)$'s cover V, there is relation

$$1 = c_1 g_1^n + \cdots + c_r g_r^n,$$

which yields

$$b = \sum_j c_j g_j^n b = \sum c_j a_{ij} b_{ij}.$$

□

9

Projective schemes

The projective varieties are fundamental in the theory of varieties, not just because they are interesting objects of study, but also because they in many aspects are better behaved than non-projective ones. In the scheme world, there is a construction extending the notion of projective varieties; from any positively graded ring R one constructs a scheme $\operatorname{Proj} R$ called the *projective spectrum*. The construction is somewhat parallel to that of the prime spectrum of a ring, but there are several key differences between the two. For instance, and perhaps most strikingly, $\operatorname{Proj} R$ does not depend functorially on R in the sense that maps between graded rings not always give maps between the projective spectra. Moreover, different R's may yield isomorphic projective spectra.

9.1 Graded rings

In this book a *graded ring* will refer to rings R which are graded by the non-negative integers, i.e. rings admitting a decomposition

$$R = \bigoplus_{n \geqslant 0} = R_0 \oplus R_1 \oplus \cdots$$

as an abelian group such that $R_m \cdot R_n \subset R_{m+n}$ for each $m, n \geqslant 0$. Occasionally, we will also discuss $\mathbb{Z}$-*graded rings*, where we allow negative degrees as well.

A ring map $\phi\colon R \to S$ between two graded rings R and S is said to be a *map of graded rings* if it respects the grading, that is, if $\phi(R_n) \subset S_n$ for all n.

Example 9.1. The simplest examples of graded rings are the polynomial rings $R = A[t_0, \ldots, t_r]$. They have a *standard grading* wherein each variable t_i has degree 1 and elements from A has degree 0. The graded piece R_n is a free module over $R_0 = A$ with the monomials of degree n as a basis.

Note that R_0 is a subring of R and that R is an algebra over R_0. Moreover, each R_n is an R_0-module. The elements in R_n are said to be *homogeneous of degree* n, and one writes $\deg x = n$ when $x \in R_n$. (Note that 0 has no well-defined degree, but is considered homogeneous of any degree.) Every non-zero element $x \in R$ can be expressed uniquely as a finite sum $x = \sum_n x_n$ with $x_n \in R_n$, and the non-zero terms in the sum are called the *homogeneous components* of x.

An R-module M is *graded* if it has a similar decomposition $M = \bigoplus_{n \in \mathbb{Z}} M_n$ as an abelian group and $R_m M_n \subset M_{m+n}$ for all n and m. Note that we allow also elements of negative degrees. A *map of graded R-modules* is an R-linear map $\phi\colon M \to N$ satisfying

$\phi(M_n) \subset N_n$ for all $n \in \mathbb{Z}$. With this notion of morphisms, the graded R-modules form a category, denoted GrMod_R.

As for graded rings, a non-zero element $x \in M$ is *homogeneous* of degree n if it lies in M_n. Any element $x \in M$ may be expressed in a unique way as a finite sum $x = \sum_n x_n$ with each x_n in M_n, and the non-zero terms are called the *homogeneous components* of x.

Most of the familiar definitions for modules carry over to the graded setting. For instance, the direct sum of a family of graded modules $\bigoplus_i M_i$ is graded in a natural way such that canoncal inclusions $M_j \hookrightarrow \bigoplus_i M_i$ preserve the grading. Likewise, the kernel and the cokernel of a map of graded modules are also graded in a natural way. One has decompositions $\operatorname{Ker}\phi = \bigoplus_{i\geqslant 0} \operatorname{Ker}\phi|_{M_n}$ and $\operatorname{Coker}\phi = \bigoplus_{i\geqslant 0} N_n/\phi(M_n)$.

A sequence of graded modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

is exact if it is exact as sequence of ordinary modules. As maps of graded modules preserve the grading, this is equivalent to saying that each of the sequences

$$0 \longrightarrow M_n' \longrightarrow M_n \longrightarrow M_n'' \longrightarrow 0.$$

are exact (as a sequence of R_0-modules).

An ideal $\mathfrak{a} \subset R$ is *homogeneous* if the homogeneous components of each element in $\mathfrak{a}$ belongs to $\mathfrak{a}$. In other words, we may write $\mathfrak{a} = \bigoplus_n \mathfrak{a}_n$ with $\mathfrak{a}_n = \mathfrak{a} \cap R_n$. An ideal $\mathfrak{a}$ is homogeneous if and only if it is generated by homogeneous elements (see Exercise 9.1.3). It is readily verified that radicals, intersections, sums and products of homogeneous ideals are homogeneous. If $\mathfrak{a}$ is an homogeneous ideal, the quotient $R/\mathfrak{a}$ inherits a grading from R and $R/\mathfrak{a} = \bigoplus_n R_n/\mathfrak{a}_n$.

We will write R_+ for the sum $\bigoplus_{n>0} R_n$; this is naturally a homogeneous ideal of R, which we call the *irrelevant ideal*.

Example 9.2. The irrelevant ideal of a polynomial ring $R = A[t_0, \ldots, t_r]$ is equal to $R_+ = (t_0, \ldots, t_r)$.

Example 9.3 (Veronese rings)**.** Common examples of graded rings are the so-called Veronese rings associated with a graded ring R. For any natural number d, the *Veronese ring* $R^{(d)}$ is the subring of R given by $\bigoplus_{n\geqslant 0} R_{nd}$.

Example 9.4. The ideal $\mathfrak{a} = (y - x, x^2)$ in the polynomial ring $k[x, y, z]$ is a homogenous ideal, and the quotient $R = k[x, y, z]/\mathfrak{a}$ is graded. The surjection $k[x, y, z] \to k[x, z]$ that sends y to x is a map of graded rings

$$k[x, y, z]/(y - x, x^2) \to k[x, z]/(x^2)$$

since it maps $\mathfrak{a}$ into the ideal (x^2). One verifies without difficulties that this is an isomorphism.

If $S \subset R$ is a multiplicative system consisting of homogeneous elements, and M is a graded module, the localization $S^{-1}M$ is naturally a graded R-module with degree n part equal to

$$(S^{-1}M)_n = \{\, m/s \in S^{-1}M \mid m \in M \text{ homogeneous}, s \in S \text{ and } \deg m - \deg s = n \,\}.$$

In particular, if f is a homogeneous element of positive degree, the localization R_f is a ($\mathbb{Z}$-graded) ring. As we will see, the degree 0 part $(R_f)_0$ will play a crucial role in the Proj-construction.

Example 9.5. In the polynomial ring $R = A[t_0, \ldots, t_n]$, with the standard grading, the elements of degree zero in the localization R_{t_j} are polynomials in the monomials $t_0 t_j^{-1}, \ldots, t_n t_j^{-1}$, so the piece of degree zero $(R_{t_j})_0$ is the polynomial ring

$$(R_{t_j})_0 = A\left[\frac{t_0}{t_j}, \ldots, \frac{t_n}{t_j}\right].$$

Exercises

Exercise 9.1.1. Let $\mathfrak{a} \subset k[x,y,z]$ be the ideal (xy, xz, yz). Show that $A = R/\mathfrak{a}$ is graded ring and describe each homogeneous component A_n.

Exercise 9.1.2. A polynomial ring $k[t_0, \ldots, t_n]$ can be given a non-standard grading by declaring the degree of each t_i to be any given natural number d_i. For instance, give $R = k[t_0, t_1]$ a grading by letting $\deg t_0 = 2$ and $\deg t_1 = 3$.

a) Describe the homogeneous pieces R_n of degree n;
b) Let $k[u]$ have standard grading and define a map $\phi\colon R \to k[u]$ by the assignments $t_0 \mapsto u^3$ and $t_1 \mapsto u^2$. Show that ϕ is a map of graded rings.
c) Describe the kernel and the cokernel of ϕ as graded modules.

Exercise 9.1.3. Show that an ideal $\mathfrak{a}$ in a graded ring R is homogeneous if and only if it is generated by homogeneous elements.

Exercise 9.1.4. Let R be a graded ring which is not necessarily positively graded. Assume that a homogeneous element f of R is expressed as a combination $f = \sum a_i g_i$ where the g_i's are homogeneous. Show that f may be expressed as $f = \sum_i b_i g_i$, where each b_i is homogeneous of degree $\deg f - \deg g_i$. HINT: Homogeneous components are unique.

Exercise 9.1.5. Let R be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal. Show that $(R_\mathfrak{p})_0$ is a local ring with maximal ideal equal to $\mathfrak{m} = \{\, fg^{-1} \mid f \in \mathfrak{p}, g \in S(\mathfrak{p}), \deg f = \deg g \,\}$.

Exercise 9.1.6. Let R be a graded ring and $\mathfrak{p}$ a homogeneous ideal in R. Show that $\mathfrak{p}$ is prime if and only if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ for all homogeneous elements x and y.

Exercise 9.1.7. Let R and S be graded rings and $\phi\colon R \to S$ a map of graded rings. Show that the inverse image $\phi^{-1}\mathfrak{p}$ of an ideal $\mathfrak{p} \subset S$ is homogeneous whenever $\mathfrak{p}$ is.

Exercise 9.1.8. Let R a graded ring. Show R is Noetherian if and only if R_0 is Noetherian and R_+ is finitely generated.

9.2 The Proj construction

Motivated by the discussion of projective varieties in Chapter 1, where homogeneous ideals play a fundamental role, we make the following definition:

Definition 9.6. Let R be a graded ring. We denote by $\operatorname{Proj} R$ the set of homogeneous prime ideals of R that do not contain the irrelevant ideal R_+. It is called the *projective spectrum* of R.

We can endow $\operatorname{Proj} R$ with a topology by letting the closed sets be sets of the form

$$V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{a} \subset \mathfrak{p} \}$$

with $\mathfrak{a}$ a homogeneous ideal. This topology is called the *Zariski topology*. The three topology axioms follow from the identities in the next lemma. The proof is exactly the same as Lemma 2.2 for $\operatorname{Spec} R$ (the arguments there are not disturbed by the conditions that primes are homogeneous and do not contain the irrelevant ideal); the key point is that sums, products and radicals persist being homogeneous when the involved ideals are.

Lemma 9.7. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\{\mathfrak{a}_i\}_{i\in I}$ be homogeneous ideals. Then:

(i) If $\mathfrak{a} \subset \mathfrak{b}$, then $V(\mathfrak{b}) \subset V(\mathfrak{a})$;
(ii) $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$;
(iii) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$;
(iv) $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$.

The reason behind the name 'the irrelevant ideal' is that $V(R_+) = \varnothing$, by definition. The following lemma shows that when constructing the closed sets $V(\mathfrak{a})$, it suffices to work with ideals contained in the irrelevant ideal. In fact, we can take $\mathfrak{a}$ lying in any prescribed power of the irrelevant ideal.

Lemma 9.8. Let $\mathfrak{a}$ and I be homogeneous ideals in the graded ring R.

(i) If $\sqrt{I} = R_+$, it holds that $V(\mathfrak{a}) = V(\mathfrak{a} \cap I)$;
(ii) $V(\mathfrak{a}) = \varnothing$ if and only if $R_+ \subset \sqrt{\mathfrak{a}}$.

Proof Proof of (i): since $V(R_+) = \varnothing$, condition (iv) of Lemma 9.7 above implies that $V(I) = \varnothing$, and condition (iii) of the same lemma then gives that $V(\mathfrak{a}\cap I) = V(\mathfrak{a})\cup V(I) = V(\mathfrak{a})$.

Proof of (ii): if $R_+ \subset \sqrt{\mathfrak{a}}$, it follows from (i) and (iv) of Lemma 9.7 that $V(\mathfrak{a}) = \varnothing$. Conversely, assume that $V(\mathfrak{a}) = \varnothing$; or in other words, that $R_+ \subset \mathfrak{p}$ for every homogeneous prime ideal $\mathfrak{p}$ with $\mathfrak{a} \subset \mathfrak{p}$. But then R_+ is contained in the intersection $\bigcap \mathfrak{p} = \sqrt{\mathfrak{a}}$, and hence $R_+ \subset \sqrt{\mathfrak{a}}$. □

Incidentally, we do not get more closed sets if we allow all ideals $\mathfrak{a}$ and not just the homogeneous ones. Any given ideal $\mathfrak{a}$ has a corresponding 'homogenization': the ideal generated by all homogeneous components of the elements in $\mathfrak{a}$. This 'homogenization' defines the exact same closed subset of $\operatorname{Proj} R$ as $\mathfrak{a}$ itself. In fact, a homogenous prime ideal contains $\mathfrak{a}$ if and only if all homogenous components of elements in $\mathfrak{a}$ lie in it. Consequently, the Zariski topology on $\operatorname{Proj} R$ can simply be understood as the induced topology from $\operatorname{Spec} R$, via the inclusion $\operatorname{Proj} R \subset \operatorname{Spec} R$.

Distinguished open subsets

As for affine schemes, there are distinguished open sets in $\operatorname{Proj} R$.

Definition 9.9 (Distinguished open sets)**.** For each $f \in R$ which is homogeneous of positive degree, we define the distinguished open set $D_+(f)$ as

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} R \,|\, \mathfrak{p} \not\ni f\}$$

In other words, $D_+(f)$ is the set of homogeneous prime ideals in R that do not contain the irrelevant ideal R_+, and do not contain f. It is clear that $D_+(f)$ is an open set, as the complement of $D_+(f)$ equals the closed set $V(f)$.

The next result says that the distinguished open sets form a basis for the topology on $\operatorname{Proj} R$. This fact will be essential when we define the scheme structure on $\operatorname{Proj} R$.

Lemma 9.10. Let R be a graded ring.

(i) If f and g are two homogeneous elements of positive degree, it holds that $D_+(f) \cap D_+(g) = D_+(fg)$.

(ii) The $D_+(f)$'s form a basis for the topology on $\operatorname{Proj} R$ when f runs through the homogeneous elements of R of positive degree.

Proof The first part is clear by the definition of a prime ideal. The second follows as in the affine case: $V(\mathfrak{a})$ is the intersection of the $V(f)$'s for the homogeneous $f \in \mathfrak{a} \cap R_+$, so $\operatorname{Proj} R - V(\mathfrak{a})$ is the union of the corresponding $D_+(f)$'s. Hence every open set is a union of sets of the form $D_+(f)$. □

Exercise 9.2.1. Let R be a graded ring and let f and $\{f_i\}_{i\in I}$ be homogenous elements from R all of positive degree. Show that the distinguished open sets $D_+(f_i)$ cover $D_+(f)$ if and only if a power of f lies in the ideal generated by the f_i's.

Dehomogenization and homogenization

Recall that for a distinguished open set $D(f)$ of an affine scheme $\operatorname{Spec} A$, there is a canonical homeomorphism between $D(f)$ which associates a prime $\mathfrak{p} \in D(f)$ with the prime ideal $\mathfrak{p}A_f$. In analogy with this, we will show below that the map $\mathfrak{p} \mapsto (\mathfrak{p}R_f)_0$ defines a homeomorphism between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$, where $(R_f)_0$ denotes the degree 0 part of the localization R_f. Before embarking on the proof, let us see an example that illustrates the underlying approach of the proof.

Example 9.11. As we saw in Chapter 1, the structure of $\mathbb{P}^n(k)$ as a variety is based on the isomorphisms $D_+(t_i) \simeq \mathbb{A}^n(k)$ given by

$$(t_0 : \cdots : t_n) \mapsto (t_0/t_i, \ldots, 1, \ldots, t_n/t_i), \tag{9.1}$$

(defined for $t_i \neq 0$). Therefore the most natural coordinates on $D_+(t_i)$ are the n quotients $u_1 = t_0/t_i, \ldots, u_n = t_n/t_i$ (where we exclude the term t_i/t_i).

Let us for simplicity consider the case $i = 0$. We would like to find a scheme analogue of the map (9.1) which works for prime ideals, not just closed points. This will involve passing from the ring $R = k[t_0, \ldots, t_n]$ to the degree 0 part $(R_{t_0})_0$ of the localization of R_{t_0}, that is,

$$(R_{t_0})_0 = k[t_1/t_0, \ldots, t_n/t_0] = k[u_1, \ldots, u_n]$$

If $G(t_0, \ldots, t_n) \in R$ is a homogeneous polynomial of degree d, we can consider its *dehomogenization* with respect to t_i, namely $g = t_0^{-d} G$. Note that g is a polynomial in the $u_1, \ldots, u_n$ and therefore defines a regular function on $D_+(t_0)$.

Conversely, given a polynomial $g \in k[u_1, \ldots, u_n]$, there is a straightforward way to make it homogeneous, namely to consider $G = t_0^d g$ where $d = \deg g$. This will almost always be an inverse to the dehomogenization process. There is an exception however: any power t_0^d will dehomogenize to 1, and there is no way of recovering t_0^d without knowing d.

In any case, the dehomogenization process allow us to understand the homogeneous prime ideals in R contained in $D_+(t_0)$. There is a map

$$D_+(t_0) \to \operatorname{Spec}(R_{t_0})_0 \tag{9.2}$$

which sends a homogeneous prime $\mathfrak{p}$ to $(\mathfrak{p} R_f)_0$. Note that the latter is a prime ideal in $(R_f)_0$. Concretely, if $\mathfrak{p} = (G_1, \ldots, G_r)$ where each G_j is a homogeneous polynomial of degree d_i, then in the localized ring R_{t_0}, we have

$$\mathfrak{p} R_{t_0} = (G_1, \ldots, G_r) = (t_0^{-d_1} G_1, \ldots, t_0^{-d_r} G_r) = (g_1, \ldots, g_r)$$

where the g_j's are the dehomogenizations of the G_j. Moreover, as each g_j has degree zero, the above equality in fact holds in $(R_{t_0})_0$. Hence the dehomogenizations form the generators for the ideal $(\mathfrak{p} R_{t_0})_0$ in $(R_{t_0})_0$. For instance, if $\mathfrak{p} = (t_1 - a_1 t_0, \ldots, t_n - a_n t_0)$, the dehomogenization produces the maximal ideal $(u_1 - a_1, \ldots, t_n - a_n)$.

We would like to find an inverse to the map (9.2). Given a prime ideal $\mathfrak{q} \in k[u_1, \ldots, u_n]$, say generated by $g_1, \ldots, g_r$, one can consider the ideal generated by the homogenizations of the g_j's. For instance, if $\mathfrak{q} = (u_1 - a_1, \ldots, u_n - a_n)$ corresponds to a closed point in $\operatorname{Spec} k[u_1, \ldots, u_n]$, homogenizing the generators gives $(t_1 - a_1 t_0, \ldots, t_n - a_n t_0)$, which indeed produces a closed point of $\mathbb{P}_k^n$ contained in $D_+(t_0)$.

However, this idea does not quite work in general, as the homogenized ideals may fail to be prime. For instance, the ideal of the affine twisted cubic curve $(u_2 - u_1^2, u_3 - u_1^3)$ homogenizes to

$$(t_2 t_0 - t_1^2, t_3 t_0^2 - t_1^3) = (t_0^2, t_0 t_1, t_1^2 - t_0 t_2) \cap (t_1^2 - t_0 t_2, t_1 t_2 - t_0 t_3, t_2^2 - t_1 t_3) \tag{9.3}$$

The issue is caused by the first ideal, which is contained in (t_0). The fix is to consider instead the ideal $\sqrt{\mathfrak{q} R_{t_0}} \cap R$, which results in the second ideal in (9.3). In fact, ideals of the form $\sqrt{\mathfrak{q} R_f} \cap R$ are always prime in R, as we will see below. Once this is established, the map $\mathfrak{q} \mapsto \sqrt{\mathfrak{q} R_{t_0}} \cap R$ gives an inverse to the map (9.2) and we get a one-to-one correspondence between all points of $\mathbb{A}_k^n$ and those in $D_+(t_0)$.

The general set up of the homeomorphism $D_+(f) \simeq \operatorname{Spec}(R_f)_0$ follows the pattern in the example. Basically one dehomogenizes elements of the ideals with respect to f (and homogenizes to get them back). It is only slightly more involved for general rings, e.g., because f needs not have degree 1.

Proposition 9.12. Let R be a graded ring and let $f \in R$ be homogeneous of degree $d \geqslant 1$. There is a canonical map $\phi\colon D_+(f) \to \operatorname{Spec}(R_f)_0$ defined by

$$\phi(\mathfrak{p}) = (\mathfrak{p}R_f)_0$$

This has the following properties:

(i) ϕ is a homeomorphism;

(ii) Open distinguished sets: for any homogeneous element $g \in R$ such that $D_+(g) \subset D_+(f)$, letting $u = g^d f^{-\deg g} \in (R_f)_0$, we have

$$\phi(D_+(g)) = D(u);$$

(iii) Closed sets: if $\mathfrak{a} \subset R$ is a homogeneous ideal, then we have that

$$\phi(V(\mathfrak{a}) \cap D_+(f)) = V((\mathfrak{a}R_f)_0).$$

Proof Note that ϕ is the restriction of the map $\operatorname{Spec} R_f \to \operatorname{Spec}(R_f)_0$ induced by the inclusion $(R_f)_0 \subset R_f$. Therefore it is continuous. Once we have proved (iii), we can conclude that it is also a closed map, hence a homeomorphism.

We begin by proving that ϕ has an inverse map $\psi\colon \operatorname{Spec}(R_f)_0 \to D_+(f)$ defined by

$$\mathfrak{q} \mapsto \sqrt{\mathfrak{q}R_f} \cap R.$$

First of all, we should check that the ideal on the right is a homogeneous prime ideal in R. The ideal $\mathfrak{q}R_f$ is in any case a $\mathbb{Z}$-graded ideal of R_f. First we claim that $(\mathfrak{q}R_f)_0 = \mathfrak{q}$. One inclusion $\mathfrak{q} \subset (\mathfrak{q}R_f)_0$ is clear, as $\mathfrak{q} \subset (R_f)_0$. Conversely, pick an element $g \in (\mathfrak{q}R_f)_0$ and express it as a sum

$$g = a_1 g_1 + \cdots + a_r g_r$$

where the g_i's are elements from $\mathfrak{q}$ and the a_i's are homogeneous elements from R_f whose degree is $\deg a_i = \deg g - \deg g_i$ (Exercise 9.1.4). But as both g and g_i have degree zero, we must have $\deg a_i = 0$ as well. Therefore $a_i \in (R_f)_0$, and hence $g \in \mathfrak{q}$.

Suppose that $ab \in \mathfrak{q}R_f$, with a and b homogeneous. Then

$$a^d b^d / f^{\deg(a)+\deg(b)} \in (\mathfrak{q}R_f)_0 = \mathfrak{q},$$

and since $\mathfrak{q}$ is prime, either $a^d/f^{\deg(a)} \in \mathfrak{q}$ or $b^d/f^{\deg(b)} \in \mathfrak{q}$. It follows that either $a^d \in \mathfrak{q}R_f$ or $b^d \in \mathfrak{q}R_f$. This shows that $\sqrt{\mathfrak{q}R_f}$ is a $\mathbb{Z}$-graded prime ideal of R_f. Therefore $\sqrt{\mathfrak{q}R_f} \cap R$ is a homogeneous prime ideal of R, and so ψ is well defined.

It remains to check that ψ is the inverse of ϕ. To prove that $\psi \circ \phi = \mathrm{id}_{D_+(f)}$, we first note that for each homogeneous prime $\mathfrak{q} \subset R_f$, it holds that $\mathfrak{q} = \sqrt{\mathfrak{q}_0 R_f}$. Indeed, the inclusion $\sqrt{\mathfrak{q}_0 R_f} \subset \mathfrak{q}$ is immediate. Conversely, let $a \in \mathfrak{q}$ be a homogeneous element. Then $a^d/f^{\deg a}$ has degree 0, and belongs to $\mathfrak{q}_0$. It follows that $a \in \sqrt{\mathfrak{q}_0 R_f}$. If $\mathfrak{p} \in D_+(f)$, this implies (with $\mathfrak{q} = \mathfrak{p}R_f$) that $\mathfrak{p}R_f = \sqrt{(\mathfrak{p}R_f)_0 R_f}$, and we see that $\mathfrak{p} = \sqrt{(\mathfrak{p}R_f)_0 R_f} \cap R = \psi(\phi(\mathfrak{p}))$.

The argument for why $\phi \circ \psi = \mathrm{id}_{\operatorname{Spec}(R_f)_0}$ is similar and is left to the reader.

Proof of (ii): let $g \in R$ be an element with $D_+(g) \subset D_+(f)$. Then for $\mathfrak{p} \in D_+(f)$, the following series of equivalences hold true because $\mathfrak{p}R_f$ is a prime ideal:

$$\begin{aligned}\mathfrak{p} \in D_+(g) &\Leftrightarrow g^d f^{-\deg g} \notin \mathfrak{p}R_f \\ &\Leftrightarrow g^d f^{-\deg g} \notin (\mathfrak{p}R_f)_0 = \phi(\mathfrak{p}).\end{aligned}$$

Hence $\phi(D_+(g)) = D(u)$.

Proof of (iii): Let $\mathfrak{p} \in V(\mathfrak{a}) \cap D_+(f)$, so that $\mathfrak{a} \subset \mathfrak{p}$ and $f \notin \mathfrak{p}$. Then $(\mathfrak{a}R_f)_0 \subset (\mathfrak{p}R_f)_0 = \phi(\mathfrak{p})$, which gives one of the inclusions. Conversely, given a prime ideal $\mathfrak{p} \subset (R_f)_0$ such that $(\mathfrak{a}R_f)_0 \subset \mathfrak{p}$, its preimage $\mathfrak{p}' = \mathfrak{p} \cap R$ will be a homogeneous prime ideal in R not containing f, and so $(\mathfrak{a}R_f)_0 \subset \phi(\mathfrak{p}') = (\mathfrak{p}'R_f)_0$. This completes the proof. □

Proj as a scheme

We now explain the scheme structure on $\operatorname{Proj} R$. For this, we need to define the structure sheaf $\mathcal{O}_{\operatorname{Proj} R}$ on $\operatorname{Proj} R$, and check that the resulting locally ringed space is locally affine. The construction of $\mathcal{O}_{\operatorname{Proj} R}$ parallels that of the structure sheaf on $\operatorname{Spec} A$, using distinguished open sets in its definition.

Let $\mathscr{B}$ be the basis for the topology on $\operatorname{Proj} R$ consisting of the distinguished open subsets. For each $D_+(f)$, we set[1]

$$\mathcal{O}(D_+(f)) = (R_f)_0. \tag{9.4}$$

When f and g are homogeneous and $D(g) \subset D(f)$, the localization map $R_f \to R_g$ will preserve the gradings.[2] Hence $(R_f)_0$ is mapped into $(R_g)_0$, and we may use the degree zero part of the localization maps as restriction maps $\mathcal{O}(D_+(f)) \to \mathcal{O}(D_+(g))$.

In this way, we obtain a $\mathscr{B}$-presheaf $\mathcal{O}$. We next show that this is a $\mathscr{B}$-sheaf. If $\{D_+(f_i)\}$ is a finite cover of $D_+(f)$, with the f_i's homogeneous, the distinguished open subsets $D(f_i)$ of $\operatorname{Spec} R$ will cover $D(f)$, and consequently the standard sequence

$$0 \longrightarrow R_f \xrightarrow{\ \alpha\ } \textstyle\prod_i R_{f_i} \xrightarrow{\ \beta\ } \prod_{i,j} R_{f_i f_j}, \tag{9.5}$$

which is an exact sequence of graded R-modules, will be exact simply because $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf. Taking degree zero parts is an exact operation, and applied to (9.5) it yields the exact sequence

$$0 \longrightarrow (R_f)_0 \xrightarrow{\ \alpha_0\ } \textstyle\prod_i (R_{f_i})_0 \xrightarrow{\ \beta_0\ } \prod_{i,j} (R_{f_i f_j})_0, \tag{9.6}$$

which exactly says that $\mathcal{O}$ is a $\mathscr{B}$-sheaf. The structure sheaf $\mathcal{O}_{\operatorname{Proj} R}$ on $\operatorname{Proj} R$ is then defined to be the unique sheaf extension of $\mathcal{O}$. This is a sheaf such that $\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) = (R_f)_0$ over any distinguished open set.

According to Proposition 9.12 on the preceding page, there is a canonical homeomorphism $D_+(f) \simeq \operatorname{Spec}(R_f)_0$, which sends a distinguished open subset $D_+(g) \subset D_+(f)$ to the subset $D(u) \subset \operatorname{Spec}(R_f)_0$ where $u = g^{\deg f} f^{-\deg g}$. Because u has degree zero, it holds that $(R_g)_0 \simeq \big((R_f)_0\big)_u$, which means that $\mathcal{O}$ restricts to the $\mathscr{B}$-sheaf induced by the

[1] There is a canonical localization which only depends on the open set $D_+(f)$, see Section XXX.

[2] As explained in Lemma 2.22, if $D(g) \subset D(f)$, it holds that $g^n = cf$ for some $c \in R$ and some $n > 0$, and the localization map is given by $af^{-r} \mapsto c^r a g^{-nr}$, and this preserves degrees.

structure sheaf on $\operatorname{Spec}(R_f)_0$. Hence $\mathcal{O}_{\operatorname{Proj} R}$ restricts to $\mathcal{O}_{\operatorname{Spec}(R_f)_0}$. The locally ringed space $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ is therefore locally affine; in other words, it is a scheme.

Definition 9.13. For a graded ring R, we call the scheme $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ the *projective spectrum* of R.

The projective spectrum $\operatorname{Proj} R$ is in a natural way a scheme over $\operatorname{Spec} R_0$. The structure map $\pi\colon \operatorname{Spec} R \to \operatorname{Spec} R_0$ restricts to a continuous map on $\operatorname{Proj} R$, which turns out to be a morphism. To check this, it suffices to show that its restriction to $D_+(f)$ is a morphism for each homogeneous f. Under the identification $\phi\colon D_+(f) \simeq \operatorname{Spec}(R_f)_0$ from Proposition 9.12, this restriction turns into the composition $\pi|_{D_+(f)} \circ \phi^{-1}$, which matches the structure map $\operatorname{Spec}(R_f)_0 \to \operatorname{Spec} R_0$. Precisely, we have that

$$\phi(\mathfrak{p}) \cap R_0 = (\mathfrak{p}R_f)_0 \cap R_0 = \mathfrak{p} \cap R_0.$$

Indeed, one inclusion is obvious, and if for some $x \in \mathfrak{p}$ it holds that $y = f^{-n}x \in R_0$, we find that y lies in $\mathfrak{p}$ since $x = f^n y$ lies there, but f does not.

Example 9.14 (Projective spaces again)**.** Among the most prominent varieties are the projective spaces, and in Section 7.6 we constructed analogues $\mathbb{P}^n_A$ over any ring. These were obtained by gluing together schemes shaped like prime spectra $\operatorname{Spec} A[t_0 t_i^{-1}, \ldots, t_n t_i^{-1}]$. In the present general setting the $\mathbb{P}^n_A$'s resurface as Proj's of standard graded polynomial rings $A[t_0, \ldots, t_n]$.

Proposition 9.15. For each ring A and each non-negative integer n, it holds true that $\mathbb{P}^n_A = \operatorname{Proj} A[t_0, \ldots, t_n]$.

Proof The only remark needed is that if $R = A[t_0, \ldots, t_n]$, it holds that $D_+(t_i) = (R_{t_i})_0 = \operatorname{Spec} R_i$ with $R_i = A[t_0/t_i, \ldots, t_n/t_i]$; indeed, these are precisely the open pieces joined together to form $\mathbb{P}^n_A$, and the gluing data are also the same because the intersections $D_+(t_i) \cap D_+(t_j)$ are equal to $D_+(t_i t_j) = \operatorname{Spec}(R_{t_i t_j})_0$, and

$$(R_{t_i t_j})_0 = R_i[t_i/t_j] = R_j[t_j/t_i].$$

□

Example 9.16. The scheme $\mathbb{P}^0_A = \operatorname{Proj} A[t_0]$ merits a comment. In this case the structure map is an isomorphism $\operatorname{Proj} A[t_0] \simeq \operatorname{Spec} A$ (so when A is a field, $\mathbb{P}^0_A$ is just a point).

Indeed, since the irrelevant ideal $A[t_0]_+$ is generated by t_0, it follows that $\operatorname{Proj} A[t_0] = D_+(t_0)$, and on the other hand, it holds that $D_+(t_0) = \operatorname{Spec}(A[t_0]_{t_0})_0$, and $(A[t_0]_{t_0})_0 = A[t_0, t_0^{-1}]_0 = A$.

Example 9.17. Consider $R = k[x, y]/(xy)$ with the natural grading. Geometrically, $\operatorname{Spec} R - V(x, y)$ represents the union of the x- and y-axes, excluding the origin. Therefore, we expect $\operatorname{Proj} R$ to consist of only two points. Besides the irrelevant ideal $R_+ = (x, y)$, there are only two homogeneous prime ideals, (x) and (y). Thus, $\operatorname{Proj} R$ indeed consists of just two points.

Here are some basic properties of $\operatorname{Proj} R$:

Proposition 9.18 (Properties of Proj). Let R be a graded ring.

(i) If R is an integral domain, then $\operatorname{Proj} R$ is integral;
(ii) If R is reduced, then $\operatorname{Proj} R$ is reduced;
(iii) If R is Noetherian, then $\operatorname{Proj} R$ is a Noetherian scheme.
(iv) If R is of finitely generated as an R_0-algebra, then $\operatorname{Proj} R$ is of finite type over $\operatorname{Spec} R_0$.

Proof The two first properties can both be checked on an open affine cover, and $\operatorname{Proj} R$ is covered by the open affines $\operatorname{Spec}(R_f)_0$ with $f \in R_+$. Provided R is an integral domain (or a reduced ring), the rings R_f are integral domains (or reduced rings), and $(R_f)_0$ being a subring of R_f, the same holds for $(R_f)_0$.

For the third and fourth properties, note that when R is Noetherian, R_+ is finitely generated, say by elements $f_1, \ldots, f_r$. Each of the the rings $(R_{f_i})_0$ are Noetherian, so $\operatorname{Proj} R$ is covered by finitely many affine schemes $\operatorname{Spec}(R_{f_i})_0$, and so it is Noetherian. Finally, if R is finitely generated over R_0, then so is each $(R_{f_i})_0$, as we will prove later in Lemma 9.39 on page 150. □

Example 9.19. When R is not Noetherian, it may very well happen that $\operatorname{Proj} R$ is not quasi-compact. This is in stark contrast with the case of affine schemes; a prime spectrum $\operatorname{Spec} A$ is always quasi-compact whatever the ring A is.

An explicit example is the polynomial ring $R = k[t_1, t_2, \ldots]$ in infinitely many variables. Then $\operatorname{Proj} R$ is covered by the distinguished opens $D_+(t_1), D_+(t_2), \ldots$, but this cover can not be reduced to a finite one. (See also Exercise 2.5.6 on page 31.)

This situation is somewhat counterintuitive, given the usual heuristic that complex projective varieties (i.e. closed subsets of the compact space $\mathbb{CP}^n$) are compact, whereas affine varieties (e.g. $\mathbb{A}^n$ or $\mathbb{A}^1 - 0$) are not. The explanation is that the usual notions of 'compactness' do not behave so well in the Zariski topology; there are other notions like 'properness' which better capture the properties we want.

Exercises

Exercise 9.2.2. Let R be a graded ring and let he $\pi\colon \operatorname{Proj} R \to \operatorname{Spec} R_0$ be the structure map. Show that for each $f \in R_0$, the inverse image $\pi^{-1}D(f)$ is isomorphic to $\operatorname{Proj} R_f$.

Exercise 9.2.3. Let R be a one-dimensional graded ring, with $R_0 = k$ a field, and assume that R is finitely generated as a k-algebra. Show that $\operatorname{Proj} R$ is a finite set. HINT: the maximal ideal R_+ contains all homogeneous prime ideals.

Exercise 9.2.4. If R is a graded integral domain, show that the function field of $X = \operatorname{Proj} R$ is given by

$$k(X) = \{ \tfrac{g}{h} \mid g \in R, h \in R, \deg g = \deg h \} \subset k(R) \qquad (9.7)$$

Exercise 9.2.5. Show that $\operatorname{Proj} R$ is empty if and only if every element in R_+ is nilpotent.

Exercise 9.2.6. Give examples of a non-Noetherian graded ring R such that $\operatorname{Proj} R$ is Noetherian, of an R that is not of finite type over a field k, but $\operatorname{Proj} R$ is, and an R which is not an integral domain, but whose projective spectrum $\operatorname{Proj} R$ is integral. HINT: The irrelevant ideal is irrelevant.

9.3 Functoriality

In contrast to the Spec-construction, the Proj-construction is not entirely functorial. A map of graded rings $\phi\colon R \to S$ does not always induce a morphism between the projective spectra $\operatorname{Proj} S$ and $\operatorname{Proj} R$, because prime ideals in S might pull back to prime ideals in R that contain the irrelevant ideal R_+. However, discarding the badly behaved primes, we find an open set where a morphism can be defined.

This is not the only functorial deficiency of the Proj construction. There are maps between Proj's that are not induced by maps of the graded rings (see for instance Proposition 9.26).

The base locus

Given a map of graded rings $\phi\colon R \to S$, we introduce the *base locus* of ϕ as the closed set

$$\operatorname{Bs}(\phi) = V(\phi(R_+)) \subset \operatorname{Proj} S.$$

Proposition 9.20. Let $\phi\colon R \to S$ be a map of graded rings. Then there is a morphism of schemes

$$F\colon \operatorname{Proj} S - \operatorname{Bs}(\phi) \longrightarrow \operatorname{Proj} R,$$

which on the level of topological spaces is given by $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$.

Proof As ϕ is a map of graded rings, $\phi^{-1}\mathfrak{p}$ is a homogeneous prime ideal whenever $\mathfrak{p}$ is one.

Note that the set $U = \operatorname{Proj} S - \operatorname{Bs}(\phi)$ is open in $\operatorname{Proj} S$ and has a canonical scheme structure. Moreover, if $\mathfrak{p} \in U$, it holds by definition that $R_+ \not\subset \phi^{-1}(\mathfrak{p})$, and $\phi^{-1}\mathfrak{p}$ therefore is a well-defined point of $\operatorname{Proj} R$. Therefore, on the level of topological spaces, the map F is well-defined, and continuous.

Next, we need a map of sheaves of rings $\mathcal{O}_{\operatorname{Proj} R} \to F_*\mathcal{O}_U$. We define this map using $\mathscr{B}$-sheaves on the open sets $D_+(f)$. That is, we need to specify ring maps $\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) \to F_*\mathcal{O}_U(D_+(f))$, one for each $f \in R$ homogeneous of positive degree, such that they are compatible with the restrictions to $D_+(g)$'s contained in $D_+(f)$.

Now, there is a sheaf map $\mathcal{O}_{\operatorname{Spec} R} \to F_*\mathcal{O}_{\operatorname{Spec} S}$ (abusing language, we let F also denote the map between the Spec's). In view of the equality $F^{-1}D(f) = D(\phi(f))$, this map when restricted to $D(f)$, is simply the localization map

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f \longrightarrow \mathcal{O}_{\operatorname{Spec} S}(D(\phi(f))) = S_{\phi(f)}. \tag{9.8}$$

This is a map of $\mathbb{Z}$-graded rings, and on the homogeneous pieces of degree zero it induces the desired map

$$\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) \longrightarrow \mathcal{O}_{\operatorname{Proj} S}(D_+(\phi(f))) = \mathcal{O}_U(F^{-1}D_+(f)). \tag{9.9}$$

Since $\mathcal{O}_{\operatorname{Spec} R} \to F_* \mathcal{O}_{\operatorname{Spec} S}$ is a map of sheaves, the maps in (9.8) coincide on intersections, and the same then holds for those in (9.9). This concludes the proof. □

Projections

Example 9.21 (Projection from a point)**.** Consider the polynomial rings $R = \mathbb{Z}[t_0, t_1]$ and $S = \mathbb{Z}[t_0, t_1, t_2]$, both equipped with standard grading, and the natural inclusion $\iota\colon R \hookrightarrow S$. Then $R_+ = (t_0, t_1) \subset R$ and $\iota(R_+)S = (t_0, t_1)S$, so the base locus is $V(t_0, t_1) \subset \operatorname{Proj} S$.

The counterpart of this example in the world of varieties (that is, on k-points with k a field) is the projection $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$ from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$. The base locus consists of the point $(0 : 0 : 1)$, where the projection is not defined.

Example 9.22 (Projection from a linear subspace)**.** Generalizing Example 9.21, one may project from any linear subspace of $\mathbb{P}^n = \operatorname{Proj} \mathbb{Z}[t_0, \dots, t_n]$; for instance, $V(t_0, \dots, t_r)$. The appropriate map of graded rings is then the inclusion $\mathbb{Z}[t_0, \dots, t_r] \hookrightarrow \mathbb{Z}[t_0, \dots, t_n]$, and the base locus equals the subscheme $V(t_0, \dots, t_r) \subset \mathbb{P}^n$. The projection is the corresponding map

$$\mathbb{P}^n - V(t_0, \dots, t_r) \to \mathbb{P}^t,$$

which on k-points acts by just keeping the $r+1$ first homogeneous coordinates and forgetting the others.

Example 9.23. Consider the map $\phi\colon \mathbb{Z}[u, v] \to \mathbb{Z}[x, y]$ of graded k-algebras defined by the two assignments $u \mapsto x^n$ and $v \mapsto y^n$ where n is a natural number (to make this a map of graded rings, we let u and v have degree n). The base locus $\operatorname{Bs}(\phi)$ equals $V(x^n, y^n) = V(x, y)$, which is the empty subscheme of $\mathbb{P}^1$. Hence the map ϕ gives rise to a morphism $\mathbb{P}^1 \to \mathbb{P}^1$. If k is an algebraically closed field, the map on k-points is given by $(a : b) \mapsto (a^n : b^n)$.

Exercise 9.3.1 (Cremona transformation)**.** Let A be a ring and consider the map of graded rings $\phi\colon \mathbb{Z}[u_0, u_1, u_2] \to \mathbb{Z}[x_0, x_1, x_2]$ defined by the three assignments $u_i \to x_j x_k$ where the indices satisfy $\{i, j, k\} = \{1, 2, 3\}$.

Determine the base locus $\operatorname{Bs}(\phi)$ and describe the k-points of $V(\operatorname{Bs}(\phi))$ when k is a field.

Closed embeddings

If $\mathfrak{a}$ is a homogeneous ideal in the graded ring R, the quotient map $\phi\colon R \to R/\mathfrak{a}$ is a map of graded rings, and it holds that $\phi(R_+) = (R/\mathfrak{a})_+$. The base locus $\operatorname{Bs}(\phi)$ is therefore empty, and the corresponding map of schemes is defined everywhere. Hence we obtain a morphism

$$\iota\colon \operatorname{Proj} R/\mathfrak{a} \to \operatorname{Proj} R$$

whose image is $V(\mathfrak{a})$. We contend that ι is a closed embedding. It will suffice to verify this on an open cover of $\operatorname{Proj} R$, so let $f \in R$ be a homogeneous element. It holds that

$\iota^{-1}D_+(f) = D_+(\phi(f))$, and the restriction of ι to $\iota^{-1}D_+(f)$ may be identified with the morphism

$$\operatorname{Spec}((R/\mathfrak{a})_{\phi(f)})_0 \to \operatorname{Spec}(R_f)_0$$

induced by the degree zero part of the localization $R_f \to (R/\mathfrak{a})_f$ of ϕ. But the latter is obviously surjective, and we infer that $\iota|_{\iota^{-1}D_+(f)}$ is a closed embedding. In fact, under mild assumptions on the graded ring R, every closed embedding into $\operatorname{Proj} R$ arises in this way, as we shall prove in Chapter **??**.

Example 9.24 (Homogeneous coordinates)**.** In Chapter 1, we saw that over an algebraically closed field k, the points $a \in \mathbb{P}^n(k)$ have *homogeneous coordinates* $a = (a_0 : \cdots : a_n)$, and that the homogeneous prime ideal corresponding to a is generated by the 2×2-minors of the matrix

$$\begin{pmatrix} t_0 & t_1 & \dots & t_n \\ a_0 & a_1 & \dots & a_n \end{pmatrix}. \tag{9.10}$$

There is an analogue of this for projective spaces over any ring A. For an $(n+1)$-tuple $a = (a_0, \dots, a_n)$ of elements of A so that the $a_0, \dots, a_n$ generate the unit ideal in A, we can construct the subscheme $\operatorname{Proj}(R/\mathfrak{a})$ of $\mathbb{P}^n_A$ defined by the same equations as above, i.e., the homogeneous ideal

$$\mathfrak{a} = (a_i t_j - a_j t_i \,|0 \leqslant i, j \leqslant n). \tag{9.11}$$

in the ring $R = A[t_0, \dots, t_n]$. We claim that the structure map $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ restricts to an isomorphism $\operatorname{Proj}(R/\mathfrak{a}) \to \operatorname{Spec} A$. Taking the inverse, we obtain an A-point σ_a : $\operatorname{Spec} A \to \mathbb{P}^n_A$. This is even an A-point over A, meaning that $\pi \circ \sigma_a = \mathrm{id}_{\operatorname{Spec} A}$, so in other words, σ_a is a 'section' of π.

As the a_i generate the unit ideal, the distinguished open sets $D(a_i)$ cover $\operatorname{Spec} A$. It will therefore suffice to see that the restriction $\pi|_{\pi^{-1}D(a_i)}$ is an isomorphism for every i. By Exercise 9.2.2, $\pi^{-1}D(a_i) = \operatorname{Proj}((R/\mathfrak{a})_{a_i})$, so replacing A by A_{a_i}, we may assume that one of the a_i's, say a_0, is invertible in A. Since $a_0 t_i - a_i t_0$ belongs to $\mathfrak{a}$, we deduce that $t_i - a_i a_0^{-1} t_0 \in \mathfrak{a}$, and hence $A[t_0, \dots, t_n]/\mathfrak{a} = A[t_0]$. By Example 9.16, it follows that the structure map restricts to an isomorphism on $V(\mathfrak{a})$.

If we multiply all the a_i by a unit $\lambda \in A$, this does not change the ideal $\mathfrak{a}$, and hence we get the same A-point $\sigma_a : A \to \mathbb{P}^n_A$. It is therefore natural to use the notation $(a_0 : \cdots : a_n)$ for the A-point σ_a.

It is not true in general that all A-points of $\mathbb{P}^n_A$ are of the 'homogeneous coordinate form' $(a_0 : \cdots : a_n)$. However, locally near each point of Spec A, they are, so in particular if A is a local ring (e.g. a field) it is true. Later we shall give a general description of morphisms into projective spaces.

Lemma 9.25. Assume that A is a local ring. Then every section $\operatorname{Spec} A \to \mathbb{P}^n_A$ of the structure map is given by $(a_0 : \cdots : a_n)$ where at least one a_i is a unit. Another such tuple $(a'_0 : \cdots : a'_n)$ gives the same map if and only if $a'_i = \lambda a_i$ for a unit $\lambda \in A$.

Proof Assume that a section $\sigma\colon \operatorname{Spec} A \to \mathbb{P}^n_A$ of the structure map is given. Then the image of the closed point of $\operatorname{Spec} A$ lies in $D_+(t_\nu)$ for some ν. This means that σ induces a map $\sigma^\sharp$ of rings from $A[t_0t_\nu^{-1},\ldots,t_nt_\nu^{-1}]$ to A; the images $a_i = \sigma^\sharp(t_it_\nu^{-1})$ are elements in A and $(a_0 : \cdots : 1 : \cdots : a_n)$, with the 'one' in the ν-th slot, will be the homogeneous coordinates giving the desired section. □

Exercise 9.3.2. Let A be ring and $\sigma\colon \operatorname{Spec} A \to \mathbb{P}^n_A$ a section of the structure map $\mathbb{P}^n_A \to \operatorname{Spec} A$. Let $x \in \operatorname{Spec} A$ be a point. Show that there is an open affine neighbourhood $U = \operatorname{Spec} A'$ of x and elements a'_i in A' such that $\sigma|_U = \sigma_{a'}$ with $a' = (a'_0,\ldots,a'_n)$.

The Veronese embedding

As we mentioned in the introduction, a significant difference between the Proj-construction and the Spec-construction, is that many different graded rings can lead to isomorphic Proj's. The Veronese embeddings provide infinitely many examples; for any natural number d, the schemes $\operatorname{Proj} R$ and $\operatorname{Proj} R^{(d)}$ are isomorphic, but the rings R and $R^{(d)}$ are typically not isomorphic. These also provide examples of morphisms between Proj 's that are not induced by graded ring maps.

Let R be a graded ring and let d be a positive integer. In Example 9.3 we introduced the Veronese ring $R^{(d)}$ associated with R as the ring $\bigoplus_n R_{dn}$. In this section we aim at showing that the inclusion $\phi\colon R^{(d)} \to R$ induces an isomorphism

$$v_d\colon \operatorname{Proj} R \longrightarrow \operatorname{Proj} R^{(d)}.$$

First, let us check that v_d is a morphism. The irrelevant ideal of $R^{(d)}$ is generated by all elements in R whose degree is positive and divisible by d. Note that $\phi(R^{(d)}_+)$ defines the empty set, since any prime $\mathfrak{p} \subset R$ such that $R_+ \cap R^{(d)} \subset \mathfrak{p}$ must contain all of R_+: if $a \in R_+$, it holds that $a^d \in R_+ \cap R^{(d)}$, and so $a \in \mathfrak{p}$ as well. The map v_d is called the *Veronese embedding*, or the *d-uple embedding* of $\operatorname{Proj} R$.

Proposition 9.26. The Veronese embedding v_d is an isomorphism.

Proof The key observation is that for a homogeneous element $f \in R_+$ the inclusion $R^{(d)} \subset R$ induces an equality of the degree 0 parts of the localizations

$$(R^{(d)}_{f^d})_0 = (R_f)_0. \tag{9.12}$$

The inclusion $(R^{(d)}_{f^d})_0 \subset (R_f)_0$ is clear. Conversely, let $g/f^s \in (R_f)_0$, and write it as $g/f^s = gf^t/f^{s+t}$ where $t \geqslant 0$ is such that $s+t$ is divisible by d. From $\deg g = s \deg f$, it holds that $g/f^s \in (R^{(d)}_f)_0)$, so we have the opposite inclusion as well.

Consequently the morphism v_d restricts to an isomorphism over open subschemes $D_+(f) \simeq D_+(f^d)$, illustrated with the commutative diagram

$$\begin{array}{ccc} \operatorname{Proj} R & \xrightarrow{v_d} & \operatorname{Proj} R^{(d)} \\ \uparrow & & \uparrow \\ D_+(f) & \xrightarrow{\simeq} & D_+(f^d). \end{array}$$

As f runs over the elements of R_+, the distinguished opens $D_+(f^d)$ cover $\operatorname{Proj} R^{(d)}$. Indeed, letting I be the ideal in $R^{(d)}$ generated by all d-powers f^d for $f \in R_+$, we have $R_+ \subset \sqrt{I}$, and so $\bigcap_{f\in R_+} V(f^d) = V\left(\sqrt{I}\right) = \varnothing$ by Lemma 9.8.

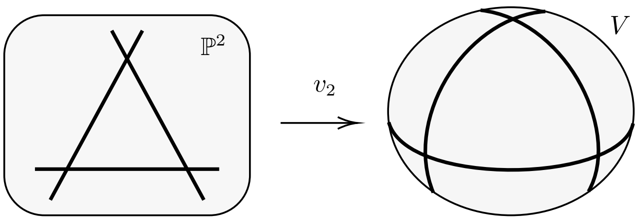
This means that v_d restricts to an isomorphism of schemes over an open covering of $\operatorname{Proj}(R^{(d)})$, and hence it is an isomorphism.

□

Example 9.27 (Veronese varieties)**.** Consider the maps $\mathbb{P}^n(k) \to \mathbb{P}^N(k)$ given by a basis for the space of homogeneous forms of degree d in the polynomial ring $R = k[x_0, \dots, x_n]$ (so $N + 1$ is the dimension of that space). For instance, the map $\mathbb{P}^2(k) \to \mathbb{P}^5(k)$ that acts on a point with homogeneous coordinates $(x_0 : x_1 : x_2)$ as

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2),$$

is one of the sort. Its image is the famous Veronese surface, which we met already in Example 1.43 in Chapter 1.



To describe a Veronese embedding in the Proj-terminology and in an absolute setting, let $\{M_i\}_{1\leqslant i\leqslant N}$ be the set of monomials of degree d in $\mathbb{Z}[x_0, \dots, x_n]$, and define a map of graded rings

$$\mathbb{Z}[t_0, \dots, t_N] \to \mathbb{Z}[x_0, \dots, x_n] = R$$

by sending a variable t_i to the monomial M_i. The image of this map is precisely the Veronese ring $R^{(d)}$, and therefore it induces, according to Section 9.3, a closed embedding of $\mathbb{P}^n = \operatorname{Proj} R^{(d)}$ into $\operatorname{Proj} R = \mathbb{P}^N$.

Example 9.28 (Rational normal curves)**.** The rational normal curves from Example 1.42 in Chapter 1 are other examples of Veronese embeddings. In that case $n = 1$, and the absolute version of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is given by the surjection

$$\begin{aligned} \mathbb{Z}[t_0, \dots, t_d] &\to \mathbb{Z}[x_0, x_1] \\ t_i &\mapsto x_0^{d-i}x_1^i. \end{aligned}$$

This example has appeared several times before. For $d = 2$, the map v_2 embeds $\mathbb{P}^1$ as the conic $V(t_1^2 - t_0t_2)$ in $\mathbb{P}^2_k$.

For $d = 3$, the image of v_3 is the projective twisted cubic curve $V(I) \subset \mathbb{P}^3$ from Example XXX.

Example 9.29. The rational normal curve of degree $d = 4$ is also interesting. The image C of the map $v_4 : \mathbb{P}^1_k \to \mathbb{P}^4_k$ is defined by the 2×2-minors of the matrix

$$\begin{pmatrix} t_0 & t_1 & t_2 & t_3 \\ t_1 & t_2 & t_3 & t_4 \end{pmatrix}$$

Let us consider the map $\mathbb{P}^1_k \to \mathbb{P}^3$ defined by the four monomials u^4, u^3v, uv^3, v^4. Writing t_0, t_1, t_3, t_4 for the coordinates on $\mathbb{P}^3_k$, the image X is defined by ideal

$$I = \left(t_1t_3 - t_0t_4, t_3^3 - t_1t_4^2, t_0t_3^2 - t_1^2t_4, t_1^3 - t_0^2t_3\right)$$

In terms of varieties, one can say that X arises as the projection of the rational normal curve from the point $(0:0:1:0:0)$.

Exercise 9.3.3. Show that the inverse of ν_d is not induced by a map of graded rings $R \to R^{(d)}$.

Cones

Let R be a graded ring which generated over R_0 by finitely many elements $t_0, \dots, t_n$. Then there is a morphism of schemes

$$\pi : \operatorname{Spec}(R) - V(R_+) \longrightarrow \operatorname{Proj} R$$

which generalizes the usual quotient construction of $\mathbb{P}^n$ from $\mathbb{A}^{n+1}_k - V(t_0, \dots, t_n)$. This morphism is constructed from the maps of affine schemes $\operatorname{Spec}(R_f) \to \operatorname{Spec}((R_f)_0)$, which one can check glue to the morphism π.

The affine scheme $\operatorname{Spec}(R)$ is called the *affine cone* of $X = \operatorname{Proj} R$, and it is commonly denoted by $C(X)$. The origin $V(t_0, \dots, t_n)$ defines a closed point in $C(X)$, called the *vertex* of the cone.

The schemes X and $C(X)$ share a relationship similar to that of $\mathbb{P}^n_k$ and $\mathbb{A}^{n+1}_k$.

On the level of k-points, the map π sends $(a_0, \dots, a_n)$ to the associated point in X with homogeneous coordinates $(a_0 : \cdots : a_n)$. (See Exericse 9.4.7 for a discussion on non-closed points.)

Example 9.30. Consider the ring $R = \mathbb{Z}[x_0, x_1, x_2]/(x_0^2 + x_1^2 - x_2^2)$. The affine cone $\operatorname{Spec} R$ represents a quadric surface in $\mathbb{A}^3_{\mathbb{Z}}$ defined by the equation $x_0^2 + x_1^2 = x_2^2$. It is therefore a cone in the usual since, at least on the level of $\mathbb{R}$-points.

To the graded ring R one can also form the *projective cone* defined by $\operatorname{Proj}(R[t])$ where we adjoin an extra variable t of degree 1. Note that

$$D(t) = \operatorname{Spec}\left(R[t, t^{-1}]\right)_0 \simeq \operatorname{Spec} R$$

This means that the affine cone $\operatorname{Spec} R$ embeds as an open subset of the projective cone. The complement of this open set is given by $V(t)$. Note that $\operatorname{Proj} R$ embeds as the closed subscheme

$$\operatorname{Proj}(R[t]/(t)) \subseteq \operatorname{Proj} R[t].$$

Example 9.31.

Weighted projective spaces

Our main source of examples will be the *weighted projective spaces*, which are defined in terms of polynomial rings with non-standard gradings:

Definition 9.32. For a ring A and a sequence $d_0, \ldots, d_n$ of natural numbers, we define the *weighted projective space*ver A as

$$\mathbb{P}_A(d_0, \ldots, d_n) = \operatorname{Proj} A[t_0, \ldots, t_n]$$

where $\deg t_i = d_i$ for each i.

While this definition resembles that of a traditional projective space (where all d_i are equal to 1), the weighted projective spaces give a surpringly diverse and rich class of examples.

One of the benefits of weighted projective spaces is that they provide concrete models for projective schemes. If R is generated by elements $t_0, \ldots, t_n$, of degrees $d_0, \ldots, d_n$ respectively, there is a graded surjection

$$R_0[t_0, \ldots, t_n] \to R$$

and hence $\operatorname{Proj} R$ embeds as a closed subscheme of the weighted projective space $\mathbb{P}(d_0, \ldots, d_n)$ over R_0. See Example 9.35 for a concrete example.

Example 9.33 (The weighted projective spaces $\mathbb{P}(p, q)$)**.** Let k be a field and p and q two relatively prime numbers. Consider the polynomial ring $R = k[x, y]$ with grading given by $\deg x = p$ and $\deg y = q$. We claim that $\operatorname{Proj} R \simeq \mathbb{P}^1_k$.

The idea is to consider the Veronese subring $R^{(d)}$ where $d = pq$. Observe that the homogeneous elements in $R^{(d)}$ are linear combination of monomials $x^\alpha y^\beta$ with $p\alpha + q\beta = dn$. Hence q divides α and p divides β and so $\alpha' + \beta' = n$ with $\alpha = q\alpha'$ and $\beta = p\beta'$.

If we consider the polynomial ring $k[u, v]$ where the grading where u and v have degree d, there is a graded ring map $\phi : k[u, v] \to R^{(d)}$ which sends $u \to x^q$ and $v \to y^p$. ϕ is clearly injective, because x^q and y^p are algebraically independent. It is also surjective: we just saw that $(R^{(d)})_{dn}$ has a basis consisting of the monomials $x^{q\alpha} y^{p\beta}$ with $\alpha + \beta = n$; these are the images of the monomials $u^\alpha v^\beta$ in A. (See also Exercise 9.4.2.) From this we conclude that ϕ induces an isomorphism

$$\operatorname{Proj} R \simeq \operatorname{Proj} R^{(d)} \simeq \operatorname{Proj} k[u, v] = \mathbb{P}^1_k.$$

Example 9.34 (The weighted projective space $\mathbb{P}(1, 1, d)$)**.** We begin with a polynomial ring $R = k[x, y, z]$ endowed with the grading $\deg x = \deg y = 1$ and $\deg z = d$ for some natural number d. We consider the weighted projective space $\mathbb{P}(1, 1, d) = \operatorname{Proj} k[x, y, z]$.

The scheme $\mathbb{P}(1, 2, 3)$ has a cover consisting of the three open affine subschemes $D_+(x)$, $D_+(y)$ and $D_+(z)$. Both $D_+(x)$ and $D_+(y)$ are isomorphic to the affine plane $\mathbb{A}^2_k$: it is straightforward to verify that $(R_x)_0 = k[yx^{-1}, zx^{-d}]$ and $(R_y)_0 = k[xy^{-1}, zy^{-d}]$, and that these are isomorphic to polynomial rings.

However, the third distinguished open affine $D_+(z)$ is different. The monomials $x^{d-i} y^i z^{-1}$, for $0 \leqslant i \leqslant d$, are clearly homogeneous elements of degree zero in R_z, and it is readily verified that they generate $(R_z)_0$, so that

$$(R_z)_0 = k[x^{d-i} y^i z^{-1} \mid 0 \leqslant i \leqslant d].$$

Let us take a closer look at the map induced by the (degree preserving) inclusion $S = k[x, y] \hookrightarrow k[x, y, z] = R$. Its corresponding base locus is the closed set $V(x, y)$ which is just a closed point p. Hence we have a map of schemes

$$f\colon \mathbb{P}(1,1,d) - \{p\} \longrightarrow \mathbb{P}^1_k.$$

Note that $\mathbb{P}(1,1,d) - \{p\}$ is covered by the open sets $D_+(x)$ and $D_+(y)$, both isomorphic to $\mathbb{A}^2_k$. One checks that $f|_{D_+(x)}$ maps $D_+(x) \subset \mathbb{P}(1,1,d)$ into $\mathbb{A}^1_k = D_+(x) \subset \mathbb{P}^1_k$, and similarly for $D_+(y)$. Moreover, when restricted to each of the $\mathbb{A}^2_k$'s, f is given by the projection onto the first coordinate axis.

Now, $D_+(x) = \operatorname{Spec} k[yx^{-1}, zx^{-d}]$ and $D_+(y) = \operatorname{Spec} k[xy^{-1}, zy^{-d}]$ are two copies of $\mathbb{A}^2_k$ glued together over the distinguished open sets $D(yx^{-1})$ and $D(xy^{-1})$ respectively. Over the overlaps, the gluing map is given by multiplication by x^d/y^d. So we recognize $f\colon X - \{p\} \to \mathbb{P}^1_k$ as being the line bundle L_d from Section 7.7.

Example 9.35. Recall the hyperelliptic curves from Example XXX. These were defined in terms of a homogeneous polynomial $f(x_0, x_1)$ of degree $2d$. The equation $y^2 = f(x_0, x_1)$ does not define a closed subscheme in $\mathbb{P}^2_k$, but it does so in the projective space $\mathbb{P}(1,1,d) = \operatorname{Proj} k[x_0, x_1, y]$ (where y has degree d). In fact, the curve in Example XXX is isomorphic to the closed subscheme X defined by $y^2 - f(x_0, x_1)$.

Note that $\mathbb{P}(1,1,d)$ has an affine covering consisting of three open sets, $D_+(x_0), D_+(x_1)$ and $D_+(y)$. For the subscheme X however, only two are needed, because X is contained in $D_+(x_0) \cap D_+(x_1) = \mathbb{P}(1,1,d) - V(x_0, x_1)$.

Example 9.36. Consider the weighted projective space $\mathbb{P}(1,2,3) = \operatorname{Proj}(R)$, where $R = k[u,v,w]$ with $\deg(u) = 1, \deg(v) = 2, deg(w) = 3$. Then

$$R^{(6)} = k[u^6, v^3, w^2, u^4v, u^3w, u^2v^2, uvw]$$

is generated by its degree six part as an an k-algebra. We therefore get a closed embedding $\mathbb{P}(1,2,3) \to \mathbb{P}^6_k$. See Exercise 9.4.6 for more on the affine covering of $\mathbb{P}(1,2,3)$.

Two blow-ups

Example 9.37 (The blow-up of the plane as a Proj)**.** Consider the polynomial ring $A = k[x,y]$ and the ideal $I = (x,y)$. Let R be the graded ring

$$R = \bigoplus_{i \geqslant 0} I^i t^i,$$

where as indicated, the graded piece of degree i equals $I^i t^i$. The irrelevant ideal R_+ is generated by xt and yt, and consequently $\operatorname{Proj} R$ is the union of the two open affine subschemes $\operatorname{Spec}(R_{xt})_0$ and $\operatorname{Spec}(R_{yt})_0$.

Note that there is a map of graded rings $\phi\colon A[u,v] \to R$, where both u and v are of degree one, given by the assignments $u \mapsto xt$ and $v \mapsto yt$. This is surjective since I is generated by x and y. Note also that the kernel contains the element $xv - yu$, and by Exercise 9.4.4

$$R \simeq A[u,v]/(xv - yu). \tag{9.13}$$

From this description we see that $\operatorname{Proj} R$ is covered by the two distinguished open sets $D_+(u) = \operatorname{Spec}(R_u)_0$ and $D_+(v) = \operatorname{Spec}(R_v)_0$, and it holds that $(R_u)_0 \simeq k[x, vu^{-1}]$

and $(R_v)_0 \simeq k[y, uv^{-1}]$. These are glued together along their intersection, which equals $\operatorname{Spec}(R_{uv})_0 \simeq (A[u,v]_{uv}/(xv - yu))_0$, and one finds

$$(A[u,v]_{uv}/(xv-yu))_0 = k[x,y,uv^{-1},vu^{-1}]/(xvu^{-1}-y) \simeq k[x,uv^{-1},vu^{-1}].$$

Thereforem $\operatorname{Proj} R$ coincides with the previous blow-up construction in Section 7.5 on page 97.

Example 9.38 (A general blow-up)**.** The previous examples is a specific example of a very general construction. Let A be a ring and let $\mathfrak{a}$ be an ideal in A. Consider the graded ring

$$R = \bigoplus_{i \geqslant 0} \mathfrak{a}^i t^i,$$

where t is a variable, i.e. R is the subring $A[t]$ of polynomials $\sum_{i\geqslant 0} a_i t^i$ with $a_i \in \mathfrak{a}^i$. In the ring R, t has degree 1 whereas the elements of A have degree 0. As $R_0 = A$, $\operatorname{Proj} R$ is a scheme over $\operatorname{Spec} A$ with structure morphism

$$\pi\colon \operatorname{Proj} R \longrightarrow \operatorname{Spec} A,$$

(this was introduced just after Definition 9.13 on page 139). We claim that π is an isomorphism outside the closed subset $\pi^{-1}V(\mathfrak{a})$, and so π merits to be called the 'blow up' of $V(\mathfrak{a})$. Indeed, if $f \in \mathfrak{a}$, it holds that $\mathfrak{a}A_f = A_f$ and consequently $\mathfrak{a}^i A_f = A_f$ for all i. Therefore, we have the equality $R_f = A_f[t]$. By Exercise 9.2.2 and Example 9.16, we then find that π induces an isomorphism

$$\pi^{-1}D(f) = \operatorname{Proj} R_f = \operatorname{Proj} A_f[t] \simeq \operatorname{Spec} A_f = D(f).$$

9.4 Rings generated in degree one

We will often work with the assumption that the graded ring R is *generated in degree one*; that is, R is generated as an R_0-algebra by the elements from R_1. This is the same thing as saying that there is surjective map of graded rings

$$R_0[t_0, \ldots, t_r] \longrightarrow R,$$

where $R_0[t_0, \ldots, t_r]$ is a polynomial ring with standard grading; in other words, that $\operatorname{Proj} R$ admits an embedding as a closed subscheme of $\mathbb{P}^r_{R_0}$.

When considering such rings, many arguments become simpler, as the grading more closely resembles the standard grading. For instance, one can say that $\operatorname{Proj} R$ will be covered by open affine subschemes of the form $D_+(t)$ where t is of degree one.

The assumption that R is generated in degree one is in fact not very restrictive. If R is a finitely generated as an algebra over R_0, then its Proj is isomorphic to the Proj of a ring generated in degree 1. This is because, for finitely generated R, some Veronese subring $R^{(d)}$ will have all generators in a single degree. Since $\operatorname{Proj} R^{(d)}$ is isomorphic to $\operatorname{Proj} R$, replacing R with $R^{(d)}$ doesn't alter the Proj (see Exercise 9.4.2 below).

Here is a basic lemma which will be useful in Chapter 16. It basically says that, when f has degree one, going to distinguished open $D_+(f)$ is the same thing as 'setting f equal to 1'.

Lemma 9.39. Let R be a graded ring and let $f \in R$ be homogeneous of degree one. Then there is a canonical isomorphism $(R_f)_0 \simeq R/(f-1)R$.

Proof There is a well defined map $R_f \to R/(f-1)R$ that sends xf^n to the class of x, and our map will be its restriction to $(R_f)_0$. It is surjective, because every element in $R/(f-1)R$ is a sum of homogeneous elements and, when x is homogeneous, the element $xf^{-\deg x}$ maps to the class of x. Assume then that $xf^{-\deg x}$ maps to zero, which means that $x = (f-1)y$ for some y. Letting $y = \sum_{s \leqslant i \leqslant t} y_i$ be the expansion of y in homogeneous components, with y_s and y_t non-zero, we find

$$x = -y_s + \sum_{i=s}^{t-1} (fy_i - y_{i+1}) + fy_t.$$

Since $y_s \neq 0$, it not only holds that $x = -y_s$, but also that $fy_i = y_{i+1}$ and $fy_t = 0$. A straightforward induction yields that $0 = fy_t = f^{t-s+1}y_s = -f^{t-s+1}x$, and so x is killed by a power of f and therefore vanishes in R_f. □

Exercises

Exercise 9.4.1. Show that $\mathbb{P}(1, \ldots, 1, d)$ is isomorphic to the cone over the Veronese variety $V_{n,d}$.

Exercise 9.4.2. Let $\{t_i\}$ be a finite set of generators for the graded ring R and let $d_i = \deg t_i$.

a) Let D be the least common multiple of the d_i and set $D_i = D/d_i$. Show that the Veronese ring $R^{(D)}$ is generated by elements of degree D.
b) Show that $\operatorname{Proj} R$ embeds as a closed subscheme of the weighted projective space $\mathbb{P}_{R_0}(d_0, \ldots, d_n)$ over R_0.

Exercise 9.4.3. Let x and y be two points in $\mathbb{P}^n_k$. Prove there is an open affine $U \subset \mathbb{P}^n_k$ containing both x and y.

Exercise 9.4.4. Show that equation (9.13) holds.

Exercise 9.4.5 (The weighted projective space $\mathbb{P}(1, 1, p)$)**.** Let R be as in the Example 9.34 above, and let $A = k[x, y, w]$ with the usual grading. Furthermore, let $\alpha\colon R \to A$ be the map of graded rings that sends z to w^p, while leaving x and y unchanged.

a) Show that α is a map of graded rings and induces a morphism $\pi\colon \mathbb{P}^2_k \to \operatorname{Proj} R$.
b) Describe the fibres of π over closed points in case k is algebraically closed.

Exercise 9.4.6. Let $R = k[x, y, z]$ be the polynomial ring with grading given by $\deg x = 1$, $\deg y = 2$ and $\deg z = 3$, and consider $\operatorname{Proj} R$ (which also is denoted $\mathbb{P}(1, 2, 3)$). The aim of the exercise is to describe the three covering distinguished subschemes $D_+(x)$, $D_+(y)$ and $D_+(z)$.

a) Show that $(R_x)_0 = k[yx^{-2}, zx^{-3}]$ and that $D_+(x) \simeq \mathbb{A}^2_k$.
b) Show that $(R_y)_0 \simeq k[x^2y^{-1}, z^2y^{-6}, xzy^{-2}]$. Show that the map of graded rings $k[u, v, w] \to (R_y)_0$ given by the assignments $x \mapsto yx^{-2}$, $v \mapsto z^2y^{-6}$ and $w \mapsto xzy^{-2}$ induces an isomorphism $k[u, v, w]/(w^2 - uv) \simeq (R_y)_0$.

Hence $D_+(y)$ is a hypersurface in $\mathbb{A}^3_k$; the so-called 'cone over a quadric'. Show it is not isomorphic to $\mathbb{A}^2_k$ (check the local ring at the origin).

c) Show that $R_z = k[x^3z^{-1}, y^3z^{-2}, xyz^{-1}]$ and that the map $k[u, v, w] \to (R_z)_0$ defined by the assignments $x \mapsto x^3z^{-1}$, $v \mapsto y^3z^{-2}$ and $w \mapsto xyz^{-1}$ induces an isomorphism $k[u, v, w]/(w^3 - uv) \simeq (R_z)_0$. Show that it is not isomorphic to $\mathbb{A}^2_k$.

d) Show that the map $R \to k[U, V, W]$ sending $x \mapsto U$, $y \to V^2$ and $z \mapsto W^3$ induces a map $\mathbb{P}^2_k \to \operatorname{Proj} R$, and describe the fibres over closed points.

Exercise 9.4.7. Let $R = k[x_0, x_1]$ where k is a field, and consider the morphism

$$\pi : \operatorname{Spec} R - V(x_0, x_1) \to \mathbb{P}^1_k = \operatorname{Proj} R$$

from page **??**

a) Show that π maps a k-point (a, b) to $(a : b)$

b) Show that π maps each height 1 prime $\mathfrak{p} = (f(x_0, x_1))$ to the generic point in $\mathbb{P}^1_k$.

10

Fibre products

10.1 Introduction

The fact that fibre products exist is one of the most important properties of the category of schemes, and one can argue that it is the definitive reason for transitioning from varieties to schemes; the fibre product of two varieties is in general not a variety, but it is a scheme.

The general fibre product is moreover extremely useful in many situations and takes on astonishingly versatile roles. At the end of the chapter we shall explain some of the various contexts where fibre products appear, including *base change* and *scheme theoretic fibres*.

We begin the chapter by recalling the definition of the fibre product of sets, then transition into a very general situation to discuss fibre products in general categories, and then finally, return to the context of schemes. We will construct the fibre product first when X, Y and S are affine schemes, and subsequently, by using several gluing constructions, show that it exists in general. The majority of the chapter will be devoted to going through the steps of this gluing procedure. Towards the end, we will treat the main applications and see a series of examples.

Fibre products of sets.

As a warm-up, we recall the fibre product in the category of sets. Given two maps of sets $f_X \colon X \to S$ and $f_Y \colon Y \to S$, their fibre product $X \times_S Y$ is the subset of the Cartesian product $X \times Y$ consisting of the pairs whose components have the same image in S; that is,

$$X \times_S Y = \{\, (x, y) \mid f_X(x) = f_Y(y) \,\}.$$

Phrased differently, the fibre product is the union of the products $f_X^{-1}(s) \times f_Y^{-1}(s)$ as s runs through S, and this is the reason for the name 'fibre product'; the fibres of the map $X \times_S Y \to S$ are the products of the fibres of the two maps f_X and f_Y.

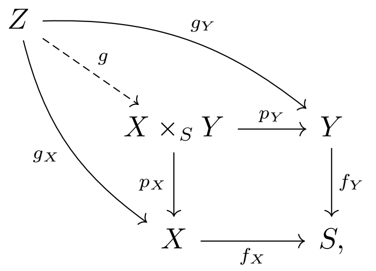
The fibre product fits into the commutative diagram below, where p_X and p_Y denote the two projections $p_X(x, y) = x$ and $p_Y(x, y) = y$:

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{f_X} & S. \end{array} \tag{10.1}$$

One also says that this diagram is a *Cartesian diagram* or a *Cartesian square*.

The fibre product enjoys the following universal property. Given two maps $g_X \colon Z \to X$

and $g_Y: Z \to Y$ such that $f_X \circ g_X = f_Y \circ g_Y$, there is a unique map $g: Z \to X \times_S Y$ satisfying $p_X \circ g = g_X$ and $p_Y \circ g = g_Y$. Indeed, just let g have components g_X and g_Y; that is, put $g(z) = (g_X(z), g_Y(z))$. The situation is described with the commutative diagram



(10.2)

where as usual the dashed arrow indicates a map required to exist.

Exercise 10.1.1. With the notation as above, show that:

a) If Y is a subset of S and f_Y is the inclusion, then $X \times_S Y$ equals the preimage $f_X^{-1}(Y)$;
b) If also X is a subset of S, more strikingly, the fibre product $X \times_S Y$ will be equal to the intersection $X \cap Y$;
c) When S has one element, $X \times_S Y$ is just the usual Cartesian product $X \times Y$.

The fibre product in general categories

The notion of a fibre product, formulated as the solution to a universal problem as above, is meaningful in any category C. Although our main concern will be the category of schemes, we give the definition in a general setting:

Definition 10.1 (Fibre product)**.** A *fibre product* of two arrows $f_X: X \to S$ and $f_Y: Y \to S$ from a category C, is a triple consisting of an object $X \times_S Y$ and two arrows $p_X: X \times_S Y \to X$ and $p_Y: X \times_S Y \to Y$ which have the following universal property:
For any two arrows $g_X: Z \to X$ and $g_Y: Z \to Y$ in C such that $f_X \circ g_X = f_Y \circ g_Y$, there is a *unique* arrow $g: Z \to X \times_S Y$ satisfying $p_X \circ g = g_X$ and $p_Y \circ g = g_Y$.

The universal property may naturally be expressed through commutative diagrams, like we did in (10.2) for sets, and the notions of *Cartesian diagrams* and *Cartesian squares* are carried over to any category.

When the fibre product exists, it is unique up to a unique isomorphism, as is true for solutions to any universal problem. However, it is a good exercise to check this in detail in this specific situation. The precise meaning is that if we have two products, say W and W', then there is exactly one isomorphism $\theta: W \to W'$ respecting the projections; that is, one that makes the diagram below commutative:

$$\begin{array}{ccccc} & & W & & \\ & \swarrow^{p_X} & \big\downarrow \theta & \searrow^{p_Y} & \\ X & & & & Y. \\ & \nwarrow_{p'_X} & & \nearrow_{p'_Y} & \\ & & W' & & \end{array}$$

For this reason, we allow ourself to speak about *the* fibre product.

It is not so hard to come up with examples of categories where fibre products do *not* exist. For instance, the fibre product does not exist in the familiar 'geometric' category of differentiable manifolds, neither does it in the category of affine varieties. This is yet another reason why we need to make the transition from varieties to schemes.

In the addition to the set up above, assume we are given two arrows $f\colon Z \to X$ and $g\colon W \to Y$ in the category C. Composing f with f_X, respectively g with f_Y, we obtain arrows $Z \to S$ and $W \to S$, and so the fibre product $Z \times_S W$ is meaningful. The compositions $f \circ p_Z$ and $g \circ p_W$ are arrows from $Z \times_S W$ to respectively X and Y, and the universal property of the fibre product implies that there is an arrow $f \times g\colon Z \times_S W \to X \times Y$ such that $p_X \circ (f \times g) = f$ and $p_Y \circ (f \times g) = g$.

10.2 Fibre products of schemes

A fundamental property of the category of schemes is that fibre products exist. Most of this chapter is devoted to proving this.

Theorem 10.2 (Existence of fibre products). Let $X \to S$ and $Y \to S$ be schemes over a scheme S. Then the fibre product $X \times_S Y$ exists.

The projections from the fibre product to X and Y will frequently be denoted by p_X and p_Y respectively.

When the base scheme S is affine, say $S = \operatorname{Spec} A$, the fibre product $X \times_S Y$ will sometimes be denoted by $X \times_A Y$.

The proof of the theorem consists of a series of reductions to the affine case, and the affine case is settled by means of the tensor product. The reductions rely heavily on the gluing techniques developed in Chapter 6.

One cannot construct the fibre product $X \times_S Y$ by defining a structure sheaf on the fibre product of the sets. In fact, as several later examples will show, the underlying set of a product of schemes can be very different from the product of the underlying sets of X and Y. This may sound counterintuitive at first, but is in fact a typical feature of the fibre products of schemes (see the examples in Section 10.3).

Products of affine schemes

We start by the constructing fibre products of affine schemes. The main observation is that the category of affine schemes is equivalent to the category of rings with arrows reversed, and that the tensor product of algebras enjoys a universal property *dual* to the one of the fibre product.

To be precise, assume that B_1 and B_2 are A-algebras, i.e. we have maps of rings $\alpha_i \colon A \to B_i$ for $i = 1, 2$. There are maps $\beta_i \colon B_i \to B_1 \otimes_A B_2$ that respectively send $b_1 \in B_1$ to $b_1 \otimes 1$ and $b_2 \in B_2$ to $1 \otimes b_2$. These are both ring maps as $bb' \otimes 1 = (b \otimes 1)(b' \otimes 1)$ and $1 \otimes bb' = (1 \otimes b)(1 \otimes b')$. Moreover, they fit into the commutative diagram

$$\begin{array}{ccc} B_1 \otimes_A B_2 & \xleftarrow{\beta_2} & B_2 \\ {\scriptstyle \beta_1}\uparrow & & \uparrow{\scriptstyle \alpha_2} \\ B_1 & \xleftarrow[\alpha_1]{} & A \end{array} \tag{10.3}$$

because $\alpha_1(a) \otimes 1 = 1 \otimes \alpha_2(a)$ by definition of the tensor product $B_1 \otimes_A B_2$ (this is the significance of the tensor product being taken over A; one can move elements coming from A from one side of the $\otimes$-glyph to the other).

Moreover, the tensor product is *universal* among diagrams such as (10.3). More presisely, assume that $\gamma_i \colon B_i \to C$ are maps of A-algebras, i.e. $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$; or phrased differently, they fit into a commutative diagram analogous to (10.3), but with the β_i's replaced by the γ_i's. The association $b_1 \otimes b_2 \to \gamma_1(b_1)\gamma(b_2)$ is A-bilinear and hence extends to an A-algebra homomorphism $\gamma \colon B_1 \otimes_A B_2 \to C$, which obviously has the property $\gamma \circ \beta_i = \gamma_i$, as expressed in the following commutative diagram:

$$\begin{array}{cccc} C & & \xleftarrow{\gamma_2} & \\ & \overset{\gamma}{\nwarrow} & & \\ & B_1 \otimes_A B_2 & \xleftarrow{\beta_2} & B_2 \\ {\scriptstyle \gamma_1}\nwarrow & {\scriptstyle \beta_1}\uparrow & & \uparrow{\scriptstyle \alpha_2} \\ & B_1 & \xleftarrow[\alpha_1]{} & A. \end{array} \tag{10.4}$$

Applying the Spec-functor to (10.3), we arrive at the diagram

$$\begin{array}{ccc} \operatorname{Spec}(B_1 \otimes_A B_2) & \xrightarrow{p_2} & \operatorname{Spec} B_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ \operatorname{Spec} B_1 & \longrightarrow & \operatorname{Spec} A, \end{array} \tag{10.5}$$

and $\operatorname{Spec}(B_1 \otimes_A B_2)$ enjoys the property of being universal among affine schemes sitting in a diagram like (10.5). Hence $\operatorname{Spec}(B_1 \otimes_A B_2)$ equipped with the two projections p_1 and p_2, serves as the fibre product in the category AffSch of affine schemes. One even has the stronger statement: it is the fibre product in the larger category Sch of schemes.

Proposition 10.3. Given morphisms $f_i \colon \operatorname{Spec} B_i \to \operatorname{Spec} A$ for $i = 1, 2$. Then the spectrum $\operatorname{Spec}(B_1 \otimes_A B_2)$ with the two projection p_1 and p_2 defined as above, is a fibre product of the $\operatorname{Spec} B_i$'s in the category Sch.

Unravelled, the conclusion reads: if Z is any scheme and $g_i \colon Z \to \operatorname{Spec} B_i$ are morphisms with $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique morphism $g\colon Z \to \operatorname{Spec}(B_1 \otimes_A B_2)$ such that $p_i \circ g = g_i$ for $i = 1, 2$.

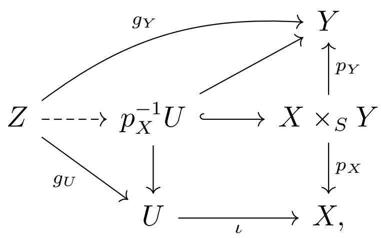
Proof To check the universal property, we rely on Theorem 6.5 about maps into affine schemes. The morphisms g_i give maps of A-algebras $B_i \to \mathcal{O}_Z(Z)$. By the universal property of the tensor product, these induce a map of A-algebras $B_1 \otimes_A B_2 \to \mathcal{O}_Z(Z)$, which in turn gives the desired map $g\colon Z \to \operatorname{Spec}(B_1 \otimes_A B_2)$ of schemes over $\operatorname{Spec} A$ by Theorem 6.5. By construction, this map satisfies $p_i \circ g = g_i$ for $i = 1, 2$, and it is unique by the uniqueness part of Theorem 6.5 and the universal property of the tensor product. □

Products of general schemes

Recall that any open subset U of a scheme X has a canonically defined scheme structure as an open subscheme with the structure sheaf being the restriction $\mathcal{O}_X|_U$. Hence, if $f\colon X \to Y$ is any morphism and $V \subset Y$ is an open subscheme, the inverse image $f^{-1}V$ is an open subscheme of X, and any morphism $g\colon Z \to X$ such that $f \circ g$ factors through V, will factor through $f^{-1}V$.

Lemma 10.4. If $X \times_S Y$ exists and $U \subset X$ is an open subscheme, then $U \times_S Y$ exists and is canonically isomorphic to the open subscheme $p_X^{-1}U$ with the two restrictions $p_X|_{p_X^{-1}U}$ and $p_Y|_{p_X^{-1}U}$ as projections.

Proof Write $\iota\colon U \to X$ for the open embedding. The situation is displayed in the following diagram



and we need to verify that $p_X^{-1}U$ together with the restriction of the two projections satisfies the universal property. If Z is a scheme and $g_U : Z \to U$ and $g_Y : Z \to Y$ are two morphisms over S, the composition $g_X = \iota \circ g_U$ is a map into X, and g_X and g_Y induce a map of schemes $g\colon Z \to X \times_S Y$ with $g_X = p_X \circ g$ and $g_Y = p_Y \circ g$. Clearly $p_X \circ g = \iota \circ g_U$ takes values in U. Therefore g takes values in $p_X^{-1}U$, and we get an induced morphism $g\colon Z \to p_X^{-1}U$, which is unique (Exercise 10.2.1 below). □

Exercise 10.2.1. Assume that $U \subset X$ is an open subscheme and let $\iota\colon U \to X$ be the inclusion map. Let f and g be two maps from a scheme Z to U and assume that $\iota \circ f = \iota \circ g$. Show that $f = g$.

The following proposition is the key point in the construction of the fibre product by gluing.

Proposition 10.5. Let $f_X : X \to S$ and $f_Y : Y \to S$ be two morphisms and assume that there is an open cover $\{U_i\}_{i \in I}$ of X such that $U_i \times_S Y$ exists for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open cover of $X \times_S Y$. Moreover, the projections restrict to projections.

Proof The proof involves gluing together the different schemes $U_i \times_S Y$ and verifying that the result indeed is a product $X \times_S Y$. We begin with introducing some notation: let $U_{ij} = U_i \cap U_j$ be the intersections of the U_i's and U_j's, and let $p_i : U_i \times_S Y \to U_i$ denote the projections.

By Lemma 10.4 the inverse images $p_i^{-1}(U_{ij})$ serve as fibre products $U_{ij} \times_S Y$ with the restrictions of p_i and p_Y as projections. Hence there are unique isomorphisms $\theta_{ji} : p_i^{-1}(U_{ij}) \to p_j^{-1}(U_{ij})$ making the diagrams

$$\begin{array}{ccc} p_i^{-1}(U_{ij}) & \xrightarrow[\simeq]{\theta_{ji}} & p_j^{-1}(U_{ij}) \\ & {\scriptstyle p_i}\searrow \quad \swarrow{\scriptstyle p_j} & \\ & U_{ij} & \end{array} \tag{10.6}$$

commute. To be able to glue together the $p_i^{-1}(U_i)$'s using the θ_{ij}'s, we need to verify the conditions of Theorem 6.3 on page 86. The two first follow readily. For the third, note that by Lemma 10.4 the preimages $p_i^{-1}(U_{ijk})$ serve as products $U_{ijk} \times_S Y$ with the restrictions of p_i and p_Y as projections. Moreover, the restrictions of the θ_{ij}'s live in diagrams

$$\begin{array}{ccccc} p_i^{-1}(U_{ijk}) & \xrightarrow[\simeq]{\theta_{ji}} & p_j^{-1}(U_{ijk}) & \xrightarrow[\simeq]{\theta_{kj}} & p_k^{-1}(U_{ijk}) \\ & {\scriptstyle p_i}\searrow & \downarrow{\scriptstyle p_j} & \swarrow{\scriptstyle p_k} & \\ & & U_{ijk}. & & \end{array}$$

The two minor triangles commute, so the big one commutes as well, and it follows by uniqueness that $\theta_{ki} = \theta_{kj} \circ \theta_{ji}$. The third gluing condition is thus fulfilled, and we can glue the $p_i^{-1}(U_i)$'s together to a scheme $X \times_S Y$. Moreover, in view of the commutative diagram (10.6) and Proposition 6.4 on page 87, the p_i's patch together to a map $p_X : X \times_S Y \to X$. The projections $U_i \times_S Y \to Y$ are essentially unaffected by the gluing process and glue together to a morphism $p_Y : X \times_S Y \to Y$. It is straightforward to check that $X \times_S Y$ with these two projections has the required universal property. □

An immediate consequence of Proposition 10.5 is that fibre products exist when the base S is affine.

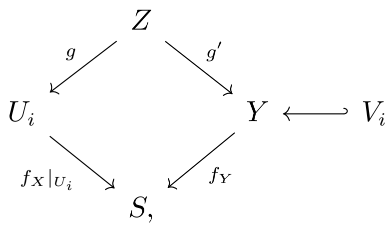
Lemma 10.6. Assume that S is affine, then $X \times_S Y$ exists.

Proof First, if Y is affine as well, we are done: cover X by open affine subschemes U_i; then each $U_i \times_S Y$ exists by the affine case, and we may apply Proposition 10.5 above. In general, cover Y by affine open subschemes V_i. As we just verified, the products $X \times_S V_i$ all exist, and applying Proposition 10.5 once more, we conclude that $X \times_S Y$ exists. □

Finally, with Lemma 10.6 established, what remains to prove, that fibre products exist in general, is a reduction to the case with an affine base. To that end, let $\{S_i\}$ be an open affine cover of S and let $U_i = f_X^{-1}(S_i)$ and $V_i = f_Y^{-1}(S_i)$. By Lemma 10.6 the products $U_i \times_{S_i} V_i$ all exist, and using the following lemma and once more Proposition 10.5, the proof will be complete.

Lemma 10.7. With the notation just introduced, $U_i \times_{S_i} V_i$ serves as the product $U_i \times_S Y$ with projections being p_{U_i} and $p_Y|_{V_i}$.

Proof We contend that $U_i \times_{S_i} V_i$ satisfies the universal product property of $U_i \times_S Y$. Consider the commutative diagram



where g and g' are two given morphisms. If a point follows the left path from Z to S in the diagram, it ends up in S_i, and the same must hold when it follows the right path. But then, V_i being equal to the inverse image $f_Y^{-1}(S_i)$, it follows that g' factors through V_i, and by the universal property of $U_i \times_{S_i} V_i$, there is a morphism $Z \to U_i \times_{S_i} V_i$ with the requested properties. □

Here are some of the basic properties of the fibre product. It is possible to deduce them directly using gluing arguments, but with the so-called 'functor of points', which we will introduce in Section 10.7, the proofs will become simple and natural.

Proposition 10.8 (Basic formulas). Let X, Y, Z and T be schemes over S. There are unique canonical isomorphisms over S, all compatible with projections:

(i) (Reflectivity) $X \times_S S \simeq X$;
(ii) (Symmetry) $X \times_S Y \simeq Y \times_S X$;
(iii) (Associativity) $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.
(iv) (Transitivity) $(X \times_S T) \times_T Y \simeq X \times_S Y$.

In the last claim Y is supposed to be a scheme over T, and $X \times_S T$ is considered a scheme over T via the projection onto T.

10.3 Examples

As noted in the introduction, the fibre product of schemes can exihibt unexpected behaviour in some situations, differing from what we are used to in set theory or topology. The main difference is that the underlying set is almost never the product of the underlying sets of the factors. The next few examples illustrate this.

Example 10.9. For a ring R and non-negative integers m, n, we have

$$R[x_1, \dots, x_m] \otimes_R R[y_1, \dots, y_n] \simeq R[x_1, \dots, x_m, y_1, \dots, y_n],$$

and so

$$\mathbb{A}^m_R \times_R \mathbb{A}^n_R \simeq \mathbb{A}^{m+n}_R.$$

Even when $R = \mathbb{C}$, the affine space $\mathbb{A}^{m+n}_{\mathbb{C}}$ has an underlying set which is different from the Cartesian product $\mathbb{A}^m_{\mathbb{C}} \times \mathbb{A}^n_{\mathbb{C}}$, and the topology is not equal to the product topology (see Example 2.16).

Even fibre products of spectra of *fields* can exhibit unexpected behaviour, as the next few examples show.

Example 10.10. A simple but illustrative example is the product $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$. This scheme has *two* distinct closed points, even if both factors are singletons. Note also that the product is not integral, not even connected. So the product of integral schemes is not necessarily integral.

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is in fact isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$ of two copies of the complex field $\mathbb{C}$. One sees this using that $\mathbb{C} = \mathbb{R}[t]/(t^2 + 1)$, which gives

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2 + 1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2 + 1) = \mathbb{C}[t]/(t - i)(t + i) = \mathbb{C} \times \mathbb{C},$$

where the last equality follows from the Chinese Remainder Theorem and that the rings $\mathbb{C}[t]/(t \pm i)$ both are isomorphic to $\mathbb{C}$.

Example 10.11. The fibre product $\operatorname{Spec} \mathbb{F}_2 \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_3$ is empty. Indeed, it is the specturm of the ring

$$\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 \simeq \mathbb{Z}/(\gcd(2, 3)) = 0$$

See Exercise 10.3.1 for a generalization.

Example 10.12. The fibre product $\operatorname{Spec} \mathbb{C}(x) \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}(y)$ is even more extreme: it has infinitely many points! This is because the tensor product $A = \mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ is a ring of Krull dimension one, and it contains infinitely many maximal ideals.

To prove this, we note that the ring A can be written as the localization $S^{-1}\mathbb{C}[x, y]$ of the polynomial ring $\mathbb{C}[x, y]$ in the multiplicative set

$$S = \{\, p(x)q(y) \mid p(x)q(y) \neq 0 \,\}.$$

Thus if $\mathfrak{p} \subset A$ is a prime ideal, it is of the form $\mathfrak{p} = S^{-1}\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset \mathbb{C}[x, y]$ that does not intersect S. Bearing in mind that the maximal ideals in $\mathbb{C}[x, y]$ are of the form $(x - a, y - b)$ with $a, b \in \mathbb{C}$, we find that $\mathfrak{q}$ is not maximal, and hence of height of at most 1. We must also have that $\mathfrak{q} \cap \mathbb{C}[x] = 0$ and $\mathfrak{q} \cap \mathbb{C}[y] = 0$, and we find that either $\mathfrak{q} = (0)$ or $\mathfrak{q} = (f(x, y))$, where f is an irreducible polynomial neither lying in $\mathbb{C}[x]$ nor in $\mathbb{C}[y]$.

In conclusion, all non-zero primes in A are therefore maximal, and so A has dimension one. Moreover, A has infinitely many maximal ideals, in fact, uncountably many.

Example 10.13 (Fibre products of varieties)**.** On a positive note, the fibre product $X \times_k Y$ is better behaved in the situation when X and Y are integral schemes of finite type over an algebraically closed field k. This includes the schemes arising from the varieties of Chapter 1.

In this case, $X \times_k Y$ is again integral (see Theorem 12.26 on page 197) and on the level of k-points, we have

$$(X \times_k Y)(k) = X(k) \times Y(k).$$

It turns out that in this case the k-points are precisely the closed points (this follows from the Nulstellensatz) and so the set of closed points in the product equals the Cartesian product of the sets of closed points of the factors.

Of course, the fibre product may additionally have many non-closed points which do not come from the closed points in each factor (Example 10.9).

Exercises

Exercise 10.3.1. Let p and q be two different prime numbers. Show the following identities:

a) $\operatorname{Spec} \mathbb{F}_p \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_q = \varnothing$;
b) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)} = \operatorname{Spec} \mathbb{Z}_{(p)}$;
c) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(q)} = \operatorname{Spec} \mathbb{Q}$.

Exercise 10.3.2. Example 10.10 can be generalized as follows. Let K/k be a finite Galois extension of fields with Galois group G. Show that the map $x \otimes y \mapsto (xg(y))_{g \in G}$ defines an isomorphism

$$K \otimes_k K \to \prod_{g \in G} K.$$

Hint: Write $K = k[x]/(f(x))$ for a minimal polynomial $f(x)$ and compute $K \otimes_k K$ using the Chinese Remainder Theorem and the fact that f factors in K.

Deduce that $\operatorname{Spec} K \times_k \operatorname{Spec} K$ has an underlying set with $|G|$ points.

Exercise 10.3.3. This exercise goes along the same lines as Exercise 10.3.2 and gives an example that a fibre product $X \times_k \operatorname{Spec} L$ may not be reduced even if X is.

Let $k = \mathbb{F}_p(a)$ for a prime number p and let $L = k[x]/(x^p - a)$. Show that

$$L \times_k L \simeq L[t]/(t^p - a) \simeq L[t]/(t - x)^p.$$

Conclude that $\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L$ is not reduced.

Exercise 10.3.4. Let X and Y be schemes over S with open affine covers $\{U_i\}$ and $\{V_j\}$. Show that $U_i \times_S V_j$ is an open cover of $X \times_S Y$.

10.4 Scheme theoretic-intersections

If X is a scheme and Y, Z are two closed subschemes we define their *scheme-theoretic intersection* as the fibre product

$$Y \times_X Z$$

of the closed embeddings $i : Y \to X$ and $j : Z \to X$.

In the special case when $X = \operatorname{Spec} A$ and Y and Z are closed subschemes given by ideals I and J respectively, then

$$Y \times_X Z = \operatorname{Spec}(A/I \otimes_A A/J) = \operatorname{Spec}(A/(I + J)).$$

Thus $Y \times_X Z$ is the closed subscheme associated to the ideal $I + J$.

Using this local model, one can prove that $Y \times_X Z$ is in general naturally a closed subscheme of X with underlying topological space is homeomorphic to $i(Y) \cap j(Z)$ in X (Exericse 10.4.1).

Note that the ideal $I + J$ may fail to be a prime ideal even if I and J are prime. Moreover, the scheme-theoretic intersection fail to be both reduced and irreducible even if both Y and Z are are. This is very natural and important: the scheme-theoretic intersection $Y \times_X Z$ is designed to capture the multiplicities of an intersection, e.g. as in Bezout's theorem. See for instance Examples 5.28 and 5.29. This important point is yet another reason for transitioning from varieties to schemes.

Exercise 10.4.1. In the setting above, show that the scheme-theoretic intersection is naturally a closed subscheme of X, with underlying topological space equal to the intersection $i(Y) \cap j(Z)$ in X.

10.5 Scheme theoretic fibres II

Suppose that $f\colon X \to Y$ is a morphism of schemes and that $y \in Y$ is a point. One of the first applications of the fibre product is to define a scheme structure on the preimage $f^{-1}(y)$. Having the fibre product at our disposal, inspired by part *a*) of Exercise 10.1.1, nothing is more natural than defining the fibre to be the fibre product $X_y = \operatorname{Spec} k(y) \times_Y X$. It appears in the diagram

$$\begin{array}{ccc} X_y = X \times_Y \operatorname{Spec} k(y) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k(y) & \longrightarrow & Y, \end{array}$$

where $\operatorname{Spec} k(y) \to Y$ is the map corresponding to the point y. Recall that the field $k(y)$ is given as $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$, and that the 'point-map' $\operatorname{Spec} k(y) \to Y$ is the composition $\operatorname{Spec} k(y) \to \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ of the two canonical maps.

Note that the fibre X_y satisfies the following universal property: a morphism $g\colon Z \to X$ factors through X_y if and only if $f \circ g$ factors through $\operatorname{Spec} k(y) \to Y$ (topologically this means it maps Z to $y \in Y$).

It is common usage to write X_y for the scheme-theoretic fibre and reserve the notation $f^{-1}(y)$ for the preimage as a topological space. In any case, the next proposition shows that the underlying topological space of X_y is equal to $f^{-1}(y)$.

Proposition 10.14. Let X and Y be schemes and $f\colon X \to Y$ a morphism. Let $y \in Y$ be a point. Then the inclusion $X_y \to X$ of the scheme theoretic fibre is a homeomorphism onto the topological fibre $f^{-1}(y)$.

Proof We may assume that Y is affine, say $Y = \operatorname{Spec} A$.

We first treat the case where X is also affine, say $X = \operatorname{Spec} B$ and $f\colon X \to Y$ is induced by a ring map $\phi : A \to B$. In this situation Proposition 2.34 states that the fibre $f^{-1}(\mathfrak{p})$ over a point $\mathfrak{p} \in \operatorname{Spec} A$ is homeomorphic to the spectrum of the ring $(B/\mathfrak{p}B)_\mathfrak{p}$. On the other hand, standard formulas for tensor products give the equality

$$(B/\mathfrak{p}B)_\mathfrak{p} = B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = B \otimes_A k(\mathfrak{p}),$$

and the Zariski topology on the spectrum $\operatorname{Spec}(B/\mathfrak{p}B)_\mathfrak{p}$ (i.e. the induced topology on $f^{-1}(\mathfrak{p})$) coincides with the Zariski topology on $\operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$ (i.e. the topology on the scheme X_y), and hence the proposition holds when X is affine.

In the general case let U be open and affine in X. Denote by ι the inclusion $\iota\colon X_y \to X$; that is, the projection $X \times_Y \operatorname{Spec} k(y) \to X$. According to Lemma 10.4 on page 156, it holds that $U \times_Y \operatorname{Spec} k(y) = \iota^{-1}U$ (equipped with the unique open scheme structure on the open set $\iota^{-1}U$), and clearly $\iota^{-1}U = U \cap X_y$. By the affine case, the two topologies we examine agree on $X_y \cap U$, and as U can be any open affine, the two topologies share a basis and must be equal. □

Example 10.15 (The fibre product is the fibre product)**.** Let $f\colon X \to S$ and $g\colon Y \to S$ be two morphisms of schemes and let $\iota\colon \operatorname{Spec} k(s) \to S$ be a point. Denote by $h\colon X \times_S Y \to S$ the structure map, i.e. $h = f \circ p_X = g \circ p_Y$. Then the scheme theoretic fibre of h is the fibre product of the scheme theoretic fibres of f and g:

$$(X \times_S Y)_s = X_s \times_{k(s)} Y_s.$$

This is immediate, applying associativity and transitivity of the fibre product (formulas (iii) and (iv) of Proposition 10.8 on page 158):

$$\begin{aligned}(X \times_S \operatorname{Spec} k(s)) \times_{k(s)} (Y \times_S \operatorname{Spec} k(s)) &= X \times_S (Y \times_S \operatorname{Spec} k(s)) \\ &= (X \times_S Y) \times_S \operatorname{Spec} k(s).\end{aligned}$$

10.6 Base change

Given a set of equations over some ring, it is often fruitful to consider the solutions in some ring extension. For instance, while $x^2 + y^2 + 1 = 0$ has no solutions over $\mathbb{R}$, there are plenty if we regard the same equation over $\mathbb{C}$. More formally, we can start with the spectrum $X = \operatorname{Spec} A$ of the ring

$$A = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$$

and consider the tensor product $A \times_\mathbb{R} \mathbb{C} = \mathbb{C}[x, y]/(x^2 + y^2 + 1)$. Note that the inclusion $\mathbb{R} \subset \mathbb{C}$ induces a morphism $\operatorname{Spec}(A \otimes_\mathbb{R} \mathbb{C}) \to \operatorname{Spec} A$. One says that the scheme $X_\mathbb{C} = \operatorname{Spec}(A \otimes_\mathbb{R} \mathbb{C})$ is obtained from $\operatorname{Spec} A$ via base change. The scheme $X = \operatorname{Spec} A$ is a scheme with no $\mathbb{R}$-points, wheras $X_\mathbb{C}$ has infinitely many $\mathbb{C}$-points.

This idea of ‘changing the base field’ has a vast generalization as follows. Let X be a scheme over S and let $T \to S$ be a morphism. The fibre product $X \times_S T$ is then naturally a scheme over T. Considering $T \to S$ as a change of base schemes, one frequently writes X_T for $X \times_S T$ and says that X_T is obtained from X by *base change*.

Taking a base change is a functorial construction. If $f\colon X \to Y$ is a morphism over S, there is induced a morphism $f_T = f \times \mathrm{id}_T$ from X_T to Y_T over T, and one easily checks that $f_T \circ p_T$ coincides with the natural projection map $X_T \to T$ (or in other words, the outer rectangle in the diagram below is Cartesian).

$$\begin{array}{ccccc} X_T & \xrightarrow[f_T]{} & Y_T & \xrightarrow[p_T]{} & T \\ \downarrow & & \downarrow{\scriptstyle p_Y} & & \downarrow \\ X & \xrightarrow{f} & Y & \longrightarrow & S. \end{array}$$

If $\mathcal{P}$ is a property of morphisms, one says that $\mathcal{P}$ is *stable under base change* if for any T over S, the map f_T has the property $\mathcal{P}$ whenever f has it. The same convention applies to properties of schemes.

Examples 10.17 and 10.20 below show that neither being irreducible nor being reduced are properties stable under base change. On the other hand, one way of phrasing Lemma 10.4 on page 156, is to say that being an open embedding is stable under base change. The same applies to closed and locally closed embeddings.

Proposition 10.16 (Embeddings and base change). Consider a Cartesian diagram of schemes

$$\begin{array}{ccc} Z_Y & \longrightarrow & Z \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & X. \end{array}$$

If the morphism $f\colon Z \to X$ is a closed, open or locally closed embedding, then the morphism $f_Y\colon Z_Y \to Y$ is respectively closed, open or locally closed.

Proof The case of open embeddings is already taken care of, and the case of locally closed embeddings follow directly from the two others, so only the statement about closed embeddings needs a proof.

Assume first that X and Y are both affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. When $Z \subset \operatorname{Spec} A$ is a closed subscheme, it holds that $Z = \operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ (Proposition 5.10 on page 71), and therefore $Z_Y = Z \times_X Y = \operatorname{Spec} A/\mathfrak{a} \otimes_A B = \operatorname{Spec} B/\mathfrak{a}B$. Hence Z_Y is a closed subscheme of Y.

In general, the statement is local on Y (Exercise 5.3.1 on page 71), so assume that $U \subset Y$ is an open affine that maps into an open affine $V \subset X$ (one may cover Y by such by first covering X by affine opens and subsequently cover each of their inverse images in Y by affine opens). Then by Lemma 10.7 on page 158 one has $f^{-1}V \times_X Y = f^{-1}V \times_V U$, and by the affine case this is a closed subscheme of U. □

If one identifies Z with its image in X, the scheme Z_Y is what one calls *the scheme theoretic inverse image* of Z. If k is a field, the k-points in Z_Y are exactly the k-points that map into Z.

Example 10.17 (Being irreducible is not stable under base change)**.** Consider the $\mathbb{R}$-algebra $A = \mathbb{R}[x,y]/(x^2+y^2)$. Over $\mathbb{R}$, the polynomial x^2+y^2 is irreducible, so $X = \operatorname{Spec} A$ is an irreducible $\mathbb{R}$-scheme. The base change to $\mathbb{C}$ however is not irreducible, because

$$A \otimes_{\mathbb{C}} \mathbb{C} \simeq \mathbb{C}[x,y]/(x-iy) \times \mathbb{C}[x,y]/(x+iy)$$

and so $X_{\mathbb{C}}$ is the union of two conjugate lines in $\operatorname{Spec}\mathbb{C}[x,y]$

Example 10.18 (Being reduced is not stable under base change)**.** An example was already given in Exercise 10.3.3. Consider the scheme $X = \operatorname{Spec}\mathbb{Z}[x,y]/(x^2-2y^2)$, viewed as a scheme over $\operatorname{Spec}\mathbb{Z}$. Clearly X is integral, as x^2-2y^2 is irreducible. However, if we take the base change via the morphism $\operatorname{Spec}\mathbb{F}_2 \to \operatorname{Spec}\mathbb{Z}$, the resulting scheme is non-reduced:

$$X_{\mathbb{F}_2} = \operatorname{Spec}(\mathbb{F}_2[x,y]/(x^2))$$

Example 10.19. For a related example, consider the polynomial $T^4 - 10T^2 + 1$, which is the minimal polynomial of $\sqrt{2}+\sqrt{3}$. This polynomial has the interesting property that it is irreducible over $\mathbb{Q}$, but its reduction modulo p factors for every prime p. This means that for the morphism

$$\operatorname{Spec}\mathbb{Z}[T]/(T^4-10T^2+1) \longrightarrow \operatorname{Spec}\mathbb{Z},$$

the fibre over the generic point is irreducible, but all of the closed fibres are reducible.

Example 10.20 (Being reduced is not stable under base change)**.** An example was already given in Exercise 10.3.3. Consider the scheme $X = \operatorname{Spec}\mathbb{Z}[x,y]/(x^2-2y^2)$, viewed as a scheme over $\operatorname{Spec}\mathbb{Z}$. Clearly X is integral, as x^2-2y^2 is irreducible. However, if we take the base change via the morphism $\operatorname{Spec}\mathbb{F}_2 \to \operatorname{Spec}\mathbb{Z}$, the resulting scheme is non-reduced:

$$X_{\mathbb{F}_2} = \operatorname{Spec}(\mathbb{F}_2[x,y]/(x^2))$$

Exercises

Exercise 10.6.1. Prove statements (i) and (iv) in Proposition 10.8.

Exercise 10.6.2. Let $A = \mathbb{R}[x,y]/(x^2+y^2+1)$ and let $X = \operatorname{Spec}A$. Show that the base-change $X_{\mathbb{C}}$ is isomorphic to $\mathbb{A}^1_{\mathbb{C}} - V(t)$, but X is not isomorphic to $\mathbb{A}^1_{\mathbb{R}} - V(t)$.

Exercise 10.6.3. Show that if B is an A-algebra, then $\mathbb{A}^n_B \simeq \mathbb{A}^n_A \times_A \operatorname{Spec}B$ and that $\mathbb{P}^n_B \simeq \mathbb{P}^n_A \times_A \operatorname{Spec}B$.

Exercise 10.6.4. Let $L_m \to \mathbb{P}^1_k$ be the line bundle constructed in Section 7.7 on page 102, and let $f_n \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the map $u \mapsto u^n$. Show that $L_m \times_{\mathbb{P}^1_k} \mathbb{P}^1_k = L_{nm}$.

Exercise 10.6.5 (Finite type and base change)**.**

a) Show that being of finite type (respectively being finite or being locally of finite type) is a property stable under base change;
b) Show that the product of two morphisms of finite type (respectively of finite or locally of finite type) is of finite type (respectively of finite or locally of finite type).

10.7 Yoneda's Lemma*

As the examples in Section 10.3 show, the fibre product product $X \times_S Y$ can be somewhat elusive when it comes to the underlying topological space. In this section, we clarify the picture using the so-called 'functor of points'. This is an important concept in algebraic geometry, and it is often very useful for proving statements about schemes (e.g., Proposition 10.8).

Recall from Section 5.4, that for a scheme X and a ring R, the set of R-valued points $X(R)$ is the set of all scheme maps $\operatorname{Spec}(R) \to X$. There is a generalization of this, where we consider the set of all morphisms $T \to X$ from a fixed scheme T into X. Formally, we define the *functor of points* associated with a scheme X to be the contravariant functor $h_X \colon \mathsf{Sch} \to \mathsf{Sets}$ defined by

$$h_X(T) = \operatorname{Hom}_{\mathsf{Sch}}(T, X).$$

This functor sends a morphism $f \colon S \to T$ to the map of sets

$$\begin{aligned} h_X(f) \colon h_X(T) &\to h_X(S) \\ g &\mapsto g \circ f. \end{aligned}$$

If $f \colon X \to Y$ is a morphism of schemes, there is for each T an induced map $h_f(T) \colon h_X(T) \to h_Y(T)$ defined by sending $g \colon T \to X$ to $f \circ g$. It is readily checked to be a *natural transformation* of functors $h_f \colon h_X \to h_Y$. Recall that a natural transformation $\eta \colon F \to G$ between two contravariant functors $F, G \colon \mathsf{C} \to \mathsf{Sets}$ is a collection of morphisms $F(T) \to G(T)$, one for each object T, such that whenever $h \colon S \to T$ is a morphism in C there should be a commutative diagram (of sets)

$$\begin{array}{ccc} F(T) & \xrightarrow{F(h)} & F(S) \\ \downarrow{\scriptstyle \eta(T)} & & \downarrow{\scriptstyle \eta(S)} \\ G(T) & \xrightarrow{G(h)} & G(S) \end{array}.$$

A natural question is whether the scheme X is determined by the functor h_X. The answer is 'yes', and this is essentially the content of Yoneda's Lemma. In short, the lemma says that there is a bijection between the set of scheme morphisms $X \to Y$ and the set of natural transformations of functors $h_X \to h_Y$.

Lemma 10.21 (Yoneda). For each X and Y there is a functorial bijection

$$\operatorname{Hom}_{\mathsf{Sch}}(X, Y) \longrightarrow \operatorname{Hom}_{\text{nat.transf.}}(h_X, h_Y) \tag{10.7}$$

sending $X \to Y$ to the natural transformation $h_X \to h_Y$. Thus every natural transformation $h_X \to h_Y$ is of the form h_f for a unique morphism $f \colon X \to Y$.

Proof Let $\eta \colon h_X \to h_Y$ be a natural transformation. Applying η to the scheme X, we get a map

$$\eta(X) \colon h_X(X) = \operatorname{Hom}_{\mathsf{Sch}}(X, X) \to \operatorname{Hom}_{\mathsf{Sch}}(X, Y) = h_Y(X).$$

If there is an $f \colon X \to Y$ such that $h_f = \eta$, then we must have

$$\eta(X)(\mathrm{id}_X) = h_f(X)(\mathrm{id}_X) = f \circ \mathrm{id}_X = f.$$

Therefore f is determined by η, and hence (10.7) is injective.

For surjectivity, we put $f = \eta(X)(\mathrm{id}_X)$ and will check that $h_f = \eta$. This means that for any scheme Z, the map of sets

$$\eta(Z)\colon h_X(Z) \to h_Y(Z)$$

is equal to the map of sets that sends $g\colon Z \to X$ to $g \circ f$. Since η is a natural transformation, we have for any $g\colon Z \to X$, a commutative diagram

$$\begin{array}{ccc} h_X(X) & \xrightarrow{h_X(g)} & h_X(Z) \\ \downarrow{\scriptstyle \eta(X)} & & \downarrow{\scriptstyle \eta(Z)} \\ h_Y(X) & \xrightarrow{h_Y(g)} & h_Y(Z) \end{array}$$

Going through the diagram clockwise, we see that id_X gets sent to $\eta(Z)(g)$, while going counterclockwise, id_X gets sent to $g \circ f$. Hence

$$\eta(Z)(g) = g \circ f = h_f(Z)(g),$$

and so $\eta = h_f$. □

In particular, we have the following consequences:

Corollary 10.22. For two schemes X and Y, we have:
(i) h_X and h_Y are isomorphic (as contravariant functors from Sch to Sets), if and only if $X \simeq Y$:
(ii) If a functor F is the same as h_X for some scheme X, then X is determined up to isomorphism.

Replacing the scheme X with its associated functor of points h_X, may at this point seem like just yet another jump in abstraction, but the nice thing is that you can work with functors whose values are good old sets. For instance, by the Yoneda lemma, we see that giving a morphism $f\colon X \to Y$ of schemes, is the same thing as for each scheme Y giving a map of sets $f(T)\colon X(T) \to Y(T)$ which is functorial in T (i.e. a natural transformation). In fact, using that schemes are locally affine, and that morphisms of schemes glue together, it is even sufficient to test this condition on affine schemes $T = \operatorname{Spec} B$.

Another important consequence of this is that instead of specifying a scheme explictly, say by giving a projective embedding and a homogeneous ideal, we can simply specify a functor equivalent to h_X, and this will precisely pin down what scheme we are talking about. Many schemes are in the first place defined as solutions to universal problems (e.g. fibre products), and often the functor perspective can clarify and simplify computations.

Example 10.23 (The functor of points of $\mathbb{A}^1$)**.** By Theorem 6.5 on page 88 to give a morphism from T into the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$ is the same thing as giving an element of $\mathcal{O}_T(T)$. Therefore, $h_{\mathbb{A}^1}$ is isomorphic to the functor

$$F(T) = \mathcal{O}_T(T)$$

More generally, $\mathbb{A}^n$ represents the functor $F(T) = \Gamma(T, \mathcal{O}_T)^n$; this is just a fancy way of saying that a morphism $X \to \mathbb{A}^n$ is the same thing as an n-tuple of regular functions.

Example 10.24 (The functor of points of $\operatorname{Spec} A$)**.** More generally, let A be a ring and consider the functor $F\colon \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(T) = \operatorname{Hom}_{\mathsf{Rings}}(A, \Gamma(T, \mathcal{O}_T)).$$

From Theorem 6.5, we deduce that $F \simeq h_{\operatorname{Spec} A}$.

Exercise 10.7.1. Show that functor of points of $\mathbb{A}^1 - \{0\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is isomorphic to

$$F(T) = \Gamma(T, \mathcal{O}_T)^{\times}.$$

Exercise 10.7.2. Show that the functor $T \mapsto GL_n(\mathcal{O}_T(T))$ is represented by the scheme $\mathbb{GL}_n = \operatorname{Spec} \mathbb{Z}[t_{ij}, \det(t_{ij})^{-1}]$.

The fibre product in terms of the functor of points

There is a nice way to explain the universal property of fibre products of two S-schemes X and Y in terms of the functors of points h_X, h_Y and h_S. For a scheme T, it translates into the following: the set $\operatorname{Hom}_{\mathsf{Sch}}(T, X \times_S Y)$ is the fibre product of the two sets $\operatorname{Hom}_{\mathsf{Sch}}(T, X)$ and $\operatorname{Hom}_{\mathsf{Sch}}(T, Y)$ over $\operatorname{Hom}_{\mathsf{Sch}}(T, S)$. In other words, there is a natural bijection of sets (!)

$$h_{X \times_S Y}(T) \to h_X(T) \times_{h_S(T)} h_Y(T). \tag{10.8}$$

By uniqueness, these bijections are functorial in T, and we conclude that the functor of points of the fibre product $X \times_S Y$ is isomorphic to the fibre product functor $h_X \times_{h_S} h_Y$, which assigns the set $h_X(T) \times_{h_S(T)} h_Y(T)$ to a scheme T. Thus the fibre product of schemes is not so mysterious after all; it is essentially forced upon us by the universal property of fibre products of sets.

Setting $T = \operatorname{Spec} R$ in (10.8), we get:

Corollary 10.25. For any ring R, there is a natural bijection

$$(X \times_S Y)(R) = X(R) \times_{S(R)} Y(R), \tag{10.9}$$

where the right-hand side is the fibre product of sets.

Once we know the functor of points of $X \times_S Y$, Yoneda's Lemma implies that many computations involving fibre products reduce to ones involving sets only. To illustrate this, we give a proof of Proposition 10.8

Proof of Proposition 10.8 By Yoneda's lemma, it suffices to verify the corresponding statements for sets, and this is elementary: note that the assignments $(b, a) \mapsto b$; $(b, c) \mapsto (c, b)$; and $((b, c), d) \mapsto (b, (c, d))$ give natural bijections of sets

$$\begin{aligned} &B \times_A A \simeq B && (b, a) \mapsto b \\ &B \times_A C \simeq C \times_A B && (b, c) \mapsto (c, b) \\ &(B \times_A C) \times_C D \simeq B \times_A (C \times_C D) && ((b, c), d) \mapsto (b, (c, d)). \end{aligned}$$

These translate into natural isomorphisms of functors

$$h_{X\times_S S} \simeq h_X \tag{10.10}$$
$$h_{X\times_S Y} \simeq h_{X\times_S Y} \tag{10.11}$$
$$h_{(X\times_S Y)\simeq SZ} \simeq h_{X\times_S (Y\times_S Z)}$$

and by Yoneda's Lemma, we have the isomorphisms between the corresponding fibre products as well. □

Exercise 10.7.3. Let A, B and C be sets.

a) Suppose $f_B\colon B \to A$, $f_{A'}\colon A' \to A$ and $f_C\colon C \to A$ are maps with $f_C = g \circ f_{A'}$ for $g\colon A' \to A$. Show that there is a bijection
$$B \times_A C \simeq (B \times_A A') \times_{A'} C$$
induced by $(b, c) \mapsto ((b, g(c)), c)$.

b) Deduce claim (iv) in Proposition 10.8.

10.8 Proj and products

In this section we will need the tensor product of to graded algebras. If $R = \bigoplus_{n\geqslant 0} R_n$ and $R' = \bigoplus_{n\geqslant 0} R'_n$ are two graded rings with the same degree zero piece A, the tensor product $R \otimes_A R'$ has a natural grading induced from the gradings of R' and R; indeed, the tensor product commutes with arbitrary direct sums, so $R \otimes_A R'$ decomposes as

$$R \otimes_A R' = \bigoplus_{i,j\geqslant 0} R_i \otimes_A R'_j.$$

Grouping together parts with $i + j = n$, we find

$$R \otimes_A R' = \bigoplus_{n\geqslant 0} \bigoplus_{i+j=n} R_i \otimes_A R'_j,$$

and this defines the induced grading. The homogeneous tensors which are decomposable, are of the form $x \otimes y$ with x and y homogeneous and $\deg x \otimes y = \deg x + \deg y$. General homogeneous elements are A-linear combinations of such.

Base change

Let $R = \bigoplus_{n\geqslant 0} R_n$ be a graded ring. Forming tensor products commutes with formimg direct sums, so if B is any R_0-algebra, we have $R \otimes_{R_0} B = \bigoplus_{n\geqslant 0} R_n \otimes_{R_0} B$ and this gives $R \otimes_{R_0} B$ a grading. In this setting the Proj-construction behaves well:

Proposition 10.26. Forming Proj commutes with base change. That is, if R is a graded ring and $\operatorname{Spec} B \to \operatorname{Spec} R_0$ is a morphism, there is a canonical isomorphism

$$\operatorname{Proj} R \times_{R_0} \operatorname{Spec} B \simeq \operatorname{Proj} R \otimes_{R_0} B.$$

Proof The salient point is that for each homogeneous element $f \in R_+$, there is a canonical[1]

[1] Characterised by inducing the identity on $R \otimes_{R_0} B$

identification $R_f \otimes_{R_0} B \simeq (R \otimes_{R_0} B)_{f \otimes 1}$. It relies on the identity $f^{-n} \otimes 1 = (f \otimes 1)^{-n}$, which is valid in $R_f \otimes_{R_0} B$, but the right side is meaningful in both rings. Clearly homogeneous elements of degree zero correspond, and hence we have canonical identifications

$$(R_f \otimes_{R_0} B)_0 \simeq \big((R \otimes_{R_0} B)_{f \otimes 1}\big)_0.$$

Translated into geometry, these provide canonical isomorphisms

$$D_+(f) \times_{R_0} \operatorname{Spec} B \simeq D_+(f \otimes 1). \tag{10.12}$$

As f runs through the homogeneous elements of R_+, the open subschemes on the left in (10.12) yield an open cover of the product $\operatorname{Proj} R \times_{R_0} \operatorname{Spec} B$. The irrelevant ideal $R \otimes_{R_0} B$ equals $R_+ \otimes_{R_0} B$ and is therefore generated by the elements $f \otimes 1$ with f running through R_+. Thus the open subschemes on the right in (10.12) constitute an open affine cover of $\operatorname{Proj} R \otimes_{R_0} B$. It only remains to observe that the isomorphisms in (10.12) being canonical, coincide on the intersections $D_+(f) \cap D_+(f') = D_+(ff')$, and hence patch together to a global isomorphism. □

Example 10.27. In Example 9.38 on page 149 we introduced the blow-up of an ideal $\mathfrak{a}$ in a ring A as the structure map $\pi\colon \operatorname{Proj} \bigoplus_{i \geqslant 0} \mathfrak{a}^i t^i \to \operatorname{Spec} A$. In that example we also showed that π is an isomorphism outside the inverse image $\pi^{-1} V(\mathfrak{a})$. Proposition 10.26 yields a closer description of the scheme-theoretic inverse image of $V(\mathfrak{a})$ in that

$$(\bigoplus_{i \geqslant 0} \mathfrak{a}^i t^i) \otimes_A A/\mathfrak{a} = \bigoplus_{i \geqslant 0} \mathfrak{a}^i/\mathfrak{a}^{i+1},$$

and hence $\pi^{-1} V(\mathfrak{a}) = \operatorname{Proj} \bigoplus_{i \geqslant 0} \mathfrak{a}^i/\mathfrak{a}^{i+1}$.

Exercise 10.8.1. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ the origin in $\mathbb{A}^n_k$.

(i) Show that the graded k-algebra $\bigoplus_{i \geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to the polynomial ring $k[t_1, \ldots, t_n]$, where t_i denotes the class of x_i in $\mathfrak{m}/\mathfrak{m}^2$.

(ii) Let $\pi\colon X \to \mathbb{A}^n$ be the blow up of the origin 0 in $\mathbb{A}^n_k$ (that is, of $\mathfrak{m}$). Show that $X_0 = \mathbb{P}^{n-1}_k$.

The Segre embedding

In the world of varieties, the *Segre embedding* is an embedding of the product of two projective spaces $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ into the projective space $\mathbb{P}^{nm+n+m}(k)$, which is given by all products of coordinates:

$$(u_0 : \cdots : u_n) \times (v_0 : \cdots : v_m) \mapsto (u_0 v_0 : u_1 v_0 : \cdots : u_i v_j : \cdots : u_n v_m),$$

or in terms of coodinates w_{ij} on $\mathbb{P}^{nm+m+n}(k)$, it is given by $w_{ij} = u_i v_j$.

Note that there are $(n+1)(m+1)$ different products. Scaling the u_i's simultaneously and the v_j's simultaneously, will scale the products simultaneously, and obviously, if at least one of the u_i's and one of the v_j's are non-zero, one of the products will be non-zero as well. Thus we obtain, at least set-theoretically, a well-defined map

$$\sigma\colon \mathbb{P}^n(k) \times \mathbb{P}^m(k) \longrightarrow \mathbb{P}^{nm+n+m}(k).$$

The map σ is injective and the image is closed; if $u_r v_s \neq 0$, we recover the coordinates of the points in $\mathbb{P}^n(k)$ and $\mathbb{P}^m(k)$ where respectively $u_r \neq 0$ and $v_s \neq 0$ as $v_i/v_s = u_r v_i/u_r v_s$ and $u_i/u_r = u_i v_s/u_r v_s$. One verifies easily that the image is the vanishing locus of the quadrics $w_{ij}w_{lm} - w_{il}w_{jm}$ for all choices of four different indices i, j, l, m. If one organize the coordinates w_{ij} into a matrix $M = (w_{ij})$, these quadrics are precisely the 2×2-minors of M; in other words, the image of σ is the locus where M has rank one.

Finally, note that $\sigma^{-1}D_+(w_{rs}) = D_+(u_r) \times D_+(v_s)$; indeed, $u_r v_s \neq 0$ precisely when both $u_r \neq 0$ and $v_s \neq 0$.

There is a scheme analogue of this which works in greater generality, however, we confine ourselves to the following simpler version.

Proposition 10.28 (The Segre embedding). Given a ring A and natural numbers m and n, there is a closed embedding

$$\sigma_{m,n} \colon \mathbb{P}^m_A \times_A \mathbb{P}^n_A \longrightarrow \mathbb{P}^{mn+m+n}_A.$$

Proof We start by chosing coordinates bt letting

$$\mathbb{P}^n = \operatorname{Proj} A[u_i | 0 \leqslant i \leqslant n]$$
$$\mathbb{P}^m = \operatorname{Proj} A[v_j | 0 \leqslant j \leqslant m],$$

and moreover we let

$$\mathbb{P}^{nm+n+m} = \operatorname{Proj} A[w_{ij} | 0 \leqslant i \leqslant n, 0 \leqslant j \leqslant m].$$

All three rings are polynomial rings, and $A[u_i|i]$ and $A[v_j|j]$ have the natural gradings, but $A[w_{ij}|ij]$ has the grading with each w_{ij} of degree two.

As the indices r and s trace the appropriate index sets, the open distinguished sets $D_+(u_r)$, $D_+(v_s)$ and $D_+(w_{rs})$ form open covers of the corresponding projective spaces. The strategy of the proof is, for each choice of r and s, to construct closed embeddings

$$f_{rs} \colon D_+(u_r) \times D_+(v_s) \to D_+(w_{rs})$$

that match on the different intersections. According to Exercise 10.3.4 the sets $D_+(u_r) \times D_+(v_s)$ form an open affine cover of $\mathbb{P}^m \times \mathbb{P}^n$, and we conclude that the f_{rs}'s may be glued together to give a morphism $\sigma \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{nm+m+n}$. It has the property that $\sigma^{-1}D_+(w_{rs}) = D_+(u_r) \times D_+(v_s)$ and hence is closed embedding (Definition **??**).

Recall that $(A[u_i|i]_{u_s})_0 = A[u_i u_s^{-1}|i]$ so that

$$D_+(u_r) = \operatorname{Spec} A[u_i u_r^{-1}|i].$$

Similarily, for the other distinguished open sets we have equalities

$$D_+(v_s) = \operatorname{Spec} A[v_j v_s|j]$$
$$D_+(w_{rs}) = \operatorname{Spec} A[w_{ij} w_{rs}^{-1}|i, j].$$

Note further the equality $A[u_i u_s^{-1}|i] \otimes_A A[v_j v_r^{-1}|i] = A[u_i u_s^{-1}, v_j v_s^{-1}|i, j]$ so to have the morphisms f_{rs}, we need surjective algebra maps

$$\phi_{rs} \colon A[w_{ij} w_{rs}^{-1}|i, j] \to A[u_i u_s^{-1}, v_j v_s^{-1}|i, j]$$

with the appropriate gluing properties. There is an obvious candidate, namly the one given by the assignments $w_{ij}w_{rs}^{-1} \mapsto u_iv_ju_r^{-1}v_s^{-1}$.

Note that ϕ_{rs} arises as the degeree zero part of the localization of in w_{rs} and u_sv_r of the natural map

$$S = A[w_{ij}|i,j] \to A[u_i, v_j|i,j]$$

that sends w_{ij} to u_iv_j.

To prove that ϕ_{rs} is surjectivity, observe that $A[u_iu_s^{-1}, v_jv_s^{-1}|i,j]$ is generated over A by elements shaped like $pu_r^{-a}qv_s^{-b}$ where p and q are homogeneous monomials with respective degrees a and b in the u_i's and the v_j's, and so it suffices to see that each of these belongs to image. Replacing p with $pu_r^{\deg q}$ and q by $qv_s^{\deg p}$, we may assume that $a = b$. Then p and q will be homogeneous monomials of the same degree, and we may match each occurrence of one of the u_i's in p with an occurrence of one of v_j's in q, and in this way form a monomial P in the w_{ij} of degree a so that Pw_{rs}^{-a} maps to $pu_r^{-a}qv_s^{-a}$.

For the gluing process to work we need that f_{rs} and $f_{r's'}$ restrict to the same map

$$D_+(u_ru_{r'}) \times D_+(v_sv_{s'}) \to D_+(w_{rs}w_{r's'});$$

or what amounts to the same, that the maps ϕ_{rs} and $\phi_{r's'}$ localize to the same map

$$A[w_{ij}w_{rs}^{-1}, w_{rs}w_{r's'}^{-1}|i,j] \to A[u_iu_r^{-1}, u_ru_{r'}^{-1}, v_jv_s^{-1}, v_sv_{s'}^{-1}|i,j].$$

Both arise from the map (10.8) through sucsessive localizations and takings of degree zero parts, and the order does not matter in view of the general formula

$$((S_w)_0)_{w'w^{-1}} = ((S_{w'})_0)_{ww'^{-1}} = (S_{ww'})_0$$

where S is any graded ring and w and w' homogeneous elements of the same degree.

It remains to see that

$$\sigma^{-1}D_+(w_{rs}) = D_+(u_r) \times D_+(v_s).$$

This will follow from the equalities

$$f_{rs}^{-1}D_+(w_{rs}) \cap D_+(w_{r's'}) = D_+(u_ru_{r'}) \times D_+(v_sv_{s'}),$$

which hold true since the inverse images $f_{rs}^{-1}D_+(w_{rs}) \cap D_+(w_{r's'})$ equal $D(\phi(w_{r's'}w_{rs}^{-1})$ inside $D_+(u_s) \times D_+(v_r) = \operatorname{Spec} A[u_iu_s^{-1}, v_jv_s^{-1}|i,j]$ because of the identity

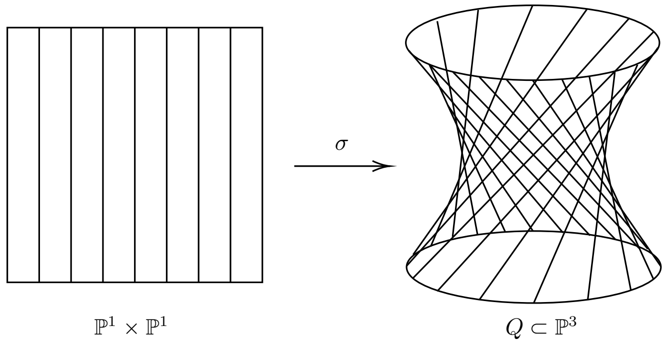
$$A[u_iu_r^{-1}, u_ru_{r'}^{-1}, v_jv_s^{-1}, v_sv_{s'}^{-1}|i,j] = A[u_iu_r^{-1}, v_jv_s^{-1}, u_rv_su_{r'}^{-1}v_{s'}^{-1}|i,j].$$

□

Example 10.29. Consider the cased that $R = k[x_0, x_1]$ and $R' = k[y_0, y_1]$ where k is a field. The assignment $z_{ij} \mapsto x_i \otimes y_j$ yields an isomorphism

$$k[z_{00}, z_{01}, z_{10}, z_{11}]/(z_{00}z_{11} - z_{01}z_{10}) \xrightarrow{\simeq} S = \textstyle\bigoplus_{n\geqslant 0}(R_n \otimes R'_n),$$

and we recover the usual embedding of $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$ as a quadric surface in $\mathbb{P}_k^3$.


σ
$\mathbb{P}^1 \times \mathbb{P}^1$
$Q \subset \mathbb{P}^3$

11

Separated schemes

We have seen several examples showing that the topology on schemes behaves very differently from the usual Euclidean topology. In particular, schemes are essentially never Hausdorff – the open sets in the Zariski topology are simply too large. Still we would like to find an analogous property that can serve as a satisfactory substitute, so that we have good properties such as ‘uniqueness of limits’. This leads to the notion of ‘separatedness’.

The route we take to defining separatedness involves the *diagonal morphism*. The motivation comes from the following basic fact from point set topology.

Proposition 11.1. A topological space X is Hausdorff if and only if the diagonal $\Delta = \{ (x, x) \mid x \in X \}$ is a closed subset of $X \times X$ (in the product topology).

Proof The diagonal $\Delta \subset X \times X$ is closed if and only if the complement $X \times X - \Delta$ is open, and with the product topology, this is equivalent to any point $(x, y) \in X \times X$ with $x \neq y$ being contained in $U \times V$ where $U, V \subset X$ are open and $U \times V \subset X \times X - \Delta$. But this is equivalent to $U \cap V \neq \varnothing$. □

Even for the affine line $X = \mathbb{A}^1_k$ over a field, the usual Hausdorff condition does not hold; any open set will contain the generic point (0) (or even in the context of varieties, two non-open subsets intersect). On the other hand, the Zariski topology on a product is typically much finer than the product topology on the underlying sets. For instance, for $\mathbb{A}^1_k$, we have $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$, and it makes perfect sense to talk about the subset $V(x - y) \subset \mathbb{A}^2_k$ of points on the ‘diagonal’, and this is indeed a Zariski closed subset.

It turns out that the ’diagonal perspective’ gives a completely satisfactory notion of ‘Hausdorffness’ for schemes. In fact, it works for relative schemes X/S as well, and thus we will speak of a *morphism* $X \to S$ being separated, rather than the scheme itself.

The freedom to glue schemes together leads to many examples of non-separated schemes, but they are not commonly encountered in practice. For instance, all affine schemes and all projective schemes are separated. More importantly, some very nice and advantageous properties hold only for separated schemes, and this legitimates the notion. For instance, in a separated scheme, the intersection of two affine subsets is again affine (this is a property which will be important later on).

11.1 Separated schemes

Let X/S be a scheme over S. There is a canonical map $\Delta_{X/S}\colon X \to X \times_S X$ of schemes over S called the *diagonal map* or the *diagonal morphism*. The two component maps of $\Delta_{X/S}$ are both equal to the identity id_X; in other words, the defining properties of $\Delta_{X/S}$ are $p_i \circ \Delta_{X/S} = \mathrm{id}_X$ for $i = 1, 2$ where the p_i's denote the two projections.

The following little lemma gives intuition for the diagonal morphism. In particular, it says that if $p_1, p_2 \in X(K)$ are two K-points (K a field), the K-point $p_1 \times p_2\colon \operatorname{Spec} K \to X \times_S X$ lies in the diagonal precisely exactly whenever $p_1 = p_2$.

Lemma 11.2. A morphism $f\colon Z \to X \times_S X$ factors through the diagonal if and only if $p_1 \circ f = p_2 \circ f$.

Proof If f factors, the equality holds by definition of the diagonal. If the equality hold, we just put $g = p_1 \circ f\colon Z \to X$, and the unicity part of the universal property gives that $\Delta_{X/S} \circ g = f$. □

In the case that X and S are affine schemes, say $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$, the diagonal has a simple and natural interpretation in terms of algebras; it corresponds to the most natural map, namely the multiplication map:

$$\mu\colon B \otimes_A B \longrightarrow B.$$

The multiplication map sends $b \otimes b'$ to the product bb', and then extends to $B \otimes_A B$ by linearity. The projections correspond to the two algebra homomorphisms $\beta_i\colon B \to B \otimes_A B$ that send B to $b \otimes 1$ respectively to $1 \otimes b$. Clearly it holds that $\mu \circ \beta_i = \mathrm{id}_B$, and on the level of schemes this translates into the defining relations for the diagonal map. Moreover, μ is clearly surjective, so we have established the following:

Proposition 11.3. If X is an affine scheme over the affine scheme S, then the diagonal $\Delta_{X/S}\colon X \to X \times_S X$ is a closed embedding.

The conclusion here is not generally true for schemes, and shortly we shall give counterexamples. However, from the proposition we just proved, it follows readily that the image $\Delta_{X/S}(X)$ is always *locally closed*, i.e. the diagonal is locally a closed embedding:

Proposition 11.4. The diagonal $\Delta_{X/S}$ is locally a closed embedding.

Proof Begin with covering S by open affine subsets and subsequently cover each of their inverse images in X by open affines as well. In this way one obtains a cover of X by affine open subsets U_i whose images in S are contained in affine open subsets S_i. The products $U_i \times_{S_i} U_i = U_i \times_S U_i$ are open and affine, and their union is an open subset containing the image of the diagonal. By Proposition 11.3 above the diagonal restricts to a closed embedding of U_i in $U_i \times_{S_i} U_i$. □

With this in place, we are ready to give the general definition of separatedness:

Definition 11.5. One says that the scheme X/S is *separated* over S, or that the structure map $X \to S$ is separated, if the diagonal map $\Delta_{X/S}\colon X \to X \times_S X$ is a closed embedding. One says for short that X is separated if it is separated over $\operatorname{Spec} \mathbb{Z}$.

Recall that being a closed embedding is a local property on the target. Translating this to the case of $\Delta_{X/S}$, a morphism $f\colon X \to S$ is separated if and only if for some open cover $\{S_i\}$ of S it holds that all the restrictions $f|_{f^{-1}S_i}$ are separated.

In fact, since $\Delta_{X/S}$ is a locally closed embedding, it suffices to check that the image $\Delta_{X/S}(X)$ is a closed subset of $X \times_S X$. In particular, this means that being separated is a condition that only involves the underlying topological part of the map $f\colon X \to S$.

Example 11.6. Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is separated, by Proposition 11.3. More generally, any affine morphism $f : X \to Y$ is separated.

Example 11.7 (Monomorphisms)**.** Recall that a morphism $f\colon X \to Y$ is called a *monomorphism* if it satisfies the following property: if $g_i\colon T \to X$ for $i = 1$, are morphisms such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. For monomorphisms, the fibre product $X \times_Y X$ is in fact equal to the diagonal; that is $\Delta_{X/Y} = X \times_Y X$. Indeed, one readily verifies that the square

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

is Cartesian. So, monomorphisms are separated.

Uniqueness of limits

A very useful property that separated schemes have, and which we referred to in the introduction as 'uniqueness of limits', is that morphisms into separated schemes are determined on open dense subschemes, at least when the source is reduced:

Proposition 11.8 (Uniquness of limits). Let X and Y be two schemes over S and let $f, g\colon X \to Y$ with be two morphisms over S. Assume that

- X is reduced
- Y is separated over S

Then if there is a dense open subscheme $U \subset X$ such that $f|_U = g|_U$, then $f = g$.

Proof We may assume that X is affine, say $X = \operatorname{Spec} A$. The two maps f and g gives a morphism $H\colon X \to Y \times_S Y$. We want to show that H factors through the diagonal $Y \to Y \times_S Y$; then $f = g$.

Taking the pullback of the diagonal $\Delta_{Y/S}$, we obtain the square in the following diagram:

$$\begin{array}{ccccc} & \nearrow & E & \longrightarrow & Y \\ & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta_{Y/S}} \\ U & \xrightarrow{\iota} & X & \xrightarrow{H} & Y \times_S Y \end{array}$$

Here E is the 'equalizer' of the two morphisms, and informally $j(E)$ is the subscheme of points in X where the morphisms are equal (see Exercise **??**). Now, pullbacks of closed embeddings are closed embeddings, hence the image $j(E)$ is closed, and by Proposition 5.10 on page 71, it is isomorphic to a subscheme of the form $\operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a}$. On the other hand, saying that $f|_U = g|_U$ means that there is a lift $U \to E$ of ι, and hence the image $j(E)$ contains the dense set U and therefore is equal to X. Thus $\mathfrak{a}$ is contained in the nilradical of A, which is zero as A is reduced. Consequently, j is an isomorphism, H factors through the diagonal and it follows that $f = g$. □

As examples shortly will show, the above proposition fails when X is not separated.

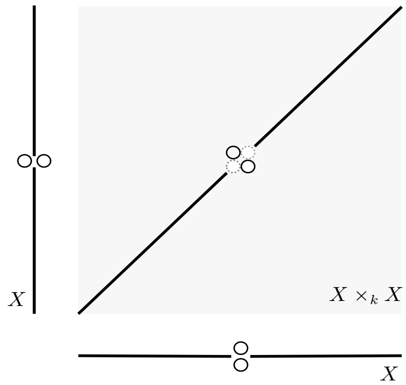
Example 11.9. Likewise, it may fail when the scheme Y is not reduced. One example can be $Y = \operatorname{Spec} k[x, y]/(y^2, xy)$ with the two maps $f_j : Y \to \operatorname{Spec} k[u]$, $j = 1, 2$ defined by $u \mapsto x$ and $u \mapsto x + y$ respectively. These agree over the distinguished open set $D(x)$, but they are different.

Example 11.10. The affine line X with two origins constructed in Section 7.3 on page 95 is not separated over $\operatorname{Spec} k$. Recall that X was constructed as the union of two copies of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[u]$ glued together along their common open subset $\operatorname{Spec} k[u, u^{-1}]$. We let $g_i : \mathbb{A}^1_k \to X$ be the two open embeddings that result from the gluing. The scheme X has two 'origins'; the images 0_1 and 0_2 of the origin $0 \in \mathbb{A}^1_k$ under respectively g_1 and g_2.

Already now, Proposition 11.8 tells that X is not separated; we have two different maps agreeing on an open dense set; but it is instructive to understand the diagonal a bit more.

In the product there are four 'origins', the images $0_i \times 0_j$ of 0 under the four maps $g_{ij} : \mathbb{A}^1_k \to X \times_k X$ with components g_i and g_j. Over the complement of the origin, these maps coincide and equal the diagonal map.

According to Lemma 11.2, only $0_1 \times 0_1$ and $0_2 \times 0_2$ lie on the diagonal. But all four lie in the closure of the diagonal: consider $0_1 \times 0_2$, for instance, which lies in the image of the map g_{12}. If V is an open subset containing $0_1 \times 0_2$, the inverse image $g_{12}^{-1} V$ will be a non-empty open, and hence meets $\mathbb{A}^1_k - \{0\}$. But then V meets $g_{12}(\mathbb{A}^1_k - \{0\})$, which is open in the diagonal.



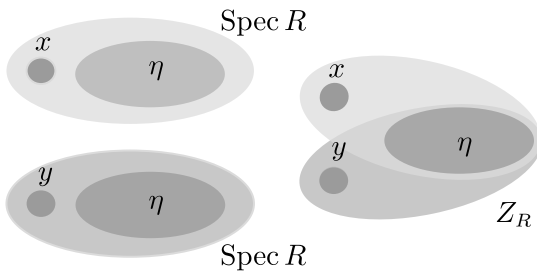
Heuristically, the maps g_{ij} are equal on $\mathbb{A}^1 - \{0\}$, but they bridge the gap at 0 differently, namely by passing over different points $0_i \times 0_j$; thus all four lie in the closure. The diagonal bridges the gap by passing over $0_1 \times 0_1$ and $0_2 \times 0_2$, but avoids the two others.

Example 11.11. An even more basic example of a scheme that is not separated is obtained by gluing the prime spectrum of a discrete valuation ring to itself along the generic point.

To give more details, let R be a DVR with fraction field K. Then $\operatorname{Spec} R = \{x, \eta\}$ where x is the closed point and η is the generic and open point. Citing the gluing lemma for schemes (Proposition 6.3 on page 86), we may glue two copies of $\operatorname{Spec} R$ together by identifying the generic points; that is, the open subschemes $\operatorname{Spec} K$ in the two copies.

In this manner we construct a scheme Z_R together with two open embeddings $g_i\colon \operatorname{Spec} R \to Z_R$. They send the generic point η to the same point, which is an open point in Z_R, but they differ on the closed point x. It follows Z_R is not separated; the principle of uniqueness of limits is violated.

The similar-looking examples of Examples 7.3 and 7.4 are separated however, because they are affine.



11.2 Properties of separated schemes

We introduce separatedness mostly because they give good formal properties. In some sense the schemes category is still a little bit ‘too large’, and separated schemes have properties that make them closer to varieties. In this section we survey a few of these properties.

Proposition 11.12. The following hold true:

(i) (Embeddings) Locally closed embeddings are separated, in particular open and closed embeddings are;

(ii) (Composition) Let $f\colon T \to S$ and $g\colon X \to T$ be morphisms. If both f and g are separated, the composition $g \circ f$ is separated as well. If X is separated over S, it is separated over T;

(iii) (Base change) Being separated is a property stable under base change: if $f\colon X \to S$ is separated and $T \to S$ is any morphism, then $f_T\colon X \times_S T \to T$ is separated;

Proof To prove (i), notice that both open and closed embeddings are monomorphisms, hence they are separated (Example 11.7). A locally closed embedding is the composition of an open and and closed embedding, and so (i) follows from (ii).

Proof of (ii): let the two separated morphisms be $f\colon X \to T$ and $g\colon T \to S$. The point is that the following diagram is Cartesian:

$$\begin{array}{ccc} X \times_T X & \xrightarrow{h} & X \times_S X \\ \downarrow & & \downarrow{\scriptstyle f\times f} \\ T & \xrightarrow{\Delta_{T/S}} & T \times_S T, \end{array} \tag{11.1}$$

where h is the canonical map being the identity on both components. This is straightforward and left to the reader (Exercise 11.3.6).

Note that $\Delta_{X/S} = h \circ \Delta_{X/T}$. Assume first that $T \to S$ is separated, then $\Delta_{T/S}$ is a closed embedding, and h will also be one as being a closed embedding and is stable under pullbacks (Proposition 10.16 on page 163). It follows that $\Delta_{X/S} = h \circ \Delta_{X/T}$ is a closed embedding (composition of closed embeddings are closed embeddings), and so X is separated over S. For the second part of the statement, assume that X is separated over S. Then the composition $h \circ \Delta_{X/T}$, being equal to $\Delta_{X/S}$, is a closed embedding, hence $\Delta_{X/T}$ is a closed embedding as well, according to Exercise 11.3.10.

When proving statement (iii), it suffices to cite Exercise 11.3.7 on page 182, that diagonals pull back to diagonals, and again Proposition 10.16, that pullbacks of closed embeddings are closed embeddings.

□

Intersection of affines

Proposition 11.13. Assume that X is a separated scheme over an affine scheme $S = \operatorname{Spec} A$, and assume that U and V are two affine open subscheme of X. Then the intersection $U \cap V$ is also affine, and the natural multiplication map

$$\mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$$

is surjective.

Proof The product $U \times_S V$ is an open and affine subset of $X \times_S X$, and $U \cap V = \Delta_{X/S}(X) \cap (U \times_S V)$. So if the diagonal is closed, $U \cap V$ is a closed subscheme of the affine scheme $U \times_S V$, hence affine (Proposition 5.10). By the construction of the fibre product of affine schemes one has

$$\Gamma(U \times_S V, \mathcal{O}_{U \times_S V}) = \Gamma(U, \mathcal{O}_U) \otimes_A \Gamma(V, \mathcal{O}_V),$$

and as $U \cap V$ is a closed subscheme of $U \times_S V$, the restriction map

$$\Gamma(U \times_S V, \mathcal{O}_{U \times_S V}) \to \Gamma(U \cap V, \mathcal{O}_{U \cap V})$$

is surjective, as we wanted to show. □

Conversely, we have

Proposition 11.14. Let X be a scheme over $\operatorname{Spec} A$, and let $\{U_i\}_{i \in I}$ be an open affine cover of X such that
(i) all intersections $U_i \cap U_j$ are affine;
(ii) $\mathcal{O}_X(U_i) \otimes_A \mathcal{O}_X(U_j) \longrightarrow \mathcal{O}_X(U_i \cap U_j)$ is surjective for each $i, j \in I$.
Then X is separated over S.

Proof Let $p_1, p_2 \colon X \times_S X \to X$ be the two projections and let $\Delta \colon X \to X \times_S X$ denote the diagonal morphism $\Delta_{X/S}$. Let $U_i = \operatorname{Spec} B_i$ and $U_j = \operatorname{Spec} B_j$ be two open subschemes belonging to the cover $\{U_i\}$. We have

$$\Delta^{-1}(p_1^{-1}(U_i) \cap p_2^{-1}(U_j)) = \Delta^{-1}(p_1^{-1}(U_i)) \cap \Delta^{-1}(p_2^{-1}(U_j)) = U_i \cap U_j, \qquad (11.2)$$

Also, from the universal property of the fibre product it follows that $p_1^{-1}(U_i) \cap p^{-1}(U_j) = U_i \times_S U_j \subset X \times_S X$, and from this we deduce that Δ is a closed embedding if each restriction

$$\Delta_{ij} \colon U_i \cap U_j \to U_i \times_S U_j$$

of Δ is a closed embedding. But this follows from the assumptions: by (i) the intersection $U_i \cap U_j$ is affine, say $U_i \cap U_j = \operatorname{Spec} C_{ij}$, and by (ii) the ring homomorphism $B_i \otimes_A B_j \to C_{ij}$ is surjective. Hence Δ_{ij} is a closed embedding for each i, j, and the proof is complete. □

Example 11.15. The above provides us with a convenient criterion to check that a scheme is separated, given an affine covering. For instance, let us show that the projective line $\mathbb{P}^1_k$ is separated. $\mathbb{P}^1_k$ is covered by the two affine subsets $U_1 = \operatorname{Spec} k[x]$ and $U_2 = \operatorname{Spec} k[x^{-1}]$, which have affine intersection $\operatorname{Spec} k[x, x^{-1}]$. To conclude, we need only check that the map

$$k[x] \otimes_k k[x^{-1}] \to k[x, x^{-1}]$$

is surjective, and it is.

Example 11.16 ($\operatorname{Proj} R$ is separated)**.** More generally, for each graded ring R it holds that $\operatorname{Proj} R$ is separated. Indeed, $\operatorname{Proj} R$ is covered by the affine open sets $D_+(f)$ where f runs over the homogeneous elements of R^+. These open sets are clearly affine (Proposition 9.12), and so is their intersection: $D_+(f) \cap D_+(g) = D_+(fg)$. Thus to prove that $\operatorname{Proj} R$

is separated, we need only check condition (ii) of Proposition 11.14 above, namely that $(R_f)_0 \otimes (R_g)_0 \to (R_{fg})_0$ is surjective for any $f, g \in R^+$, which it is.

Example 11.17. Here is a non-separated scheme where two affine open subsets have non-affine intersection. We glue two copies of the affine plane $\mathbb{A}_k^2$ together along the complement $U_{12} = \mathbb{A}_k^2 - V(x, y)$ of the origin. If U_1 and U_2 denote the two open embeddings of the affine plane, then $U_1 \cap U_2 = U_{12}$, but the open set U_{12} is not affine (see the example in Section 5.6 on page 69). In this example, the multiplication map in the proposition coincides with $k[x, y] \otimes k[x, y] \to \Gamma(U_{12}, \mathcal{O}_{U_{12}})$, which *is* surjective.

Examples of diagonals

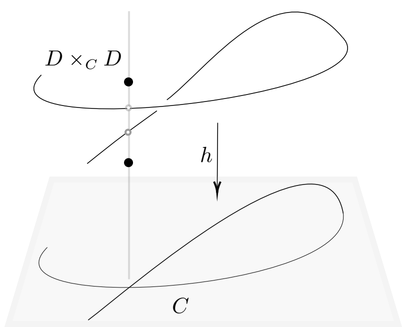
When the fibres of a morphism $f: X \to Y$ vary in regular and uniform way, the fibre product $X \times_Y X$ has a regular behaviour. For instance, if $f: L_m \to \mathbb{P}_k^1$ is one of the line bundles from Section 7.7, the product $L_m \times_{\mathbb{P}_k^1} L_m$ will be what one might call a 'plane bundle', all its fibres are affine planes $\mathbb{A}_k^2$, and the diagonal is just L_m, with each fibre sitting diagonally in each $\mathbb{A}_k^2$-fibre.

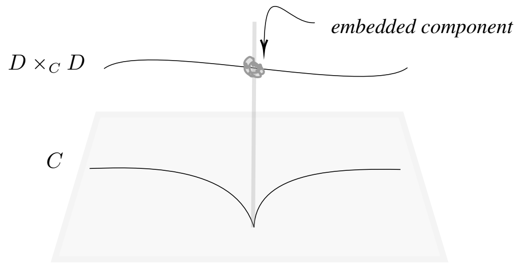
When the morphism has irregular fibres however, the product $X \times_Y X$ also show irregular behaviour. We shall illustrate this by a few examples. They are all birational; i.e. f is an isomorphism on an open dense subscheme $U \subset Y$.

Quite generally, over any open $U \subset Y$ the fibre product $U \times_Y U$ is an open subset of $X \times_Y X$, and when $f|_{f^{-1}U}$ is an isomorphism, the projection $f^{-1}U \times_Y f^{-1}U \to U$ will be an isomorphism, and U will be an dense open subscheme of the diagonal Δ in $X \times_Y X$. In each of these cases, which is typical for birational morphisms, the diagonal will be an irreducible component of the product. In the examples h will denote the canonical map $h: X \times_Y X \to Y$, and k will be an algebraically closed field.

Example 11.18. Let $f: X \to \mathbb{A}_k^2$ be the blow-up of a point p. The fibre of f over p is a projective line, and by Example **??**, the fibre of h over p will then be $\mathbb{P}_k^1 \times_k \mathbb{P}_k^1$. By the transitivity of pull backs, this intersects diagonal in the fibre of the diagonal over p (even scheme theoretical), and in the identification of the diagonal with X, this corresponds to the exceptional divisor.

So $X \times_{\mathbb{A}_k^2} X$ has two components, the diagonal X and a copy of $\mathbb{P}^1 \times \mathbb{P}_k^1$, and they meet along a $\mathbb{P}_k^1$ which is the exceptional divisor in one component and the diagonal in the other.





Example 11.19. Consider the *nodal cubic curve* C given in the affine plane $\mathbb{A}^2_k$ by the equation $v^2 = u^3$. The nodal cubic may be parameterized by the map $f\colon D = \operatorname{Spec} k[t] \to C$ identifying C as $\operatorname{Spec} k[t^2 - 1, t(t^2 - 1)]$. We claim that the fibre product $D \times_C D$ is the disjoint union of the diagonal D and two closed isolated points lying over the origin.

Away from the fibre $h^{-1}(0)$ the canonical map h is an isomorphism since f restricts to an isomorphism $D - h^{-1}(0) \simeq C - \{0\}$, and this shows that the diagonal D is an irreducible component of $D \times_C D$.

The fibre D_0 of f over the origin is given as

$$D_0 = \operatorname{Spec} k[t]/(t^2 - 1, t(t^2 - 1)) = \operatorname{Spec} k[t]/(t^2 - 1),$$

and it decomposes as the disjoint union $x_1 \cup x_2$ where each $x_i = \operatorname{Spec} k$. According to Example **??** the fibre of h over 0 then consists of the four points $x_i \times x_j$ with $1 \leqslant i, j \leqslant 2$, each being a copy of $\operatorname{Spec} k$.

Two of these ($x_1 \times x_1$ and $x_2 \times x_2$) are absorbed in the diagonal, but the others must be isolated point in the product, indeed, they are closed, as the fibre is closed, and their union is the complement of the diagonal, which is closed.

Example 11.20. Next, consider the *cuspidal cubic curve* $C = \operatorname{Spec} k[t^2, t^3]$. It is parameterized by $D = \operatorname{Spec} k[t]$, the map $f\colon D \to C$ being induced by the inclusion $k[t^2, t^3] \subset k[t]$. This is a homeomorphism, and away from the origin 0 it is an isomorphism. The scheme theoretic fibre over the origin equals $D_0 = \operatorname{Spec} k[t]/(t^2)$.

The closed points of $D \times_C D$ are equal to $D(k) \times_{C(k)} D(k) = D(k)$ since f is bijective, and so h is bijective as well, and set-theoretic it equals the diagonal. since f is an isomorphism away from $f^{-1}(0)$, the map h will be an isomorphism away from $h^{-1}(0)$. However, the fibre over 0 is large:

$$(D \times_C D)_0 = D_0 \times_k D_0 = \operatorname{Spec} k[t](t^2) \otimes_{k[t]} k[u]/(u^2) = \operatorname{Spec} k[t, u]/(t^2, u^2).$$

The algebra $k[u, t]/(t^2, u^2)$ is of length four, twice the length of the fibre of f, so something is going on at the origin: the product $D \times_C D$ has an embedded component there. You will find further details in Execerise 11.3.14.

11.3 Exercises

Exercise 11.3.1. In the setting of the proof of Proposition 11.4, show that $\Delta_{X/S}|_{U_i} = \Delta_{U_i/S}$.

Exercise 11.3.2. Let $X = \operatorname{Spec} \mathbb{C}$ and $S = \operatorname{Spec} \mathbb{R}$. Recall that the product $X \times_S X$

consists of two (closed) points. Which one is the diagonal? Can you find another $\mathbb{R}$-algebra A so that if $Y = \operatorname{Spec} A$ it holds that $Y \times_S Y \simeq X \times_S X$ and the diagonal is the other point?

Exercise 11.3.3. Recall that a morphism $\phi\colon X \to Y$ is said to be *affine* if for some cover $\{U_i\}$ of Y of open affine sets, the inverse images $\phi^{-1}(U_i)$ are affine (Definition 8.20 on page 122). Show that affine morphisms are separated.

Exercise 11.3.4. Show that if a scheme X is separated (over $\mathbb{Z}$), then for every scheme Y and every morphism $f\colon X \to Y$, the morphism f is separated.

Exercise 11.3.5. Let X and Y be schemes separeted over a scheme S. Show that their product $X \times_S Y$ is separated over S.

Exercise 11.3.6. Let $T \to S$ be a morphism and let X and Y be two schemes over T. Show that there is a Cartesian diagram

$$\begin{array}{ccc} X \times_T X & \xrightarrow{\iota} & X \times_S X \\ \downarrow & & \downarrow{\scriptstyle f \times f} \\ T & \xrightarrow{\Delta_{T/S}} & T \times_S T, \end{array}$$

and conclude that the natural map $\iota\colon X \times_T Y \to X \times_S Y$ is a locally closed embedding. Hint: Use the functor of points to reduce to a statement of sets.

Exercise 11.3.7 (Pullback of diagonals)**.** Let $X \to S$ and $T \to S$ be morphisms between schemes, and as usual, let $X_T = X \times_S T$. Show that the diagonal $\Delta_{X/S}$ pulls back to the diagonal $\Delta_{X_T/T}$; in other words, that there is a canonical Cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{\Delta_{X_T/T}} & X_T \times_T X_T \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

Exercise 11.3.8. Let X/S be a scheme and let $\iota : W \to X$ be a an open subscheme or a closed subscheme (over S). Show that the diagram below is Cartesian

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \downarrow{\scriptstyle \Delta_{W/S}} & & \downarrow{\scriptstyle \Delta_{X/S}} \\ W \times_S W & \longrightarrow & X \times_S X \end{array}$$

Conclude that W/S is separated if X/S is.

Exercise 11.3.9 (The graph of a morphism)**.** A morphism $\phi\colon X \to Y$ over S has a *graph* $\Gamma_\phi\colon X \to X \times_S Y$; it is the pullback of the diagonal $\Delta_{Y/S}$ under the morphism $\phi \times \mathrm{id}_Y : X \times Y \to Y \times_S Y$. Show that the graph is a closed embedding when Y is separated.

Exercise 11.3.10 (Closed embeddings)**.** Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes.

a) Assume that g is separated. Show that if the composition $g \circ f$ is a closed embedding, then f is a closed embedding. HINT: Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y & \longrightarrow & Y \\ & & \big\downarrow & & \big\downarrow{\scriptstyle g} \\ & & X & \xrightarrow[g \circ f]{} & Z \end{array}$$

where the square is Cartesian and Γ_f is the graph of f.

b) Show by an example that in general f is not necessarily a closed embedding even if $g \circ f$ is. HINT: For one of the copies of $\mathbb{A}^1$, say U_1, in the affine line X with two origins constructed on page 95 in Chapter **??**, exhibit a morphism $X \to \mathbb{A}^1$ that restricts to the identity on U_1.

Exercise 11.3.11. Let R and S be two DVR's with the same fraction field, and denote by $\mathfrak{m}_R$ and $\mathfrak{m}_S$ the two maximal ideals. Assume that R and S different in the sense that $\mathfrak{m}_R \cap S \nsubseteq \mathfrak{m}_S$ and $\mathfrak{m}_S \cap R \nsubseteq \mathfrak{m}_R$. Let Z be the scheme obtained by gluing $\operatorname{Spec} R$ and $\operatorname{Spec} S$ together along the generic points. Show that Z is affine, more precisely, show that Z is isomorphic to $\operatorname{Spec}(R \cap S)$.

Exercise 11.3.12 (Equalizers)**.** Let X and Y be schemes over S and f_1 and f_2 two morphisms from Y to X. Let $f\colon Y \to X \times_S X$ be the morphism whose components are the f_i's; that is, $f_i = \pi_i \circ f$ (as usual, the π_i's are the two projections). The pullback $f^{-1}\Delta_{X/S}$ is called the *equalizer* of the f_i's, and we shall denote it by $\eta\colon E \to Y$. In other words, the diagram below is Cartesian:

$$\begin{array}{ccc} E & \xrightarrow{\eta} & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

a) Show that a morphism $g\colon Z \to Y$ satisfies $f_1 \circ g = f_2 \circ g$ if and only if g factors via η;

b) Show that X is separated if and only if all equalizers of maps into$X$ are closed.

Exercise 11.3.13. Let A be a B-algebra. Show that the kernel of the multiplication map $\mu\colon A \otimes_B A \to A$ is generated by the elements of the form $a \otimes 1 - 1 \otimes a$. HINT: $\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1 - 1 \otimes a_i) \cdot 1 \otimes b_i + \sum_i 1 \otimes a_i b_i$.

Exercise 11.3.14. This exercise connects up with Example 11.20 and explains the embedded component appearing the $D \times_C D$ in the product of the normalisation D of the cuspidal plane cubic C over C. Consider the ring $A = k[t] \otimes_{k[t^2,t^3]} k[t]$

a) Show that $\alpha = t \otimes 1 - 1 \otimes t$ is nilpotent and generates a prime ideal; HINT: a generates the kernel of the multiplication map. Compute α^3 by the binomial theorem.

b) show that $\mathfrak{m} = (t \otimes 1, 1 \otimes t)$ is a maximal ideal; HINT: Consider the image of $\mathfrak{m}$ under the multiplication map.

c) Show that the ideal generated by $t \otimes t$ is $\mathfrak{m}$-primary; HINT: consider $\mathfrak{m}^3$.

d) Show that $(0) = (t \otimes 1 - 1 \otimes t) \cap (t \otimes t)$ is a primary decomposition of the zero-ideal (0). HINT: All $t^i \otimes t^j$ with either $i \geqslant 2$ or $j \geqslant 2$ kill α.

Exercise 11.3.15. With reference to Example 11.18, check by hand that $X \times_{\mathbb{A}^2_k} X$ has two components by covering X with two affine opens.

Exercise 11.3.16. The aim of this exercise is to show that infinite products $\prod_{n=1}^{\infty} X_i$ may fail to exist in the category of schemes. That is, there is no scheme that has the universal property of the product for schemes.

a) Show that if X and Y are schemes, the set of points of Y where two morphisms $Y \to X$ agree is a locally closed subset of Y.
b) Let Z denote the affine line with the doubled origin. Suppose that $\prod_{n=1}^{\infty} Z$ is represented by a scheme X. Let $Y = \operatorname{Spec} A$ be an affine scheme. Show that every countable intersection of distinguished open sets of Y occurs as the locus where two maps $Y \to X$ agree. Show that this gives a contradiction, e.g., for $Y = \operatorname{Spec} \mathbb{Z}$, so that X is not a scheme.

12

Algebraic varieties

In the introductory chapter we gave a temporary and restricted definition of a variety, and there we only spoke about either affine or projective varieties. With the theory of schemes sufficiently developed, we are now ready for the full truth

Definition 12.1. A variety over a field k is an integral, separated scheme of finite type over k.

The literature sees a varying terminology at this point. Some authors do not require varieties to be irreducible (but they are always reduced), and many require the base field to be algebraically closed. It is also convenient to accept the empty scheme as a variety (over any field k).

Example 12.2. The schemes

$$\mathbb{A}^1_{\overline{\mathbb{Q}}} = \operatorname{Spec} \overline{\mathbb{Q}}[t], \qquad \operatorname{Spec} \mathbb{C}[x,y]/(x^2-y^3), \qquad \operatorname{Spec} \overline{\mathbb{F}}_p[x,y,z]/(x^2-yz),$$

are affine varieties, whereas the following schemes are not:

$$\operatorname{Spec} \overline{\mathbb{Q}}[t]/t^2, \qquad \operatorname{Spec} \mathbb{C}[x,y]/(xy), \qquad \operatorname{Spec} \mathbb{Z}.$$

In the introductory chapter we did not introduce maps between varieties (we did it for affine varieties, but not for projective). Now, quite naturally, a map between two varieties is a morphism between the schemes. In this way, the varieties constitute a full subcategory Var/k of the category Sch/k of k-schemes.

Subvarieties

The notion of subschemes has a counterpart in the notion of subvarieties:

Proposition 12.3 (Subvarieties). Let X be a variety over the field k.

(i) (Open subvarieties) Every open subscheme $U \subset X$ is a variety;
(ii) (Closed subvarieties) Every closed, integral subscheme $Y \subset X$ is a variety;
(iii) Every closed irreducible subset $Y \subset X$ has a unique structure as closed subvariety.

Proof Open subschemes of integral schemes are integral by Proposition 5.22 on page 76, and open embeddings are separated (Proposition 11.12 on page 178), so U is integral and separated. Finally, U is of finite type over k; indeed, U is covered by finitely many open affine subschemes since X is, and we conclude by Corollary 8.19.

For the second statement, according to Example 8.16 the subscheme Y is locally of finite type, and since X is quasi compact, it is of finite type. By hypothesis, it is integral, and it is separated by Proposition 11.12.

As to the last claim, each closed subset carries a unique reduced scheme structure, which is integral when the subset is irreducible. The rest follows from (ii). □

Example 12.4 (Affine varieties)**.** Prime spectra $\operatorname{Spec} A$ of integral algebras of finite type over k are varieties since all prime spectra are separated (Proposition 11.3 on page 174). In particular the affine spaces $\mathbb{A}^n_k = \operatorname{Spec} k[t_1, \ldots, t_n]$ will all be varieties.

An *affine variety* is a variety which is isomorphic to a prime spectrum, and by Corollary 8.19 these are precisely the varieties that are affine schemes. The affine varieties form a full subcategory AffVar/k of Sch/k, and the relative version of The Main Theorem for Affine Schemes (Theorem 5.2 on page 68) yields that AffVar/k is equivalent to the opposite of the category of integral domains finitely generated over k.

When k is algebraically closed, the category of 'old style varieties' and polynomial maps is equivalent to AffVar/k with the functor $Z \mapsto \operatorname{Spec} A(Z)$ being an equivalence. Note that, except when Z is a point, $\operatorname{Spec} A(Z)$ is much larger than the 'old style variety' Z. It contains all prime ideals of $A(Z)$ and not only the maximal ones. In a way, $\operatorname{Spec} A(Z)$ carries information about all subvarieties of Z.

Example 12.5 (Projective varieties)**.** The projective spectrum $\operatorname{Proj} R$ of a graded integral domain R with $R_0 = k$ which is finitely generated over k, is a variety. Proposition **??** tells us that $\operatorname{Proj} R$ is of finite type over k, it is integral by 9.18 and separated by Example **??**. In particular, the projective spaces $\mathbb{P}^n_k$ are varieties.

From 12.3 above, it follows that each closed integral subscheme $Z \subset \mathbb{P}^n_k$ is a variety. Such varieties are called *projective varieties*. One also has the notions of *quasi projective varieties* and *quasi affine varieties*, which are varieties that are isomorphic to open subvarieties of either projective or affine varieties.

12.1 Noether's Normalization Lemma

We now turn to one of the key results in the theory of varieties, the Normalization Lemma of Emmy Noether. It relates the dimension of an affine variety X to the transcendence degree r of its function field over the base field, and in some sense it is the closest one comes to having global coordinates on affine varieties. In geometric terms it states that projection of a closed subvariety $X \subset \mathbb{A}^n_k$ onto a general linear subspace $\mathbb{A}^r_k$ of $\mathbb{A}^n_k$ is a *finite* morphism.

Transcendence degree

The notion of 'transcendence degree' of a field extension $k \subset K$ plays a central role, so we begin with quickly recalling a few facts. Elements $a_1, \ldots, a_r$ from K are said to *algebraically independent* if for every polynomial $P(t_1, \ldots, t_r)$ with coefficients from k it holds that $P(a_1, \ldots, a_r) \neq 0$; or in other words, that sending indeterminates t_i to a_i yields a k-algebra isomorphism $k[t_1, \ldots, t_r] \simeq k[a_1, \ldots, a_r]$. Likewise, one says that a possibly infinite subset $S \subset K$ is *algebraically independent* when every finite collection of distinct members of S are algebraically independent.

A *transcendence basis for K over k* is a maximal algebraically independent set $S \subset K$. For independent element $a_1, \ldots, a_n$ to form a transcendence basis it is necessary and sufficient that the field extension $k(a_1, \ldots, a_n) \subset K$ is algebraic, and one may prove that all transcendence bases have the same cardinality. This common cardinality is called the *transcendence degree of K over k* and is denoted by $\operatorname{trdeg}_k K$. In general, the transcendence degree may be infinite, but for finitely generated field extensions it will be always finite. Note that if $A \subset B$ is an extension of domains with B of finite type over A, then the associated extension of fraction fields will be a finitely generated field extension with a finite transcendence degree.

The Normalization Lemma

With minor modifications, the standard proof of the classical version of the Normalization Lemma yields a somewhat more general result:

Theorem 12.6. Let $A \subset B$ be two domains with B of finite type over A and let n be transcendence degree of the quotient field $K(B)$ over $K(A)$. Then there are elements $x_1, \ldots, x_n$ in B which are algebraically independent over A and an element $f \in A$ such that B_f is a finite module over $A_f[x_1, \ldots, x_n]$.

When A is a field, the localization is unnecessary, and the classical Normalization Lemma ensues.

Corollary 12.7 (Noether's Normalization Lemma). Let k be a field and let B be a domain of finite type over k and denote by n the transcendence degree of $K(B)$ over k. Then there are algebraically independent elements $x_1, \ldots, x_n$ in B such that B is a finite module over $k[x_1, \ldots, x_n]$.

The proof of the theorem goes by induction on the number of generators that A requires. The inductive step hinges on the following lemma of purely algebraic content:

Lemma 12.8. Let $p(t_1, \ldots, t_m)$ be a polynomial over a domain A, and let $u_2, \ldots, u_m$ be new variables. Then for s a sufficiently large integer, the leading coefficient of $p(t_1, u_2 + t_1^s, \ldots, u_m + t_1^{s^{m-1}})$ as a polynomial in t_1 will be a non-zero element of A.

Proof The substitutions $t_i = u_i + t_1^{s^{i-1}}$ for $i \geqslant 2$ in a monomial $t_1^{\alpha_1} \ldots t_m^{\alpha_m}$ result in a polynomial in t_1 whose leading term is of degree $\alpha_1 + \alpha_2 s + \cdots + \alpha_m s^{m-1}$, and whose leading coefficient is one.

Now, the crucial point is that for $s >> 0$, the expressions $\alpha_1 + \alpha_2 s + \cdots + \alpha_m s^{m-1}$ will all be different, so the term of highest degree in t_1 appears only once when one develops

$p(t_1, u_2 + t_1^s, \ldots, u_m + t_1^{s^{m-1}})$ in powers of t_1, and hence it is not cancelled. Indeed, for any pair of distinct monomials an equality

$$\alpha_1 + \alpha_2 s + \cdots + \alpha_m s^{m-1} = \alpha_1' + \alpha_2' s + \cdots + \alpha_m' s^{m-1}$$

holds only for finitely many s since non-zero polynomials merely have finitely many zeros. And as there are only finitely many pairs of monomials terms in f, we are through. □

Proof of the theorem Choose generators $w_1, \ldots, w_m$ for B as an algebra over A. Then $K(B)$ is generated as a field over $K(A)$ by the w_i's as well. It follows that $n \leqslant m$, and in case of equality, that $w_1, \ldots, w_m$ are algebraically independent over A. Hence in that case $B = A[w_1, \ldots, w_m]$ is a polynomial ring, and the induction can start.

If $n < m$, there is a non-zero polynomial $p(t_1, \ldots, t_m)$ with coefficients in A such that $p(w_1, \ldots, w_m) = 0$. We introduce new variables $t_i - t_1^{s^{i-1}} = u_i$, where s is a natural number, and set

$$q(t_1, u_2, \ldots, u_m) = p(t_1, u_2 + t_1^s, \ldots, u_m + t_1^{s^{m-1}}).$$

According to the lemma we may chose $s >> 0$ so that the leading coefficient g of q as a polynomial in t_1 lies in A. With $z_i = w_i - w_1^{s^{i-1}}$ for $i \geqslant 2$ it holds true that $q(w_1, z_2, \ldots, z_m) = 0$, and since $g^{-1}q$ is monic in t_1, it ensues that B_g is a finite module over the subalgebra $B' = A_g[z_2, \ldots, z_m]$.

Now, $K(B) = K(B_g)$ is algebraic over $K(B')$ so that the two fields have the same transcendence degree over $K(A) = K(A_g)$. Moreover, by construction, B' is generated by less than m elements over A_g. Induction applies, and there is a $h \in A_g$ and algebraically independent elements $x_1, \ldots, x_n$ so that B_h' is finite over $A_{gh}[x_1, \ldots, x_n]$. Now, g is invertible in B', so that $B_h' = B_{gh}$, and taking $f = gh$ we are done. □

Example 12.9. Consider 'the hyperbola' $X = V(xy - 1) \subset \mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$ and the projection $X \to \mathbb{A}_k^1 = \operatorname{Spec} k[x]$ onto the x-axis, which is induced by the inclusion $k[x] \subset k[x, 1/x]$. The algebra $k[x, 1/x]$ is not finite over of $k[x]$; it requires all powers $1/x$ as generators. However, for any elements a, b of k with $ab \neq 0$, it holds that $k[x, x^{-1}]$ is finite over $k[ax + bx^{-1}]$; indeed, $k[x, 1/x]$ is generated by x over $k[ax + bx^{-1}]$, and x satisfies the equation

$$x^2 - xa^{-1}(ax + bx^{-1}) + ba^{-1} = 0.$$

Turning Theorem 12.6 into geometry, we arrive at the following description of the generic behaviour of a morphism locally of finite type between integral schemes.

Theorem 12.10 (Generic structure of morphisms of locally finite type). Let X and Y be integral schemes and $f\colon X \to Y$ a dominating morphism locally of finite type. Then there are open affine subsets $U \subset Y$ and $V \subset X$ so that $f(V) = U$ and so that $f|_V$ factors as

$$V \xrightarrow{\ g\ } U \times \mathbb{A}^n \xrightarrow{\ p\ } U$$

where g is finite and p is the projection and where $n = \operatorname{trdeg}_{k(Y)} k(X)$.
If X and Y are affine, we may take V to be the inverse image of a distinguished open set.

Note that $\mathbb{A}^n$ is the absolute affine space $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n]$, and the product $U \times \mathbb{A}^n$ is the product over $\mathbb{Z}$. If X, Y and f are defined over a ring R, the product may be replaced by (and in fact, coincides with) the product $U \times_R \mathbb{A}^n_R$.

Proof Choose two open affine subschemes $\operatorname{Spec} A \subset Y$ and $\operatorname{Spec} B \subset X$ such that the inclusion $f(\operatorname{Spec} B) \subset \operatorname{Spec} A$ holds true. According to Proposition 8.18, the A-algebra B will of finite type. Moreover, since f is dominating, it holds that $f^\sharp\colon A \to B$ is injective, and we may as well assume that $A \subset B$. Applying Theorem 12.6 to the extension $A \subset B$, we can find algebraically independent elements $x_1, \ldots, x_n$ and an $g \in A$ such that B_g is finite over $A_g[x_1, \ldots, x_n]$. Let $U = D(g) \subset \operatorname{Spec} A$ and $V = D(g) \subset \operatorname{Spec} B$, and note that $\operatorname{Spec} A_g[x_1, \ldots, x_n] = \operatorname{Spec} A_g \times \mathbb{A}^n$.

Finally, by construction $f(V) \subset U$, and since both g, being finite and dominant (by 'Lying–Over', Proposition 8.27 on page 123), and p are surjective, it ensues that $f(V) = U$. Note that the projection p is surjective since $(U \times \mathbb{A}^n)(k) = U(k) \times \mathbb{A}^n(k)$ and $\mathbb{A}^n(k) \neq \varnothing$ for every field k. □

12.2 The Nullstellensatz

As a first application of the Normalization Lemma, we give short proofs of the versions of Nullstellensatz cited in Chapter 1. The first out is the The Weak Nullstellensatz (Theorem 1.10 on page 6), and the full Nullstellensatz (Theorem 1.9 on page 6) follows suit, after a basic result about the density of closed points of a variety.

Corollary 12.11 (Weak Nullstellensatz). If X is a scheme of locally of finite type over a field k, and $x \in X$ is a closed point, then $k(x)$ is a finite extension of k.

When k is algebraically closed, it follows that $k(x) = k$; in other words, the k-points and the closed points of X coincide. In particular, for $X = \mathbb{A}^n_k$ this is precisely the content of statement (ii) of Theorem 1.10. In case X is the spectrum of a field, the corollary if often called Zariski's Lemma .

Proof The point x is contained in an affine open subscheme $\operatorname{Spec} A$ of X with A of finite type over k. The residue field $k(x)$, being a quotient of A, is of finite type as an algebra over k as well, and the Normalization Lemma implies that $k(x)$ is a finite extension of k. Indeed, it says that $k(x)$ is finite over a polynomial ring A over k. But Going–Up (Exercise 8.2.5

on page 124) then implies that A is a field, and no genuine polynomial ring is field. Hence $A = k$. $\square$

Corollary 12.12 (Density of closed points). Let X and Y be schemes over a field k with X locally of finite type over k.

(i) If $f\colon X \to Y$ is a morphism over k, then $f(x)$ is a closed point for each closed point $x \in X$;

(ii) The closed points in X form a dense subset.

That X be locally of finite type is essential; for instance, the statements fail for local rings. If e.g. A is a domain and $\mathfrak{p} \in \operatorname{Spec} A$ is a prime ideal which is not maximal, the image in $\operatorname{Spec} A$ under the canonical map of the single closed point in $\operatorname{Spec} A_{\mathfrak{p}}$, is equal to the prime $\mathfrak{p}$, which is not closed. Note also that if A is a local domain, but not a field, $\operatorname{Spec} A$ has just one closed point, which is not dense as there are other points.

Proof To prove (i), we may assume that X and Y are affine, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and that B is of finite type over k. The point x corresponds to a maximal ideal $\mathfrak{m}$ in B, and $k(x) = B/\mathfrak{m}$ is a finite extension of k according to the Weak Nullstellensatz. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to $f(x)$; that is, $\mathfrak{p}$ is the preimage of $\mathfrak{m}$ under the map $f^\sharp\colon A \to B$. This map induces an injection $A/\mathfrak{p} \hookrightarrow B/\mathfrak{m} = k(x)$. Now $k(x)$ is integral over k, hence *a fortiori* integral over $A/\mathfrak{p}$, and it follows from Going–Up (Exercise 8.2.5 on page 124) that $B/\mathfrak{p}$ is a field.

Proof of (ii): it suffices to see that each open subset of X contains a closed point. From Corollary 8.18 follows that X, being locally of finite type over k, has a basis consisting of open affines of finite type over k, and each of these have closed points. If an open subscheme $U \subset X$ is of finite type over k and $x \in U$ is closed in U, it ensues from (i) that x is closed in X as well, and we are done. $\square$

The full version of the Nullstellensatz takes the following form in a setting over an arbitrary field.

Corollary 12.13. Let A be an algebra of finite type over a field k, and $\mathfrak{a} \subset A$ an ideal. It then holds that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a}\subset\mathfrak{m}} \mathfrak{m}$, the intersection extending over all maximal ideals containing $\mathfrak{a}$.

To draw the line back to Hilbert's Nullstellensatz as formulated in Chapter 1, assume that k is algebraically closed and let $\mathfrak{a}$ be an ideal in $k[t_1, \dots, t_n]$. To say that a polynomial f vanishes at all k-points in $Z(\mathfrak{a})$, is to say that f lies in all maximal ideals that contain $\mathfrak{a}$, and consequently, by the corollary, it belongs to $\sqrt{\mathfrak{a}}$.

Proof The radical $\sqrt{\mathfrak{a}}$ is equal to the intersection of all prime ideals containing $\mathfrak{a}$, so we may as well assume that $\mathfrak{a}$ is prime. Replacing A by $A/\mathfrak{a}$ it suffices to see that the intersection of all maximal ideals in a domain of finite type over k is reduced to the zero ideal. But by Proposition 2.11 on page 25 this is equivalent to the close points of $\operatorname{Spec} A$ being dense, which holds true according to Corollary 12.12. $\square$

12.3 The dimension of schemes of finite type over a field

In a general setting the definition of the dimension of a scheme in terms of the Krull dimension, suffers from several deficiencies. The most troublesome is that maximal chains of closed integral subschemes situated between two fixed subschemes, are not always of the same length. In particular, the codimension of an integral subscheme Y in X, as the length of a maximal chain ascending from Y, does not always equal the 'intuitive' codimension $\dim X - \dim Y$.

For varieties over a field, however, these occult phenomena does not take place; all maximal chains are of the same length, and the dimension behaves as one expects. One underlying reason is that the dimension of a variety coincides with the transcendence degree of its function field. This follows from Normalization Lemma and Going–Up, once it holds for affine space itself, and so to establish this will be our first task.

Dimension of affine space

The transcendence degree of the function field $k(\mathbb{A}^n_k)$ over k is by definition equal to n, and one might be tempted to take for granted that affine space $\mathbb{A}^n_k$ is of dimension n, but this is in fact slightly subtle. What is is obvious, is that $\dim \mathbb{A}^n_f \geqslant n$ since there are chains of linear subspaces of length n, however, the converse inequality requires some effort.

Lemma 12.14. Let k be a field and n a natural number.
(i) $\dim \mathbb{A}^n_k = n$;
(ii) for each non-constant, irreducible polynomial $f \in k[t_1, \dots, t_n]$, it holds that $\dim V(f) = n - 1$.

Proof The proof goes by induction on n, and the case $n = 1$ is clear. Consider a polynomial f in $A = k[t_1, \dots, t_n]$ which is not constant. As in Lemma 12.8, let $u_i = t_i - t_1^s$ with $s >> 0$. Then

$$f(t_1, \dots, t_n) = f(t_1, u_2 + t_1^s, \dots, u_n + t_1^s)$$

is a monic polynomial in t_1 with coefficients in $B = k[u_2, \dots, u_n] \subset A$, and the u_i's are algebraically independent. By induction $\dim B = n - 1$.

Consider now the algebra $A/(f)A$. The algebra B maps injectively into $A/(f)A$; a polynomial in the kernel depends only on the u_i's, but it also is a multiple of f (which depends on t_1), hence it must vanish. The extension

$$B \subset A/(f)A$$

is integral since $A/(f)A$ is generated over B by the class of t_1, which is integral since f is monic. Going-Up then yields that $\dim A/(f)A = n - 1$.

As to (i), let $0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_r$ be a saturated chain in A, and chose an irreducible polynomial $f \in \mathfrak{p}_1$ (in fact, $\mathfrak{p}_1$ is a principal ideal since A is a UDF). By (ii), it holds that $\dim A/(f) = n - 1$, and so $r - 1 \leqslant n - 1$, and we infer that $\dim \mathbb{A}^n_k \leqslant n$. □

Exercise 12.3.1. Let A be a ring and $\mathfrak{m} \subset A[t]$ a maximal ideal. Let $\mathfrak{m}_0 = A \cap \mathfrak{m}$ and $k = A/\mathfrak{m}_0$.

a) Show that $A[t]/\mathfrak{m}_0 A[t] \simeq k[t]$;
b) Show that if $\mathfrak{m}_0$ is maximal and generated by r elements, then $\mathfrak{m}$ is generated by $r+1$ elements. HINT: $k[t]$ is a principal ideal domain;
c) Show by induction on the number of variables that each maximal ideal in a polynomial ring $k[t_1, \ldots, t_r]$ over a field k is generated by r elements;
d) (Alternative proof that $\dim \mathbb{A}^n_k = n$) Show that if A is an algebra of finite type over a field k, then $\dim A[t] = \dim A + 1$. HINT: Claim (i) of Corollary 12.12 is useful;
e) If X is a variety over k, show that $\dim X \times_k \mathbb{A}^n_k = \dim X + n$.

Dimension and transcendence degree

We have now come to the main result about the dimension of a variety:

Theorem 12.15 (Dimension and transcendence degree). Let X be a variety over the field k.

(i) $\dim X = \operatorname{trdeg}_k k(X)$;
(ii) For each non-empty open subvariety $U \subset X$, it holds that $\dim U = \dim X$;
(iii) If $Y \subset X$ is a closed subvariety, all maximal chain of irreducible subvarieties
$$Y \subset Z_1 \subset \ldots \subset Z_r \subset X$$
have the same length;
(iv) $\operatorname{codim}(Y, X) = \dim X - \dim Y$.

Note that with Y the empty subvariety, claim (iii) says that all maximal chains in X are of the same length. In particular, it holds that $\dim \mathcal{O}_{X,x} = \dim X$ for all closed points $x \in X$.

Proof In view of Lemma 8.30 on page 125, the general case follows from the affine case, so we may assume that X is affine, say $X = \operatorname{Spec} A$. The Normalization Lemma tells us that there is a finite surjective morphism $p\colon X \to \mathbb{A}^n_k$ where $n = \operatorname{trdeg}_k k(X)$. Applying Going–Up (Proposition 8.32 on page 125) and Lemma 12.14, we infer that $\dim X = \dim \mathbb{A}^n_k = n$.

Statement (ii) follows since U has the same function field as X.

To prove (iii), consider a maximal chain $0 \subset \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in A, and chose algebraically independent elements $t_1, \ldots, t_n$ such that A is finite over $B = k[t_1, \ldots, t_n]$. The ideal $\mathfrak{p}_0$ is minimal among the non-zero prime ideals in A, and Going–Down (part (iv) of Theorem A.17 on page 422) ensures that $\mathfrak{q} = \mathfrak{p}_0 \cap B$ is minimal among the non-zero prime ideals in B. Hence $\mathfrak{q} = (f)$ for some $f \in B$, as polynomial rings are UFD's. Now, $B/\mathfrak{q} \subset A/\mathfrak{p}_0$ is an integral extension, and by Lemma 12.14 we have $\dim B/\mathfrak{q} = n-1$. Hence $\dim A/\mathfrak{p}_0 = n-1$ by Going–Up; induction applies, and the chain $\{\mathfrak{p}_i/\mathfrak{p}_0\}$ in $A/\mathfrak{p}_0$ is of length $n-1$, which implies that the original chain has length n.

Finally, claim (iv) is a direct consequence of (iii). □

Example 12.16. The projective space $\mathbb{P}^n_k = \operatorname{Proj} k[t_0, \ldots, t_n]$ is a variety of dimension n. It has open subvarieties isomorphic to affine n-space $\mathbb{A}_k^n$, namely the distinguished subvarieties $D_+(t_i)$.

Example 12.17. The quadric cone $Q = \operatorname{Spec} k[x,y,z]/(x^2 - yz)$ of Example 5.25 on page 76 has dimension 2. This follows directly from (ii) of Lemma 12.14. More generally, for any irreducible non-constant polynomial $f \in k[t_1,\ldots,t_n]$, the closed subvariety $V(f) = \operatorname{Spec} k[t_1,\ldots,t_n]$, where f vanishes, is of dimension $n-1$.

In an analogues manner, an irreducible homogeneous polynomial $f \in k[t_0,\ldots,t_n]$ defines a closed subscheme $Z = \operatorname{Proj} k[t_0,\ldots,t_n]/(f)$ of $\mathbb{P}_k^n$, which is a closed subvariety of dimension $n-1$. Indeed, at least one distinguished open set, say $D(t_i)$, meets Z in a non-empty open subscheme $U_i = D(t_i) \cap Z$, which equals $\operatorname{Spec} k[t_0 t_i^{-1},\ldots,t_n t_i^{-1}]/(F)$, where F is f dehomogenized with respect to t_i; that is, it equals $f(t_0 t_i^{-1},\ldots,t_n t_i^{-1})$ (see Sections 1.3 and 9.2). Hence $\dim U_i = n-1$ and so also $\dim Z = n-1$.

The subvarieties described in this example are respectively called *affine* and *projective hypersurfaces*.

There is a generalization of the notion of 'hypersurfaces' which is meaningful for any scheme X. A subscheme is said to be *locally given by one equation* if one may find an open affine cover $\{U_i\}$ of X and non-zerodivisors $f_i \in \mathcal{O}_X(U_i)$ so that $Z \cap U_i = V(f_i)$.

In the Noetherian case the codimension of Z will be one according to the Hauptidealsatz, for each generic point of Z it holds that $\dim \mathcal{O}_{Z,\eta} = 1$, and one could be tempted to expect that $\dim Z = n-1$. However this is not always true even in the Noetherian case; there are examples of Noetherian domains of any Krull dimension having principal maximal ideals. This pathology, which is due to maximal chains of prime ideals being of varying length, does not occur in the realm of varieties, so for those, intuition concords with reality:

Proposition 12.18. Let X be variety over k and let $Z \subset X$ be a closed subvariety locally defined by one equation. Then $\dim Z = \dim X - 1$.

For schemes which are not integral, but of finite type over k, we still have a good control over the dimension. First of all, the dimension of X is the same as of X_{red}, so we may assume that X is reduced. Then, if $X = \bigcup X_i$ is the decomposition into irreducible components, each X_i is integral, and $\dim X$ is the maximum of all $\dim X_i$.

Example 12.19. Consider $\mathbb{A}_k^3 = \operatorname{Spec} k[x,y,z]$ and $Y = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = (xy - x, x^2, y^2 z - z, y^3 - y, xy^2 - xy) = (z,y,x) \cap (y-1, x^2) \cap (y+1, x).$$

The associated primes of $\mathfrak{a}$ are $\mathfrak{p}_1 = (x, y+1)$, $\mathfrak{p}_2 = (x, y-1)$ and $\mathfrak{p}_3 = (x,y,z)$. So Y has three components: $L = V(x, y+1)$, $M = V(x, y-1)$ (two lines), and $P = V(x,y,z)$ (the origin). The dimension of Y equals the largest of the dimension of each component, and $\dim L = 1, \dim M = 1, \dim P = 0$, so $\dim Y = 1$. The codimension of Y in $\mathbb{A}_k^3$ equals the maximum of the heights of the associated primes of $\mathfrak{a}$; i.e. $\operatorname{ht}(\mathfrak{p}_1) = 2$. So the codimension of Y equals 2.

Dimension of fibres

When investigating a morphism $f\colon X \to Y$, understanding the fibres over closed points is a must, and a first step in that direction is to survey how the dimension of a fibre X_y varies with the closed point y. There are some general principles which we will explain, and which involves the 'relative dimension' $r = \dim X - \dim Y$.

Heuristically, one would believe that the dimension of a fibre should be equal to the relative dimension. However, this is not generally true, but still holds for most fibres. The fibre dimension does not vary arbitrarily, all components of each fibre is of dimension at least the relative dimension, and we begin with with proving this. The argument is based on Krull's Hauptidealsatz combined with the fact that all maximal ideals in $k[t_1, \ldots, t_n]$ are generated by n elements.

Proposition 12.20. Let $f\colon X \to Y$ be a dominant morphism between varieties over a field k. For every closed point $y \in Y$ in the image of f and every irreducible component Z of the fibre X_y, it holds that $\dim Z \geqslant \dim X - \dim Y$.

Proof Replacing Y by some open affine neighbourhood U of y and X by some open affine subscheme that meets Z and maps into U, we may assume that X and Y both are affine; say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$.

We first treat the essential case that $Y = \mathbb{A}^n_k$. So, let $\mathfrak{m}$ be the maximal ideal in the polynomial ring $k[t_1, \ldots, t_n]$ that corresponds to y. It is generated by n elements $g_1, \ldots, g_n$. Consequently, the fibre X_y is given as

$$X_y = \operatorname{Spec} B/\mathfrak{m}B = \operatorname{Spec} B/(g_1, \ldots, g_n),$$

and the actual component Z of the fibre X_y equals $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ minimal over $(g_1, \ldots, g_n)$. Citing the Hauptidealsatz, we infer that $\operatorname{codim}(Z, X) = \dim B_{\mathfrak{p}} \leqslant n$. Hence by (iv) of 12.15 we conclude that $\dim X - \dim Z \leqslant n = \dim \mathbb{A}^n_k$; or on other words, that $\dim Z \geqslant \dim X - \dim \mathbb{A}^n_k$.

Attacking the general case, we appeal to the Normalization Lemma to find a finite and dominant morphism $p\colon Y \to \mathbb{A}^n_k$, and consider the composition $h = p \circ f\colon X \to \mathbb{A}^n_k$. The point is that $z = p(y)$ is closed in $\mathbb{A}^n_k$, and that Z is a component of the fibre $h^{-1}(z)$; indeed, the fibre $p^{-1}(z)$ is finite and discrete. □

Theorem 12.10 combined with Going–Up gives the following;

Proposition 12.21 (Dimension of generic fibres). Let X and Y be varieties over k and let $f\colon X \to Y$ be a dominant morphism. There is an open dense subset $U \subset Y$ so that for all closed points $y \in Y$ and all irreducible components Z of X_y, it holds that $\dim Z = \dim X - \dim Y$.

Proof We may clearly assume that Y is affine, and we cover X by finitely many open affine subschemes $\{W_i\}$.

For each W_i we choose open affines $V_i \subset W_i$ and $U_i \subset Y$ such that $f_i = f|_{V_i}$ factors as in Theorem 12.10; that is, as the composition of two maps

$$V_i \xrightarrow{g_i} U_i \times \mathbb{A}^r \xrightarrow{p_i} U_i$$

with g_i finite and p_i the projection and $r = \dim V_i - \dim U_i$. Note that $r = \dim X - \dim Y$ by (ii) of Theorem 12.15. We claim that the set $U = \bigcap_i U_i$ will be as required. Indeed, consider a closed point $y \in U$ and a component Z of the fibre X_y. At least one of the W_i meets the given component Z in an open dense set, and hence the corresponding V_i meets Z as well. Then $Z_i = Z \cap W_i$ is open and dense in Z, and $\dim Z = \dim Z_i$ by (ii) of Theorem 12.15. The restriction $g_i|_{Z_i} \colon Z_i \to p_i^{-1}(y) = y \times_k \mathbb{A}_k^r$ is a finite map, and so by Going–Up, the closure of the image is of the same dimension as Z_i; hence $\dim Z = \dim Z_i \leqslant r$. The converse inequality is just Proposition 12.20, so $\dim Z = r$. □

Proposition 12.22 (Semicontinuity of the fibre dimension). Let X and Y be varieties over k and let $f \colon X \to Y$ be a surjective morphism. Then for all integers s the set $F_s(f) = \{\, y \in Y \mid \dim X_y \geqslant s \,\}$ is closed in Y.

Proof The proof goes by induction on $\dim Y$. The case $\dim Y = 0$ is trivial, so assume that $\dim Y > 0$. If $s \leqslant r = \dim X - \dim Y$, it holds that $F_s(f) = X$ by Proposition 12.20 (remember that f is surjective). Suppose then that $s > r$, and let $U \subset Y$ be an open set as in Proposition 12.21. Let Z_i be the components of $Y - U$ and let W_{ij} be the components of $f^{-1}Z_i$. Then $\dim Z_i < \dim Y$, and by induction each $F_s(f|_{W_{ij}})$ is closed in Z_i. We contend that

$$F_s(f) = \bigcup_{ij} F_s(f|_{W_{ij}}), \tag{12.1}$$

and this will imply that $F_s(f)$ is closed since Z_i is closed in Y.

As to (12.1), note that for all points in $y \in U$, each component W of X_y has $\dim W = r < s$, and hence the inclusion $F_s(f) \subset \bigcup_{ij} F_s(f|_{W_{ij}})$ holds true. Then pick a point $y \in F_s(f|_{W_{ij}})$. Each component of $f|_{W_{ij}}^{-1}(y)$ is contained in a component of $f^{-1}(y)$, so we infer that $\dim f^{-1}(y) \geqslant \dim f|_{W_{ij}}^{-1}(y) \geqslant s$. □

Images and constructible sets

Images of morphisms A subset E of a topological space is *locally closed* if it is the intersection of an open and a closed set. When X is Noetherian, a *constructible set* is defined to be a finite union of locally closed sets. It is easy to verify that finite unions and finite intersections of constructible sets are constructible, and that a subset which is constructible in a closed subspace, is constructible in the surrounding space.

The main interest in constructible sets lies in the fact that images of morphisms, which in general are neither closed nor open, are constructible; at least when the morphisms are of finite type and the schemes are Noetherian.

Example 12.23. The standard example is the map $\mathbb{A}_k^2 \to \mathbb{A}_k^2$ (where k is algebraically closed), that acts on closed points as $(x, y) \mapsto (xy, y)$. The image is the union of the complement of the x-axis and the origin. Indeed, the only possible points in the preimage of a point on the x-axis are points on the x-axis, but all these map to the origin. For points (a, b) off the x-axis, $b \neq 0$ and (ab^{-1}, b) is a preimage.

Theorem 12.24 (Chevalley's constructibility theorem). Let X and Y be Noetherian schemes and let $f\colon X \to Y$ be a morphism of finite type. Then the image $f(X)$ is constructible.

Proof Since Y and Y_{red} are homeomorphic and since being constructible is a purely topological property, we may assume that Y is reduced. The proof will be by Noetherian induction. Consider the set

$$\Sigma = \{\, Z \subset Y \mid Z \text{ is closed and } f(f^{-1}Z) \text{ is not constructible} \,\}.$$

If $f(X)$ is not constructible, Σ is non-empty (it contains Y), and since Y is Noetherian, Σ has a smallest member. Replacing Y with this smallest 'crook', we may assume that $f(f^{-1}Z)$ is constructible for all proper closed subsets of Y. If f is not dominant, we are through, so we may assume that f is dominant. By Theorem 12.10 there is an open non-empty set $U \subset f(X)$, and for all irreducible components Z_i of the complement $Y - U$ (which are finite in number since Y is Noetherian), it holds that $f(f^{-1}Z_i)$ is constructible. But as

$$f(X) = U \cup \bigcup_i f(f^{-1}Z_i),$$

it ensues that $f(x)$ is constructible. □

One easily extends the theorem to images of constructible sets:

Corollary 12.25. Let X and Y be Noetherian schemes and let $f\colon X \to Y$ be a morphism of finite type. For each constructible subset $E \subset X$ the image $f(E)$ is constructible.

Proof If E is locally closed, we give E the unique reduced scheme structure, which is Noetherian and such that $f|_E$ is a morphism of finite type. Then $f|_E$ has constructible image equal to $f(E)$. The corollary then follows since $f(E \cup F) = f(E) \cup f(F)$ for all sets. □

Exercise 12.3.2. Show that the constructible sets in a topological space form the smallest Boolean algebra containing the open (or the closed) sets. Show inverse images under continuous maps of contructible sets are constructible.

Exercise 12.3.3. Let X be a scheme and $x \in X$ a point. One says that a point $y \in X$ is a *specialization* of x if $y \in \bar{x}$, and that y is a generalization of x if $x \in \bar{y}$.

One says that a subset $E \subset X$ is *closed under specialization* if specializations of points in E belong to E. Likewise, E is said to be *closed under generalization* if generalizations of points in E belong to E.

a) Show that E is closed under specializations if and only if the complement $X - E$ is closed under generalizations;
b) Show that E is closed under specialization if and only if it has the following property: if $x \in E$ and $Z \subset X$ is a closed irreducible set with $x \in E$ then $Z \subset E$;

c) Show that closed sets are closed under generalization and that open sets are closed under generalization;
d) Show that if E set closed under specialization and $x \notin E$, then each irreducible component Z of X containing x is disjoint from E;
e) Show that in a Noetherian scheme, a constructible subset E is closed if it is closed under specialization and that it is open if it is closed under generalization;
f) Give example that the Noetherian hypothesis is necessary. HINT: Consider the spectrum in Exercise **??**.

Products of varieties

In section**??** gave examples of domains of finite type over k such that the tensor product $A \otimes_k B$ is not a domain — in the examples A and B were even fields. In other words, and in geometric terms, the product $X \times_k Y$ of two varieties needs not be a variety; it will be separated and of finite type, but not necessarily integral. But, as we are about to see, such things occur only when the base field is not algebraically closed.

Theorem 12.26 (Product of varieties). If X and Y are two varieties over an algebraically closed field k, then $X \times_k Y$ is a variety.

Proof The product of to separated schemes of finite type over k is separated (Exercise 11.3.5 on page 182) and of finite type. So the crucial point is to see that the product is integral. To that end, one easily reduces the proof to the affine case and so to prove that the tensor product $A \otimes_k B$ of two domains finitely generated over k is a domain.

Suppose that $f = \sum a_i \otimes b_i$ and $g = \sum c_i \otimes d_i$ are two elements such that $fg = 0$. We may arrange it so that the a_i's and the c_i's are linearly independent over k. Let $\mathfrak{b}$ be the ideal in B generated by the b_i's and $\mathfrak{d}$ the one generated by the d_i's.

For a maximal ideal $\mathfrak{m}$ in B and an element $b \in B$, let $\bar{b}$ denote the class of b in $B/\mathfrak{m}$. By the Nullstellensatz $B/\mathfrak{m} = k$ and so $A \otimes_k B/\mathfrak{m} = A$. Clearly $\bar{f}\bar{g} = 0$. As A is a domain and the a_i's and the c_i's are linearly independent, either $\bar{f} = \sum \bar{b}_i a_i = 0$, and all $b_i \in \mathfrak{m}$, or $\bar{g} = \sum \bar{d}_i c_i = 0$, and all $d_i \in \mathfrak{m}$. Hence $\mathfrak{b} \cap \mathfrak{d} \subset \mathfrak{m}$. This holds for all maximal ideals $\mathfrak{m} \subset B$, and according to Corollary 12.12 the intersection of all maximal ideals in B equals 0, hence it holds that $\mathfrak{b} \cap \mathfrak{d} = 0$. As B is a domain, it ensues that either $\mathfrak{b} = 0$ or $\mathfrak{d} = 0$, which means that either $f = 0$ or $g = 0$. □

Corollary 12.27. The product of two projective varieties X and Y over an algebraically closed field k, is a projective variety.

Proof The product is a variety by the theroem. The Segre embedding (Proposition 10.28 on page 170) realizes $\mathbb{P}^n_k \times \mathbb{P}^m_k$ as a projective variety, and if $X \subset \mathbb{P}^n_k$ and $Y \subset \mathbb{P}^m_k$ are closed embeddings, then $X \times_k Y \subset \mathbb{P}^n_k \times_k \mathbb{P}^m_k$ is a closed embedding. □

Proposition 12.28 (Dimension of a product). If X and Y are varieties over the algebraically closed field k, it holds that $\dim X \times_k Y = \dim X + \dim Y$.

Proof Replacing X and Y with non-empty open subsets, we may assume that both X and Y are affine. Chose finite surjective morphisms $f\colon X \to \mathbb{A}^n_k$ and $g\colon Y \to \mathbb{A}^m_k$, where n and m are the dimensions of X and Y respectively. The morphism $f \times g\colon X \times_k Y \to \mathbb{A}^n_k \times_k \mathbb{A}^m_k = \mathbb{A}^{n+m}_k$ is finite and surjective, hence $\dim X \times_k Y = n + m$. □

Exercise 12.3.4. Let X and Y be schemes of finite type over an algebraically closed field k. Show that if both are irreducible, then the product $X \times_k Y$ is irreducible. Show that if both are reduced, then the product $X \times_k Y$ is reduced.

Exercise 12.3.5 (Alternative proof of Theorem 12.26)**.** This exercise presents a proof of a slightly stronger version of Theorem 12.26. If X and Y are two varieties over k and k is closed in the function field $k(X)$, then $X \times_k Y$ is integral. (A subfield $k \subset K$ is closed in K if any root in K of a polynomial with coefficients in k lies in k; or equivalently every irreducible polynomial over k is irreducible over K.)

It suffices to do the affine version: let A be a domain of finite type over the field k. Assume that the ground field k is algebraically closed in the fraction field K of A.

a) If $L = k(t)$, show that $A \otimes_k k(t) = S^{-1}A[t]$ where S is the multiplicative set of non-zero polynomials in $A[t]$ with coefficients in k. Conclude that $A \otimes_k k(t)$ is a domain.
b) If $L = k[t]/(f)$ with f irreducible, show that $A \otimes_k L = A[t]/(f)$ and that $A[t]/(f)$ is integral. HINT: $K[t]/(f)$ is a field.
c) Show by induction on the number of generators over k required by L, that $A \otimes_k L$ is integral for all finitely generated field extensions L of k.
d) Show that $A \otimes_k B$ is a domain for all integral k-algebras B of finite type. HINT: The tensor product is contained in $A \otimes_k K(B)$ which is integral.

12.4 Birational vs biregular geometry

Two varieties are said to be *birationally equivalent* if they they have isomorphic open subsets. This is a much weaker relation than being isomorphic; for instance, blowing up a point in $\mathbb{P}^2_k$ yields a variety which is birationally equivalent with but not isomorphic to $\mathbb{P}^2_k$.

Rational maps

Let us be precise about what a rational map from X to Y is. Heuristically, just like rational functions, it is a morphism $U \to Y$ where U is an open non-empty subset of X. To avoid the ambiguity in the domain of definition U, one introduces an equivalence relation between such pairs (U, f), and says that two pairs (U, f) and (U', f') are equivalent if $f|_{U\cap U'} = f'|_{U\cap U'}$. A *rational map* is then an equivalence class of such pairs. However, it follows immediately from Proposition 6.4 about gluing morphisms that there is a preferred member in each class for which the open set U is maximal, and this is another way of resolving the ambiguity. A rational map is denoted with a dashed arrow $f\colon X \dashrightarrow Y$ (with the set of definition tacitly understood).

One says that a rational map $f\colon X \dashrightarrow Y$ is *dominant* if $f(U)$ is dense in Y where U is some open set where f is defined (if true for one U, it holds for all). Let $g\colon Y \dashrightarrow Z$ be another rational map say defined on $V \subset Y$. The open set $f^{-1}(V)$ is non-empty since $f(U)$ being dense entails that $f(U) \cap V \neq \varnothing$, and on $f^{-1}(V)$ the composition $g \circ f$ is defined. We conclude that dominant rational maps can be composed, and so the varieties over k together with the dominant rational maps form a category Rat_k.

A map dominant rational map $f\colon X \dashrightarrow Y$ is *birational* if it is an isomorphism in Rat_k; or in clear text, if there is dominant rational map $g\colon Y \dashrightarrow X$ so that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. One says that X and Y are *birationally equivalent* if there is birational map between them.

Example 12.29. Sending $(u_0 : u_1 : u_2)$ to $(u_1u_2 : u_0u_2 : u_0u_1)$ is a rational map from $\mathbb{P}^2_k$ to $\mathbb{P}^2_k$ defined away from the three coordinate points $(0:1:0)$, $(1:0:1)$ and $(1:1:0)$. It is birational with itself as inverse.

Example 12.30. Sending $(u_0 : u_1) \times (v_0 : v_1)$ to $(u_0v_0 : u_1v_0 : u_1v_1)$ is a rational map $\mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$. Defined away from $(1:0) \times (0:1)$. It is also birational with $(t_0 : t_1 : t_2) \mapsto (t_0 : t_1) \times (t_1 : t_2)$ as inverse; this map is defined away from $(0:0:1)$ and $(1:0:0)$.

The main theorem of birational geometry

A fundamental truth is that the study of of dominant rational maps, basically is reduced to the study of extensions of function fields:

Theorem 12.31. Let X and Y be two varieties over k. Then there is a one-to-one correspondence between rational dominant maps $X \dashrightarrow Y$ and k-algebra homomorphisms $k(Y) \subset k(X)$. In particular, two varieties are birationally equivalent if and only if their function fields are isomorphic as k-algebra.

We need a little lemma.

Lemma 12.32. Let A and B be two domains of finite type over a field k and denote their fraction fields by K and L respectively. Assume that $\phi\colon K \to L$ is a k-algebra homomorphism. Then there is some element $d \in B$ so that $\phi(A) \subset B_d$.

Proof Let $a_1, \ldots, a_r$ generate A over k. Each $\phi(a_i)$ is of the form $\phi(a_i) = b_i/c_i$ with $b_i, c_i \in A$. Then $d = c_1 \ldots c_r$ does the job. □

Recall also that when A and B are domains, a morphism $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ being dominant is equivalent to the associated map $f^\sharp\colon A \to B$ being injective; this is just Proposition 2.29 on page 33 bearing in mind that $\sqrt{0} = 0$ in B. Note further that a rational map $f\colon X \dashrightarrow Y$ being dominant means that it maps the generic point of X to the generic point of Y.

Proof of the theorem Let $U = \operatorname{Spec} B \subset X$ and $V = \operatorname{Spec} A \subset Y$ be open affine subsets. Then $k(Y)$ is the fraction field of A and $k(X)$ that of X.

Given a dominant rational map $f \colon X \dashrightarrow Y$, we may chose U and V so that f is defined on U and maps U into V. The induced k-homomorphism $A \to B$ is injective since f is dominant and extends to a k-homomorphism $k(Y) \to k(X)$. This does not depend on the choice of open affines; indeed, it is the map between stalks at the generic points induced by f.

For the converse, if a k-homomorphism $\phi \colon k(Y) \to k(X)$ is given, there is according to Lemma 12.32 an element $d \in B$ so that $\phi(A) \subset B_d$; then ϕ induces a morphism $\operatorname{Spec} B_d \to \operatorname{Spec} A = V \subset Y$ hence a rational map $X \dashrightarrow Y$. Evidently, A maps injectively into B_d so the morphism is dominant.

One leisurely verifies that the two assignments are mutually inverses (the key comment is that all maps between coordinate rings of affines are restrictions of $f^\sharp \colon k(Y) \to k(X)$) □

Associating X to the function field $k(X)$ defines a functor from the category Rat_k of varieties over k and dominant rational maps to the category of fields of finite type over k and k-homomorphism. Theorem 12.31 tells us that it is fully faithful; that is, it holds that $\operatorname{Hom}_{\mathsf{Rat}_k}(X, Y) \simeq \operatorname{Hom}_{\mathsf{Alg}_k}(k(X), k(Y))$. In fact, as we shortly will see, it is also essentially surjective: every field K of finite type over k is of the form $k(X)$ for some variety X. So it makes the two categories 'essentially equivalent', but there is no natural functor that serves as the inverse functor — there is no good, systematic way to pick out one particular model for each field K. A variety X so that $k(X) \simeq K$ is called a model for the field K.

Theorem 12.33 (Main theorem of birational geomtry). The assignment $X \mapsto k(X)$ is fully faithful and essentially surjective functor between the following categories:

(i) The category of projective varieties and dominant rational maps;
(ii) The category of finitely generated field extensions of k and k-algebra homomorphisms.

Proof Given a field K of finite type over k Assume that $K = k(t_1, \ldots, t_r)$ and let A be the subring of K generated by the t_i's; that is, $A = k[t_1, \ldots, t_r]$. To get a projective projective variety, embed $X = \operatorname{Spec} A$ in affine space $\mathbb{A}^r_k$ and close it up in $\mathbb{P}^r_k$. □

Note, to obtain a non singular model X for each field X is highly desirable, but extremely difficult. An illustrious result of Hironaka's is that it is true in characteristic zero, but in positive characteristic it is still un-known, except in low dimensions.

Exercise 12.4.1. Let $Q(x_0, \ldots, x_n)$ be a homogeneous quadratic polynomial. Show that the subvariety of $\mathbb{P}^{n+2}_k$ given by $x_{n+1}x_{n+2} + Q(x_2, \ldots, x_n)$ is birational to $\mathbb{P}^{n+1}_k$.

13

Local properties

13.1 Tangent spaces

Consider an affine variety $X \subset \mathbb{A}_k^n$, say $X = V(I)$ where $I = (f_1, \ldots, f_r)$. For a k-point $p \in X$, the tangent space T_pX is usually defined as the sub-vector space of k^n given by the null space of the *Jacobian matrix*

$$J(f_1, \ldots, f_r)(p) = \left(\frac{\partial f_i}{\partial x_j}(p) \right)_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant n}}. \tag{13.1}$$

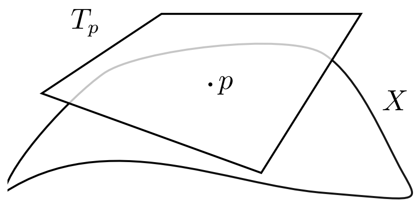
It is easily verified using the chain rule that T_pX does not depend on the choice of generators for I.

The dimension of T_pX is given by

$$\dim T_pX = n - \operatorname{rank} J(f_1, \ldots, f_r)(p). \tag{13.2}$$

As it is defined, T_pX is a subspace of k^n. One sometimes also talks about the *affine tangent space* at a point $a = (a_1, \ldots, a_n)$ as the subvariety defined by the linear equations (in $\mathbb{A}_k^n$)

$$J(f_1, \ldots, f_r)(p) \cdot (x - a) = 0.$$



Example 13.1. Consider the cuspidal cubic curve $X = V(x^3 + y^2)$ in $\mathbb{A}_{\mathbb{C}}^2$. Then the Jacobian at a closed point $p = (a, b)$ is given by $J = (3a^2, 2b)$. Therefore, T_pX has dimension 2 at the origin $p = (0, 0)$ and dimension 1 for every other point.

There is an intrinsic description of the tangent space T_pX, which is independent of the affine embedding of X, and which will be the inspiration for the general definition.

Suppose for simplicity that $p = (0, \ldots, 0)$ is the origin (we may always arrange this by a linear change of coordinates), and write $\mathcal{M} = (x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$ for the maximal ideal at p. For a polynomial $f \in k[x_1, \ldots, x_n]$, we consider its *linearization at* p, given by

$$Df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0) x_i.$$

This is just the linear part of the Taylor expansion at p. Note that the coordinates $x_1, \ldots, x_n$ give a basis for the dual space $(k^n)^\vee = \operatorname{Hom}_k(k^n, k)$. Hence we may view Df as a linear functional on k^n, and in this way we get a k-linear map

$$D\colon \mathfrak{M} \to (k^n)^\vee.$$

It is clear that D is surjective, since $D(x_i) = x_i$. A polynomial f lies in kernel of D precisely when all terms are of degree at least two, or phrased differently, the kernel of D equals $\mathfrak{M}^2$. Hence D induces an isomorphism of k-vector spaces

$$\mathfrak{M}/\mathfrak{M}^2 \simeq (k^n)^\vee.$$

Returning to the variety X and the tangent space T_pX, we take the dual of the inclusion $T_pX \subset k^n$, to obtain a surjection

$$(k^n)^\vee \to (T_pX)^\vee.$$

Concretely, this map is given by restricting a linear functional on k^n to the subspace T_pX. The composition

$$\theta : \mathfrak{M}/\mathfrak{M}^2 \to (k^n)^\vee \to (T_pX)^\vee$$

is also surjective.

We claim that $\operatorname{Ker}\theta = \mathcal{M}^2 + I$. Indeed, note that $f \in \operatorname{Ker}\theta$ if and only if Df restricts to 0 on T_pX. This happens if and only if $Df = Dg$ for some $g \in I$ (since T_pX is the zero locus of Dg for all $g \in I$); that is, if and only if $f - g \in \operatorname{Ker} D = \mathfrak{M}^2$, or equivalently, $f \in \mathfrak{M}^2 + I$.

It follows that we have isomorphisms of k-vector spaces

$$(T_pX)^\vee \simeq \mathcal{M}/(\mathcal{M}^2 + I) \simeq \mathfrak{m}/\mathfrak{m}^2. \tag{13.3}$$

where $\mathfrak{m} \subset \mathcal{O}_{X,p}$ is the maximal ideal. Taking duals, we now have:

Proposition 13.2. There is a natural isomorphism

$$T_pX \simeq \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k). \tag{13.4}$$

Tangent spaces in general

Taking Proposition 13.2 as motivation, we make the following definition of tangent spaces of general schemes.

Definition 13.3. Let X be a scheme and let $p \in X$ be a point.

(i) The *cotangent space* is defined the $k(p)$-vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$, where $\mathfrak{m}_p$ is the maximal ideal in the local ring $\mathcal{O}_{X,p}$.

(ii) The *tangent space* is defined as the dual $k(p)$-vector space

$$T_pX = \operatorname{Hom}_{k(p)}(\mathfrak{m}_p/\mathfrak{m}_p^2, k(p))$$

The cotangent space is functorial in the following sense. Let $f\colon X \to Y$ be a morphism and let $y = f(x)$. The map of local rings $f^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ takes the maximal ideal into the maximal ideal, and being a ring map, it sends $\mathfrak{m}_y^2$ into $\mathfrak{m}_x^2$. Therefore it induces a map of $k(y)$-vector spaces

$$f_x^\sharp\colon \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2.$$

Moreover, for each morphism g which is composable with f one has

$$(g \circ f)_x^\sharp = f_x^\sharp \circ g_{f(x)}^\sharp$$

since $(g \circ f)^\sharp = f^\sharp \circ g^\sharp$.

The map $f_x^\sharp$ is, however, just a map of $k(y)$-vector spaces. In general, there is no way to make $\mathfrak{m}_y/\mathfrak{m}_y^2$ a $k(x)$-vector space, and for this reason the tangent spaces are not functorial in general; the required duals will be with respect to different fields.

One exception is when X and Y are varieties over some field k, and x and y both are k-points. Then $k(x) = k(y) = k$, and we are permitted to take duals to get a map

$$df\colon T_x X \to T_y Y.$$

Once the tangent maps are defined, they behave functorially:

$$d(g \circ f)_x = dg_y \circ df_x$$

when $g\colon Y \to Z$ is a map of k-schemes and x is a k-point.[1]

Zariski tangent spaces and the ring of dual numbers

When X is a scheme over a field k, there is an interesting relation between the Zariski tangent space at k-points and the ring $k[\epsilon]/(\epsilon^2)$. This ring is called the *ring of dual numbers* over k, which is often written $k[\epsilon]$, tacitly understanding that $\epsilon^2 = 0$ in this ring. The spectrum of $k[\epsilon]$ is a very simple scheme: its underlying topological space is a single point. However, the non-reduced structure on $\operatorname{Spec} k[\epsilon]$ shows that it is more interesting than $\operatorname{Spec} k$. We picture it as a point ε with a vector 'sticking out of it'.

Proposition 13.4. Let X be a scheme over k. To give a k-morphism $\operatorname{Spec}(k[\epsilon]) \to X$ is equivalent to giving a k-rational point $x \in X(k)$, and an element of $T_x X$.

Before proving the proposition, let us mention that there are other interesting tiny algebras related to $k[\epsilon]$. If V any vector space over k, one may form the 'infinitesimal' k-algebra $D_V = k \oplus V$ where V is as a maximal ideal with square zero; that is, the multiplication is $(a + w) \cdot (b + v) = ab + (aw + bv)$. The important property of D_V is that k-algebra maps $D_V \to k[\epsilon]$ correspond bijectively to linear functionals on V; in other words, there is an isomorphism

$$\operatorname{Hom}_{\mathsf{Alg}_k}(D_V, k[\epsilon]) \simeq \operatorname{Hom}_k(V, k).$$

Indeed, if $\alpha\colon D_V \to k[\epsilon]$ is given, the restriction $\alpha|_V$ is k-linear and takes values in $(\epsilon) = k$.

[1] Note that this only works if $\dim_{k(y)} T_y Y$ is finite; this subtle point is another reason why the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is preferable to the tangent space.

For the inverse map, if $\alpha\colon V \to k$ is a given functional, the assignment $a + v \mapsto a + \alpha(v)\epsilon$ defines a k-algebra map.

Proof of Proposition 13.4 Fix a k-point $p \in X(k)$. Every map $\operatorname{Spec} k[\epsilon] \to X$ that sends the point (ϵ) to p, must factor through each open affine neighbourhood of p, and so we may well assume that X is affine, say $X = \operatorname{Spec} A$. Let $\mathfrak{m} = \mathfrak{m}_p$. A homomorphism $\alpha\colon A \to k[\epsilon]$ corresponds to a morphism $\operatorname{Spec} k[\epsilon] \to X$ that sends (ϵ) to p, precisely when the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & k[\epsilon] \\ \downarrow & \swarrow & \\ k & & \end{array}$$

commutes (where $A \to k$ and $k[\epsilon] \to k$ are the quotient maps associated to $\mathfrak{m}_p$ and ϵ respectively). Such maps α factor in a unique manner through the canonical map $A \to A/\mathfrak{m}^2$ (since $\alpha(\mathfrak{m}) \subset (\epsilon)$ and $\epsilon^2 = 0$). Now, the reduction map $A/\mathfrak{m}^2 \to A/\mathfrak{m} = k$ splits as an algebra homomorphism, the structure map $k \to A/\mathfrak{m}^2$ being a section, and $A/\mathfrak{m}^2$ decomposes as an k-algebra into $A/\mathfrak{m}^2 = k \oplus (\mathfrak{m}/\mathfrak{m}^2)$; in other words, $A/\mathfrak{m}^2 = D_{\mathfrak{m}/\mathfrak{m}^2}$ in the terminology above. It follows that we have our desired isomorphism

$$\operatorname{Hom}_{\mathsf{Alg}_k}(A, k[\epsilon]) \simeq \operatorname{Hom}_{\mathsf{Alg}_k}(A/\mathfrak{m}^2, k[\epsilon]) \simeq \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

□

Exercise 13.1.1. Let V and W be two vector spaces over k. Show that there is a functorial isomorphism $\operatorname{Hom}_{\mathsf{Alg}_k}(D_V, D_W) \simeq \operatorname{Hom}_k(V, W)$.

13.2 Normal schemes

Recall that an integral domain A is said to be *normal* if it is integrally closed in its fraction field $K = k(A)$. In other words, any element $z \in K$ which satisfies a monic equation with coefficients in A, is already contained in A. Here are a few examples of normal rings:

Example 13.5. Any UFD is normal (e.g., $\mathbb{Z}$, $\mathbb{Z}[x_1, \dots, x_n]$).

To see this, take any element $u/v \in K$. If there is a monic relation of the form

$$(u/v)^n + a_{n-1}(u/v)^{n-1} + \cdots + a_0 = 0 \tag{13.5}$$

with the $a_i \in A$, then multiplying by v^n shows that v divides u^n. But then if we assume that u have no common factors, we must have $u = a \cdot v$ for some $a \in A$, hence $u/v \in A$.

Example 13.6. If A is normal, then so is $A[x]$.

Example 13.7. Any localization $S^{-1}A$ of a normal integral domain A is normal.

The last part has a converse: An integral comain A is normal if and only if $A_{\mathfrak{p}}$ is normal for all prime ideals $\mathfrak{p}$, if and only if $A_{\mathfrak{m}}$ is normal for all maximal ideals $\mathfrak{m}$.

Motivated by all the desirable algebraic properties of normal rings, we make the following definition:

Definition 13.8. Let X be a scheme. We say that X is *normal* if for each point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed integral domain.

The primary example of a normal scheme is $X = \operatorname{Spec} A$, where A is a normal integral domain. Note however, that in the definition of a normal scheme we do not make the assumption that X is integral. However, if $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integral domain, and normality implies that there is a unique irreducible component X_i of X containing x, and X_i, with its induced scheme structure is integral. In any case, any normal scheme is reduced.

Example 13.9. $\mathbb{A}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_{\mathbb{Z}}$ are normal schemes, because the local rings is isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ which is a localization of an UFD, hence normal.

Althogh it is not obvious from the definition, the notion of normality is related to regularity. This is because of the algebraic fact that local regular rings are unique factorization domains **?**, hence they are normal (Example 13.5). From this we conclude:

Proposition 13.10. Any regular scheme is normal.

Example 13.11. More generally, a scheme which is *locally factorial* (meaning that all stalks $\mathcal{O}_{X,x}$ are UFD's), is also normal.

We will see an example below of a normal scheme which is non-regular. While normal schemes are more general than regular schemes, they still have several nice properties. For instance, if X is a normal variety, then:

(i) The singular locus of X has codimension at least 2 in X (Theorem 13.22);
(ii) Any finite birational morphism $Y \to X$ is an isomorphism (Proposition 13.27);
(iii) Any rational function defined outside a closed set of codimension at least 2, extends to a regular function on all of X (Theorem 13.19).

Normalization

In this section, we will construct the *normalization* of a scheme. This produces a normal scheme $\overline{X}$ together with a dominant morphism $\pi : \overline{X} \to X$. We construct the normalization $\overline{X}$ because has better properties than X, e.g., $\overline{X}$ typically has a smaller singular locus than X. When X is a variety, the normalization morphism π is birational, so $\overline{X}$ can be viewed as a sort of 'mild resolution of singularities' of X. In fact, when X is a curve, being normal is the same as being regular, so $\overline{X}$ is indeed the desingularization of X (cf. XXX).

Theorem 13.12 (Normalization). For an integral scheme X, there is a normal scheme $\overline{X}$, and a morphism $\pi\colon \overline{X} \to X$ satisfying the following universal property: For any dominant morphism $h\colon Y \to X$ from a normal scheme Z, there is a unique morphism $\overline{h}\colon Z \to \overline{X}$ such that $h = \pi \circ \overline{h}$.

$$\begin{array}{ccc} & & \overline{X} \\ & \overset{\overline{h}}{\nearrow} & \downarrow \pi \\ Z & \xrightarrow{h} & X \end{array}$$

Proof If $X = \operatorname{Spec} A$ is affine, define $\overline{X} = \operatorname{Spec} \overline{A}$ where $\overline{A} \supseteq A$ is the integral closure of A in $K = k(X)$, and $\pi_X : \operatorname{Spec} B \to \operatorname{Spec} A$ is the morphism induced by the inclusion. Note that the scheme $\overline{X}$ is normal, because all the local rings are given by localizations $B_{\mathfrak{p}}$ which are normal in $k(\overline{A}) = K$ by assumption. Moreover, $Y(X)$ is integral, because $\overline{A}$ is an integral domain.

Next we verify the universal property. Let $h : Z \to X$ be a dominant morphism from an integral normal scheme Z. This means that the map $h^\sharp : A \to \mathcal{O}_Z(Z)$ is injective. As $\mathcal{O}_Z(Z)$ is normal, the ring map $A \to \mathcal{O}_Z(Z)$ factors via $\overline{A}$ as $A \to \overline{A} \to \mathcal{O}_Z(Z)$. Hence h factors via $Y(X)$, and we are done.

Now suppose X is a general integral scheme. For an affine subset $U = \operatorname{Spec} A \subset X$, we set $Y(U) = \operatorname{Spec} \overline{A}$ and check that the collection of morphisms $\pi_U : Y(U) \to U$ satisfy the conditions of Proposition 24.1, so that they glue to a morphism $\pi_X : Y(X) \to X$.

If U, V are two affines with $V \subset U$, we can consider the open subscheme $W = \pi_U^{-1}(V) \subset Y(V)$. By assumption, this scheme is affine (since π_U is an affine morphism), integral and normal, being an open set in $Y(U)$. Note that

$$\mathcal{O}_W(W) = \bigcap_{\mathfrak{p} \in W} \mathcal{O}_{Y(U),\mathfrak{p}}.$$

The intersection takes place inside $K = k(W) = k(X)$. As the local rings $\mathcal{O}_{Y(U),\mathfrak{p}}$ are integrally closed, we see that $\mathcal{O}_W(W)$ is normal. By Exercise 13.5.1, we see that $\mathcal{O}_W(W)$ coincides with the integral closure of V in K. In other words, $Y(V)$ is canonically identified with $\pi_U^{-1}(V) = Y(U) \times_U V$. Finally, if $W \subset V \subset U$ are three affines, the map $Y(W) \to Y(U)$ clearly factors via $Y(V)$.

Finally, we prove that the scheme $\overline{X}$ and $\pi_X : \overline{X} \to X$ satisfy the universal property. So let $h : Z \to X$ be a dominant morphism from a normal integral scheme Z. Over each U_i, we have an induced morphism $h^{-1}(U_i) \to U_i$, which by the universal property over the U_i must factor uniquely via $\overline{U_i}$ via $g_i : h^{-1}(U_i) \to \overline{U_i}$. Again the uniqueness in the universal property tells us that these maps must agree over the overlaps $h^{-1}(U_{ij})$. Since the $h^{-1}(U_i)$ form an open cover of Z, these maps glue to a map $g : Z \to \overline{X}$ factoring h. □

Proposition 13.13. For a Noetherian integral scheme X, the normalization $\overline{X}$ has the following properties:

(i) $\pi\colon \overline{X} \to X$ is surjective;
(ii) $\overline{X}$ and X have the same dimension;
(iii) There is a dense open subset $U \subset X$ so that π restricted to $\pi^{-1}(U)$ is an isomorphism;
(iv) If X is of finite type over a field or over $\mathbb{Z}$, then $\pi\colon \overline{X} \to X$ is a finite morphism.

Proof All of these properties are 'local on X'. Thus by the gluing construction used in the construction of $\overline{X}$, we reduce to $X = \operatorname{Spec} A$ and $\overline{X} = \operatorname{Spec} \overline{A}$ and π is induced by the inclusion $A \subset \overline{A}$. Here the points (i)–(iv) follow from basic properties of integral ring extensions. For instance, both statements (i) and (ii) follow from the Going-Up theorem.

The statement (iii) holds true because by construction, X and $\overline{X}$ have the same fraction field K, and π maps the generic point $\overline{\eta} = \operatorname{Spec} K$ of $\overline{X}$ maps to the generic point of X.

Finally, the statement (iv) follows from Theorem A.18, which tell us that with our assumptions, $\overline{A}$ is *finite* as an A-module.

□

Corollary 13.14 (Being normal is a generic property). Let X be a Noetherian integral scheme. Then there is a non-empty open subscheme $U \subset X$ which is normal.

13.2.1 Examples

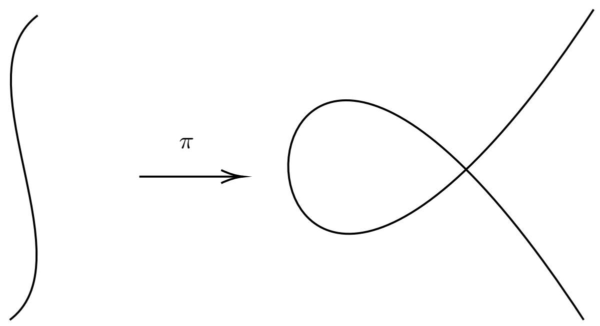
Example 13.15 (Cuspidal cubic)**.** Let k be a field, and let $X = \operatorname{Spec} A$ where $A = k[x, y]/(y^2 - x^3)$. This is the *cuspidal cubic curve* in $\mathbb{A}^2_k$.

There is an isomorphism of k-algebras $A \xrightarrow{\simeq} k[t^2, t^3]$ given by sending $x \mapsto t^2$ and $y \mapsto t^3$. It is clear that $k[t^2, t^3]$ is an integral domain with fraction field $K = k(t)$. On the other hand this ring is visibly not normal, as $t \notin k[t^2, t^3]$ but yet it satisfies the monic equation $T^2 - t^2 = 0$. The normalization of A equals $\overline{A} = k[t]$. The inclusion $A \subset \overline{A}$ induces the normalization morphism $\pi : \mathbb{A}^1_k \to X$, and this is an isomorphism over the open set $D(t) \subset \mathbb{A}^1_k$.

Example 13.16 (Nodal cubic)**.** Let now $X = \operatorname{Spec} A$ with A being the ring $A = k[x, y]/(y^2 - x^3 - x^2)$, where k now is a field whose characteristic is not two (if the characteristic is two, we are back in previous cuspidal case). This is the *nodal cubic curve* in $\mathbb{A}^2_k$. Here it is a little bit tricker to find the normalization, but it helps to think about it geometrically.

If we think of the corresponding affine variety $\{ (x, y) \mid y^2 = x^3 + x^2 \} \subset \mathbb{A}^2(k)$, we see that the origin $(0, 0)$ is a special point: a line $l \subset \mathbb{A}^2_k$ through the closed point $(0, 0) \in X$ (with equation $y = tx$) will intersect X at $(0, 0)$ and at one more point (with $x = t^2 - 1$), and this gives a parameterization of the curve, which is generically one-to-one.

Back in the scheme world, we imitate this by introducing the parameter $t = yx^{-1}$ in the function field K of X, the equation $y^2 = x^3 - x^2$ then reduces to $t^2 = 1 + x$ after



being divided by x^2. Moreover, the element t is integral, since it satisfies the monic equation $T^2 - x - 1 = 0$ (which has coefficients in A). Since $x = t^2 - 1$ and $y = x \cdot y/x = t^3 - t$, we see that

$$A = k[t^2 - 1, t^3 - t] \subset k[t] \subset K = k(t),$$

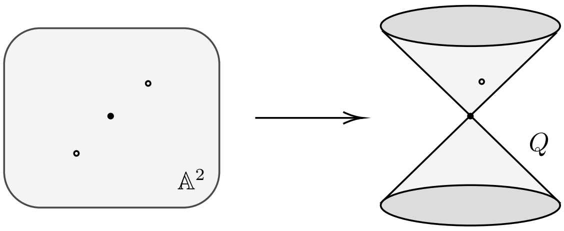
and since $k[t]$ is integrally closed, any element in K which is integral over A, can be written as a polynomial in t. So $\overline{A} = k[t]$ is the integral closure of A in $k(t)$. The normalization map $\pi : \operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is an isomorphism outside the origin $(0,0) \in X$. Geometrically the map π identifies two points $(t+1)$ and $(t-1)$ in $\mathbb{A}^1_k$ to the origin in X.

Example 13.17 (The quadratic cone)**.** Consider the affine scheme $X = \operatorname{Spec} A$ where $A = \mathbb{C}[x,y,z]/(xy - z^2)$. Note that this is not a factorial scheme as $xy = z^2$ and one easily checks that x, y and z all are irreducible elements, so we cannot immediately conclude that A is normal. However, there is an isomorphism of rings

$$\phi \colon A \to \mathbb{C}[u^2, uv, v^2],$$

and the latter algebra is normal in $\mathbb{C}(u^2, uv, v^2)$. Indeed, note that if $T = p/q \in \mathbb{C}(u^2, uv, v^2)$

satisfies a monic equation with coefficients in $\mathbb{C}[u^2, uv, v^2]$, then $T \in \mathbb{C}[u, v]$ is a polynomial (as $\mathbb{C}[u, v]$ is integrally closed). Therefore, $T \in \mathbb{C}[u^2, uv, v^2]$.



For another proof, see Exercise 13.2.1.

Example 13.18. In general, the normalization map of a scheme $\pi : \overline{X} \to X$ needs not be finite in the sense of Definition 8.20 on page 122. The first examples of Noetherian integral domains A whose integral closure is not finite over A were found by Yasuo Akizuki and Friedrich Karl Schmidt in the 1930s.

Exercise 13.2.1. Prove directly that $A = \mathbb{C}[x, y, z]/(z^2 - xy)$ is normal as follows. Let $B = \mathbb{C}[x, y]$, so that $A = B[z]/(z^2 - xy)$.

a) Show that A is a finite B-module of rank 2, with basis $1, z$.
b) Show that $K(B) = \mathbb{C}(x, y)$ and the field extension $K(B) \subset K(B)$ has degree 2.
c) Show that $w = u + vz \in A$ satisfies the monic polynomial
$$T^2 - 2uT + (u^2 - xyv^2) = 0.$$
d) Show that if w is integral over B, then $u \in \mathbb{C}[x, y]$; $xyv^2 \in \mathbb{C}[x, y]$ and hence $v \in \mathbb{C}[x, y]$. Conclude that $w \in A$.

13.3 Normality and rational functions

Theorem 13.19 ("Algebraic Hartogs's theorem"). Let X be a Noetherian normal scheme, and let $U \subseteq X$ be an open subset with $\operatorname{codim}_X(X - U) \geqslant 2$. Then the restriction map
$$\mathcal{O}_X(X) \to \mathcal{O}_X(U) \tag{13.6}$$
is an isomorphism.

In other words, every regular function $f \in \mathcal{O}_X(U)$ on U extends uniquely to all of X.

Proof We begin by proving the theorem for the case when X is affine, say $X = \operatorname{Spec} A$, where A is a normal integral domain. If we view $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X(U)$ as subrings of the function field $k(X)$, the restriction map (13.6) is simply an inclusion $A \subset \mathcal{O}_X(U)$. As $X - U$ is assumed to be of codimension at least 2, U contains all points x corresponding to prime ideals $\mathfrak{p}$ of height 1. This means that $\mathcal{O}_X(U) \subset \mathcal{O}_{X,x} = A_{\mathfrak{p}}$ for every such $\mathfrak{p}$.

Therefore, by Proposition XXX, we conclude that

$$\mathcal{O}_X(U) \subset \bigcap_{\text{ht }\mathfrak{p}=1} A_\mathfrak{p} = A.$$

Next, suppose X is a general Noetherian normal scheme, and let $\{U_i\}$ be an affine cover of X. Consider the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_X(X) & \longrightarrow & \prod_i \mathcal{O}_X(U_i) & \longrightarrow & \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathcal{O}_X(U) & \longrightarrow & \prod_i \mathcal{O}_X(U_i \cap U) & \longrightarrow & \prod_{i,j} \mathcal{O}_X(U_i \cap U_j \cap U) \end{array}$$

By the affine case, the middle vertical arrows are isomorphisms. Therefore, by a diagram chase, we would be able to say that the left-most vertical map is an isomorphism if the right-most map is. The issue is that the intersections $U_{ij} = U_i \cap U_j$ need not be affine. Nevertheless, fix i and j and let U_{ijk} be a covering of U_{ij} consisting of affine open sets which are distinguished in both U_i and U_j. Again, by the affine case, we get that the restriction map

$$\mathcal{O}_{U_{ijk}}(U_{ijk}) \to \mathcal{O}_{U_{ijk}}(U_{ij} \cap U)$$

is an isomorphism. Moreover, in this case, the intersections $U_{ijk} \cap U_{ijk'}$ are now affine, so by the diagram above applied to $X = U_{ij}$ and the covering U_{ijk}, we conclude that $\mathcal{O}_X(U_{ij}) \to \mathcal{O}_X(U_{ij} \cap U)$ is an isomorphism, and we are done. □

Example 13.20. The assumption that the codimension is at least 2 can not be removed: For the open set $D(t) \subset \mathbb{A}^1_k$ we have $\mathcal{O}_{\mathbb{A}^1_k}(D(t)) = k[t, t^{-1}]$ whereas $\mathcal{O}_{\mathbb{A}^1_k} = k[t]$.

By the way, the Proposition gives another way to see the why $\mathcal{O}_U(U) = k[u, v]$ for the open set $U = \mathbb{A}^2_k - V(u, v)$ in $\mathbb{A}^2_k$ (Example XXX).

There is a converse to this result, known as *Serre's Criterion*. It gives a more geometric characterisation of the property of 'normality' (which is fundamentally an algebraic notion).

Theorem 13.21 (Serre's Criterion). Let X be a Noetherian integral scheme. Then X is normal if and only

(i) The set of singular points, sing(X), has codimension at least 2 in X.
(ii) Whenever $U \subset X$ is an open set whose complement has codimension at least 2, the restriction map (13.6) is an isomorphism.

In particular, we get:

Corollary 13.22. If X is a normal variety, then the singular set sing(X) has codimension at least 2.

Example 13.23 (Curves)**.** A curve X is normal if and only it is regular.

Example 13.24 (Hypersurfaces)**.** Let X be a regular variety and let $Y \subset X$ be a hypersurface defined by $f \in \mathcal{O}_X(X)$. Then the condition (ii) in Theorem 13.21 is automatically satisfied (this is a non-trivial fact; see **?**). Thus Y is normal if and only if $\text{sing}(X)$ has codimension at least 2.

We have seen several examples of non-normal schemes which do not satisfy condition (i) of Serre's criterion. Here is one where the second condition fails:

Example 13.25. Let X be the scheme obtained by gluing together two copies of $\mathbb{A}^2_k$ at the origin (see Example 24.12 on page 405). Then X is an integral scheme of dimension 2, and the singular locus consists of a single point p. However, consider now the complement $U = X - p$, which consists of two disjoint copies of $\mathbb{A}^2_k - p$. The regular function $f \in \mathcal{O}_X(U)$ which takes the value 0 on one component and 1 on the other clearly does not extend to all of X.

See Exercise 13.5.3 for another example.

13.4 Normality and finite birational morphisms

Birational morphisms $f : Y \to X$ are isomorphisms over an open set, but they need not be global isomorphisms. For instance, when f is the blow-up of $\mathbb{A}^2_k$ at a point, there is a whole $\mathbb{P}^1_k$ which is collapsed to a point. But what if we in addition assume that f is finite – is f an isomorphism then? In general, the anwer is no; here is a counterexample.

Example 13.26. Consider

$$f : \operatorname{Spec} k[x, y]/(y^2 - x^3) \to k[x]$$

given by Example XXX. The map is a homeomorphism and birational, but not an isomorphism in a neighborhood of the origin. Even worse, the map

$$f : \operatorname{Spec} k[x, y]/(y^2 - x^3 - x^2) \to k[x]$$

of Example XXX is not even bijective.

This type of phenomenon does not occur if the target is a normal scheme. The examples above are not normal schemes, and the failure of being an isomorphism is entirely concentrated at the singular point at the origin.

Proposition 13.27. Let X and Y be integral schemes, and let $f : Y \to X$ be a finite, birational morphism. If X is normal, then f is an isomorphism.

Proof Since the property of being an isomorphism is local on the target, and finite morphisms are affine, we may reduce to the case where both X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and f is induced by a ring map $\phi : A \to B$.

As f is a finite morphism, the map ϕ makes B into a finite A-module. In addition, if f is birational, ϕ induces an isomorphism of the function fields $\phi_K : K(A) \to K(B)$. Therefore, since A is integrally closed, the map $\phi : A \to B$ must be an isomorphism, being finite hence integral. Therefore, f is an isomorphism of schemes. □

13.5 Exercises

Exercise 13.5.1. Let A be an integral domain with fraction field K. Let $x \in K$ be an element. Show that the following are equivalent:

a) x is integral over A
b) $A[x]$ is a finite A-module
c) There exists a subalgebra $A' \subset A$ such that $x \in A'$ and R is a finite A'-module.

Exercise 13.5.2 (The cone over a rational quartic curve)**.** Consider $X = \operatorname{Spec} A$, where A is the $\mathbb{C}$-algebra

$$A = \mathbb{C}[u^4, u^3v, uv^3, v^4] \simeq \mathbb{C}[t_0, t_1, t_3, t_4]/(t_0t_4 - t_1t_3, t_1^3 - t_0^2t_3, t_3^3 - t_1t_4^3).$$

a) Show that X is a variety of dimension 2.
b) Show that X is non-singular outside the origin $p = V(t_0, t_1, t_3, t_4)$.
c) Show that

$$\frac{t_1^2}{t_0} = u^2v^2 = \frac{t_3^2}{t_4}$$

defines a regular function on $X - p$, but it does not extend to all of X. Conclude that X satisfies (i) but not (ii) of Serre's criterion.

d) Show that the ideal (t_0) is not principal in A. HINT: A primary decomposition of (t_0) is given by

$$(t_0) = (t_0, t_1^2) \cap (t_0, t_4)$$

Exercise 13.5.3. Consider $X = \operatorname{Spec} A$, where A is the $\mathbb{C}$-algebra

$$A = \mathbb{C}[s^4, s^3t, st^3, t^4] \simeq \mathbb{C}[x, y, z, w]/(xw - yz, y^3 - x^2z, z^3 - yw^3).$$

a) Show that X is a variety of dimension 2.
b) Show that X is non-singular outside the origin $p = V(x, y, z, w)$.
c) Show that

$$\frac{y^2}{x} = s^2t^2 = \frac{z^2}{w}$$

defines a regular function on $X - p$, but it does not extend to all of X.

d) Conclude that X satisfies (i) but not (ii) of Serre's criterion.

Exercise 13.5.4. Show that the normalization of the scheme $X = \operatorname{Spec} \mathbb{Z}[6i]$ is given by $\operatorname{Spec} \mathbb{Z}[i]$.

14

Sheaves of Modules

In this section we develop the theory of sheaves in greater detail. For a scheme X, the category of sheaves on X is a particularly nice category which behaves very much like the category of modules over a ring. One is able to form kernels, cokernels and images of maps, direct sums and products of sheaves and there is the the notion of exact sequences. In short, the category $\mathsf{AbSh_X}$ of sheaves on X is an abelian category with arbitrary products and direct sums.

Kernels

For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we define its *kernel* as follows:

Definition 14.1. The *kernel* $\operatorname{Ker}\phi$ of ϕ is the subsheaf of $\mathcal{F}$ defined by

$$(\operatorname{Ker}\phi)(U) = \operatorname{Ker}\phi_U$$

for each open $U \subset X$. In other words, $(\operatorname{Ker}\phi)(U)$ consists of the sections in $\mathcal{F}(U)$ that map to zero under $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

The kernel is clearly a presheaf, because $\phi_V(s|_V) = \phi_U(s)|_V$ for any section $s \in \mathcal{F}(U)$ and any open $V \subset U$ (the diagram (3.3) commutes).

We check the two sheaf axioms. The Locality axiom for $\operatorname{Ker}\phi$ is inherited from the Locality axiom for $\mathcal{F}$. For the Gluing axiom, suppose we are given a cover $\{U_i\}$ of an open set U and sections $s_i \in (\operatorname{Ker}\phi)(U_i)$ that agree on the overlaps. One may glue together the s_i's to a section s of $\mathcal{F}$ over U, and one has $\phi(s)|_{U_i} = \phi(s|_{U_i}) = \phi(s_i) = 0$. By the Locality axiom for $\mathcal{G}$, it then follows that $\phi(s) = 0$, and hence $s \in (\operatorname{Ker}\phi)(U)$.

Lemma 14.2. For each point $x \in X$, one has $(\operatorname{Ker}\phi)_x = \operatorname{Ker}\phi_x$.

Proof The inclusion $(\operatorname{Ker}\phi)_x \subset \operatorname{Ker}\phi_x$ is clear. Conversely, an element in $\operatorname{Ker}\phi_x$ is the germ s_x of a section s of $\mathcal{F}$ over some open neighbourhood U of x, such that the germ $\phi_U(s)_x$ of $\phi_U(s)$ equals zero. This means that for some open $V \subset U$ it holds that $\phi_U(s)|_V = 0$. Hence $s|_V \in (\operatorname{Ker}\phi)(V)$, and therefore $s_x \in (\operatorname{Ker}\phi)_x$. □

A map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is said to be *injective* if $\operatorname{Ker}\phi = 0$; this is equivalent to ϕ_U being injective for each open U. In light of the previous lemma, it is also equivalent to the condition that $\operatorname{Ker}\phi_x = 0$ for all x; that is, all stalk maps ϕ_x are injective. One often expresses this by saying that 'ϕ is injective on stalks'.

Images

Defining the *image* of a map $\phi\colon \mathcal{F} \to \mathcal{G}$ between sheaves is more subtle than defining the kernel. It might be tempting to define $\operatorname{Im}\phi$ over an open set U by

$$\operatorname{Im}(\phi_U) = \left\{ \phi_U(s) \in \mathcal{G}(U) \;\middle|\; s \in \mathcal{F}(U) \right\}, \tag{14.1}$$

but this will in general not be a sheaf. It is however a presheaf, as ϕ is compatible with restrictions. Gluing sections of the form $\phi_{U_i}(s_i)$ for a cover $\{U_i\}$ of U can be done inside $\mathcal{G}(U)$, but for the result to lie in $\operatorname{Im}\phi(U)$, one must make sure that the s_i's come from an element s in $\mathcal{F}(U)$. However, unless ϕ is injective, there is no reason to expect that the s_i's should agree on the intersections $U_i \cap U_j$. Here is a concrete example where this fails:

Example 14.3. Let Z be the closed subscheme given by the 'x-axis' in $\mathbb{A}^2_k$. That is, $Z = \operatorname{Spec} k[x]$ inside $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. Let $\iota\colon Z \to \mathbb{A}^2_k$ denote the inclusion, and consider the associated map of sheaves

$$\iota^\sharp\colon \mathcal{O}_{\mathbb{A}^2_k} \longrightarrow \iota_*\mathcal{O}_Z.$$

We claim that the naive image presheaf $\mathcal{G}$ given by $\mathcal{G}(W) = \operatorname{Im}(\iota^\sharp(W))$ is not a sheaf. To see why, let $U = D(x)$ and $V = D(y)$. For these open sets, we have $U \cap Z = Z - V(x)$ and $V \cap Z = \varnothing$. Over these open sets, the map $i^\sharp$ is given by

$$\begin{array}{ll} i^\sharp_U: & \mathcal{O}_{\mathbb{A}^2}(U) = k[x,y]_x \longrightarrow \mathcal{O}_Z(\iota^{-1}U) = k[x]_x \\ i^\sharp_V: & \mathcal{O}_{\mathbb{A}^2}(V) = k[x,y]_x \longrightarrow \mathcal{O}_Z(\iota^{-1}V) = 0 \\ i^\sharp_{U\cap V}: & \mathcal{O}_{\mathbb{A}^2}(U\cap V) = k[x,y]_{xy} \longrightarrow \mathcal{O}_Z(\iota^{-1}U\cap V) = 0. \\ i^\sharp_{U\cup V}: & \mathcal{O}_{\mathbb{A}^2}(U\cup V) = k[x,y] \longrightarrow \mathcal{O}_Z(\iota^{-1}(U\cup V)) = k[x]_x. \end{array}$$

Here we have used Example XXX for $\mathcal{O}_{\mathbb{A}^2}(U \cup V) = k[x,y]$. Now note that the elements $x^{-1} \in \mathcal{G}(U)$ and $0 \in \mathcal{G}(V)$ both restrict to 0 in $\mathcal{G}(U \cap V) = 0$. However, they do not glue together to a section over $U \cup V$, because there is no element of $k[x,y]$ that maps to x^{-1} in $k[x]_x$.

To define the image sheaf, we need to add in all sections that can be obtained by gluing together local sections of the form $\phi_{U_i}(s_i)$ as above. In other words, we take the sections of $\mathcal{G}(U)$ which are 'locally images of ϕ'. This will then be a subsheaf of $\mathcal{G}$; it is the smallest subsheaf of $\mathcal{G}$ containing the images of ϕ. For a later applications, we allow $\mathcal{F}$ to be simply a presheaf.

Definition 14.4. For a map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, we define the *image sheaf* $\operatorname{Im}\phi$ by

$$(\operatorname{Im}\phi)(U) = \left\{ t \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \text{there is a cover } U_i \text{ of } U \text{ and sections} \\ s_i \in \mathcal{F}(U_i) \text{ such that } t|_{U_i} = \phi(s_i) \end{array} \right\}.$$

This is a presheaf because ϕ is compatible with restrictions (the diagram (3.3) commutes). The Locality axiom holds for free because $\mathcal{G}$ is a sheaf. As for the Gluing axiom, suppose we are given an open cover $\{U_i\}$ of an open set U and sections $t_i \in (\operatorname{Im}\phi)(U_i)$ that agree on the overlaps. Since $\mathcal{G}$ is a sheaf, the t_i's glue together to a section $t \in \mathcal{G}(U)$, and t is by construction locally an image because each t_i is.

Unlike the situation for kernels, $(\operatorname{Im}\phi)(U)$ is not always equal to $\operatorname{Im}\phi_U$ (see Example 14.3). In general, all we can say is that $\operatorname{Im}\phi_U \subset (\operatorname{Im}\phi)(U)$ (any section of the form $t = \phi(s)$ clearly lies in $\operatorname{Im}\phi$). But in the particular case of *injective* maps, i.e. when each map ϕ_U is injective, the sheaf $\operatorname{Im}\phi$ coincides with the naive presheaf as in (14.1), and we have:

Lemma 14.5. If $\phi : \mathcal{F} \to \mathcal{G}$ is injective, then taking the image commutes with taking sections; that is, $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ for all U.

The situation for stalks is better: in general forming images commutes with forming stalks.

Lemma 14.6. For each $x \in X$ we have $(\operatorname{Im}\phi)_x = \operatorname{Im}\phi_x$.

Proof Let $t_x \in \operatorname{Im}\phi_x$ and pick an $s_x \in \mathcal{F}_x$ with $\phi_x(s_x) = t_x$. We may extend these germs to sections s and t over some open neighbourhood V, so that $\phi_V(s) = t$, and t is a section of $\operatorname{Im}\phi$ over V. This shows that $\operatorname{Im}\phi_x \subset (\operatorname{Im}\phi)_x$. Conversely, if t is a section of $\operatorname{Im}\phi$ over an open U containing x, the restriction $t|_V$ lies in $\operatorname{Im}\phi_V$ for some smaller neighbourhood V of x; hence the germ t_x lies in $\operatorname{Im}\phi_x$. □

A map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is said to be *surjective* if the *image sheaf* $\operatorname{Im}\phi$ equals $\mathcal{G}$. By the lemma below, this is equivalent to all the stalk maps ϕ_x being surjective (one says that 'ϕ is surjective on stalks'). However, we underline that this does not imply that the maps ϕ_U are surjective for every open U.

Lemma 14.7. Two subsheaves $\mathcal{H}, \mathcal{G}$ of a sheaf $\mathcal{F}$ are equal if and only if $\mathcal{H}_x = \mathcal{G}_x$ (as subgroups of $\mathcal{F}_x$) for all $x \in X$.

Proof Only the 'if-part' needs an argument, so let $U \subset X$ be open and let $s \in \mathcal{G}(U)$ be a section all whose germs s_x lie in $\mathcal{H}_x$. Extend each s_x to a section of $\mathcal{H}$ over some neighbourhood U_x of x; these extensions coincide on intersections $U_x \cap U_y$, and hence they patch together to a section in $\mathcal{H}(U)$, which by the Locality axiom equals s. This shows that $\mathcal{H} \subset \mathcal{G}$ as subsheaves of $\mathcal{F}$, and the same argument with $\mathcal{G}$ and $\mathcal{H}$ switched gives $\mathcal{H} = \mathcal{G}$. □

In the special case when X is affine, and $\mathcal{F}$ and $\mathcal{G}$ are of 'tilde-type', we have the following:

Proposition 14.8. If $X = \operatorname{Spec} A$ is an affine scheme, and M and N are A-modules, then the following are equivalent:

(i) $\phi : \widetilde{M} \to \widetilde{N}$ is surjective (resp. injective)
(ii) $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is surjective (resp. injective) for every $\mathfrak{p} \in \operatorname{Spec} A$.
(iii) $\phi_X : M \to N$ is surjective (resp. injective).

Example 14.9. The map $\iota^\sharp : \mathcal{O}_{\mathbb{A}^2_k} \to \iota_* \mathcal{O}_Z$ of Example 14.3 is surjective, as a map of sheaves, even though it is not surjective over every open set. To see this, note that $\mathbb{A}^2_k$ is covered by the two opens $U = D(x)$ and $U' = D(x-1)$. We already showed that $\iota^\sharp_U$ is surjective, as this is given by the quotient map $k[x,y]_x \to k[x]_x$. By Example 14.8, $\iota^\sharp_p$ is surjective for all $p \in U$. A similar argument applies to U', where the map $\iota^\sharp_{U'}$ is given by

$$\mathcal{O}_{\mathbb{A}^2_k} = k[x,y]_{x-1} \to \mathcal{O}_Z(\iota^{-1}U') = k[x]_{x-1}.$$

For a map $\phi : \mathcal{F} \to \mathcal{G}$ to be an *isomorphism*, the situation is better:

Proposition 14.10. Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then the following four conditions are equivalent.

(i) The map ϕ is an isomorphism;
(ii) For every $x \in X$, the map on stalks $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism;
(iii) One has $\operatorname{Ker} \phi = 0$ and $\operatorname{Im} \phi = \mathcal{G}$;
(iv) For all open subsets $U \subset X$ the map on sections $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

Proof (i) ⇒ (ii). This implication is clear.

(ii) ⇒ (iii). $\operatorname{Ker} \phi = 0$ follows by Lemma 14.2 and $\operatorname{Im} \phi = \mathcal{G}$ follows by Lemma 14.6 and (14.7).

(iii) ⇒ (iv). As $\operatorname{Ker} \phi = 0$, it follows that ϕ is injective, in which case taking images commutes with taking sections (Lemma 14.5), and so we have $\operatorname{Im} \phi_U = (\operatorname{Im} \phi)(U)$. But by assumption, $\operatorname{Im} \phi = \mathcal{G}$, so we are done.

(iv) ⇒ (iii). If ϕ_U is an isomorphism for every U, the inverse maps $\psi_U = \phi_U^{-1}$ gives an inverse $\psi : \mathcal{G} \to \mathcal{F}$. □

Exercise 14.0.1. Fill in the details of the proof of Lemma 14.5.

14.1 Exact sequences

Exact sequences are essential in the study of modules over a ring. There is an analogous notion of exactness for sequences of sheaves, which very much resembles the definition for modules.

A sequence of maps of sheaves

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \tag{14.2}$$

is said to be *exact* if $\operatorname{Im} \phi = \operatorname{Ker} \psi$ (as subsheaves of $\mathcal{G}$).

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0 \tag{14.3}$$

where we have exactness at all stages. This is just a convenient way of simultaneously saying that ϕ is injective, that ψ is surjective and that $\operatorname{Im}\phi = \operatorname{Ker}\psi$.

Exactness for a sequence of sheaves is a purely local condition; the sequence (14.2) is exact if and only if for each $x \in X$ the sequence induced on stalks

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \tag{14.4}$$

is exact. This follows from Lemma 14.7 applied to $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$.

The following proposition will be very important:

Proposition 14.11 (Taking sections is left exact). Given a short exact sequence as in (14.3). Then for each open subset $U \subset X$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

is exact.

Proof As ϕ is injective, we have that ϕ_U is injective, and also that $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ by Lemma 14.5. By definition, we have $\operatorname{Ker}\phi_U = (\operatorname{Ker}\phi)(U)$. Combining these, we find

$$(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U = \operatorname{Ker}\psi_U = (\operatorname{Ker}\psi)(U),$$

and hence the above sequence is exact. □

One way of phrasing Proposition 14.11 is to say that taking sections over an open set U is a *left exact functor*. This functor, however, is not exact in general. The defect of lacking surjectivity is a fundamental problem in every part of mathematics where sheaf theory is used, and to cope with it one has developed cohomology. (We will explore this in Chapter 17.)

Example 14.12. Consider the two points $p = (0 : 1)$ and $q = (1 : 0)$ in $\mathbb{P}^1_k$ and let $\iota : Z \to \mathbb{P}^1_k$ be the closed embedding given by their union. Let $\mathcal{I}$ be the kernel of the map $\iota^\sharp : \mathcal{O}_{\mathbb{P}^1_k} \to \iota_*\mathcal{O}_Z$. $\mathcal{I}$ fits into the following sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \xrightarrow{\iota^\sharp} \iota_*\mathcal{O}_Z \longrightarrow 0. \tag{14.5}$$

We first claim that this sequence is exact, i.e., that $\iota^\sharp$ is surjective. For this, it suffices to check that the map is surjective locally. If $U_0 = \mathbb{P}^1_k - p \simeq \operatorname{Spec} k[s]$, then $(\iota_*\mathcal{O}_Z)(U) = k[s]/s$ and the map $\iota^\sharp(U)$ is given by the quotient map $k[s] \to k[s]/s$, which is surjective. A similar argument shows that $\iota^\sharp$ is surjective over $U_1 = \mathbb{P}^1_k - q$. Hence the sequence (14.5) is exact.

Evaluating (14.5) on global sections, we get $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_k}) = k$ by Proposition 7.1 and $\iota_*\mathcal{O}_Z(\mathbb{P}^1_k) = \mathcal{O}_Z(Z) = k \oplus k$ and the sequence becomes

$$0 \longrightarrow \Gamma(\mathbb{P}^1_k, \mathcal{I}) \longrightarrow k \longrightarrow k \oplus k,$$

showing that the global evaluation map can not be surjective.

Example 14.13 (Sheaf version of the sheaf sequence)**.** For each inclusion $\iota\colon U \to X$ of an open subset U into a topological space X and each sheaf $\mathcal{F}$ on X there is a canonical map $\mathcal{F} \to \iota_*\mathcal{F}|_U$. Over an open $V \subset X$ it simply given by the restriction map $\mathcal{F}(V) \to \mathcal{F}(V \cap U)$. The fundamental sheaf sequence (3.2) on page 44 has a sheafy version involving these 'restriction maps'.

Given a finite open cover $\{U_i\}_{i\in I}$ of X, there is an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_i \iota_{i*}\mathcal{F}|_{U_i} \longrightarrow \prod_{i,j} \iota_{ij*}\mathcal{F}|_{U_{ij}}$$

where $\iota_i\colon U_i \to X$ denotes the inclusion map, where $U_{ij} = U_i \cap U_j$ and $\iota_{ij}\colon U_{ij} \to X$ also denotes the inclusion. Indeed, over an open $U \subset X$, the corresponding sequence of sections appears as

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U \cap U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U \cap U_{ij})$$

which being the fundamental sequence (3.2) for the cover $\{U \cap U_i\}$ of U is exact.

14.2 The sheaf associated to a presheaf

Essentially any construction for abelian groups, such as forming kernels, cokernels, tensor products, direct sums etc. have analogues for sheaves. For these constructions, one typically starts by writing down a naive presheaf and then proceeds to show that it satisfies the two sheaf axioms. This works well in some cases (e.g., for the kernel sheaf in the previous section), but in general, it can fail to be a sheaf (as for images). To obtain an actual sheaf, we sometimes need to replace this naive presheaf with a sheaf which in some sense best approximates it; in other words, as one says, we *sheafify* it. More precisely, to any presheaf $\mathcal{F}$, we shall build a sheaf $\mathcal{F}^+$ and a map of presheaves

$$\kappa_{\mathcal{F}}\colon \mathcal{F} \longrightarrow \mathcal{F}^+$$

which is universal among maps from $\mathcal{F}$ into a sheaf. The map kills the sections which are 'locally zero', that is, those whose stalks are all zero, and $\mathcal{F}^+$ 'enriches' $\mathcal{F}$ by including the results of all possible gluing processes.

The main properties of $\mathcal{F}^+$ and $\kappa_{\mathcal{F}}$ are summarised in the following proposition.

Proposition 14.14. Given a presheaf $\mathcal{F}$ on X, there is a sheaf $\mathcal{F}^+$ and a natural presheaf map $\kappa_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}^+$ satisfying the following:

(i) $\kappa_{\mathcal{F}}$ is functorial in $\mathcal{F}$: a map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a map of sheaves $\phi^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ such that $\phi^+ \circ \kappa_{\mathcal{F}} = \kappa_{\mathcal{G}} \circ \phi$;

(ii) $\kappa_{\mathcal{F}}$ enjoys the universal property that any map of presheaves $\mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, factors through $\mathcal{F}^+$ in a unique way. This property characterizes $\mathcal{F}^+$ up to a unique isomorphism. In other words, if $\mathcal{G}$ is a sheaf, there is a natural isomorphism

$$\operatorname{Hom}_{\mathsf{AbPrSh}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{Sh}_X}(\mathcal{F}^+, \mathcal{G}), \tag{14.6}$$

where on the left hand side $\mathcal{G}$ is considered as a presheaf;

(iii) $\kappa_{\mathcal{F}}$ induces an isomorphism on stalks: $\mathcal{F}_x \simeq \mathcal{F}^+_x$ for every $x \in X$.

We will now explain how to construct $\mathcal{F}^+$ and $\kappa_{\mathcal{F}}$ from $\mathcal{F}$. If you find the construction a bit daunting, don't worry, we will never need the explicit construction again. All of the arguments using $\mathcal{F}^+$ in this book use only the three properties in the Proposition 14.14. This is a good illustration of the slogan: "ask not what the thing is, but what it does".

The construction uses the so-called *Godement sheaf* $\Pi(\mathcal{F})$ associated with $\mathcal{F}$. For a presheaf $\mathcal{F}$, the sections of this sheaf is defined by

$$\Pi(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x = \{\, (t_x)_{x\in U} \mid t_x \in \mathcal{F}_x \,\}. \tag{14.7}$$

In other words, the sections are sequences $(t_x)_{x\in U}$ of arbitrary germs[1] at the various points x in U. The restriction maps are the projections: $(t_x)_{x\in U}|_V = (t_x)_{x\in V}$ for open subsets $V \subset U$. The first thing to check is that this indeed yields a sheaf:

Lemma 14.15. $\Pi(\mathcal{F})$ is a sheaf.

Proof Locality holds: if $\{U_i\}$ is an open cover of U, and $s = (s_x)_{x\in U}$ is a section of $\Pi(\mathcal{F})$ over U such that $s|_{U_i} = 0$ for each i, then $s_x = 0$ for every $x \in U_i$. Hence, if $t|_{U_i} = 0$ for all i, it follows that $t = 0$.

Gluing holds: Suppose we are given an open cover $\{U_i\}$ of U and sections $t_i = (t^i_x)_{x\in U_i}$ of $\Pi(\mathcal{F})$ over U_i matching on the intersections $U_i \cap U_j$. Saying that the sections agree over the overlaps, means that the component of t_i at a point $x \in U_i \cap U_j$ is the same as that of t_j. Hence we get a well-defined section $t \in \Pi(\mathcal{F})(U)$ by using this common component as the component of t at x. It is clear that $t|_{U_i} = t_i$. □

There is a canonical map

$$\sigma_{\mathcal{F}}\colon \mathcal{F} \longrightarrow \Pi(\mathcal{F})$$

that sends a section $s \in \mathcal{F}(U)$ to the sequence of all its germs; that is, to the element $(s_x)_{x\in U}$ of the product in (14.7). This map kills the sections of $\mathcal{F}$ which are 'locally zero'. Indeed, the kernel consists exactly of the sections with all germs equal to zero.

[1] The notation is not ideal: t_x is a germ at x, but at the same time, x serves as an index.

The map $\sigma_{\mathcal{F}}$ depends functorially on $\mathcal{F}$. For any map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we may define $\Pi(\phi)\colon \Pi(\mathcal{F}) \to \Pi(\mathcal{G})$ over an open U as the appropriate product of all the stalk-maps $\mathcal{F}_x \to \mathcal{G}_x$ with $x \in U$. In other words, $\Pi(\phi)_U$ sends $(s_x)_{x\in U}$ to $(\phi_x(s_x))_{x\in U}$. There is thus a commutative diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\sigma_{\mathcal{F}}} & \Pi(\mathcal{F}) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\Pi(\phi)} \\ \mathcal{G} & \xrightarrow[\sigma_{\mathcal{G}}]{} & \Pi(\mathcal{G}). \end{array} \tag{14.8}$$

It is not hard to check that $\Pi(\mathrm{id}_{\mathcal{F}}) = \mathrm{id}_{\Pi(\mathcal{F})}$ and that $\Pi(\psi \circ \phi) = \Pi(\psi) \circ \Pi(\phi)$ for two composable morphisms between presheaves on X, so that Π is a functor from the category of presheaves on X to the category of sheaves on X.

Definition 14.16. For a presheaf $\mathcal{F}$ on X, we define the *sheaf associated to* $\mathcal{F}$, or the *sheafification* of $\mathcal{F}$, as the image sheaf $\mathcal{F}$ of the map $\sigma_{\mathcal{F}}\colon \mathcal{F} \to \Pi(\mathcal{F})$. The map $\kappa_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}^+$ is just $\sigma_{\mathcal{F}}$, but considered to take values in $\mathcal{F}^+$.

Explicitly, a section of $\mathcal{F}^+(U)$ is a sequence $t = (t_x)_{x\in U}$ of elements in the $t_x \in \mathcal{F}_x$ that locally come from sections of $\mathcal{F}$, that is, there is an open cover $\{U_i\}$ and sections $s_i \in \mathcal{F}(U_i)$ so that $(s_i)_x = t_x$ for $x \in U_i$.

Taking the associated sheaf is a functorial operation. To each map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ there is a map of sheaves $\phi^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ that lives in the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\kappa_{\mathcal{F}}} & \mathcal{F}^+ & \longrightarrow & \Pi(\mathcal{F}) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi^+} & & \downarrow{\scriptstyle\Pi(\phi)} \\ \mathcal{G} & \xrightarrow[\kappa_{\mathcal{G}}]{} & \mathcal{G}^+ & \longrightarrow & \Pi(\mathcal{G}). \end{array} \tag{14.9}$$

Indeed, a section of $\mathcal{F}^+$ is a section of $\Pi(\mathcal{F})$ which locally comes from $\mathcal{F}$; that is, it is of the form $\sigma_F(s)$. But then $\Pi(\phi)(\sigma_{\mathcal{F}}(s))$ locally comes from $\mathcal{G}$ as well, because $\Pi(\phi)(\sigma_{\mathcal{F}}(s)) = \sigma_{\mathcal{G}}(\phi(s))$. Thus $\Pi(\mathcal{F})$ maps $\mathcal{F}^+$ into $\mathcal{G}^+$, and we let ϕ^+ be the restriction of $\Pi(\phi)$ to $\mathcal{F}^+$.

Proof of Proposition 14.14 Assertion (i) has already been taken care of.

As for (ii), the main observation is that when $\mathcal{G}$ is a sheaf, $\kappa_{\mathcal{G}}$ is an isomorphism; indeed, the Locality axiom then causes κ_G to be injective. On the other hand, $\mathrm{Im}\,\kappa_{\mathcal{G}}$ is the smallest subsheaf containing the 'naive presheaf image' of $\mathcal{G}$, which equals $\mathcal{G}$ itself when $\mathcal{G}$ is a sheaf. This means that $\phi^+ \circ \kappa_G^{-1}$ provides the wanted factorization.

Finally, let us prove claim (iii), starting with the surjectivity. An element $\mathcal{F}_x^+$ is the germ t_x of a section t of $\mathcal{F}^+$ over some open neighbourhood U of x. The section t comes locally from $\mathcal{F}$, so its restrictions to the open sets belonging to some open cover $\{U_i\}$ of the neighbourhood are of the form $t|_{U_i} = \kappa_{\mathcal{F}}(s_i)$ with $s_i \in \mathcal{F}(U_i)$. Now x lies in one of the U_i's, and hence the corresponding germ $(s_i)_x$ maps to t_x. The injectivity follows since the kernel of $\kappa_{\mathcal{F}}$ consists of sections with all germs vanishing, but tautologically, these vanish already in $\mathcal{F}_x$. □

Example 14.17. A presheaf $\mathcal{F}$ which is contained in a sheaf $\mathcal{G}$ is particularly easy to sheafify. The sheafification $\mathcal{F}^+$ equals the image of the inclusion map $\mathcal{F} \to \mathcal{G}$. The sections in $\mathcal{F}^+(U)$

are the sections in $\mathcal{G}(U)$ that locally lie in $\mathcal{F}$; that is, sections s so that $s|_{U_i} \in \mathcal{F}(U_i)$ for some open cover $\{U_i\}$ of U.

Exercise 14.2.1. Prove that the sheafification is unique up to a unique isomorphism.

Cokernels and quotients

We follow the strategy outlined above and define cokernels and quotient sheaves using the sheafification procedure.

For a map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, we define the *cokernel* $\operatorname{Coker}\phi$ to be the sheaf associated to the presheaf

$$(\operatorname{Coker}'\phi)(U) = \mathcal{G}(U)/\operatorname{Im}\phi(U).$$

For a subsheaf $\mathcal{G} \subset \mathcal{F}$ of a sheaf $\mathcal{G}$, the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ is the sheaf associated to the presheaf

$$(\mathcal{F}/\mathcal{G})'(U) = \mathcal{F}(U)/\mathcal{G}(U).$$

In other words, $\mathcal{F}/\mathcal{G}$ is the cokernel of the inclusion map $\mathcal{G} \to \mathcal{F}$.

Note that over an open set U, the cokernel presheaf is simply given by $\operatorname{Coker}\phi_U$. Composing ϕ with the canonical map $\operatorname{Coker}'\phi \to \operatorname{Coker}\phi$ we obtain a map $\mathcal{G} \to \operatorname{Coker}\phi$. It sits in the sequence

$$\mathcal{F} \xrightarrow{\ \phi\ } \mathcal{G} \longrightarrow \operatorname{Coker}\phi \longrightarrow 0. \tag{14.10}$$

Example 14.18. In the sequence (14.5) the subsheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^1_k}$ identifies with the sections of $\mathcal{O}_{\mathbb{P}^1_l}$ vanishing along the subscheme Z. By the uniqueness of the cokernel, we get an isomorphism of sheaves $\mathcal{O}/\mathcal{I} \simeq \iota_*\mathcal{O}_Z$. Even in this example it is necessary to sheafify, as the 'naive' quotient sheaf on global sections satisfies $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)/\mathcal{I}(\mathbb{P}^1) = k$, whereas $(\iota_*\mathcal{O}_Z)(\mathbb{P}^1) = \mathcal{O}_Z(Z) = k \oplus k$.

Exercise 14.2.2. Show that the sequence (14.10) is exact. HINT: Show that it is exact on stalks.

Exercise 14.2.3 (Universal properties)**.** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves.

a) Show that $\operatorname{Ker}\phi$ satisfies the following universal property: Any map of sheaves $\nu : \mathcal{H} \to \mathcal{F}$ such that $\nu \circ \phi = 0$ factors via a unique map $\overline{\nu} : \mathcal{H} \to \operatorname{Ker}\phi$.
b) Show that $\operatorname{Im}\phi$ satisfies the following universal property: Given a map of sheaves $\alpha : \mathcal{F} \to \mathcal{H}$ and $\beta : \mathcal{H} \to \mathcal{G}$ such that $\beta \circ \alpha = \phi$, there is a unique morphism $t : \mathcal{H} \to \operatorname{Im}\phi$ factoring β.
c) Show that $\operatorname{Coker}\phi$ satisfies the following universal property: Given a map $\psi : \mathcal{G} \to \mathcal{H}$ with $\psi \circ \phi = 0$, there is a unique map $t : \operatorname{Coker}\phi \to \mathcal{H}$ factoring ψ.

HINT: The arguments in each case are rather different. For b), use the explicit description of $\operatorname{Im}\phi$. For c), the universal property of sheafification may be helpful.

14.3 Direct sums and products

The category of sheaves also has direct sums. For a finite collection $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of sheaves, the presheaf given by

$$\Gamma(U, \bigoplus_{i=1}^{n} \mathcal{F}_i) = \bigoplus_{i=1}^{n} \mathcal{F}_i(U)$$

is a sheaf. Indeed, restrictions are given componentwise and Locality holds because if $s = (s_1, \ldots, s_n) \in \bigoplus_{i=1}^{n} \mathcal{F}_i(U)$ restricts to 0 on a covering, then all $s_1 = \cdots = s_n = 0$ by locality for the $\mathcal{F}_i$'s. Likewise, given local sections matching on the overlaps, one can glue componentwise.

This all works well for finitely many sheaves, but for a general collection of sheaves $\{\mathcal{F}_i\}_{i \in I}$ one has to sheafify in order to define the direct sum. That is, we define $\bigoplus_{i \in I} \mathcal{F}_i$ to be the sheaf *associated to* the presheaf

$$\left(\bigoplus_{i \in I} \mathcal{F}_i \right)' (U) = \bigoplus_{i \in I} \mathcal{F}_i(U). \tag{14.11}$$

For the collection $\mathcal{F}_i$, one can also form the *direct product*, denoted $\prod_i \mathcal{F}_i$, which is defined by

$$\Gamma\left(U, \prod_{i \in I} \mathcal{F}_i \right) = \prod_{i \in I} \mathcal{F}_i(U). \tag{14.12}$$

This is again a presheaf with componentwise restriction maps. It is not necessary to sheafify $\prod_i \mathcal{F}_i$; gluing can be done componentwise.

Example 14.19. Here is an example showing that it is necessary to sheafify in the definition of the direct sum. Let $X = \coprod_{n=1}^{\infty} \operatorname{Spec} \mathbb{C}$ be the disjoint union of countably many copies of $\operatorname{Spec} \mathbb{C}$. The topology on X is the discrete topology. For each $n \in \mathbb{N}$, let $\iota_n \colon p_n = \operatorname{Spec} \mathbb{C} \to X$ be the open embedding of the n-th copy of $\operatorname{Spec} \mathbb{C}$ and let $\mathcal{F}_n = \iota_{n*}\mathbb{C}$ the skyscraper sheaf at p_n.

We let $\mathcal{F} = \bigoplus_{n=1}^{\infty} \mathcal{F}_n$ and claim that $\mathcal{F}(X) \neq \bigoplus_{n=1}^{\infty} \mathcal{F}_n(X)$. Note that the right hand side is just the countable sum $\bigoplus_{n=1}^{\infty} \mathbb{C}$. On the other hand, X is covered by the open sets $U_n = \{p_n\}$ and the elements $x_n = 1 \in \mathcal{F}_n(U_n)$ trivially agree on the (empty) intersection $U_m \cap U_n$ for $m \neq n$. Therefore the x_n's glue to an element $x \in \mathcal{F}(X)$, which, since all the x_n's are non-zero, can not lie in $\bigoplus_n \mathcal{F}_n(X)$.

Summing up what we have done so far, the category AbSh_X of sheaves on X is an abelian category. It is an additive category and every map has a kernel, a cokernel and an image, and every map α lives in an exact sequence

$$0 \longrightarrow \operatorname{Ker} \alpha \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \longrightarrow \operatorname{Coker} \alpha \longrightarrow 0$$

Exercise 14.3.1. Show that the direct product presheaf $\prod_{i \in I} \mathcal{F}_i$ defined above is a sheaf.

Exercise 14.3.2 (Universal properties of $\bigoplus$ and $\prod$)**.** Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of sheaves.

a) Show that the direct sum has canonical inclusions $\epsilon_i\colon \mathcal{F}_i \to \bigoplus_i \mathcal{F}_i$, which have the following universal property: for any family of maps $\eta_i\colon \mathcal{F}_i \to \mathcal{G}$ there is a unique map $\eta\colon \bigoplus_i \mathcal{F}_i \to \mathcal{G}$ such that $\eta_i = \eta \circ \epsilon_i$.
b) Show that the direct product has canonical projections $\pi_i\colon \prod_i \mathcal{F}_i \to \mathcal{F}_i$ having the universal property dual to the direct sum: i.e. for any family of maps $\epsilon_i\colon \mathcal{F}_i \to \mathcal{G}$ there is a map $\eta\colon \mathcal{F}_i \to \prod_i \mathcal{F}_i$ such that $\pi_i \circ \eta = \epsilon_i$.

Exercise 14.3.3. Show that the direct sum can be defined as the image sheaf of the natural map $\bigoplus_{i\in I} \mathcal{F}_i \to \prod_{i\in I} \mathcal{F}_i$ where the left-hand side is regarded as a presheaf. Hint: Use Example 14.17 and the universal property of $\bigoplus$.

14.4 Sheaves of modules

A module over a ring is an additive abelian group equipped with a multiplicative action of the ring. Loosely speaking, we can multiply elements of the module by elements from the ring. In a similar way, if X is a scheme, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ whose sections over open sets U can be multiplied by sections of $\mathcal{O}_X(U)$.

More formally, we define an *$\mathcal{O}_X$-module* as a sheaf $\mathcal{F}$ equipped with multiplication maps $\mathcal{F}(U) \times \mathcal{O}_X(U) \to \mathcal{F}(U)$, one for each open subset U of X, making the group of sections $\mathcal{F}(U)$ into a $\mathcal{O}_X(U)$-module in a manner which is compatible with restriction maps. In other words, for every pair of open subsets $V \subset U$, the diagram below is required to commute

$$\begin{array}{ccc} \mathcal{F}(U) \times \mathcal{O}_X(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) \times \mathcal{O}_X(V) & \longrightarrow & \mathcal{F}(V). \end{array} \tag{14.13}$$

Here vertical arrows represent restrictions maps and horizontal ones are multiplication maps.

A *map of $\mathcal{O}_X$-modules* is simply a map of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ such that for each open U the map $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a map of $\mathcal{O}_X(U)$-modules. The $\mathcal{O}_X$-modules on a scheme X therefore form a category, which we denote by Mod_X . We write $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G})$, or sometimes $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the set of $\mathcal{O}_X$-linear maps $\mathcal{F} \to \mathcal{G}$. Note that this is an abelian group.

Most of the constructions for modules over a ring now have analogues for $\mathcal{O}_X$-modules.

For instance, for a map of $\mathcal{O}_X$-modules $\alpha : \mathcal{F} \to \mathcal{G}$, the kernel, image and cokernel of α, as defined in Section XXX, have natural $\mathcal{O}_X$-modules structures. Here it is clear that the image and cokernel presheaves have natural $\mathcal{O}_X$-module structures, and then Exercise 14.5.2 shows that also the associated sheaves are $\mathcal{O}_X$-modules.

If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, the direct sum $\mathcal{F} \oplus \mathcal{G}$ is also an $\mathcal{O}_X$-module in a natural way, with multiplication being defined component-wise. The same is true for more general direct sums $\bigoplus_{i\in I} \mathcal{F}_i$.

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we can also define the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$T(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \tag{14.14}$$

(Here it is necessary to sheafify; see Example XXX.) We will sometimes write simply $\mathcal{F} \otimes \mathcal{G}$ for this tensor product.

If $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules, the presheaf given by $\operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ over an open set $U \subset X$ is a sheaf, denoted by $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. This is also a $\mathcal{O}_X$-module in a natural way.

For a morphism $f : X \to Y$ and an $\mathcal{O}_X$-module $\mathcal{F}$, the pushfoward $f_*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module via the natural map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. That is, for a section $s \in f_*\mathcal{F}(V)$ and $a \in \mathcal{O}_Y(V)$, we define $a \cdot s \in f_*\mathcal{F}(V)$ to be section $f^\sharp(a) \cdot s \in \mathcal{F}(f^{-1}V)$.

Example 14.20 (Ideal sheaves)**.** Ideal sheaves are important examples of $\mathcal{O}_X$-modules. Formally, a sheaf $\mathcal{I}$ is an ideal sheaf if $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ and $\mathcal{I}(U)$ is an ideal for each open set $U \subset X$. For an ideal sheaf $\mathcal{I}$, the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ associated to an ideal sheaf $\mathcal{I}$ is an $\mathcal{O}_X$-module.

The primary example is the following. Let $\iota : Y \to X$ be a closed embedding, then the kernel $\mathcal{I}$ of the map $\iota^\sharp : \mathcal{O}_X \to \iota_*\mathcal{O}_Y$ is an ideal sheaf of $\mathcal{O}_X$, and there is an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \iota_*\mathcal{O}_Y \to 0$$

We also see that $\iota_*\mathcal{O}_Y \simeq \mathcal{O}_X/\mathcal{I}$ as $\mathcal{O}_X$-modules.

See Example 14.12 for a more concrete example of an ideal sheaf.

Example 14.21. If $\mathcal{F}$ is a sheaf obtained by gluing together sheaves $\mathcal{F}_i$ defined on a cover $\mathcal{U} = \{U_i\}$, and each $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module, then $\mathcal{F}$ is an $\mathcal{O}_X$-module.

Example 14.22. Write $\mathbb{P}^1$ for the projective line over a field k, and consider the sheaves $\mathcal{O}_X(n)$ from Section 7.7. That is, $\mathcal{O}_{\mathbb{P}^1}(n)$ is the sheaf obtained by gluing $\mathcal{O}_{U_0}$ to $\mathcal{O}_{U_1}$ using the isomorphism $\mathcal{O}_{U_1}|_{U_0 \cap U_1} \to \mathcal{O}_{U_0}|_{U_0 \cap U_1}$ on $U_0 \cap U_1 = \operatorname{Spec} k[u, u^{-1}]$ given by multiplication by u^n. Then $\mathcal{O}_{\mathbb{P}^1}(n)$ is an $\mathcal{O}_{\mathbb{P}^1}$-module. The map $\phi : \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ is a map of $\mathcal{O}_{\mathbb{P}^1}$-modules, and the image of ϕ is an ideal sheaf of $\mathcal{O}_{\mathbb{P}^1}$.

Example 14.23 (Modules on spectra of DVR's)**.** Modules on the prime spectrum of a discrete valuation ring R are particularly easy to describe. Recall that the scheme $X = \operatorname{Spec} R$ has only two non-empty open sets: the whole space X itself and the $\{\eta\}$ consisting of the generic point. The singleton $\{\eta\}$ is the underlying set of the open subscheme $\operatorname{Spec} K$, where K denotes the fraction field of R.

We claim that giving an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space N and an R-module homomorphism $\rho\colon M \to N$.

Indeed, given an $\mathcal{O}_X$-module $\mathcal{F}$, we get the R-modules $M = \mathcal{F}(X)$ and $N = \mathcal{F}(\{\eta\})$, and the latter is a vector space over $K = \mathcal{O}_X(\{\eta\})$. The homomorphism ρ is just the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{\eta\})$. Conversely, given the data M, N and a map $\rho\colon \mathcal{F}(X) \to \mathcal{F}(\{\eta\})$, we can define a presheaf $\mathcal{F}$ by setting $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = N$ and use ρ as the restriction map. If we also set $\mathcal{F}(\varnothing) = 0$, we have a presheaf $\mathcal{F}$ which satisfies the two sheaf axioms. Furthermore, since M and N are modules over $\mathcal{O}_X(X) = R$ and $O_X(\{\eta\}) = K$ respectively, this makes $\mathcal{F}$ into an $\mathcal{O}_X$-module.

Note that the restriction map can be any R-module homomorphism $M \to N$. In particular, it can be the zero homomorphism, and in that case M and N can be completely arbitrary modules.

Exercise 14.4.1. Let $X = \mathbb{A}^1_{\mathbb{C}}$ and let $\mathcal{F}$ be the constant sheaf on $\mathbb{Z}$. Is $\mathcal{F}$ an $\mathcal{O}_X$-module?

Example 14.24 (Godement sheaves again)**.** We may generalize the construction of the Godement sheaf in the following way. Given any collection of abelian groups $\{A_x\}_{x \in X}$ indexed by the points x of X, we can define a sheaf $\mathcal{A}$ by

$$\mathcal{A}(U) = \prod_{x \in U} A_x,$$

and whose restriction maps to smaller open subsets are just the projections onto the corresponding smaller products.

If we suppose that each A_x be a module over the stalk $\mathcal{O}_{X,x}$, the sheaf $\mathcal{A}$ becomes an $\mathcal{O}_X$-module. Indeed, the group $\Gamma(U, \mathcal{A}) = \prod_{x \in U} A_x$ is automatically an $\mathcal{O}_X(U)$-module, as the the multiplication is defined component-wise with the help of the stalk maps $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$. Clearly these module structures are compatible with the projections, and thus makes $\mathcal{A}$ into an $\mathcal{O}_X$-module.

14.5 Exercises

Exercise 14.5.1. Let $A = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. Describe all $\mathcal{O}_X$-modules on $X = \operatorname{Spec} A$.

Exercise 14.5.2. Suppose that $\mathcal{F}$ is a *presheaf* of $\mathcal{O}_X$-modules (i.e. a presheaf satisfying the usual $\mathcal{O}_X$-module axioms). Show that the sheafification $\mathcal{F}^+$ is an $\mathcal{O}_X$-module in a natural way.

HINT: One can use the universal properties of sheafification, but the simplest way is via the explicit description of $\mathcal{F}^+$.

Exercise 14.5.3. Let $\alpha \colon \mathcal{F} \to \mathcal{G}$ be a map between two $\mathcal{O}_X$-modules.

a) Show in detail that the kernel, cokernel and image of α as a map of sheaves indeed are $\mathcal{O}_X$-modules. Moreover, show that they satisfy the universal properties of kernel, cokernel and image in the category of $\mathcal{O}_X$-modules as well.
b) Show that a sequence of $\mathcal{O}_X$-modules is exact if and only it is exact as a sequence of sheaves.

Exercise 14.5.4. Show that the category Mod_X has arbitrary products and direct sums, by showing that the products and sums in the category of sheaves AbSh_X are $\mathcal{O}_X$-modules and are the products, respectively the direct sums, in the category Mod_X.

Exercise 14.5.5. For each of the schemes below, describe the $\mathcal{O}_X$-modules on X.

a) X is the scheme obtained by gluing $\operatorname{Spec} \mathbb{Z}_{(2)}$ and $\operatorname{Spec} \mathbb{Z}_{(3)}$ along their common open subscheme $\operatorname{Spec} \mathbb{Q}$.
b) X is the scheme obtained by gluing two copies of $\operatorname{Spec} \mathbb{Z}_{(2)}$ along $\operatorname{Spec} \mathbb{Q}$.
c) Let X be the scheme obtained by gluing the schemes $X_i = \operatorname{Spec} \mathbb{Z}_{(p_i)}$ together along their common open subschemes $\operatorname{Spec} \mathbb{Q}$. Describe the $\mathcal{O}_X$-modules on X.

(Here $\mathbb{Z}_{(p)}$ denotes the localization at the prime ideal (p).)

15

Quasi-coherent sheaves

15.1 The tilde of a module

The primary example of an $\mathcal{O}_X$-module is a sheaf of the form $\widetilde{M}$ which we introduced in Section 4.4. Let us briefly recall the construction. If A is a ring, and M is an A-module, the sheaf $\widetilde{M}$ on $X = \operatorname{Spec} A$ is the sheaf extending the following $\mathscr{B}$-sheaf

$$\widetilde{M}(D(f)) = M_f.$$

The restriction maps are the canonical localization maps, which are described as follows: when $D(g) \subset D(f)$, we may write $g^n = af$ for some $a \in A$ and some $n \in \mathbb{N}$, and the localization map $M_f \to M_g$ sends mf^{-r} to $a^r mg^{-nr}$.

It is almost immediate that $\widetilde{M}$ is an $\mathcal{O}_X$-module. Over a distinguished open set $U = D(f)$, the group $\widetilde{M}(D(f)) = M_f$ is a module over A_f, and if $U \subset X$ is any open subset, we may cover it by distinguished open sets $D(f)$ and define a $\mathcal{O}_X(U)$-module structure on $\widetilde{M}(U)$ by means of the exact sequence in claim (iii) of Proposition 4.22. In the same way, one verifies that the restriction maps are $\mathcal{O}_X$-module homomorphisms. The tilde-construction therefore yields a functor from Mod_A to Mod_X, and it has very good properties, as we are going to see. We start by explaining a the universal property of $\widetilde{M}$ among $\mathcal{O}_X$-modules.

Proposition 15.1. Let $X = \operatorname{Spec} A$ be an affine scheme. For an A-module M and an $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \xrightarrow{\simeq} \operatorname{Hom}_A(M, \mathcal{F}(X))$$

that sends $\phi\colon \widetilde{M} \to \mathcal{F}$ to $\phi_X : M \to \mathcal{F}(X)$. It is functorial in both M and $\mathcal{F}$.

Proof Let $f \in A$, and consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi_X} & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ M_f & \xrightarrow{\phi_{D(f)}} & \mathcal{F}(D(f)) \end{array}$$

where the vertical maps are restriction maps. This gives the following relation:

$$\phi_{D(f)}(m|_{D(f)}) = \phi_X(m)|_{D(f)}.$$

Note that $\mathcal{F}(D(f))$ is an A_f-module, because $\mathcal{F}$ is an $\mathcal{O}_X$-module. Therefore, in the localizations at f, we have the following relation

$$\phi_{D(f)}(mf^{-n}) = \phi_X(m)|_{D(f)} \cdot f^{-n}, \tag{15.1}$$

where $mf^{-n} \in M_f$. This means that the maps $\phi_{D(f)}$ are completely determined by $\phi_X : M \to \mathcal{F}(X)$. By Proposition 3.17, the map of sheaves ϕ is completely determined once it is specified over the $D(f)$'s. Thus, ϕ is determined by ϕ_X, and the map in the proposition is injective.

For the surjectivity, suppose we are given a map of A-modules $\alpha\colon M \to \mathcal{F}(X)$. As usual, to define a map $\widetilde{M} \to \mathcal{F}$ it suffices to tell what it does to sections over the distinguished open sets $D(f)$. Inspired by (15.1), we define $\alpha_{D(f)}$ by

$$\alpha_{D(f)}(mf^{-n}) = \alpha(m)|_{D(f)} \cdot f^{-n}.$$

Thus $\alpha_{D(f)}$ is simply the composition of the two maps of A_f-modules

$$M_f \xrightarrow{\alpha_f} \mathcal{F}(X)_f \longrightarrow \mathcal{F}(D(f)),$$

where the right-hand map is induced from the restriction map $\mathcal{F}(X) \to \mathcal{F}(D(f))$ by localization (note that $\mathcal{F}(D(f))$ is an A_f-module). This is compatible with the restriction maps, so we get a well-defined map of sheaves $\phi : \widetilde{M} \to \mathcal{F}$. Taking $f = 1$, we see that we recover α from ϕ on global sections.

The statement about the functoriality follows from formula (15.1); the details are left to the reader. □

If we apply Proposition 15.1 to $M = \mathcal{F}(X)$ and consider the preimage of the identity map $\mathcal{F}(X) \to \mathcal{F}(X)$, we obtain the following corollary:

Corollary 15.2. For each $\mathcal{O}_X$-module $\mathcal{F}$ on an affine scheme X, there is a *unique* $\mathcal{O}_X$-module homomorphism

$$\beta_{\mathcal{F}}\colon \widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F} \tag{15.2}$$

that induces the identity on the spaces of global sections. The map $\beta_{\mathcal{F}}$ is functorial in $\mathcal{F}$.

In concrete terms, the map $\beta_{\mathcal{F}}$ is defined over a distinguished open subset $D(f)$ as follows. A section of the sheaf $\widetilde{\mathcal{F}(X)}$ over $D(f)$ is an element of the form sf^{-n} where $s \in \mathcal{F}(X)$. Regarding f^{-n} as a section of $\mathcal{O}_X(D(f)) = A_f$, we may send sf^{-n} to the product $s|_{D(f)} \cdot f^{-n}$, which, because $\mathcal{F}$ is an $\mathcal{O}_X$-module, defines a section of $\mathcal{F}(D(f))$.

The following proposition summarizes the basic properties of the tilde-functor.

Proposition 15.3 (Properties of the tilde-functor). Let A be a ring and let $X = \operatorname{Spec} A$. Then:

(i) The tilde-functor is additive, i.e., it takes direct sums to direct sums.

(ii) For any two A-modules M and N, the map $\alpha \mapsto \widetilde{\alpha}$ gives an isomorphism $\operatorname{Hom}_A(M,N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$, whose inverse is the map $\phi \mapsto \phi_X$;

(iii) The tilde-functor is exact.

Proof For statement (i) see Exercise **??**. Statement (ii) follows from Proposition 15.1 with $\mathcal{F} = \widetilde{N}$ and the fact that by definition $(\widetilde{\alpha})_X = \alpha$. To prove statement (iii), let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{15.3}$$

be an exact sequence of A-modules. This gives the sequence $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow 0. \tag{15.4}$$

To check that (15.4) is exact, it suffices to check that it is exact on stalks for every point $x \in X$. But if $x \in X$ corresponds to the prime ideal $\mathfrak{p} \subset A$, the stalks of (15.4) is simply the localization of (15.3) at $\mathfrak{p}$ (which is exact, because localization is an exact functor). □

Item (ii) above says that the tilde functor is fully faithful. Hence it establishes an equivalence between the category Mod_A of A-modules and a subcategory of Mod_X. This subcategory is usually a strict subcategory; most $\mathcal{O}_X$-modules are not of tilde-type.

The next result tells us that the restriction of a tilde type sheaf to an affine open is again of tilde type. More precisely, let $X = \operatorname{Spec} A$ be an affine scheme, with an open subscheme $U = \operatorname{Spec} B \subset X$, and let M be an A-module. Then the group of sections $\widetilde{M}(U)$ is a module over $\mathcal{O}_X(U) = B$, and there is a B-linear map

$$M \otimes_A B \to \left(\widetilde{M}|_U\right)(U),$$

defined by $m \otimes b \mapsto bm|_U$. Applying tilde, we get a map of $\mathcal{O}_U$-modules

$$\widetilde{M \otimes_A B} \to \widetilde{M}|_U, \tag{15.5}$$

and this turns out to be an isomorphism:

Proposition 15.4 (Restriction of tilde type to open affines). Let $X = \operatorname{Spec} A$ and let $U = \operatorname{Spec} B \subset X$ be an open affine subscheme. Then for each A-module M the canonical map in (15.5) is an isomorphism

$$\widetilde{M \otimes_A B} \simeq \widetilde{M}|_U.$$

Proof By Proposition XXX, it suffices to prove that (15.5) is an isomorphism on every stalk. If $y \in U$ corresponds to the prime ideal $\mathfrak{p} \subset A$, the induced map on stalks is given by the isomorphism

$$(M \otimes_A B)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \simeq M_{\mathfrak{p}}.$$

□

The restriction to an open affine open is a special case of a *pullback*; we will study these in more detail in Section **??**.

We end this section by describing how tilde-type modules behave when pushed forward. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Giving a map $f\colon X \to Y$ is the same thing as giving the map of rings $\phi\colon A \to B$, which in turn is equivalent to giving an A-algebra structure on B. Any B-module M is therefore also an A-module, and when wanting to emphazise the A-module structure of M, we will write M_A for M considered as an A -module. In particular, it holds for localizations in elements $g \in A$ that $M_{\phi(g)} = (M_A)_g$.

Recall Proposition 2.27 which says that $f^{-1}D(g) = D(\phi(g))$. This means that we have equalities

$$(f_*\widetilde{M})(D(g)) = \widetilde{M}(f^{-1}D(g)) = (M_A)_g;$$

the last by (ii) of Proposition 4.22, and the first by definition of pushforwards. These equalities are compatible with restrictions, and so citing Exercise 3.4.1 on page 53), we have shown:

Proposition 15.5 (Pushforward of tilde type modules). Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ be affine schemes, and let $f\colon X \to Y$ be a morphism. For each $\mathcal{O}_X$-module of tilde type $\widetilde{M}$ on X it holds that $f_*\widetilde{M} = \widetilde{M_A}$.

Example 15.6. If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $f : X \to Y$ is a morphism induced by $\phi : A \to B$, then canonical map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is the map $\widetilde{A} \to \widetilde{B}$, where we consider B as an A-module.

15.2 Quasi-coherent sheaves

The following is the most important definition in this chapter.

Definition 15.7. Let X be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *quasi-coherent* if there is an open affine covering $\{U_i\}_{i\in I}$ of X, say $U_i = \operatorname{Spec} A_i$, and modules M_i over A_i such that for each i there's an isomorphism $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ of $\mathcal{O}_X$-modules.

In particular, the modules $\widetilde{M}$ on an affine scheme $\operatorname{Spec} A$ are all quasi-coherent. Note that a priori, there could be more quasi-coherent sheaves on $\operatorname{Spec} A$. Indeed, for an $\mathcal{O}_X$-module $\mathcal{F}$ to be quasi-coherent, we require that $\mathcal{F}$ be locally of tilde-type for just *one* open affine cover. However, it turns out that this will hold for *any* open affine cover, or in other words, that $\mathcal{F}|_U$ is of tilde-type for any open affine subset $U \subset X$ (in particular for X itself when X is affine). This is a much stronger than the requirement in the definition, and is a rather important fact.

Theorem 15.8. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only for any open affine subscheme $U = \operatorname{Spec} A$ in X, the restriction $\mathcal{F}|_U$ is of tilde-type; that is, there is an A-module M and an isomorphism of $\mathcal{O}_X$-modules $\mathcal{F}|_U \simeq \widetilde{M}$.

Proof Let $\mathcal{P}$ be the following property of an open affine U in X: the canonical map

$$\beta_{\mathcal{F}|_U} \colon \widetilde{\mathcal{F}(U)} \longrightarrow \mathcal{F}|_U$$

from Corollary 15.2 is an isomorphism. The key point is that $\mathcal{P}$ is a distinguished property (as defined on page **??**). Given this, the theorem follows, by Proposition XXX.

The first requirement of Proposition XXX comes for free, because if $D(g) \subset U$ is distinguished, we have

$$\mathcal{F}|_{D(g)} \simeq \widetilde{\mathcal{F}(U)}|_{D(g)} = \widetilde{\mathcal{F}(U)}_g.$$

The second condition requires some work. Let $U \subset X$ be an open affine covered by two distinguished opens $D(g_1)$ and $D(g_2)$ both having property $\mathcal{P}$. This means that we have isomorphisms

$$\beta_{\mathcal{F}|_{D(g_i)}} \colon \mathcal{F}(\widetilde{D(g_i)}) \xrightarrow{\simeq} \mathcal{F}|_{D(g_i)}.$$

In view of $D(g_1) \cap D(g_2) = D(g_1 g_2)$, the fundamental exact sheaf sequence (3.2) on page 44 for the cover $\{D(g_1), D(g_2)\}$ of U takes the form

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}(D(g_1)) \oplus \mathcal{F}(D(g_2)) \longrightarrow \mathcal{F}(D(g_1 g_2)),$$

and there is also a corresponding exact sequence for the restrictions of the involved sheaves (as in Example 14.13)

$$0 \longrightarrow \mathcal{F}|_U \longrightarrow \iota_{1*}\mathcal{F}|_{D(g_1)} \oplus \iota_{2*}\mathcal{F}|_{D(g_2)} \longrightarrow \iota_{12*}\mathcal{F}|_{D(g_1 g_2)}.$$

where $\iota_i \colon D(g_i) \to U$ and $\iota_{12} \colon D(g_1 g_2) \to U$ denote the inclusion maps. Since the β-maps are functorial, these two sequences give rise to the following commutative diagram of $\mathcal{O}_X$-modules

$$\begin{array}{ccccccc} 0 \longrightarrow & \widetilde{\mathcal{F}(U)} & \longrightarrow & \iota_{1*}\mathcal{F}(\widetilde{D(g_1)}) \oplus \iota_{2*}\mathcal{F}(\widetilde{D(g_2)}) & \longrightarrow & \iota_{12*}\mathcal{F}(\widetilde{D(g_1 g_2)}) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathcal{F}|_U & \longrightarrow & \iota_{1*}\mathcal{F}|_{D(g_1)} \oplus \iota_{2*}\mathcal{F}|_{D(g_2)} & \longrightarrow & \iota_{12*}\mathcal{F}|_{D(g_1 g_2)} \end{array}$$

where the vertical maps are the appropriate β-maps. The upper sequence is exact since the tilde-functor is exact (bearing Proposition 15.5 in mind), and the two vertical maps to the right are isomorphism; the middle one by assumption and the rightmost one by requirement **(D1)**. We then finish the proof by appealing to the 5-lemma, which shows that the left vertical map, which equals $\beta_{\mathcal{F}|_U}$, is an isomorphism. □

Applying Theorem 15.8 to affine schemes yields the important fact that each quasi-coherent sheaf on an affine scheme $X = \operatorname{Spec} A$ is of tilde-type.

Theorem 15.9. Assume that $X = \operatorname{Spec} A$. The tilde-functor $M \mapsto \widetilde{M}$ and the global section functor $\mathcal{F} \mapsto \mathcal{F}(X)$ are mutually inverse functors giving an equivalence of the categories Mod_A and QCoh_X.

When speaking about mutually inverse functors one should be careful; often such a statement is an abuse of language. Two functors F and G are mutually inverses when there are natural transformations, both being isomorphisms, between the compositions $\mathsf{F} \circ \mathsf{G}$ and $\mathsf{G} \circ \mathsf{F}$ and the appropriate identity functors. In the present case one really has an *equality* $\Gamma(X, \widetilde{M}) = M$, so that $\Gamma \circ \widetilde{(-)} = \mathrm{id}_{\mathsf{Mod}_A}$. On the other hand, the functorial maps $\beta_{\mathcal{F}} \colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$ from Corollary 15.2 on page 227 gives only an isomorphism of functors $\widetilde{(-)} \circ \Gamma \simeq \mathrm{id}_{\mathsf{QCoh}_X}$

The theorem has the important corollary that in the setting of quasi-coherent sheaves on affine schemes, the global section is an exact functor:

Corollary 15.10. Let $X = \operatorname{Spec} A$ be an affine scheme. If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of quasi-coherent sheaves, then the sequence on global sections

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0 \tag{15.6}$$

is also exact. In other words, the global section functor is exact.

Proof Since the global section functor is left exact, we need only show that (15.6) is right exact, i.e., that the cokernel $C = \operatorname{Coker}(\mathcal{F}(X) \to \mathcal{F}''(X))$ is zero. In any case, there is an exact sequence

$$\mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow C \longrightarrow 0$$

Applying the tilde functor, which is exact (Proposition 15.3), we get

$$\mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow \widetilde{C} \longrightarrow 0$$

By assumption, the map $\mathcal{F} \to \mathcal{F}''$ is surjective, so $\widetilde{C} = 0$ and hence $C = \Gamma(X, \widetilde{C}) = 0$ as well. □

15.2.1 Examples

Example 15.11 (Quasi-coherent modules on $\mathbb{P}^1$)**.** Consider the projective line $\mathbb{P}^1_k$ over k. It is as usual covered by two affine open subschemes $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$, which are glued together along their common open set $\operatorname{Spec} k[u, u^{-1}]$.

The sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ are all quasi-coherent. This follows because $\mathcal{O}_{\mathbb{P}^1_k}(n)|_{U_i} \simeq \mathcal{O}_{U_i}$ for each $i = 1, 2$ and $\mathcal{O}_{U_i}$ is of course quasi-coherent.

More generally, we can classify all quasi-coherent sheaves on $\mathbb{P}^1_k$ as follows. A quasi-coherent sheaf on $\mathbb{P}^1_k$ is given by a triple (M_0, M_1, τ), where M_0 is a module over $\mathcal{O}_X(U_0) =$

$k[u]$, where M_1 is a module over $\mathcal{O}_X(U_1) = k[u^{-1}]$ and where

$$\tau\colon M_1 \otimes_{k[u^{-1}]} k[u,u^{-1}] \to M_0 \otimes_{k[u]} k[u,u^{-1}].$$

is an isomorphism of modules over $k[u,u^{-1}]$.

In terms of this description, the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ are given by the triples $M_0 = k[u]$, $M_1 = k[u^{-1}]$ and the map $\tau\colon k[u,u^{-1}] \to k[u,u^{-1}]$ is multiplication by u^n.

Example 15.12 (Quasi-coherent sheaves on spectra of DVR's)**.** The example of an discrete valuation ring is always useful to consider, and we continue exploring Example **??** above. A $\mathcal{O}_X$-module $\mathcal{F}$ given by the data M, N, ρ is $\mathcal{F}$ quasi-coherent if and only if $\rho \otimes \mathrm{id}_K\colon M \otimes_R K \to N$ is an isomorphism (of K-vector spaces).

If $\mathcal{F}$ is quasi-coherent, then every point has a neighbourhood on which $\mathcal{F}$ is the tilde of some module. The only neighbourhood of the unique closed point is X itself, and so $\mathcal{F} = \widetilde{M}$. Therefore, $N = \mathcal{F}(U) = M_{(0)} = M \otimes_R K$ and ρ is an isomorphism. Conversely, if $\rho \otimes \mathrm{id}_K\colon M \otimes_R K \to N$ is an isomorphism, then $\mathcal{F}$ is given by $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = M \otimes_R K$, and so $\mathcal{F} \simeq \widetilde{M}$, and it is quasi-coherent.

Exercise 15.2.1. Show that a sheaf $\mathcal{F}$ on a scheme X is quasi-coherent if and only if there is an open cover $\{U_i\}$ such that each of the restrictions $\mathcal{F}|_{U_i}$ may be presented as the cokernel of a map between free $\mathcal{O}_X$-modules; that is, they appear in exact sequences

$$\mathcal{O}_{U_i}^J \longrightarrow \mathcal{O}_{U_i}^I \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0,$$

where $\mathcal{G}^I$ stands for the direct sum $\bigoplus_{i\in I} \mathcal{G}$ of copies of a sheaf $\mathcal{G}$ (and where I and J may be infinite and dependent on i). Conclude that being quasi-coherent is a local property for an $\mathcal{O}_X$-module.

Another nice consequence of the equivalence in Theorem 15.9 is that any purely categorical construction commutes with the tilde-functor – any universal property that holds in Mod_A holds as well in QCoh_X.

Example 15.13. The tilde of the direct sum $(\bigoplus_{i\in I} M_i)^\sim$ of a family of modules equals the direct sum $\bigoplus_i \widetilde{M}_i$ in QCoh_X. Likewise, if $\{M_i\}_{i\in I}$ is a directed system of modules, we have $(\varinjlim M_i)^\sim$ is the direct limit $\varinjlim \widetilde{M}_i$ in the category QCoh_X. In both examples, the sheaf constructed in fact satisfies the universal property in Mod_X, not just QCoh_X.

Exercise 15.2.2. Let $X = \operatorname{Spec} A$ be an affine scheme and let

$$\cdots \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \cdots$$

be an exact sequence of quasi-coherent sheaves. Show that

$$\cdots \longrightarrow \mathcal{F}_{i-1}(X) \longrightarrow \mathcal{F}_i(X) \longrightarrow \mathcal{F}_{i+1}(X) \longrightarrow \cdots$$

is also exact.

Exercise 15.2.3. Let $X = \operatorname{Spec} A$ and consider an distinguished open subscheme $D(g) \simeq \operatorname{Spec} A_g$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show the following two properties:

(i) If $s \in \mathcal{F}(X)$ and $s|_{D(g)} = 0$, then $g^n s = 0$ for some $n \in \mathbb{N}$;

(ii) If s is a section of $\widetilde{M}$ over $D(g)$, then for some $n \in \mathbb{N}$ the section $g^n s$ extends to X; that is, there is a $t \in \mathcal{F}(X)$ so that $t|_{D(g)} = g^n s$.

HINT: Use that $\mathcal{F}$ is of tilde type so that $\mathcal{F}(D(g))$ is the localized A_g-module $\mathcal{F}(D(g)) = \mathcal{F}(X)_g$

Exercise 15.2.4 (Direct products of quasi-coherent sheaves on affines)**.** When $X = \operatorname{Spec} A$ is affine it is straightforward to see that arbitrary direct products exist in QCoh_X, but they are not as well behaved as direct sums. Let $\{M_i\}_{i \in I}$ be a family of A-modules.

(i) Show that $\widetilde{\prod_i M_i}$ is the direct product of the sheaves $\widetilde{M}_i$ in QCoh_X.
(ii) Show by giving examples, that if I is infinite, forming the product does not commute with restrictions. HINT: In general, $(\prod_i M_i)_f$ is different from $\prod_i (M_i)_f$. Components of an element in the latter have denominators of the form f^{-n}, whereas in the former they can be f^{-n_i} with the n_i's being unbounded.
(iii) Conclude that the direct product of the $\widetilde{M}_i$'s in the category Mod_X (as defined in Section 14.3 and Exercise 14.5.4) and in QCoh_X are different.

15.3 Properties of quasi-coherent sheaves on general schemes

In this section, we establish some basic properties of quasi-coherent sheaves on a general scheme.

Proposition 15.14. Let X be a scheme.

(i) If $\phi : \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent sheaves, then $\operatorname{Ker} \phi$, $\operatorname{Im} \phi$ and $\operatorname{Coker} \phi$ are all quasi-coherent.
(ii) (The 2-out-of-3 property) If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \qquad (15.7)$$

is a short exact sequence of $\mathcal{O}_X$-modules, and if two of $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are quasi-coherent, then the third is quasi-coherent as well.

Proof Over each open affine subset $U = \operatorname{Spec} A$ of X a map $\alpha \colon \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules may be described as $\alpha|_U = \widetilde{a}$ where $a \colon M \to N$ is a A-module homomorphism and M and N are A-modules with $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Since the tilde-functor is exact, one has $\operatorname{Ker} \alpha|_U = (\operatorname{Ker} a)\widetilde{\ }$. Moreover, by the same reasoning, it holds that $\operatorname{Coker} \alpha|_U = (\operatorname{Coker} a)\widetilde{\ }$and $\operatorname{Im} \alpha|_U = (\operatorname{Im} a)\widetilde{\ }$.

The proof of (ii) relies on a future result from the cohomological toolbox (see propxxxx), that in 15.10 only the leftmost sheaf $\mathcal{F}'$ needs to be quasi-coherent. In view of this, if an extension like (15.7) with $\mathcal{F}$ and $\mathcal{H}$ quasi-coherent is given and $U \subset X$ is an affine open subscheme, the induced sequence of sections over U is exact. The upper horizontal sequence in the diagram below is hence exact. The three vertical maps are the natural β-maps from Corollary 15.2 on page 227, and since $\mathcal{F}$ and $\mathcal{H}$ both are quasi-coherent, the two outer vertical maps are isomorphisms. The snake lemma then implies that the middle vertical map is an isomorphism as well, and $\mathcal{G}$ is quasi-coherent.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathcal{F}(U)} & \longrightarrow & \widetilde{\mathcal{G}(U)} & \longrightarrow & \widetilde{\mathcal{H}(U)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}|_U & \longrightarrow & \mathcal{G}|_U & \longrightarrow & \mathcal{H}|_U & \longrightarrow & 0 \end{array}$$

□

15.4 Constructions of Quasi-coherent sheaves

The following lemma will be useful for constructing quasi-coherent sheaves on general schemes.

Lemma 15.15. Let X be a scheme and let $\mathscr{B}$ be a basis for the topology on X consisting of the affine open subsets. Let $\mathcal{F}$ be a $\mathscr{B}$-presheaf such that

(i) $\mathcal{F}$ is a an $\mathcal{O}_X$-module on the opens in $\mathscr{B}$; that is, for each $U \in \mathscr{B}$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion $V \subset U$ in $\mathscr{B}$, the diagram (14.13) commutes.

(ii) For each $U, V \in \mathscr{B}$ with $V \subset U$ the canonical map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \longrightarrow \mathcal{F}(V) \qquad (15.8)$$

given by $s \otimes g \mapsto gs|_V$ is an isomorphism.

Then $\mathcal{F}$ is a $\mathscr{B}$-sheaf and extends to a quasi-coherent sheaf on X.

Proof We first treat the affine case and write $X = \operatorname{Spec} A$. Taking $U = X$ and $V = D(f) \subset U$, the isomorphism (15.8) shows that $\mathcal{F}(D(f)) = \mathcal{F}(X) \otimes_A A_f = \mathcal{F}(X)_f$. In other words, $\mathcal{F}$ is isomorphic to $\widetilde{\mathcal{F}(X)}$ on distinguished opens. Therefore, $\mathcal{F}$ extends to a quasi-coherent sheaf, namely $\widetilde{\mathcal{F}(X)}$.

In general case, it follows for free from the affine case that $\mathcal{F}$ is $\mathscr{B}$-sheaf. Indeed, if U belongs to $\mathscr{B}$, the locality and gluing conditions to be a $\mathscr{B}$-sheaf involves only the open sets in $\mathscr{B}$ which are contained in U, and these are fulfilled because $\mathcal{F}|_U$ equals $\widetilde{\mathcal{F}(U)}$. The extended sheaf is then by construction quasi-coherent. □

An immediate corollary is the following:

Corollary 15.16. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Then $\mathcal{F}$ is quasi-coherent if and only if for any pair $V \subset U$ open affine subsets the natural map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V) \qquad (15.9)$$

that sends $s \otimes g$ to $gs|_V$, is an isomorphism.

Proof Let $U = \operatorname{Spec} A$, and $V = \operatorname{Spec} B$ with $U \supset V$. If $\mathcal{F}$ is quasi-coherent, then $\mathcal{F}|_U \simeq \widetilde{M}$ for some and the multiplication map (15.9) is an isomorphism by Proposition 15.4. Conversely, if (15.9) holds, then $\mathcal{F}$ (considered as a $\mathscr{B}$-sheaf) satisfies the conditions of the above lemma, and extends to a quasi-coherent sheaf on X, and this extension of course coincides with $\mathcal{F}$. □

This is one explaination for the word 'coherence': the groups of sections $\mathcal{F}(V)$ of $\mathcal{F}$ over an open affine V are by no means arbitrary, they fit together with the sections of $\mathcal{F}(U)$ over any larger open affine U, in a way determined by the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Direct sums

In Section XXX, we saw the direct sum of any collection of $\mathcal{O}_X$-modules has the natural structure of an $\mathcal{O}_X$-module. Here we will apply Lemma 15.15 to show that the direct sum of a collection of quasi-coherent sheaves is again quasi-coherent.

Proposition 15.17. For any collection of quasi-coherent sheaves $\{\mathcal{F}_i\}_{i\in I}$, the direct sum $\bigoplus_i \mathcal{F}_i$ is again quasi-coherent. Moreover,
(i) For open affine subsets U it holds that $(\bigoplus_i \mathcal{F}_i)(U) \simeq \bigoplus_i \mathcal{F}_i(U)$;
(ii) On stalks it holds that $(\bigoplus_i \mathcal{F}_i)_x \simeq \bigoplus_i (\mathcal{F}_i)_x$.

We underline that (i) does not in general hold for opens that are not affine as Example 14.19 shows.

Proof Consider the direct sum presheaf given by $\mathscr{S}(U) = \bigoplus_{i\in I} \mathcal{F}_i(U)$. This defines a $\mathscr{B}$-sheaf on the basis of open affines U, and clearly each $\mathscr{S}(U)$ is an $\mathcal{O}_X(U)$-module. To check the final condition in Lemma 15.15, we need the result from commutative algebra that tensor products commute with direct sums: If N is an A-module and $\{M_i\}_{i\in I}$ is a collection of A-modules, one has a canonical isomorphism

$$(\bigoplus_{i\in I} M_i) \otimes_A N = \bigoplus_{i\in I} (M_i \otimes_A N) \tag{15.10}$$

defined by $(\sum m_i) \otimes n \mapsto \sum m_i \otimes n$ (where the sums are finite).

Applying this to $\mathscr{S}$, we get

$$\begin{aligned}\mathscr{S}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) &= \big(\bigoplus_{i\in I} \mathcal{F}_i(U)\big) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) = \bigoplus_{i\in I} \big(\mathcal{F}_i(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)\big) \\ &= \bigoplus_{i\in I} \mathcal{F}_i(V) = \mathscr{S}(V).\end{aligned}$$

Finally, (ii) follows by considering an open affine neighbourhood $\operatorname{Spec} A$ of x and again cite (15.10) with $N = A_{\mathfrak{p}}$ and $\mathfrak{p}$ the prime corresponding to x. □

Tensor products

Next, we consider the tensor product of two quasi-coherent sheaves.

Proposition 15.18. For two quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a scheme X the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ in QCoh_X is quasi-coherent.

(i) If $U \subset X$ is an open affine, there is a canonical isomorphism that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) \simeq \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

(ii) On stalks at points $x \in X$ we have canonical isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

The proof is similar to that of Proposition 15.17, checking that the $\mathscr{B}$-presheaf $T(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ satisfies two conditions of Lemma 15.15. The main fact we need is that for an A-algebra B there are a canonical isomorphisms

$$(M \otimes_A N) \otimes_A B \simeq (M \otimes_A B) \otimes_B (N \otimes_A B). \tag{15.11}$$

for A-modules M and N.

Exercise 15.4.1. Let $\{\mathcal{F}_i\}$ be a family of sheaves on X and $U \subset X$ an open set. If U is quasi-compact, show that $(\bigoplus_i \mathcal{F}_i)(U) = \bigoplus_i \mathcal{F}_i(U)$.

Exercise 15.4.2. Let X be a scheme and let $\{\mathcal{F}_i\}_{i \in I}$ be a directed system of quasi-coherent sheaves X. Show the following claims:

(i) For each open affine subscheme $U \subset X$ it holds that $(\varinjlim \mathcal{F}_i)(U) = \varinjlim \mathcal{F}_i(U)$;
(ii) If $U \subset X$ is open, then $(\varinjlim \mathcal{F}_i)|_U = \varinjlim \mathcal{F}_i|_U$;
(iii) For each $x \in X$ it holds that $(\varinjlim \mathcal{F}_i)_x = \varinjlim (\mathcal{F}_i)_x$.

HINT: : Use the description in Exercise A.1.13 and properties of the direct sum.

Exercise 15.4.3. Fill in the details of the proof of Propositition 15.18.

15.5 Pushforwards and Pullbacks

For a morphism of schemes $f : X \to Y$, there are two natural operations, the *pushforward* f_* and the *pullback* f^*, for producing sheaves on Y from sheaves on X and vice versa. We already introduced the pushforward functor in Chapter 3. This operation produces an $\mathcal{O}_Y$-module $f_*\mathcal{F}$ from an $\mathcal{O}_X$-module $\mathcal{F}$. There is an opposite operation, called the *pullback*, which produces an $\mathcal{O}_X$-module $f^*\mathcal{G}$ on X from a $\mathcal{O}_Y$-module $\mathcal{G}$ on Y. We will define this in the next section. Here we remark that the pushforward of a quasi-coherent sheaf still will be quasi-coherent for a large class of morphisms.

Recall the definition of $f_*\mathcal{F}$ for a sheaf $\mathcal{F}$: for each open set $U \subset Y$, the sections are given by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$. Furthermore, recall Proposition 15.5 on page 229 which describes the pushforward of modules of tilde-type. We rephrase it here in the terminology of quasi-coherent sheaves:

Proposition 15.19. Let X and Y be affine schemes and let $f : X \to Y$ be a morphism. For each quasi-coherent $\mathcal{F}$ on X it holds that $f_*\mathcal{F} = \widetilde{\mathcal{F}(X)}$. In particular, the sheaf $f_*\mathcal{F}$ is quasi-coherent.

The proposition generalizes to a large class of morphisms. The proof below will use only that inverse images of open affine subsets are quasi-compact and that the intersection of two affine subsets can be covered by finitely many affines. For simplicity, we state it here with the more modest hypothesis that X is Noetherian.

Theorem 15.20 (Quasi-coherence of pushforwards). Let $f\colon X \to Y$ be a morphism of schemes with X Noetherian and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the direct image $f_*\mathcal{F}$ is quasi-coherent on Y.

Proof We may assume that $Y = \operatorname{Spec} B$ as being quasi-coherent is a local property. Since X is Noetherian it is quasi-compact and may be covered by finitely many open affines U_i. Each intersection $U_i \cap U_j$ is again quasi-compact and we cover it with finitely many open affines U_{ijk}. With a slight modification of the sequence in Example 14.13 one has the following exact sequence of sheaves on X:

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \textstyle\prod_i f_{i*}\mathcal{F}|_{U_i} \longrightarrow \prod_{i,j,k} f_{ijk*}\mathcal{F}|_{U_{ijk}} \tag{15.12}$$

where $f_i = f|_{U_i}$ and $f_{ijk} = f|_{U_{ijk}}$. Now, each of the sheaves $f_{i*}\mathcal{F}|_{U_i}$ and $f_{ij*}\mathcal{F}|_{U_{ij}}$ are quasi-coherent by the affine case, and they are finite in number as the covering U_i is finite. Hence $\prod_i f_{i*}\mathcal{F}|_{U_i}$ and $\prod_{i,j} f_{ij*}\mathcal{F}|_{U_{ij}}$ are finite products of quasi-coherent sheaves and therefore they are quasi-coherent. Now, the sheaf $f_*\mathcal{F}$ equals the kernel of a homomorphism between two quasi-coherent sheaves, and so the theorem follows from Proposition 15.14 on page 233. □

Example 15.21. Consider the projective line $\mathbb{P}^1_k$ and the squaring map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$, which restricts to the squaring map $\operatorname{Spec} k[x] \to V_0 = \operatorname{Spec} k[y]$. We use the notation of Example XXX.

We claim that $f_*\mathcal{O}_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

The following example shows the proposition fails if X is not assumed to be Noetherian:

Example 15.22. Let $X = \coprod_{i\in I} \operatorname{Spec}\mathbb{Z}$ be the disjoint union of countably infinitely many copies of $\operatorname{Spec}\mathbb{Z}$ and let $f\colon X \to \operatorname{Spec}\mathbb{Z}$ be the morphism that equals the identity on each of the copies of $\operatorname{Spec}\mathbb{Z}$. Then $f_*\mathcal{O}_X$ is not quasi-coherent. Indeed, the global sections of $f_*\mathcal{O}_X$ satisfy

$$\Gamma(\operatorname{Spec}\mathbb{Z}, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i\in I}\mathbb{Z}.$$

On the other hand if p is any prime, one has

$$\Gamma(D(p), f_*\mathcal{O}_X) = \Gamma(f^{-1}D(p), \mathcal{O}_X) = \prod_{i\in I}\mathbb{Z}[p^{-1}].$$

It is not true that $\Gamma(D(p), f_*\mathcal{O}_X) = \Gamma(\operatorname{Spec}\mathbb{Z}, f_*\mathcal{O}_X)\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}]$. Indeed, elements in $\prod_{i\in I}\mathbb{Z}[p^{-1}]$ are sequences of the form $(z_i p^{-n_i})_{i\in I}$ where $z_i \in \mathbb{Z}$ and $n_i \in \mathbb{N}$. Such an element lies in $(\prod_{i\in I}\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}]$ only if the n_i's form a bounded sequence, which is not the case for general elements of the form $(z_i p^{-n_i})_{i\in I}$ when I is infinite. In particular, $f_*\mathcal{O}_X$ is not quasi-coherent.

15.5.1 Pullbacks

Let $f : X \to Y$ be a morphism of schemes. Recall we defined the pushforward functor which produces an $\mathcal{O}_Y$-module $f_*\mathcal{F}$ from an $\mathcal{O}_X$-module $\mathcal{F}$. There is an opposite operation, called the *pullback*, which produces an $\mathcal{O}_X$-module $f^*\mathcal{G}$ on X from a $\mathcal{O}_Y$-module $\mathcal{G}$ on Y. Even though this can be defined for any $\mathcal{O}_Y$-module $\mathcal{G}$, we will outline a contruction which works for quasi-coherent sheaves, where the definition is much more explicit.

The best way of understanding the pullback f^* is by how it interacts with the pushforward f_*. Namely, the sheaf $f^*\mathcal{G}$ satisfies a certain universal property with respect to f_* and $\mathcal{O}_X$-modules $\mathcal{F}$: maps of $\mathcal{O}_X$-modules $f^*\mathcal{G} \to \mathcal{F}$ are in 1-1 correspondence with maps of $\mathcal{O}_Y$-modules $\mathcal{G} \to f_*\mathcal{F}$. The precise statement is the following theorem:

Theorem 15.23. Let $f : X \to Y$ be a map of schemes and let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then there exists a quasi-coherent sheaf $f^*\mathcal{G}$ on X along with canonical functorial bijections

$$\theta_{\mathcal{F}} : \operatorname{Hom}_X(f^*\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \tag{15.13}$$

for each $\mathcal{O}_X$-module $\mathcal{F}$. The sheaf $f^*\mathcal{G}$ is unique up to isomorphism.

Here $\theta_{\mathcal{F}}$ is functorial in both $\mathcal{F}$ and $\mathcal{G}$. This means that commutative diagrams on the left induce (and are induced by) diagrams on the right:

$$\begin{array}{ccc} f^*\mathcal{G} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ f^*\mathcal{G}' & \longrightarrow & \mathcal{F}' \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} \mathcal{G} & \longrightarrow & f_*\mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{G}' & \longrightarrow & f_*\mathcal{F}' \end{array}$$

The pullback $f^*\mathcal{G}$ is determined up to isomorphism by the universal property (15.13) (by Lemma (15.26)). We will refer to any sheaf satisfying this condition as the *pullback of* $\mathcal{G}$ *by* f.

An important feature of the universal property (15.13) is that there exist canonical maps

$$\eta : \mathcal{G} \longrightarrow f_*f^*\mathcal{G} \tag{15.14}$$

and and if $f_*\mathcal{F}$ is quasi-coherent,

$$\nu : f^*f_*\mathcal{F} \longrightarrow \mathcal{F}. \tag{15.15}$$

These are obtained by applying (15.13) to the identity maps $f^*\mathcal{G} \to f^*\mathcal{G}$ and $f_*\mathcal{F} \to f_*\mathcal{F}$ respectively. If $\phi : f^*\mathcal{G} \to \mathcal{F}$ is a map on the right hand side of (15.13), then $\theta_{\mathcal{F}}(\phi)$ is obtained as the composition $\mathcal{G} \xrightarrow{\eta} f_*f^*\mathcal{G} \xrightarrow{f_*\phi} f_*\mathcal{F}$. Likewise, if $\psi : \mathcal{G} \to f_*\mathcal{F}$ is a map on the left hand side, then the corresponding map $f^*\mathcal{G} \to \mathcal{F}$ is obtained by composing $f^*\psi : f^*\mathcal{G} \to f^*f_*\mathcal{F}$ with ν.

Pullacks for X and Y affine

We begin by proving Theorem 15.23 in the most important special case, namely when $f : X \to Y$ is a map of affine schemes.

Suppose $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and $f : X \to Y$ is induced by a ring map $A \to B$. Consider a quasi-coherent sheaf of the form $\mathcal{G} = \widetilde{N}$ on Y, where N is an A-module. As B is an A-algebra, the tensor product $N \otimes_A B$ is naturally a B-module, and we define the pullback of $\mathcal{G}$ by the formula

$$f^*\widetilde{N} = \widetilde{N \otimes_A B}. \tag{15.16}$$

In other words, we define

$$f^*\mathcal{G} = \big(\mathcal{G}(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X)\big)^{\sim}. \tag{15.17}$$

This definition is motivated by the formal properties of Hom and the tensor product. More precisely, we recall the following natural bijection, which holds for all A-modules N and B-modules M:

$$\operatorname{Hom}_B(N \otimes_A B, M) = \operatorname{Hom}_A(N, M_A) \tag{15.18}$$

This bijection sends a B-linear map ϕ on the left-hand side to the A-linear map $N \to M_A$ given by $n \mapsto \phi(n \otimes 1)$. This map is functorial in M and N. (See Exercise 15.5.3.)

Now the universal property (15.13) is a consequence from the bijection (15.18). Indeed, by Proposition XXX, we have for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\begin{aligned}
\operatorname{Hom}_X(f^*\mathcal{G}, \mathcal{F}) &= \operatorname{Hom}_X(\widetilde{\mathcal{G}(Y) \otimes_A} B, \mathcal{F}) && (15.19)\\
&= \operatorname{Hom}_A(\mathcal{G}(Y) \otimes_A B, \mathcal{F}(X)) && (15.20)\\
&= \operatorname{Hom}_A(\mathcal{G}(Y), \mathcal{F}(X)_A) && (15.21)\\
&= \operatorname{Hom}_Y(\widetilde{\mathcal{G}(Y)}, f_*\mathcal{F}) && (15.22)\\
&= \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).
\end{aligned}$$

The fact that these isomorphisms are functorial in $\mathcal{F}$ and $\mathcal{G}$ follows from the functoriality of the isomorphism in (15.18) (see Exercise **??**) and the isomorphism $\beta_{\mathcal{F}}$ in Proposition XXX. The uniqueness part of the theorem follows from Lemma 15.26 below. This concludes the proof.

Example 15.24. We have $f^*\mathcal{O}_Y = \mathcal{O}_X$ for any morphism $f : X \to Y$. Indeed, $f^*\mathcal{O}_Y$ is the tilde of the tensor product $A \otimes_A B$, and $A \otimes_A B \simeq B$ (as B-modules).

Example 15.25. If $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$, we can understand the two adjunction maps (15.14) and (15.15) as follows. The map

$$\eta : \widetilde{N} \longrightarrow f_*f^*\widetilde{N}$$

is the map of $\mathcal{O}_Y$-modules induced by $N \to (N \otimes_A B)_A$, sending $n \mapsto (n \otimes 1)$. Likewise,

$$\nu : \widetilde{M} \longrightarrow f_*f^*\widetilde{M}$$

is induced by the map of B-modules $M_A \otimes_A B \to M$ sending $m \otimes b$ to bm.

The following lemma was used for the in the proof above. It is a version of the 'Yoneda lemma'.

Lemma 15.26 (Yoneda lemma for sheaves). Let X be a scheme and let $\mathcal{H}, \mathcal{H}'$ be two $\mathcal{O}_X$-modules. Assume that there are natural bijections

$$\beta_{\mathcal{F}} : \operatorname{Hom}_X(\mathcal{H}, \mathcal{F}) \to \operatorname{Hom}_X(\mathcal{H}', \mathcal{F})$$

for each $\mathcal{O}_X$-module $\mathcal{F}$. Then there is a unique isomorphism $\iota : \mathcal{H}' \to \mathcal{H}$ such that $\beta_{\mathcal{F}}(\phi) = \phi \circ \iota$.

Proof See Exercise 15.5.4. □

Here are a few nice properties of the pullback:

Theorem 15.27. Let $f\colon X \to Y$ be a morphism of affine schemes. Then:

(i) The pullback is a functor

$$f^*\colon \mathsf{QCoh}_Y \longrightarrow \mathsf{QCoh}_X;$$

(ii) If $g : W \to X$ is another morphism of affine schemes, then

$$(f \circ g)^* = g^* \circ f^*;$$

(iii) The pullback functor is additive, right exact and sends tensor products to tensor products;

(iv) $f^*\mathcal{O}_Y = \mathcal{O}_X$;

(v) For $x \in X$, we have

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

Proof Suppse $f : X \to Y$ is induced by a ring map $\phi : A \to B$.

(i): Any a map of quasi-coherent sheaves $\widetilde{N} \to \widetilde{N'}$ is induced by a map of A-modules $N \to N'$. This in turn induces a map of B-modules $N \otimes_A B \to N' \otimes_A B$ and consequently a map of $\mathcal{O}_X$-modules $f^*\widetilde{N} \to f^*\widetilde{N'}$. Hence f^* is a functor.

(ii): If $g : W \to X$ is induced by a ring map $R \to B$, and N is an A-module, then (ii) follows from the isomorphism of R-modules

$$(N \otimes_A B) \otimes_B R \simeq N \otimes_A R$$

Alternatively, one may use the universal property (15.13) (see Lemma 15.32).

(iii): This follows from the tensor product being additive, right exact and from the formula

$$(M \otimes_A N) \otimes_A B = (M \otimes_A B) \otimes_B (N \otimes_A B).$$

(iv): This was Example 15.24 above.

(v): If $x \in X$ corresponds to $\mathfrak{p} \subset B$, then $f(x) \in Y$ corresponds to $\mathfrak{q} = \phi^{-1}(\mathfrak{p}) \subset B$. Moreover,if $\mathcal{G} = \widetilde{N}$, the stalk of $f^*\mathcal{G}$ at x is given by the localization

$$(N \otimes_A B)_{\mathfrak{p}} = N \otimes_A B_{\mathfrak{p}} = (N \otimes_A A_{\mathfrak{q}}) \otimes_{A_{\mathfrak{q}}} B_{\mathfrak{p}} = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

□

There is also a way to pull back sections of a quasi-coherent sheaf. If $\mathcal{G}$ is a quasi-coherent sheaf on Y and $s \in \mathcal{G}(Y)$ is a section, the tensor product $s \otimes 1 \in \mathcal{G}(Y) \otimes_A B$ defines a section of $f^*\mathcal{G}$ over X. We call this section the *pullback of s by f*, and denote it by $f^*(s)$.

Here is a concrete example.

Example 15.28. Consider $\mathbb{A}^1_k$ and the squaring map $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ induced by $k[y] \to k[x]$ sending y to x^2. Then $f^*\mathcal{O}_{\mathbb{A}^1_k} = \mathcal{O}_{\mathbb{A}^1_k}$, and the isomorphism is simply the tilde of the isomorphism of $k[x]$-modules

$$k[y] \otimes_{k[y]} k[x] = k[x].$$

On global sections, the section $y \in \mathcal{O}_{\mathbb{A}^1_k}(\mathbb{A}^1_k)$, pulls back to $y \otimes 1 \in k[y] \otimes_{k[x]} k[x]$, which maps to x^2 via this isomorphism. Hence we write $f^*(y) = x^2$.

Example 15.29. More generally, for a section $s \in \mathcal{O}_Y(Y)$, the corresponding pull back $f^*(s) \in \mathcal{O}_X(X)$ via the isomorphism in the isomorphism $f^*\mathcal{O}_Y = \mathcal{O}_X$ is simply the element $f^\sharp_Y(s) \in \mathcal{O}_X(X)$ by the usual sheaf map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Example 15.30. In general, the pullback f^* is only right-exact. Consider for instance the ideal sheaf sequence of the origin $p \in \mathbb{A}^1_k = \operatorname{Spec} k[t]$:

$$0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_{\mathbb{A}^1_k} \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

which is the tilde of the sequence

$$0 \longrightarrow k[t] \xrightarrow{\ t\ } k[t] \longrightarrow k(p) \longrightarrow 0.$$

If $f\colon \operatorname{Spec} k(p) \to \mathbb{A}^1_k$, where $k(p) = k[t]/(t)$, is the inclusion of p, the pullback of the sequence is the tilde of the sequence

$$0 \longrightarrow k[t] \otimes_{k[t]} k(p) \xrightarrow{\ t\ } k[t] \otimes_{k[t]} k(p) \longrightarrow k(p) \otimes_{k[t]} k(p) \longrightarrow 0,$$

which is not exact because the map on the left-hand is the zero map.

Note also that in this example, the pullback of an element $h(t) \in \mathcal{O}_{\mathbb{A}^1_k}(\mathbb{A}^1_k) = k[t]$ is given by the evaluation at p: $f^*(h(t)) = h(0)$.

Pullbacks for general morphisms

In this section we construct the pullback f^* for a general morphism of schemes $f : X \to Y$. The idea is to construct the sheaf $f^*\mathcal{G}$ by a gluing procedure that resembles the construction of the fibre product in Chapter 10.

The next result is another useful special case of Theorem 15.23, for the case when f is an open embedding. In this case, the pullback is simply the restriction.

Lemma 15.31. Let $X \subset Y$ be an open subscheme, with open embedding $\iota : X \to Y$ and let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then $\iota^*\mathcal{G} := \mathcal{G}|_X$ is a pullback of $\mathcal{G}$ by i.

Proof Note first that $\iota^*\mathcal{G}$ is quasi-coherent by Lemma XXX. We need to check that this sheaf satisfies the universal property (15.13). The map $\theta_{\mathcal{F}}$ is defined as follows. Given a map of $\mathcal{O}_X$-modules $\phi : \mathcal{G}|_X \to \mathcal{F}$ and an open set $V \subset Y$, we consider the composition

$$\mathcal{G}(V) \to \mathcal{G}(V \cap X) \xrightarrow{\phi_{V\cap X}} \mathcal{F}(V \cap X) = (\iota_*\mathcal{F})(V).$$

One checks that this is a map of $\mathcal{O}_Y(V)$-modules, and that these maps are compatible with the restriction maps. We define $\theta_{\mathcal{F}}(\phi)$ to be the induced map of $\mathcal{O}_Y$-modules $\mathcal{G} \to \iota_*\mathcal{F}$.

The inverse of $\theta_{\mathcal{F}}$ is defined as follows. Let $\psi : \mathcal{G} \to \iota_*\mathcal{F}$ be a map of $\mathcal{O}_Y$-modules. For an open set $W \subset X$, we may define $\iota^*\mathcal{G} \to \mathcal{F}$ over W simply by

$$\psi_W : (\mathcal{G}|_X)(W) \longrightarrow \mathcal{F}(\iota^{-1}W) = \mathcal{F}(W).$$

This is a map of modules over the ring $\mathcal{O}_Y(W) = \mathcal{O}_X(W)$. These maps give a map of $\mathcal{O}_X$-modules $\mathcal{G}|_X \to \mathcal{F}$. It is not hard to check that these two assignments are inverses to each other. □

Lemma 15.32. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If there exists pullback functors f^* and g^*, then also $(g \circ f)^*$ exists, and $(g \circ f)^* = f^* \circ g^*$.

Proof Let $\mathcal{G}$ be a quasi-coherent sheaf on Z, and consider $f^*(g^*\mathcal{G})$, which, by our assumptions, is a quasi-coherent sheaf on X. For an $\mathcal{O}_X$-module $\mathcal{F}$, we now have

$$\begin{aligned}
\operatorname{Hom}_X(f^*(g^*\mathcal{G}), \mathcal{F}) &= \operatorname{Hom}_Y(g^*\mathcal{G}, f_*\mathcal{F}) && (15.23)\\
&= \operatorname{Hom}_Z(\mathcal{G}, g_*f_*\mathcal{F}) && (15.24)\\
&= \operatorname{Hom}_Z(\mathcal{G}, (g \circ f)_*\mathcal{F}).
\end{aligned}$$

Here all equalities are canonical bijections. □

With these two lemmas, we are ready to prove Theorem 15.23.

Proof Cover X and Y by affine open subschemes $X_i = \operatorname{Spec} B_i$ and $Y_i = \operatorname{Spec} A_i$, $i \in I$, so that f maps X_i into Y_i. That is, $f|_{X_i}$ factors as a map of affine schemes $f_i : X_i \to Y_i$ followed by an open embedding $\tau_i : Y_i \to Y$. The situation is shown in the diagram below.

$$\begin{array}{ccc}
X_i & \xrightarrow{\iota_i} & X \\
\downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f} \\
Y_i & \xrightarrow{\tau_i} & Y
\end{array}$$

By the affine case, we get quasi-coherent sheaves $f_i^*(\mathcal{G}|_{Y_i})$ on X_i, satisfying the universal property (15.13) for $\mathcal{O}_{X_i}$-modules $\mathcal{F}$. We claim that these glue to a sheaf $f^*\mathcal{G}$ on X satisfying the same universal property. The first observation is that the sheaf $f_i^*(\mathcal{G}|_{Y_i})$ satisfies the universal property of $(f|_{X_i})^*\mathcal{G}$ for the morphism $f|_{X_i} = \tau_i \circ f_i : X_i \to Y$. This follows from Lemma 15.31 and 15.32.

Fix $i, j \in I$ and let $X_{ij} = X_i \cap X_j$. Let $\iota_i : X_{ij} \to X_i$ be the i-th inclusion and let $f_{ij} = f_i \circ \iota_i : X_{ij} \to Y_i$. Lemma 15.32 shows that the restrictions $f_i^*(\mathcal{G}|_{Y_i})|_{X_{ij}}$ and

$f_j^*(\mathcal{G}|_{Y_j})|_{X_{ij}}$ to X_{ij} both satisfy the universal property (15.13) for $\mathcal{O}_{X_{ij}}$-modules $\mathcal{F}$. So by Lemma 15.26, there is an isomorphism

$$\tau_{ij} : f_j^*(\mathcal{G}|_{Y_j})|_{X_{ij}} \longrightarrow f_i^*(\mathcal{G}|_{Y_i})|_{X_{ij}}. \tag{15.25}$$

Furthermore, Lemma 15.32 implies that over the triple intersection X_{ijk}, the three restrictions $f_i^*(\mathcal{G}|_{Y_i})|_{X_{ijk}}$ $f_j^*(\mathcal{G}|_{Y_j})|_{X_{ijk}}$ and $f_k^*(\mathcal{G}|_{Y_k})|_{X_{ijk}}$ all satisfy the universal property of $(f|_{X_{ijk}})^*\mathcal{G}$. By the uniqueness part of Lemma 15.26, the isomorphisms between them, i.e., the restrictions of (15.25), must satisfy $\tau_{ik} = \tau_{ij} \circ \tau_{jk}$ when restricted to X_{ijk}. Hence the sheaves $f_i^*(\mathcal{G}|_{Y_i})|_{X_{ij}}$ glue to a sheaf on X, an $\mathcal{O}_X$-module, which we denote by $f^*\mathcal{G}$. By construction, we have $f^*\mathcal{G}|_{X_i} = f_i^*(\mathcal{G}|_{Y_i}) = \widetilde{\mathcal{G}(Y_i) \otimes_{A_i} B_i}$ on X_i, so $f^*\mathcal{G}$ is quasi-coherent.

Finally, we check that this sheaf satisfies the universal property (15.13). If $\mathcal{F}$ is any $\mathcal{O}_X$-module, and $\phi : f^*\mathcal{G} \to \mathcal{F}$ is a map of $\mathcal{O}_X$-modules, we get restrictions $\phi_i : f^*\mathcal{G}|_{X_i} \to \mathcal{F}|_{X_i}$ and $\phi_{ij} : f^*\mathcal{G}|_{X_{ij}} \to \mathcal{F}|_{X_{ij}}$. By the universal properties of $f^*\mathcal{G}|_{X_i} = f_i^*(\mathcal{G}|_{Y_i})$, these maps correspond bijectively to maps of $\mathcal{O}_{Y_i}$-modules

$$\psi_i : \mathcal{G}|_{Y_i} \longrightarrow f_{i*}(\mathcal{F}|_{X_i})$$

Using the fact that ϕ_i and ϕ_j both restrict to the same map, namely ϕ_{ij}, on X_{ij}, we see that these maps glue to a sheaf map $\psi : \mathcal{G} \to f_*\mathcal{F}$. As the ψ_i are maps of $\mathcal{O}_{Y_i}$-modules, ψ is a map of $\mathcal{O}_Y$-modules.

Conversely, given a map of $\mathcal{O}_Y$-modules $\psi : \mathcal{G} \to f_*\mathcal{F}$, consider the restriction $\psi_i : \mathcal{G}|_{Y_i} \to (f_*\mathcal{F})|_{Y_i}$. Again, by the universal property of $f_i^*(\mathcal{G}|_{Y_i})$, each ψ_i corresponds to a map of $\mathcal{O}_{X_i}$-modules $\phi_i : f_i^*(\mathcal{G}|_{Y_i}) \to \mathcal{F}|_{X_i}$. For any two indexes i, j, there is also a $\phi_{ij} : f_{ij}^*(\mathcal{G}|_{Y_{ij}}) \to \mathcal{F}|_{X_{ij}}$ on X_{ij}, by Lemma **??**. Both ϕ_i and ϕ_j restrict to this map over X_{ij}, and so they glue to a map of $\mathcal{O}_X$-modules $\phi : \mathcal{G} \to f_*\mathcal{F}$. □

The main properties of f^* in the affine case still hold in the general setting.

Theorem 15.33. The properties (i)-(v) in Theorem 15.27 hold for any morphism of schemes $f : X \to Y$.

Proof By construction, the pullback of a quasi-coherent sheaf is quasi-coherent. Item (ii) follows by Lemma 15.32.

The remaining properties follow either from the explicit construction of $f^*\mathcal{G}$ by gluing, or the universal property (15.13). For instance, to show that $f^*\mathcal{O}_Y = \mathcal{O}_X$, one can either note that the local isomorphisms $\widetilde{A \otimes_A B} \simeq B$ glue to an isomorphism $f^*\mathcal{O}_Y \simeq \mathcal{O}_X$, or verify that $\mathcal{O}_X$ satisfies the universal property of $f^*\mathcal{O}_Y$. (See Exercise 15.5.2). □

15.5.2 Pulling back sections

Given a morphism $f : X \to Y$ and a section $s \in G(V)$ of quasi-coherent sheaf $\mathcal{G}$ on Y, we can also define a pull back $f^*(s) \in \Gamma(f^{-1}V, f^*\mathcal{G})$. One way to define this is via the canonical map $\mathcal{G} \to f_*f^*\mathcal{G}$; over an open set $V \subset Y$ this is a map

$$\Gamma(V, \mathcal{G}) \to \Gamma(f^{-1}V, f^*\mathcal{G})$$

More concretely, we can define the section $f^*(s)$ as follows. Cover V and $f^{-1}V$ by affine opens $V_i = \operatorname{Spec} A_i$ and $U_i = \operatorname{Spec} B_i$ respectively such that $\mathcal{G}|_{V_i} \simeq \widetilde{N_i}$. Then s induces elements $s_i \in \Gamma(V_i, \mathcal{G}) = N_i$. We define f^*s as the section of $f^*\mathcal{G}$ given by the sections

$$s_i \otimes 1 \in \Gamma(U_i, f^*\mathcal{G}|_{U_i}) = N_i \otimes_{A_i} B_i.$$

(These match over the overlaps U_i, because the s_i agree over the V_i)

In the special case when $\mathcal{G} = \mathcal{O}_Y$, we have $f^*\mathcal{O}_Y = \mathcal{O}_X$ and the pullback of $s \in \mathcal{O}_Y(V)$ is given by

$$f^*(s) = f^\sharp(s) \in \mathcal{O}_X(f^{-1}V).$$

Example 15.34. Consider the projective line $\mathbb{P}^1_k$ and the squaring map $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$, which restricts to the squaring map $U_0 \to U_0$ on each $U_i \simeq \mathbb{A}^1_k$. We claim that $f^*\mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(2)$.

Recall that $\mathcal{O}_{\mathbb{P}^1_k}(1)$ is obtained by gluing together $\mathcal{O}_{U_0}$ and $\mathcal{O}_{U_1}$ over $U_0 \cap U_1$ via the isomorphism $\tau_{01} : \mathcal{O}_{U_1}|_{U_0 \cap U_1} \to \mathcal{O}_{U_0}|_{U_0 \cap U_1}$ given by multiplication by x. This means that $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is obtained by gluing together $\mathcal{O}_{U_0}$ and $\mathcal{O}_{U_1}$ over $\mathcal{U}_0 \cap U_1$ via the isomorphism $f^*(\tau_{01}) : \mathcal{O}_{U_1}|_{U_0 \cap U_1} \to \mathcal{O}_{U_0}|_{U_0 \cap U_1}$ given by multiplication by $f^*(x) = x^2$ (see Example 15.28). Therefore $f^*\mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(2)$.

We can also pull back sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Let $s \in \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1}(1))$ be the section given locally by $s_0 = ax + b$ on U_0 and $s_1 = a + bx^{-1}$ on U_1. Then f^*s is the section given by $f^*(s_0) = ax^2 + b$ and $f^*(s_1) = a + bx^{-2}$ on the respective open sets.

Example 15.35. Let us consider the morphism $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$, given by $(u_0 : u_1) \mapsto (u_0^2 : u_0u_1 : u_1^2)$. Over the standard covering, $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ is given by the two morphisms

$$f_0 : U_0 = \operatorname{Spec} k[t] \to V_0 = \operatorname{Spec} k[x, y]$$

given by $t \mapsto (t, t^2)$ and

$$f_1 : U_1 = \operatorname{Spec} k[s] \to V_1 = \operatorname{Spec} k[u, v]$$

given by $s \mapsto (s^2, s)$.

Over the overlap $U_0 \cap U_1 = \operatorname{Spec} k[t, t^{-1}]$,we have $u = xy^{-1}$, $v = y^{-1}$, so both morphisms agree with the one induced by $k[x, y, x^{-1}y, xy^{-1}] \to k[t, t^{-1}]$ $x \mapsto t$, $y \mapsto t^2$.

Consider the ideal sheaf $\mathcal{I}$ of the closed subscheme given by the line $V(x_0)$. Then

$$f^*\mathcal{I} \simeq f^*\mathcal{O}_{\mathbb{P}^2_k}(-1) = \mathcal{O}_{\mathbb{P}^1_k}(-2)$$

The section $f^*x_0 = u_0^2 \in \mathcal{O}_{\mathbb{P}^2}(-2)$ defines the subscheme of $\mathbb{P}^1_k$ given by the ideal (u_0^2).

Exercise 15.5.1. Let $f : X \to Y$ be a morphism of schemes; $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{G}$ an $\mathcal{O}_Y$-module.

a) Given a map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{F}$, show that $f^\sharp$ induces a map of $\mathcal{O}_Y$-modules $\mathcal{O}_Y \to f_*\mathcal{F}$.
b) Given a map of $\mathcal{O}_Y$-modules $\mathcal{O}_Y \to f_*\mathcal{F}$, show that there is an induced map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{F}$.
c) Show that the constructions in a) and b) are inverse to each other and conclude that $f^*\mathcal{O}_Y = \mathcal{O}_X$.

Exercise 15.5.2. Let $f : X \to Y$ be a morphism and let $\mathcal{G}$ a be a quasi-coherent $\mathcal{O}_Y$-module. Show that the stalk of $f^*\mathcal{G}$ at $x \in X$ is given by $\mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$. HINT: Reduce to the affine case.

Exercise 15.5.3. Show that the map in (15.18) is a bijection, and that the isomorphism is functorial in M and N.

Exercise 15.5.4. In this exercise you will prove Lemma 15.26. In the lemma, 'naturality' means that $\beta_{\mathcal{F}'}(\sigma \circ \rho) = \sigma \circ \theta_{\mathcal{F}}(\rho)$ for every morphism $\sigma : \mathcal{F} \to \mathcal{F}'$ and $\rho : \mathcal{G} \to \mathcal{F}$.

Let $\Phi = \beta_{\mathcal{H}}(\mathrm{id}_{\mathcal{H}}) : \mathcal{H}' \to \mathcal{H}$ and $\Psi : \mathcal{H} \to \mathcal{H}'$ the unique morphism such that $\beta_{\mathcal{H}}(\Psi) = \mathrm{id}_{\mathcal{H}'}$.

a) Show that $\Psi \circ \Phi = \mathrm{id}_{\mathcal{H}}$.
b) Show that $\beta_{\mathcal{H}}(\Phi \circ \Psi) = \beta_{\mathcal{H}}(\mathrm{id}_{\mathcal{H}})$.
c) Conclude using injectivity of $\beta_{\mathcal{H}}$ that $\Phi \circ \Psi = \mathrm{id}_{\mathcal{H}}$ and that Φ and Ψ are inverses to each other.
d) Check that the uniqueness requirement in Lemma 15.26 is satisfied.

15.6 Closed subschemes and closed embeddings

In Section 5.3 we gave a preliminary definition of a closed subscheme. Here we give a more extensive treatment of these. The prototype example of a closed subscheme is the affine subscheme $\operatorname{Spec} A/I \subset \operatorname{Spec} A$ defined by an ideal $I \subset A$. The general definition will involve *ideal sheaves* rather than ideals. Thus closed subschemes will correspond to ideal sheaves $\mathcal{I}$, so that $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal for each U. In order to obtain a scheme, it is important that $\mathcal{I}$ is *quasi-coherent*.

We will need the notion of the support of a sheaf:

The support of a sheaf

For a sheaf $\mathcal{F}$ on a space X we define the *support* of $\mathcal{F}$, denoted by Supp$(\mathcal{F})$, by

$$\operatorname{Supp}(\mathcal{F}) = \left\{ x \in X \,\middle|\, \mathcal{F}_x \neq 0 \right\}.$$

In a similar way, for a section $s \in \mathcal{F}(U)$ we define the *support* of $s \in \mathcal{F}(U)$, denoted by $\operatorname{Supp}(s)$, as the set of points $x \in U$ such that the germ $s_x \in \mathcal{F}_x$ of s is nonzero.

Observe that if $s \in \mathcal{F}(X)$ is a section and x is a point such that $s_x = 0$ in $\mathcal{F}_x$, then there is an open neighbourhood $V \subseteq X$ containing x such that $s_y = 0$ for all $y \in V$. It follows that the support of s is a closed subset of X. In contrast, the support of a sheaf is in general not closed (see Example 15.36 below).

Example 15.36. Let $S \subset \operatorname{Spec} \mathbb{Z}$ be an infinite set of primes not equal to the set of all primes, and consider the $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$-module $\bigoplus_{p \in S} k(p)$ where $k(p)$ denotes the skyscraper with stalk $\mathbb{Z}/p\mathbb{Z}$ at p. This is a quasi-coherent sheaf, being the tilde of the module $\bigoplus_{p \in S} \mathbb{Z}/p\mathbb{Z}$ (Proposition 15.17). The stalk at a prime p equals $\mathbb{Z}/p\mathbb{Z}$ when $p \in S$ and 0 otherwise (the stalk over $(0) \in \operatorname{Spec} \mathbb{Z}$ is also 0). Therefore, the support is equal to S, which is not closed (proper closed subsets of $\operatorname{Spec} \mathbb{Z}$ are finite).

15.6.1 Closed embeddings

Lemma 15.37. The support $\operatorname{Supp}\mathcal{O}_X/\mathcal{I}$ of a quasi-coherent ideal $\mathcal{I}$ on a scheme X is a closed subset. Equipping it with the induced Zariski topology and the sheaf $\mathcal{O}_X/\mathcal{I}$, we get a scheme $(\operatorname{Supp}\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{I})$.

For a general ideal sheaf $\mathcal{I}$, the quotient $\mathcal{O}_X/\mathcal{I}$ is always the structure sheaf of a locally ringed space, and the quasi-coherence of $\mathcal{I}$ guarantees that this locally ringed space is locally affine.

We shall denote this scheme by $V(\mathcal{I})$; this introduces a certain ambiguity in that $V(\mathcal{I})$ also denotes the closed subset $\operatorname{Supp}\mathcal{O}_X/\mathcal{I}$, but it is not more serious than the common usage of letting X stand for both a scheme and its underlying topological space.

Proof For easier notation, let $Z = \operatorname{Supp}\mathcal{O}_X/\mathcal{I}$. An ideal $\mathcal{I}$ being quasi-coherent means that for each open affine subscheme U of X the restriction $\mathcal{I}|_U$ is of tilde-type. Clearly $\mathcal{I}|_U$ is contained in $\mathcal{O}_X|_U$, so if $U = \operatorname{Spec} A$, the restriction $\mathcal{I}|_U$ will be the tilde of a unique ideal $I \subset A$. Since the tilde functor is exact, we have the equality

$$\mathcal{O}_X/\mathcal{I}|_U = \widetilde{A/I}. \tag{15.26}$$

In particular, we see that $Z \cap U = V(I)$, which is closed in U, and so Z is closed in X since the open affines form a basis for the topology. Moreover, (15.26) shows as well that $\mathcal{O}_X/\mathcal{I}|_Z$ is a sheaf of rings on Z. The stalks are moreover local rings because they are quotients of $\mathcal{O}_{X,x}$. Thus we have produced a locally ringed space $V(\mathcal{I}) = (Z, \mathcal{O}_X/\mathcal{I})$, and again by (15.26), it is locally affine and hence a scheme. □

The lemma leads to the following definition:

Definition 15.38 (Closed subschemes)**.** A *closed subscheme* of a scheme X is one of the form $V(\mathcal{I}) = (\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ above with $\mathcal{I}$ a quasi-coherent ideal.

Let us verify that this definition is in accordance with the temporary definition from Section 5.3, which relied on the notion of closed embeddings. Recall that a closed embedding $\iota\colon Z \to X$ is a morphism such that for an open affine cover $\{U_i\}$ of X each restriction $\iota|_{\iota^{-1}U_i}\colon \iota^{-1}U_i \to U_i$ is isomorphic to one of the form $\operatorname{Spec} A/I \subset \operatorname{Spec} A$. This implies that the map $\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is surjective.

The main observation is that the pushforward $\iota_*\mathcal{O}_Z$ is quasi-coherent. This follows by Theorem 15.20 and the remarks preceding it. If $U \subset X$ is affine, the inverse image $\iota^{-1}U$ is affine being closed in U, hence it is quasi-compact, and the intersection of the inverse images of two affines is affine since closed embeddings are separated (11.12). Given this, we conclude by Proposition 15.14 that the ideal $\mathcal{I} = \operatorname{Ker}\iota^\sharp$ is quasi-coherent, and hence that ι yields an isomorphism between Z and the closed subscheme $\operatorname{Spec}\mathcal{O}_X/\mathcal{I}$. Thus the two definitions agree.

In the affine case, we get a proof of 5.10, which we for completeness reproduce here:

Proposition 15.39 (Closed subschemes in the affine case). The map $I \mapsto \operatorname{Spec}(A/I)$ is a one-to-one correspondence between the set of ideals of a ring A and the set of closed subschemes of $X = \operatorname{Spec} A$. In particular, each closed subscheme of an affine scheme is also affine.

A corollary of the above reasoning is the following characterisation of closed embedding among affine maps:

Proposition 15.40. An affine morphism $f\colon X \to Y$ of schemes is a closed embedding if and only if the induced map $\iota^\sharp\colon \mathcal{O}_Y \to \iota_*\mathcal{O}_X$ is surjective.

Proof One way is clear, so assume $\iota^\sharp$ is surjective. There is induced a map $X \to V(\mathcal{I})$ with $\mathcal{I} = \operatorname{Ker} \iota^\sharp$, which locally is an isomorphism by the affine case. □

In Definition 15.38, there could *a priori* be closed subsets not supporting a scheme. However this is not the case. One may even find a reduced scheme structure on every closed subset, which in fact gives a canonical scheme structure on each closed subset.

Proposition 15.41. Each closed subset $Z \subset X$ of a scheme is the support of a unique reduced closed subscheme.

Proof Define a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ by the formula

$$\mathcal{I}(U) = \{\, s \in \mathcal{O}_X(U) \mid s(x) = 0 \text{ for all } x \in Z \cap U \,\};$$

where as usual $s(x)$ denotes the image of the germ s_x in the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{p}_x$. It is straightforward to check that $\mathcal{I}(U)$ is compatible with restrictions (forming germs is), and that $\mathcal{I}(U)$ is in fact an ideal. We contend that $\mathcal{I}$ is quasi-coherent.

If $U = \operatorname{Spec} A$ is affine, it holds that $Z \cap U = V(I)$ for a unique radical ideal I, which equals the intersection $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$ of all the prime ideals containing I. This intersection is precisely the set of elements $a \in A$ that vanish at all points in $V(I)$, and hence $\mathcal{I}(U) = I$. Being a radical ideal is a property that localizes, so $\mathcal{I}(V) = \widetilde{I}|_V$ for all distinguished open subsets V of U, and consequently $\mathcal{I}|_U = \widetilde{I}$. This shows that $\mathcal{I}$ is quasi-coherent. Dividing by a radical ideal yields a quotient without nilpotent elements, so $\mathcal{O}_X/\mathcal{I}$ will be without nilpotent sections.

The uniqueness statement is clear in the affine case, from which it follows in general: if $\mathcal{I}$ and $\mathcal{I}'$ are two quasi-coherent ideals as in the proposition, they restrict to equal subsheaves on open affines, and so the inclusion $\mathcal{I} \subset \mathcal{I} + \mathcal{I}'$ is an equality (e.g. by Lemma 14.7). Hence $\mathcal{I} = \mathcal{I}'$. □

The ideal $\mathcal{I}$ constructed above with $Z = V(\mathcal{J})$ for a quasi-coherent ideal $\mathcal{J}$, restricts to the tilde of the radical ideal $\sqrt{\mathcal{J}(V)}$ on an affine open subset $V \subset X$. So it makes sense to call it *the radical* of $\mathcal{J}$, and we will consequently denote it by $\sqrt{\mathcal{J}}$. It is the largest ideal containing $\mathcal{J}$ and defining the closed subset $V(\mathcal{J})$. A quasi-coherent ideal is said to be *radical* if $\sqrt{\mathcal{J}} = \mathcal{J}$.

Theorem 15.42. Let X be scheme.

(i) The map $\mathcal{I} \mapsto \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ sets up a bijection between radical quasi-coherent ideals and closed subsets of X;

(ii) The map $\mathcal{I} \mapsto V(\mathcal{I})$ is a bijection between quasi-coherent ideals and closed subschemes of X .

Example 15.43. The sections of the radical $\sqrt{\mathcal{I}}$ of an ideal does not alway equal the radical of the sections. An example follows: let $X_i = \operatorname{Spec} k[t_i]/(t_i^{n_i})$ and let $X = \bigcup_i X_i$ be the disjoint union. Then $\mathcal{O}_X(X) = \prod k[t_i]/(t_i^{n_i})$, and the element $t = (t_i)$ will be a global section that vanishes everywhere, but it will not be nilpotent when n_i is an unbounded sequence, so in that case t is a global section of the sheaf $\sqrt{0}$ which does not belong to the radical of $\mathcal{O}_X(X)$.

In particular, we may apply this construction to $Z = X$. We denote the resulting scheme by X_{red} and refer to it as the *reduced scheme associated with* X. The scheme X_{red} and the corresponding closed embedding $r_X \colon X_{\text{red}} \to X$ satisfy the following universal property, which among other things, entail that X_{red} depends functorially on X (see Exercise 15.9.5 below).

Proposition 15.44. Let $f \colon Y \to X$ be a morphism of schemes, with Y reduced. Then f factors uniquely through the natural map $r_X \colon X_{\text{red}} \to X$, i.e. there exists a unique morphism $g \colon Y \to X_{\text{red}}$ such that $f = r \circ g$.

Proof The question is easily reduced to case of affine schemes, where it follows from the fact that a map of rings $A \to B$ where B is without nilpotents, factors unambiguously through $A/\sqrt{0}$. □

15.7 Coherent sheaves

Just as for modules there are various ways to impose finiteness conditions on a quasi-coherent sheaf on a scheme X. In this section we will introduce the three most important ones, $\mathcal{O}_X$-modules of finite type, $\mathcal{O}_X$-modules of finite presentation and coherent $\mathcal{O}_X$-modules. All three are generalisations of properties for modules, so we begin with a short recap.

Coherent modules

Recall that a module M over a ring A is *finitely generated* if it can be generated by a finite set of elements; that is, if there is a surjective map of A-modules

$$A^n \xrightarrow{\phi} M \longrightarrow 0.$$

The module M is said to be *finitely presented* if it sits in an exact sequence of the form

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

with $n, m \in \mathbb{N}$. In other words, M is finitely generated, and the module of relations between the generators is also finitely generated. It is not too hard to see that if M is of finite presentation, then every surjective map $A^n \to M$ has a finitely generated kernel.

One says that a module M is *coherent* if every finitely generated submodule M is of finite presentation, which may be rephrased by saying that the kernel of every map $A^n \to M$, surjective or not, is finitely generated.

Contrary to what holds for the two first properties, a ring is not necessarily a coherent module over itself (see Example 15.45 below). However, over Noetherian rings the three conditions are equivalent; indeed, in that case every submodule of a free module of finite rank is finitely generated.

Example 15.45 (A ring that is not coherent)**.** The following is an almost tautological example of a module that is not coherent. Let $R = k[x, y, t_i, u_i | i \in \mathbb{N}]$ and $\mathfrak{a} = (t_i x - u_i y | i \in \mathbb{N})$. Then the R-module $A = R/\mathfrak{a}$ is not coherent: the ideal (x, y) is finitely generated, but the relations are not. Indeed, map the free module $Re \oplus Rf$ with basis e_1, e_2 into A by sending $e_1 \to x$ and $e_2 \to y$. The kernel has generators $t_i e_1 - u_i e_2$ for $i \in \mathbb{N}$ and is not finitely generated: its image in R under e.g. the first projection equals the ideal $(u_i | i \in \mathbb{N})$, which for sure is not finitely generated. It easily follows that the ring A is not coherent as a module over itself.

Exercise 15.7.1. Show that if M is of finite presentation, every surjection $A^r \to M$ has a finitely generated kernel.

Coherent sheaves

In the literature one finds slightly different versions of the finiteness conditions for $\mathcal{O}_X$-modules, some work even over general locally ringed spaces. We will be loyal to our overall policy emphasizing quasi-coherent sheaves, which we regard as sheaves that locally, on open affines, are of tilde type. Hence our definition reads as follows:

Definition 15.46. Let X be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is of *finite type* (respectively *finitely presented* or *coherent*) if for some open affine cover $\{U_i\}$ it holds that $\mathcal{F}|_{U_i} = \widetilde{M_i}$ where M_i is a finitely generated (respectively of finite presentation or coherent) module over $\mathcal{O}_{U_i}(U_i)$.

And just as for quasi-coherence, the conditions hold for any open affine cover given that it holds for one:

Proposition 15.47. If $\mathcal{F}$ is finite type (respectively of finite presentation or coherent) then for each open affine $U = \operatorname{Spec} A$ it holds that $\mathcal{F}|_U = \widetilde{M}$ with M a finitely generated A-module (respectively of finite presentation or coherent).

Proof We will show that $\mathcal{F}|_U = \widetilde{M}$ with M finitely generated is a distinguished property for open affine subschemes $U = \operatorname{Spec} A$. Obviously M_f is finitely generated when M is, so the first condition is fulfilled. For the second, assume that $\operatorname{Spec} A = D(f) \cup D(g)$ and that M_f and M_g are finitely generated over A_f and A_g respectively. Choose finite sets of elements $\{m_i\}$ and $\{n_i\}$ in M whose images generate respectively M_f and M_g. We contend that they together generate M.

Indeed, if $m \in M$, it holds that $f^r m = \sum_i a_i m_i$ and $g^r m = \sum_j b_j n_j$ for some $r \in \mathbb{N}$, and since $D(f^r) \cup D(g^r) = \operatorname{Spec} A$, there is a relation $1 = af^r + bg^r$ with $a, b \in A$. Hence

$$m = af^r m + bg^r M = \sum_i a a_i m_i + \sum_j b b_j n_j.$$

The statement regarding the two other properties follow similarly: if for the kernel K of a map $A^n \to M$ it holds that K_f and K_g are finitely generated, K will be finitely generated by the same argument. □

The full subcategory of QCoh_X consisting of the coherent sheaves will be denoted by Coh_X. It is an abelian category having finite direct sums.

Example 15.48. Let $X = \operatorname{Spec} \mathbb{Z}$. If $\mathcal{F}$ is any coherent sheaf on X, then $\mathcal{F} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module M, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^r \oplus T$, where T is a finite direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$. Thus we may write

$$\mathcal{F} = \mathcal{O}_X^r \oplus \mathcal{T} \tag{15.27}$$

where $\mathcal{T}$ is a sheaf having stalks $\mathcal{T}_p = 0$ for all but finitely many p and $T_{(0)} = 0$. (T is a *torsion sheaf*, see Exercise XXX).

Example 15.49. The argument of the previous example in fact applies over any PID A: every coherent sheaf on $X = \operatorname{Spec} A$ must have the form $\widetilde{M}$ for $M = A^r \oplus T$ where T is a finitely generated torsion module. In particular, any coherent sheaf on the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ decomposes as

$$\mathcal{F} = \mathcal{O}_X^r \oplus \mathcal{T} \tag{15.28}$$

where $\mathcal{T}$ is a torsion sheaf.

Exercise 15.7.2. Consider an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Show that M is coherent if and only if both M' and M'' are. Show that the category of coherent A-modules is abelian.

Exercise 15.7.3. If A is a coherent ring, show that every finitely generated A-module is of finite presentation, and hence that the three conditions are equivalent.

One benefit of using coherent $\mathcal{O}_X$-modules rather than finitely generated ones is that the category of coherent modules is an abelian category, even in the non-Noetherian setting. However, a problem is that coherence is very difficult to check in general, and actually, for some schemes, even affine ones, the structure sheaf $\mathcal{O}_X$ is not coherent.

Sheaves of homomorphisms

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we defined the Hom-sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, by

$$\mathscr{H}om(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}_U(\mathcal{F}||_U, \mathcal{G}|_U). \tag{15.29}$$

where the right-hand side means all the $\mathcal{O}_U$-linear maps. This sheaf is an $\mathcal{O}_X$-module is a natural way, but it is not always quasi-coherent even if both $\mathcal{F}$ and $\mathcal{G}$ are. This is due to the deficiency that Hom does not always commute with localization in general. However, if $\mathcal{F}$ is of finite presentation, one has

Proposition 15.50. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent modules on a scheme X, and assume that $\mathcal{F}$ is of finite presentation. Then $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent. $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is also coherent if $\mathcal{F}$ and $\mathcal{G}$ are. Moreover:

(i) For every open affine $U = \operatorname{Spec} A$,

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_U = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))^{\sim}.$$

(ii) The stalk at a point $x \in X$ is given by

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x = \operatorname{Hom}_{\mathcal{O}_{X_x}}(\mathcal{F}_x, \mathcal{G}_x). \tag{15.30}$$

Proof If M is a finitely presented A-module, and B is a flat A-algebra there is a canonical

$$\operatorname{Hom}_A(M, N) \otimes_A B \simeq \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B)$$

for each A-module A. (This is clear when $M = A^n$, and the general case follows by choosing a finite presentation of M.) By Lemma 15.15, the $\mathscr{B}$-sheaf given by (15.29) on affines gives a quasi-coherent sheaf.

The claim about stalks follow from Item (i) because one can compute stalks by taking the direct limit over affine subsets. □

15.8 Invertible sheaves and the Picard group

Invertible sheaves is a very important class of $\mathcal{O}_X$-modules. They are special cases of the more general class 'locally free sheaves', which we will discuss in Chapter 19. Some of the proofs will be postponed until that chapter.

We usually use the letter L for invertible sheaves. By definition, L is invertible whenever there exists a covering $\mathcal{U} = \{U_i\}$ and isomorphisms

$$\phi_i \colon \mathcal{O}_{U_i} \longrightarrow L|_{U_i}.$$

We say that $g_i = \phi_i(1) \in L(U_i)$ is a *local generator* for L. By Lemma 19.4 on page 334 a coherent $\mathcal{O}_X$-module L is invertible if and only if the stalk L_x is isomorphic to $\mathcal{O}_{X,x}$ for every $x \in X$. In particular, L is invertible if and only if every point $x \in X$ has an open neighbourhood U such that $L|_U \simeq \mathcal{O}_U$.

Proposition 15.51. Let X be a scheme and L and M two invertible sheaves on X. Then we have

(i) $L \otimes_{\mathcal{O}_X} M$ is also an invertible sheaf. If g and h are local generators for L and M respectively, then $g \otimes h$ is a local generator for $L \otimes_{\mathcal{O}_X} M$;

(ii) $\mathscr{H}om_{\mathcal{O}_X}(L, M)$ is also invertible. In particular, $\mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ is invertible, and

$$\mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X) \otimes M \simeq \mathscr{H}om_{\mathcal{O}_X}(L, M)$$

In particular,

$$\mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X) \otimes_{\mathcal{O}_X} L \simeq \mathcal{O}_X$$

This proposition explains the term 'invertible'. Indeed, the tensor product acts as a sort of binary operation on the set of invertible sheaves; $L \otimes M$ is invertible if L and M are, and the tensor product is associative. Tensoring an invertible sheaf by $\mathcal{O}_X$ does nothing, so $\mathcal{O}_X$ serves as the identity. Moreover, for an invertible sheaf L we will define $L^{-1} = \mathscr{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$; by the proposition, L^{-1} is again invertible, and serves as a multiplicative inverse of L under $\otimes$. We can make the following definition:

Definition 15.52. Let X be a scheme. The *Picard group* $\operatorname{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X under the tensor product.

Note that it is the set of isomorphism classes of invertible sheaves that form a group, not the invertible sheaves themselves: $L \otimes_{\mathcal{O}_X} L^{-1}$ is isomorphic, but strictly speaking, not *equal* to $\mathcal{O}_X$. Note also that $\operatorname{Pic}(X)$ is an *abelian* group because $L \otimes_{\mathcal{O}_X} M$ is canonically isomorphic to $M \otimes_{\mathcal{O}_X} L$.

Invertible sheaves behave nicely with respect to pullbacks:

Lemma 15.53. Let $f : X \to Y$ be a morphism of schemes and let L be an invertible sheaf on Y. Then f^*L is invertible on X. Moreover, if L and M are two invertible sheaves on X, then

$$f^*(L \otimes_{\mathcal{O}_Y} M) = f^*(L) \otimes_{\mathcal{O}_X} f^*(M)$$

Lemma 15.54. For a morphism of schemes $f\colon X \to Y$, the assignment $L \mapsto f^*L$ induces a morphism of groups

$$f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X).$$

Example 15.55. Let $X = \operatorname{Spec} \mathbb{Z}$. If $\mathcal{E}$ is any coherent sheaf on X, then $\mathcal{E} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module M, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^r \oplus T$, where T is a finite direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$. If $\mathcal{E}$ in addition is required to be locally free, it must hold that $T = 0$ (otherwise, if p is a prime factor of an n appearing in one of the summands of T, the stalk at (p) will not be free). Thus $\mathcal{E} = \widetilde{\mathbb{Z}^r} = \mathcal{O}_X^r$, and we conclude that every coherent locally free sheaf on $\operatorname{Spec}\mathbb{Z}$ is trivial. In particular, we get that

$$\operatorname{Pic}(\operatorname{Spec}\mathbb{Z}) = 0.$$

On the other hand, $\operatorname{Pic}(\mathbb{Z}[\sqrt{-5}]) \neq 0$, by Example 19.15.

15.8.1 Locally free sheaves on the affine line

The argument of the previous example in fact applies over any PID A: every coherent sheaf on $X = \operatorname{Spec} A$ must have the form $\widetilde{M}$ for $M = A^r \oplus T$ where T is a finitely generated torsion module, and if we require $\widetilde{M}$ to be locally free, the torsion part must vanish; i.e. it must hold that $T = 0$. In particular, this applies to locally free sheaves on the affine line $\mathbb{A}_k^1 = \operatorname{Spec} k[x]$:

Proposition 15.56. Any invertible sheaf on $\mathbb{A}_k^1$ is trivial. In particular, $\operatorname{Pic}(\mathbb{A}_k^1) = 0$.

We will prove more generally that $\operatorname{Pic}(\mathbb{A}_k^n) = 0$ for any n in Chapter **??**.

15.8.2 Invertible sheaves on $\mathbb{P}_k^1$

On page 94 in Chapter **??** we constructed the family $\mathcal{O}_{\mathbb{P}_A^1}(m)$ of sheaves on the projective line over a ring A. They are all invertible, as we showed in Chapter **??**, and in this section we intend to show there are no others when A is a field.

Recall that $\mathbb{P}_k^1$ is obtained by gluing together the two open affine subsets $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$ along $V = \operatorname{Spec} k[u, u^{-1}]$. Given an invertible sheaf L on $\mathbb{P}^1$, the restriction of it to each of the two opens must be trivial since $\operatorname{Pic}(\mathbb{A}_k^1) = 0$, so there are isomorphisms $\phi_i \colon L|_{U_i} \to \mathcal{O}_{U_i}$. Over the intersection $V = U_0 \cap U_1$ we thus obtain two isomorphisms $\phi_i|_V \colon L|_V \to \mathcal{O}_V$. In particular, the composition $\phi_1|_V \circ \phi_0|_V^{-1} \colon \mathcal{O}_V \to \mathcal{O}_V$ is an isomorphism. Like any such map, it is induced by a module homomorphism $k[u, u^{-1}] \to k[u, u^{-1}]$ which is just multiplication by some unit in $k[u, u^{-1}]$. But all units in $k[u, u^{-1}]$ are of the form αu^m for an integer m and non-zero scalar α, the latter can be ignored (incorporate it in one of the ϕ_i's), and we recognize L to be the sheaf $\mathcal{O}_{\mathbb{P}_k^1}(m)$ from Chapter **??**.

With the present set-up we also obtain in a natural way an isomorphism $\mathcal{O}_{\mathbb{P}_k^1}(m) \otimes \mathcal{O}_{\mathbb{P}_k^1}(m') \simeq \mathcal{O}_{\mathbb{P}_k^1}(m + m')$: the gluing map over V for the tensor product equals the tensor product of the two gluing maps (which are multiplication by s^m and $s^{m'}$ respectively), and when we identify $\mathcal{O}_V \otimes \mathcal{O}_V$ with $\mathcal{O}_V$, it becomes the product of the two; that is, it becomes multiplication by $s^{m+m'}$. In particular, it holds that $\mathcal{O}_{\mathbb{P}_k^1}(m) \otimes \mathcal{O}_{\mathbb{P}_k^1}(-m) \simeq \mathcal{O}_{\mathbb{P}_k^1}$.

Back in Chapter **??** we verified that the sheaves $\mathcal{O}_{\mathbb{P}_k^1}(m)$ are not isomorphic when $m \geqslant 0$; e.g. since they have different spaces of global sections, and what we just did, extends this to all m. We thus have shown:

Proposition 15.57. Every invertible sheaf on $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(m)$ for some $m \in \mathbb{Z}$, and sending $\mathcal{O}_{\mathbb{P}^1_k}(m)$ to m yields an isomorphism $\operatorname{Pic} \mathbb{P}^1_k \simeq \mathbb{Z}$.

We will prove a generalization of this in Proposition 18.38.

15.9 Exercises

Exercise 15.9.1. Show the following:

(i) The skyscraper sheaf of k on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ at the origin 0 is quasi-coherent;
(ii) The skyscraper sheaf of $k(t)$ on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ at the origin 0 is *not* quasi-coherent. HINT: Consider sections over $U = D(t)$.

Exercise 15.9.2. Let $\mathbb{A}^3_k = \operatorname{Spec} k[x, y, z]$ and consider the *twisted cubic curve* C given by the ideal

$$I = (y - x^2, z - x^3)$$

Let $\pi : C \to \mathbb{A}^1_k = \operatorname{Spec} k[z]$ be the projection from the line $L = V(x, y)$.

(i) Show that π is a finite morphism;
(ii) Compute $\pi_*\mathcal{O}_C$, $\pi^*\mathcal{O}_{\mathbb{A}^1_k}$ and $\pi^*\mathcal{J}$ where $\mathcal{J}$ is the ideal sheaf of the closed point $0 \in \mathbb{A}^1_k$.

Exercise 15.9.3. Let $f : X \to Y$ be a morphism of schemes and let $x \in X$ be a point. We say that:

- A quasi-coherent sheaf $\mathcal{F}$ on X is *flat over* Y at x if $\mathcal{F}_x$ is flat as a $\mathcal{O}_{Y,f(x)}$-module (where $\mathcal{F}_x$ is considered as a $\mathcal{O}_{Y,f(x)}$-module via the natural map $f^\sharp_x : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,f(x)}$);
- $\mathcal{F}$ is *flat* if it is flat at every point in X;
- f is flat if $\mathcal{O}_X$ is flat over Y

(i) Show that open embeddings are flat. What about closed immersions?
(ii) Show that a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if the map of rings $A \to B$ is flat. More generally, a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Spec} B$ is flat over $\operatorname{Spec} A$ if and only if M is flat as an A-module;
(iii) Which of the morphisms in Exercise 2.7.14 are flat?
(iv) Prove that the blow-up morphism $\pi : Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ is not flat.

Exercise 15.9.4. Prove that the morphism $r\colon X_{\text{red}} \to X$ is a closed immersion.

Exercise 15.9.5 (Functoriality of $(-)_{\text{red}}$)**.** If $f\colon X \to Y$ is a morphism, show that there is a unique morphism $f_{\text{red}}\colon X_{\text{red}} \to Y_{\text{red}}$ so that $f_{\text{red}} \circ r_X = r_Y \circ f_{\text{red}}$. Show that assignments $X \mapsto X_{\text{red}}$ and $f \mapsto f_{\text{red}}$ defines a functor Sch to RedSch which is adjoint to the inclusion functor $\mathsf{RedSch} \to \mathsf{Sch}$, where RedSch is the full subcategory of Sch whose objects are the reduced schemes.

Exercise 15.9.6. Prove Proposition 15.44

Exercise 15.9.7 (Morphisms to a closed subscheme)**.** Let Z be a closed subscheme of X given by sheaf of ideals $\mathcal{I}$. Suppose $f : Y \to X$ is a morphism of schemes. Show that f factors through a map $g : Y \to Z$ if and only if

(i) $f(Y) \subseteq Z$;

(ii) $\mathcal{I} \subseteq \operatorname{Ker}(f^\sharp\colon \mathcal{O}_X \to f_*(\mathcal{O}_Y))$.

For a morphism of schemes $f : Y \to X$, we can define the *scheme-theoretic image* of f as a subscheme $Z \subseteq X$ satisfying the universal property that if f factors through a subscheme $Z' \subseteq Z$, then $Z \subseteq Z'$. To define Z it is is tempting to use the ideal sheaf $\mathcal{I} = \operatorname{Ker}(\mathcal{O}_X \to f_*(\mathcal{O}_Y))$ — but this may fail to be quasi-coherent for a general morphism f. However, one can show that there is a largest quasi-coherent sheaf of ideals $\mathcal{J}$ contained in $\mathcal{I}$, and we then define Z to be associated to $\mathcal{J}$.

Exercise 15.9.8 (Noetherian induction)**.** Let X is a scheme. The closed subschemes form a partially ordered set when one lets $Z \subset Z'$ mean that the closed immersion $Z \hookrightarrow X$ factors through the immersion $Z' \hookrightarrow X$.

(i) Show that $Z \subset Z'$ if and only it $\mathcal{I}(Z') \subset \mathcal{I}(Z)$;

(ii) Assume X to be Noetherian. Show that any non-empty set Σ of closed subschemes contains a minimal element.

Exercise 15.9.9 (Generic freeness of coherent sheaves)**.** Assume that X is a reduced and irreducible scheme and let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is 'generically free', or phrased differently, 'up to coherent sheaves with proper support it may be approximated by a free sheaf'. In precise terms, show that there is a coherent sheaf $\mathcal{H}$ on X and a map $\alpha\colon \mathcal{F} \to \mathcal{H}$ with the two properties

(i) Both supports $\operatorname{Supp}\operatorname{Ker}\alpha$ and $\operatorname{Supp}\operatorname{Coker}\alpha$ are proper subschemes of X;

(ii) There is an integer and an inclusion $\mathcal{O}_X^r \subset \mathcal{H}$ of a free sheaf such that the quotient $\mathcal{H}/\mathcal{O}_X^r$ has proper support.

Exercise 15.9.10 (An ideal sheaf which is not quasi-coherent)**.** Let $X = \operatorname{Spec} k[T] = \mathbb{A}_k^1$ and consider the origin $P \in X = \mathbb{A}_k^1$ corresponding to the maximal ideal $(T) \subset k[T]$. Define the presheaf $\mathcal{I}$ of $\mathcal{O}_X$ by for each open subset $U \subset X$ letting$\mathcal{I}(U) \subset \mathcal{O}_X(U)$ be given as

$$\mathcal{I}(U) = \begin{cases} \mathcal{O}_X(U) & \text{if } P \notin U; \\ 0 & \text{if } P \in U. \end{cases}$$

a) Show that $\mathcal{I}$ is an ideal sheaf, and $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is not a closed subset of X.

b) Show directly that $\mathcal{I}$ is not quasi-coherent by showing that $\mathcal{I}(X) = 0$, but $\mathcal{I} \neq 0$.

Exercise 15.9.11. Show that the '2-out-of-3'-property holds for coherent sheaves. That is, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, and if two of $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are coherent, then so is the third.

Exercise 15.9.12. Show that the direct sum of two coherent sheaves is again coherent. Hint: Use Exercise 15.9.11.

Exercise 15.9.13. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of coherent sheaves. Show that $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$ and

$\operatorname{Coker} \phi$ are all coherent. Hint: Find two natural exact sequences involving these sheaves and apply Exercise 15.9.11.

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Sheaves on projective schemes

Projective schemes are to affine schemes what projective varieties are to affine varieties. The construction of the projective spectrum $\operatorname{Proj} R$ is similar to that of the affine spectrum $\operatorname{Spec} R$: the underlying topological space is defined with the help of prime ideals and the structure sheaf from localizations of R. However, there are some fundamental differences between the two: in the proj-construction one only considers *graded* rings R, and only homogeneous prime ideals that do not contain the irrelevant ideal R_+. As we saw, this reflects the construction of the projective spectrum $\operatorname{Proj} R$ as a quotient space

$$\pi\colon \operatorname{Spec} R - V(R_+) \to \operatorname{Proj} R.$$

Given this, we can pull back a quasi-coherent sheaf to $\operatorname{Spec} R - V(R_+)$ and extend it to a sheaf on $\operatorname{Spec} R$ via the inclusion map. Thus, it is natural to expect that quasi-coherent sheaves on $\operatorname{Proj} R$ should be in correspondence with ‘equivariant’ modules on $\operatorname{Spec} R$; i.e. the *graded*[1] R-modules. The irrelevant subscheme $V(R_+)$ complicates the picture and makes the classification a little bit more involved than it is for affines schemes. In particular, we will see that different graded R-modules may correspond to the same quasi-coherent sheaf on $\operatorname{Proj} R$.

Another important feature of $\operatorname{Proj} R$ is that it comes equipped with a canonical invertible sheaf which is denoted by $\mathcal{O}_{\operatorname{Proj} R}(1)$. This is the geometric manifestation of the fact that R is graded. Unlike the case of affine schemes, $\operatorname{Proj} R$ can typically not be recovered from the global sections of the structure sheaf. It is the sheaf $\mathcal{O}_{\operatorname{Proj} R}(1)$, or rather, the various tensor powers $\mathcal{O}_{\operatorname{Proj} R}(d) = \mathcal{O}_{\operatorname{Proj} R}(1)^{\otimes d}$, that will play the role of the affine coordinate ring. So it is rather from the pair $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R}(1))$ one may hope to recover R.

16.1 The graded tilde-functor

Let R be a graded ring and let GrMod_R denote the category of graded R-modules. Just as in the case of the affine spectrum $\operatorname{Spec} A$, we shall set up a tilde-construction which produces sheaves on $\operatorname{Proj} R$ from graded R-modules, and in this way gives a functor GrMod_R to $\mathsf{QCoh}_{\operatorname{Proj} R}$. However, in contrast to the affine case, this will not be an equivalence of categories.

[1] In the model case of the projective spaces, the variety $\mathbb{P}^n$ is the quotient of $\mathbb{A}^n - \{0\}$ by the group $k^\times$ acting by scalar multiplication, so in this case, the notion ‘equivariant’ is precise and pertinent.

Homogenization and dehomogenization

Back in in Chapter 9 on page 135, we used a homogenization-dehomogenization process to construct the structure sheaf on $\operatorname{Proj} R$, and we shall rely on a similar technique for modules in the tilde-construction.

For an inclusion of two distinguished open sets $D_+(g) \subset D_+(f)$, we have a relation of the form $g^r = af$ for some homogeneous $a \in R$ and some $r \in \mathbb{N}$. And as f becomes invertible in R_g, there is a canonical map $M_f \to M_g$ between the localized modules. It respects the gradings since both f and g are homogeneous, and its action on the degree zero parts yields a canonical map

$$\rho_{f,g} \colon (M_f)_0 \to (M_g)_0,$$

which sends an element mf^{-n} with x homogeneous and $\deg m = n \deg f$ to the element $a^n m g^{-nr}$.

Let $\mathscr{B}$ being the basis for the Zariski topology consisting of the distinguished open subsets. We define a $\mathscr{B}$-presheaf $\widetilde{M}$ by letting its sections over $D_+(f)$ be given by

$$\widetilde{M}(D_+(f)) = (M_f)_0,$$

and when $D_+(g) \subset D_+(f)$, letting the restriction maps $\widetilde{M}(D_+(f)) \to \widetilde{M}(D_+(g))$ be the maps $\rho_{f,g}$ above. The two requirements to be a presheaf are easily verified; for instance, by taking the degree zero part (which is an exact operation) of the fundamental sequence 3.2 for the sheaf $\widetilde{M}$ on $\operatorname{Spec} R$.

In Proposition 9.12 on page 137 we established a canonical isomorphism $D_+(f) \simeq \operatorname{Spec}(R_f)_0$. Unsurprisingly, the presheaf $\widetilde{M}$ restricted to $D_+(f)$ yields the sheaf $\widetilde{(M_f)_0}$ on $\operatorname{Spec}(R_f)_0$:

Proposition 16.1. Under the isomorphism between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$ one has $\widetilde{M}|_{D_+(f)} \simeq \widetilde{(M_f)_0}$.

A distinguished subset $D_+(g)$ of $D_+(f)$ is mapped isomorphically onto the distinguished open subset $D(u)$ of $\operatorname{Spec}(R_f)_0$ where $u = g^{\deg f}/f^{\deg g}$ (the simplest degree zero element in R_f one can create out of f and g).

The proposition follows directly from the lemma below, whose proof relies on the following two simple observations. If M is any module (graded or nor not) over a ring R, one has a canonical isomorphism[2] $(M_f)_a \simeq M_{fa}$, which follows from the universal property of localization. Secondly, if M is graded, localization in a degree zero elements commutes with taking the degree zero part: if $\deg a = 0$, we have a natural isomorphism $((M_f)_a)_0 \simeq ((M_f)_0)_a$.

Lemma 16.2. With the notation above, the canonical homomorphism $\rho_{f,g} \colon (M_f)_0 \to (M_g)_0$ induces an isomorphism $((M_f)_0)_u \simeq (M_g)_0$;

[2] In clear text, this boils down to writing $x/a^s f^t = a^{t-s}x/a^t f^t$ or $x/a^s f^t = f^{s-t}x/a^s f^s$ according to which one of s or t is the bigger.

Proof Write $g^r = af$, and consider let $u = g^{\deg f}/f^{\deg g}$. Note that in the ring R_f, where f is invertible, the multiplicative systems $\{a^i\}$ and $\{u^i\}$ have the same saturation (both being the saturation of $\{g^i\}$) so that $(M_f)_a = (M_f)_u$. From the observations preceding the lemma we infer that $(M_f)_u \simeq (M_f)_a \simeq M_{g^r} \simeq M_g$, and that $(M_g)_0 \simeq ((M_f)_u)_0 \simeq ((M_f)_0)_u$. □

As an immediate consequence of Proposition 16.1 we obtain the desired

Proposition 16.3. The $\mathscr{B}$-presheaf $\widetilde{M}$ is a $\mathscr{B}$-sheaf, and extends to a quasi-coherent sheaf on $\operatorname{Proj} R$; which we continue to denote $\widetilde{M}$.

Proof The $\mathscr{B}$-presheaf satisfies the axioms for being a $\mathscr{B}$-sheaf. Indeed, for a fixed $D_+(f)$ in $\mathscr{B}$, both $\mathscr{B}$-sheaf axioms only involve distinguished opens contained in $D_+(f)$, and the restriction $\widetilde{M}|_{D_+(f)}$ of the $\mathscr{B}$-presheaf is a sheaf by Proposition 16.1.

□

As is the case for the tilde-construction for affine spectra, the assignment $M \mapsto \widetilde{M}$ is functorial and gives a functor $\mathsf{GrMod}_R \to \mathsf{QCoh}_{\operatorname{Proj} R}$. This is close to obvious as a map $M \to N$ which is homogeneous of degree zero, persists being homogeneous of degree zero when localized, and so induces maps $(M_f)_0 \to (N_f)_0$.

Basic properties of the tilde-functor

In some aspects the projective tilde-functor behaves as the affine one, but in other aspects the behaviour deviates seriously; the most striking difference being that different modules may yield isomorphic sheaves, and this is inherent, not accidental.

The following proposition summarizes the basic properties of the tilde-functor.

Proposition 16.4. Let R be a graded ring. The functor $\mathsf{GrMod}_R \to \mathsf{QCoh}_{\operatorname{Proj} R}$ that sends M to $\widetilde{M}$ has the following properties:

(i) It is additive and exact and commutes with direct limits;
(ii) Sections over distinguished open sets: for homogeneous elements $f \in R$, it holds that $\widetilde{M}(D_+(f)) = (M_f)_0$;
(iii) Stalks: for each $\mathfrak{p} \in \operatorname{Proj} R$ it holds that $\widetilde{M}_{\mathfrak{p}} = (M_{\mathfrak{p}})_0$;
(iv) When M is finitely generated, then $\widetilde{M}$ is of finite type, and when M is of finite presentation, the same holds for $\widetilde{M}$. In particular, when R is Noetherian and M is finitely generated, $\widetilde{M}$ will be coherent.

Proof Claim (i) holds since localization and taking degree zero parts are exact operation that commutes with forming direct limits and (arbitrary) direct sums. The second claim is just the definition of the sheaf $\widetilde{M}$, and third is a consequence of $\widetilde{M}|_{D_+(f)} = \widetilde{(M_f)_0}$ (Proposition 16.1), that a prime $\mathfrak{p} \in D_+(f) \subset \operatorname{Proj} R$ corresponds to $\mathfrak{q} = (\mathfrak{p}R_f)_0$ and that we have the equality $\widetilde{M}_{\mathfrak{p}} = ((M_f)_0)_{\mathfrak{q}} = (M_{\mathfrak{q}})_0 = (M_{\mathfrak{p}})_0$.

The last statements are direct corollaries of the tilde-functor being exact. □

A consequence of (ii) is that each element $m \in M_0$ gives rise to a section $\Gamma(D_+(f), \widetilde{M})$, which is just the image of m in $(M_f)_0$. These local sections clearly agree on overlaps $D_+(fg)$ the two restrictions borth being the image of m in $(M_{fg})_0$, and so they glue together to a global section of $\widetilde{M}$. Hence we have the following

Lemma 16.5. There is a canonical map $M_0 \to \Gamma(\operatorname{Proj} R, \widetilde{M})$, which is functorial in M.

It is important to note that, unlike in the affine case, the tilde-functor is not faithful; several modules can correspond to the same sheaf. This is not so surprising and is rooted in the fact that primes in $V(R_+)$ are thrown away in the Proj-construction, which has the effect that modules supported in $V(R_+)$ necessarily give the zero sheaf when exposed to the tilde-functor.

For any integer d we let $M_{>d}$ be the R-module $M_{>d} = \bigoplus_{i>d} M_i$ (it is an R-module because of the standing hypothesis that R be positively graded).

Lemma 16.6. Assume that R is a graded ring and let M and N be two graded R-modules,

(i) If $\operatorname{Supp} M \subset V(R_+)$, then $\widetilde{M} = 0$;

(ii) Assume that $M_{>d} \simeq N_{>d}$ for some d. Then $\widetilde{M} \simeq \tilde{N}$.

If R is generated in degree one, the converse of (i) holds true.

The converse of (ii) does not hold in general even if R is generated in degree one; but as we shall see, in that case it holds for finitely generated R-modules.

Proof To prove (i), suppose that $\operatorname{Supp} M \subset V(R_+)$. Statement (iii) of Proposition 16.4 above then entails that $\widetilde{M} = 0$ since $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Proj} R$.

To prove (ii), note that the quotient $M/M_{>d}$ is killed by the power $(R_+)^d$ and consequently has support in $V(R_+)$. By (i) its tilded sheaf vanishes, and hence $\widetilde{M_{>d}} = \widetilde{M}$. As this holds for both M and N we are through.

For the converse of (i): if the support of M is not contained in $V(R_+)$, there is a homogeneous prime ideal $\mathfrak{p} \in \operatorname{Proj} R$ such that $M_{\mathfrak{p}} \neq 0$. In general, it might be that there are no elements of degree zero in $M_{\mathfrak{p}}$, but as R is generated in degree one it holds that $(M_{\mathfrak{p}})_0 \neq 0$. In that case, $\operatorname{Proj} R$ is covered by distinguished open sets $D_+(f)$ with $\deg f = 1$, and so there is an f of degree one not lying in $\mathfrak{p}$. Then for each non-zero homogeneous element $x \in M_{\mathfrak{p}}$, the element $x/f^{\deg f}$ yields a non-zero element of degree zero in $M_{\mathfrak{p}}$. □

Example 16.7. On $X = \operatorname{Proj} k[x_0, x_1]$, the module $M = k[x_0, x_1]/(x_0^2, x_1^2)$ has $\widetilde{M} = 0$, but it is non-zero.

The following simple example may be instructive. It illustrates the subtlety of the proj-construction for rings not generated in degree one.

Example 16.8. Let R be a graded ring generated in degree two, which means that all elements in R are of even degree. A graded R-module M all whose elements are of odd degree, will then have a vanishing tilde-sheaf whatever its support is, for the simple reason that an element like x/f^s can not be of degree zero when $\deg x$ is odd and $\deg f$ is even. Hence $(M_{\mathfrak{p}})_0 = 0$ for all $\mathfrak{p} \in \operatorname{Proj} R$.

A concrete minimalistic example can be $R = k[t^2]$ and $M = \bigoplus_{i\geqslant 0} k \cdot t^{2i+1}$, the submodule of $k[t]$ of polynomials all whose non-zero terms are of odd degree. Then M is not supported in $V(R_+) = \{(x^2)\}$, but $\widetilde{M} = 0$ by the above reasoning.

It may seem paradoxical that redefining the grading on $k[t^2]$ by giving t^2 degree one, the tilde-construction will be faithful for modules supported off $V(R_+)$; the explanation is that the 'counter-example' M above is no more a graded module! Well, the only sensible degree one could give t and still make the example work, would be $1/2$, which is not allowed.

The next lemma is sometimes useful when working with the localization of M when R is generated in degree one. It says essentially that we are allowed to 'substitute 1 for f' when restricting a module to an affine chart $D_+(f) \subset \operatorname{Proj} R$.

Lemma 16.9. Suppose that M is a graded R-module and that $f \in R$ homogeneous of degree one. Then there are natural isomorphisms of $(R_f)_0$-modules

$$(M_f)_0 \simeq M/(f-1)M \simeq M \otimes_R R/(f-1)R.$$

In particular, $(R_0)_f \simeq R/(f-1)R$.

Proof The element f acts as the identity on the R-module $M/(f-1)M$, so $M/(f-1)M$ is a module over R_f. Plainly sending xf^{-r} to the class of x yields an R_f-linear homomorphism $M_f \to M/(f-1)M$, as one easily verifies, and restricting it to the degree zero piece one obtains an $(R_f)_0$-homomorphism $(M_f)_0 \to M/(f-1)M$. It is surjective: the class of a homogeneous element x is the image of $xf^{-\deg x}$, and every element in $M/(f-1)M$ is the sum of classes of homogenous elements. To check it is injective, assume that $xf^{-\deg m}$ maps to zero; i.e. that $x = (f-1)y$ for some $y \in M$. Expanding y in homogeneous components we may write $y = \sum_{s\leqslant i\leqslant t} y_i$ with neither y_s nor y_t equal to zero. Then

$$x = (f-1)y = -y_s + \sum_{s}^{t-1}(fy_i - y_{i+1}) + fy_t.$$

Because x is homogeneous and $y_s \neq 0$, we may infer that $y_s = -x$, but also that $fy_t = 0$ and $y_{i+1} = fy_i$. A straightforward induction then yields equalities $y_t = f^{t-s}y_s = -f^{t-s}x$; consequently x is killed by a power of f and vanishes in M_f. □

Example 16.10. That f is of degree one is essential. To give an example where the above lemma fails, let $M = R = k[t]$ and $f = t^2$. We find $k[t]_{t^2} = k[t, t^{-2}] = k[t, t^{-1}]$ so that $(k[t]_{t^2})_0 = k$. But $k[t]/(t^2-1) \simeq k \oplus k$.

Tensor product & Hom's

Let M and N be two graded modules over the graded ring R. There is a natural way of giving the tensor product a graded structure; a decomposable tensor $x \otimes y$ is precisely homogenous when x and y are, and of course, it is of degree $\deg x + \deg y$. Homogenous tensors will be the R_0-linear combinations of decomposables of the same degree, so the graded piece of degree d of $M \otimes_R N$ will be the image of $\bigoplus_{i+j=d} M_i \otimes_{R_0} M_j$. One may check that $M \otimes_R N$ as an R_0-module is the direct sum of these graded parts (that they generate is obvious; that the pairwise intersections are zero is slightly more subtle).

The tilde-functor is in the case of affine spectra well-behaved when it comes to tensor products in that $\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{N} = \widetilde{M \otimes_A N}$. In the projective case it is not always the case. Unless R is generated in degree one, curious phenomena take place.

Example 16.11. We return to the ring $R = k[t^2]$ with t^2 of degree two and the graded R-module $M = \bigoplus_{i \geqslant 0} k \cdot x^{2i+1}$ from Exanple 16.8, wherewe saw that $\widetilde{M} = 0$. However, in the tensor product $M \otimes_R M$ all elements are of even degree (indeed, $\deg x \otimes y = \deg x + \deg y$ and both these are odd), and consequently it holds that $(M \otimes_R M)\tilde{} \neq 0$. So we have en example that $(M \otimes_R M)^{\sim} \neq \widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{M}$.

Note that the example also illustrates that the converse of Lemma 16.6 does not hold unconditionally (but, again as we shall see, it holds true when R is generated in degree one).

Let us proceed to compare $\widetilde{M \otimes_R N}$ with $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. For each homogeneous element $f \in R$ there is a canonical map

$$M_f \otimes_{(R_f)_0} N_f \to M_f \otimes_{R_f} N_f \simeq (M \otimes_R N)_f$$

which sends $x/f^n \otimes y/f^m$ to $x \otimes y/f^{n+m}$. When restricted to elements of degree zero it gives a map

$$(M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \to ((M \otimes_R N)_f)_0, \tag{16.1}$$

which one easily checks is compatible with the restriction maps induced from inclusions $D_+(g) \subset D_+(f)$, and so it is a map of $\mathscr{B}$-sheaves with $\mathscr{B}$ being the basis of distinguished open subsets. Hence it induce maps between sheaves, and we get a natural map

$$\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{N} \to \widetilde{M \otimes_R N}, \tag{16.2}$$

It is, as the Example 16.11 above shows, not always an isomorphism, but when R is generated ine degree one, it is well behaved:

Proposition 16.12. Let R be a graded ring and suppose that R is generated in degree one. For every graded R-modules M and N, the natural map

$$\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{N} \to \widetilde{M \otimes_R N}$$

is an isomorphism.

Proof By assumption, $\operatorname{Proj} R$ is covered by open affines of the form $D_+(f)$ where f has degree one. For such an f, the functor $M \to (M_f)_0$ coincides with the tensor-functor $M \mapsto M \otimes_R R/(f-1)R$ by Lemma 16.9. Furthermore, one of the standard properties of the tensor product is that the canonical map $(x \otimes a) \otimes (y \otimes b) \mapsto x \otimes y \otimes ab$ yields an isomorphism

$$(M \otimes_R R/(f-1)R) \otimes_{R/(f-1)R} (N \otimes_R R/(f-1)R) \simeq (M \otimes_R N) \otimes_R R/(f-1)R,$$

but this is just the map in (16.1). □

Exercise 16.1.1. Let $R = \mathbb{Q}[x, y, z]$ with $\deg x = 1, \deg y = 2, \deg z = 3$. Show that the map (16.2) is not an isomorphism for $M = R(1)$ and $N = R(2)$.

16.2 Serre's twisting sheaf $\mathcal{O}(1)$

Arguably the most interesting sheaf on $\operatorname{Proj} R$ is the so-called *twisting sheaf*, denoted by $\mathcal{O}_{\operatorname{Proj} R}(1)$. This is a generalization of the tautological sheaf on $\mathbb{P}^n_k$, and constitutes a geometric manifestation of the fact that R is a *graded ring*. They were introduced in the groundbreaking paper **?** by Jean-Pierre Serre. Elements in R do not define 'regular functions' on $\operatorname{Proj} R$, and we shall see that in good cases R_d will be the space of sections of the tensor power $\mathcal{O}_{\operatorname{Proj} R}(d)$ when $d \geqslant 0$, and this is a means of recovering the ring R. We already met the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(d)$ on the projective line in Section 7.2

Let M be a graded module over the graded ring R. For each integer n one defines an R-module $M(n)$ as follows: the underlying R-module of $M(n)$ is just M, but the grading is shifted:

$$M(n)_d = M_{d+n}. \tag{16.3}$$

Thus $N = M(n)$ is a graded R-module with $N_0 = M_n$, $N_1 = M_{n+1}$ and so on. Note that elements from M_d considered as element in $M(n)$ will be of degree $d - n$ /replace d by $d - n$ in (16.3)). The construction is functorial and is called the functor $M \mapsto M(n)$ is called *shift-functor* or the *twist-functor*.

In the particular case when $M = R$, this gives a graded and free R-module $R(n)$, which is generated by the element $1 \in R_{-n}$. Note the equality $M(n) = M \otimes_R R(n)$: both have M as underlying module, and the image of $\bigoplus_{i+j=d} M_i \otimes_{R_0} R(n)_j$ equals M_{d+n}.

Example 16.13. It holds that $R(n) \otimes_R R(m) = R(n + m)$.

Applying the tilde-functor to $R(n)$ gives us a quasi-coherent $\mathcal{O}_{\operatorname{Proj} R}$-module on $\operatorname{Proj} R$:

Definition 16.14. Let R be a graded ring. For each integer n, and for $X = \operatorname{Proj} R$, we define

$$\mathcal{O}_X(n) = \widetilde{R(n)}.$$

It $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X, we let $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ and call it the *twist of* $\mathcal{F}$ *by* n.

Consider an element $f \in$ of degree one. As f is invertible in R_f it holds for each $n \in \mathfrak{Z}$ that $f^n R_f = R_f$, and since f is of degree one, we find taking out the piece of degree n, that $(R(n)_f)_0 = (R_f)_n = f^n \cdot (R_f)_0$. Thus, on the distinguished affine open set $D_+(f)$ it holds true that $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_X|_{D_+(f)}$. In particular, $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_{D_+(f)} \simeq \mathcal{O}_{D_+(f)}$. Said differently, if R is generated in degree one, the $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ is an invertible sheaf. The following generalises the multiplicative properties of $\mathcal{O}_{\mathbb{P}^1_A}(m)$ from Example **??** to general projective schemes

Proposition 16.15. When R is generated in degree one, the sheaf $\mathcal{O}_X(n)$ is invertible for every $n \in \mathbb{Z}$. Moreover, there are canonical isomorphisms

$$\mathcal{O}_X(m+n) \simeq \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Proof Indeed, if R is generated in degree one, Proposition **??** shows that $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n)$ is the sheaf associated to $R(m) \otimes_R R(n) \simeq R(n+m)$; that is, it equals $\mathcal{O}_X(n+m)$. □

So this is a big difference between affine schemes and projective schemes: $\operatorname{Proj} R$ comes equipped with lots of invertible sheaves.

Example 16.16 ($\mathbb{P}^1_A$ **once more).** Recall the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(n)$ from Section 7.2, which of course are the same as the ones constructed above. Let $\mathbb{P}^1_A = \operatorname{Proj} A[u_0, u_1]$. On the distinguished open sets $D_+(u_0)$ and $D_+(u_1)$ it holds that $\mathcal{O}_{\mathbb{P}^1_A}(n)|_{D_+(u_0)} = u_0^n \mathcal{O}_{\mathbb{P}^1_A}|_{D_+(u_0)}$, and that $\mathcal{O}_{\mathbb{P}^1_A}(n)|_{D_+(u_1)} = u_1^n \mathcal{O}_{\mathbb{P}^1_A}|_{D_+(u_1)}$, so the gluing function over $D_+(u_0) \cap D_+(u_1)$ is multiplication by $(u_0/u_1)^n$, which agrees nicely with the gluing function used in Section 7.2.

As alluded to above, the main point of the sheaves $\mathcal{O}_X(d)$ is that they help us recover the ring R; for instance, while x_0^d does not correspond to a regular function on $\operatorname{Proj} k[x_0, x_1]$, it gives a section of the sheaf $\mathcal{O}_X(d)$.

16.3 The associated graded module

We have associated to a graded R-module M a sheaf $\widetilde{M}$ on $X = \operatorname{Proj} R$. To classify quasi-coherent sheaves on X we would, as in the case of affine schemes, like to give some sort of inverse to this assignment. However, as opposed to the case for $X = \operatorname{Spec} A$, simply using the global sections functor will not work. Indeed, even for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1_k}$ on $\mathbb{P}^1_k$, it holds that $\Gamma(\mathbb{P}^1_k, \mathcal{F}) = k$, from which we certainly cannot recover $\mathcal{F}$. The remedy is to look at the various Serre twists $\mathcal{F}(d)$ of $\mathcal{F}$; in fact all of them at once:

Definition 16.17. Let R be a graded ring and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X = \operatorname{Proj} R$. We define the *graded R-module associated* to $\mathcal{F}$, denoted $\Gamma_*(\mathcal{F})$ as

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{F}(d)).$$

In particular, from X alone we get the *associated graded ring*

$$\Gamma_*(\mathcal{O}_X) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)).$$

The associated graded module has the structure of an R-module defuned in the following way.

For each graded R-module M there is a homomorphism of graded R-modules

$$\alpha_M \colon M \to \Gamma_*(\widetilde{M}). \tag{16.4}$$

Indeed, for each integer d it holds that $(M(d))_0 = M_d$, and the map in Lemma 16.5 is a map $M_d \to \Gamma(X, \widetilde{M}(d))$. Summing up over all integers d then yields the map α_M. In particular, with $M = R$ we infer that every homogeneous element $h \in R_d$ gives a section $\alpha(h)$ (which we by abuse of language we also shall denote by h). In other words we have a map of sheaves

$$\mathcal{O}_X \to \mathcal{O}_X(d). \tag{16.5}$$

Tensorized by $\mathcal{F}(n)$ this map induces a map $\mathcal{F}(n) \to \mathcal{F}(n+d)$. On global sections it is a map $\Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{F}(n+d))$, and summing up over all $n \in \mathbb{Z}$, we find a map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{F})$, which is to be multiplication h.

On a distinguished open subscheme $D_+(g)$ the map in (16.5) equals the map $(R_g)_0 \to (R(d)_g)_0 = (R_g)(d)$ that acts as $x/g^r \mapsto hx/g^r$. In particular, over $D_+(h)$ we have an isomorphism

$$\mathcal{O}_X(d)|_{D_+(h)} \simeq h\mathcal{O}_{D_+(h)},$$

and in case x is of degree one, this yields an isomorphism

$$\mathcal{O}_X(d)|_{D_+(x)} \simeq x^d\mathcal{O}_{D_+(x)},$$

Proposition 16.18. Let R be a graded ring finitely generated over R_0 in degree one by elements $x_0, \ldots, x_n$ which are non-zerodivsors in R. Let $X = \operatorname{Proj} R$. Then
(i) $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^n R_{x_i} \subset K(R)$;
(ii) If each x_i is a prime element, then $R = \Gamma_*(\mathcal{O}_X)$.

Proof Cover X by the distinguished open subschemes $U_i = D_+(x_i)$. We have, since $\Gamma(D_+(x_i), \mathcal{O}(d)) \simeq (R_{x_i})_d$, that the sheaf axiom sequence takes the form

$$0 \to \Gamma(X, \mathcal{O}(d)) \to \bigoplus_{i=0}^n (R_{x_i})_d \to \bigoplus_{i,j} (R_{x_i x_j})_d,$$

which when summed over all integers m becomes

$$0 \to \Gamma_*(\mathcal{O}_X) \to \bigoplus_{i=0}^n R_{x_i} \to \bigoplus_{i,j} R_{x_i x_j}.$$

So a section of $\Gamma_*(\mathcal{O}_X)$ corresponds to an $(n+1)$-tuple $(t_0, \ldots, t_n) \in \bigoplus_{i=0}^n (R_{x_i})$ such that t_i and t_j coincide in $R_{x_i x_j}$ for each $i \neq j$. Now, the x_i are not zero-divisors in R, so the localization maps $R \to R_{x_i}$ are injective. It follows that we can view all the localizations R_{x_i} as subrings of $R_{x_0 \ldots x_n}$, and then $\Gamma_*(\mathcal{O}_X)$ coincides with the intersection

$$\bigcap_{i=0}^n R_{x_i} \subset R_0[x_0, x_0^{-1}, \cdots, x_n, x_n^{-1}].$$

In the case that the x_i's are relatively prime, this intersection is just R. □

Corollary 16.19. Let $X = \mathbb{P}_A^n = \operatorname{Proj} A[x_0, \ldots, x_n]$ for a ring A. Then

$$\Gamma_*(\mathcal{O}_X) \simeq A[x_0, \ldots, x_n]$$

In particular we can identify $\Gamma(\mathbb{P}_A^n, \mathcal{O}(d))$ with the A-module generated by homogeneous degree d polynomials.

When R is not a polynomial ring, it can easily happen that $\Gamma_*(\mathcal{O}_X)$ is different than R. Here is a concrete example:

Example 16.20 (A quartic rational space curve)**.** A systematic way of producing examples of projective schemes $X = \operatorname{Proj} R$ so that R differs from $\Gamma_*\mathcal{O}_X$, is to start with a projective scheme (or a variety if you want) $X \in \mathbb{P}^n_k$ and project it into $\mathbb{P}^{n-1}_k$. In good cases this will anew be closed embedding of X, but in this new embedding X will be represented as $\operatorname{Proj} S$ with another graded ring S.

The simplest example of this set up is the rational normal quartic curve in $\mathbb{P}^4_k$. It is given as $X = \operatorname{Proj} R$ with

$$R = k[u^4, u^3v, u^2v^2, uv^3, v^4] \subset k[u, v],$$

where all of the generators are of degree one (this is nothing but the Veronese ring $k[u, v]^{(4)}$ from Section 9.3).

Projecting into a lower projective space corresponds to discarding some of the generators, and in our example we throw the monomial u^2v^2 away and work with $X = \operatorname{Proj} S$ where

$$S = k[u^4, u^3v, uv^3, v^4].$$

Evidently S_1 is of dimension 4, and we shall se that the monomial u^2v^2 reappears in $\Gamma(X, \mathcal{O}_X(1))$, and so $\Gamma(X, \mathcal{O}_X(1))$ will be of dimension 5.

Let us compute $\Gamma(X, \mathcal{O}_X(1))$. The first observation is that X has an open affine cover consisting of $U_0 = D_+(u^4)$ and $U_1 = D_+(v^4)$. This is the case because the ideal (u^4, v^4) is primary for $S_+ = (u^4, u^3v, uv^3, v^4)$; indeed, it holds that $S_+^4 \subset (u^4, v^4)$. Moreover we have equalities $\mathcal{O}_X(U_0) = (S_{u^4})_0 = k[vu^{-1}]$ and $\mathcal{O}_X(U_1) = k[uv^{-1}]$, and the cover consisting of the U_i's tivializes the sheaf $\mathcal{O}_X(1)$ with isomorphisms $\mathcal{O}_X(1)|_{U_0} \simeq \mathcal{O}_{U_0}u^4$ and $\mathcal{O}_X(1)|_{U_1} \simeq \mathcal{O}_{U_1}v^4$. The fundamental sequence then takes the shape

$$0 \to \Gamma(X, \mathcal{O}_X(1)) \to k[vu^{-1}]u^4 \oplus k[uv^{-1}]v^4 \to k[uv^{-1}, vu^{-1}]u^4.$$

Note that $u^2v^2 = (uv^{-1})^2u^4 = (vu^{-1})^2v^4$, so the monomial u^2v^2 belongs to both the rings $k[sv^{-1}]v^4$ and $k[tu^{-1}]u^4$ and defines an element in $\Gamma(X, \mathcal{O}_X(1))$. In fact, one easily checks that

$$\Gamma(X, \mathcal{O}_X(1)) = k\{u^4, u^3v, u^2v^2, uv^3, v^4\}.$$

Thus $\Gamma(X, \mathcal{O}_X(1))$ contains all 5 monomials, while u^2v^2 is missing from S_1. In this example, the graded ring $\Gamma_*(\mathcal{O}_X) = k[u^4, u^3v, u^2v^2, uv^3, v^4]$ is the integral closure of S. Exercise 16.3.2 below shows that this is not a coincidence.

Example 16.21. Here is another example that Γ_* is not right exact. Let $\mathbb{P}^1_k = \operatorname{Proj} R$, with $R = k[u_0, u_1]$, and consider the exact sequence of graded R-modules

$$0 \longrightarrow R(-n) \xrightarrow{u_0^n} R \longrightarrow R/(u_0^n)R \longrightarrow 0.$$

When we apply the tilde functor to it, we obtain the exact sequence od sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-n) \xrightarrow{u_0^n} \mathcal{O}_{\mathbb{P}^1_k} \xrightarrow{\rho} \mathcal{A} \longrightarrow 0.$$

The sheaf $\mathcal{A}$ is a skyscraper sheaf supported at the point $x = (0 : 1)$ with stalk $\mathcal{A}_x$ at x equal to $A = k[u]/(u^n)$ where $u = u_0u_1^{-1}$. All the twists $\mathcal{A}(d)$ are also skyscraper sheaves, and their stalks at x are $\mathcal{A}(d)_x = \mathcal{A}_x \otimes \mathcal{O}_{\mathbb{P}^1_k}(d)_x = u_1^d \cdot A$ (which is isomorphic to $\mathcal{A}_x$). And, as for any skyscraper sheaf, the global sections coincides with the stalk: $\Gamma(\mathbb{P}^1_k, \mathcal{A}(d)) = u_1^d \cdot A$

The map $\rho(d)$ on global sections becomes

$$\rho_d \colon k[u_0, u_1]_d \to u_1^d \cdot A = u_1^d \cdot k[u]/(u^n)$$

which acts in the following way: write a homogeneous polynomial $p(u_0, u_1)$ of degree d as $p(u_0u_1^{-1}, 1)u_1^d$ and send it to $p_n(u_0u_1^{-1}, 1)u_1^d$ where p_n is the Taylor polynomial of p of degree n; that is, the truncated polynomial $p_n = \sum_{i \leqslant n} a_i u^i$ (when $p = \sum_{0 \leqslant i \leqslant d} a_i u^i$). One easily shows that the cokernel of this map is the k-vector space

$$B_d = \bigoplus_{d+1 \leqslant i \leqslant n-1} k \cdot u_0^i u_1^{i-d}$$

when $d \leqslant n-2$, and $B_d = 0$ when $d \geqslant n-1$.

Summing up over d and using that $\Gamma_* \mathcal{A} = \bigoplus_{i \geqslant 0} u_1^d A$ (where all elements of $A = k[u]/(u^n)$ are of degree zero), we find the exact sequence:

$$0 \longrightarrow \Gamma_* O_{\mathbb{P}^1_k}(-n) \longrightarrow \Gamma_* O_{\mathbb{P}^1_k} \xrightarrow{\Gamma_*\rho} \Gamma_* \mathcal{A} \longrightarrow \bigoplus\nolimits_{d \leqslant n-2} B_d \longrightarrow 0 \tag{16.6}$$

and $\Gamma_*\rho$ is not surjective when $n \geqslant 2$ even though ρ is.

Exercise 16.3.1. Let k be a field and let $R = k[x_0, \ldots, x_n]$. Let $\pi : \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n_k = \operatorname{Proj} R$ denote the 'quotient morphism' from Exercise **??**. Show that for a graded R-module M, we have

$$\pi_*(\widetilde{M}|_{\mathbb{A}^{n+1}_k - 0}) = \bigoplus_{n \in \mathbb{Z}} \widetilde{M}(d)$$

Exercise 16.3.2. Let R be a graded Noetherian integral domain generated in degree one. Show that $\Gamma_*(\mathcal{O}_X)$ is an integral extension of R. (Hint: Use the Cayley–Hamilton theorem.)

Exercise 16.3.3 (An exotism of QCoh_X)**.** In Exercise 15.2.4 we noted that when X is affine, QCoh_X has arbitrary direct products just defined as $\prod \widetilde{M}_i = (\prod M_i)\widetilde{\ }$. But unlike the case of modules, products of surjections in QCoh_X are not necessarily surjective: for each $n \in \mathbb{N}$ consider the map $\Gamma_*\rho$ in (16.6), give it an index and call it $\Gamma_*\rho_n$. Show that the tildes $\widetilde{\Gamma_*\rho_n}$ of the $\Gamma_*\rho_n$ are surjective, but that their product $\prod_n \widetilde{\Gamma_*\rho_n}$ is not.
HINT: The cokernel $\operatorname{Coker} \Gamma_*\rho_n$ of each map $\Gamma_*\rho_n$ is supported in $V(R_+)$ (it is of finite dimension over k), but their direct product is not (numerous elements are not killed by any power of x_0).

16.4 Quasi-coherent sheaves on $\operatorname{Proj} R$

As before, we assume that R is a graded Noetherian ring generated in degree one. The main theorem of this section says that any quasi-coherent sheaf $\mathcal{F}$ on $X = \operatorname{Proj} R$ is the tilde of some graded R-module M. Not surprisingly, this R-module is exactly the associated graded module $M = \Gamma_*(\mathcal{F})$.

Proposition 16.22. Let R be a graded ring, finitely generated in degree one over R_0. Suppose $\mathcal{F}$ is a quasi-coherent sheaf on $\operatorname{Proj} R$. Then there is a canonical isomorphism

$$\beta\colon \widetilde{M} \to \mathcal{F}. \tag{16.7}$$

where $M = \Gamma_*(\mathcal{F})$.

We will need some notation. Let $X = \operatorname{Proj} R$. Choose generators $x_1, \ldots, x_r$ of degree one for R and let let $U_i = D_+(x_i)$, then $U_i = \operatorname{Spec}(R_{x_i})_0$. Any homogenous element $f \in R$ of degree one induces a section in $\Gamma(X, \mathcal{O}_X(1))$, which we will continue denoting by f. The restriction to U_i of the invertible sheaf $\mathcal{O}_X(1)$ has x_i as generator so that $\mathcal{O}(1)|_{U_i} \simeq x_i\mathcal{O}_{U_i}$. Under this isomorphism $f|_{U_i}$ may be written as $f_i x_i$ for some $f_i \in \Gamma(U_i, \mathcal{O}_X) = (R_{x_i})_0$. Also note that $D_+(f) \cap U_i = D(f_i) \subset U_i$ is a distinguished open subset of U_i. When also f is of degree one, there are canonical isomorphisms

$$\mathcal{F}(d)|_{D_+(f)} \simeq f^d \mathcal{F}_{D_+(f)}. \tag{16.8}$$

In particular, multiplication by x_i^n gives an isomorphism

$$\mathcal{F}(n)|_{U_i} \simeq x_i^n \mathcal{F}.$$

Proof of Theorem 16.22 We begin with defining the map (16.7) over the distinguished opens $D_+(f)$ with $f \in R_1$; that is, we shall give maps

$$\beta_f\colon \widetilde{M}|_{D_+(f)} \to \mathcal{F}|_{D_+(f)}. \tag{16.9}$$

Since $D_+(f)$ is affine, it suffices to tell how β_f acts on global sections. Over $D_+(f)$ a section of $\widetilde{M}$ is represented by a fraction m/f^d where $m \in M_d = \Gamma(X, \mathcal{F}(d))$ and where d is sufficiently large. By (16.8) the section $m|_{D_+(f)}$ is of the form $f^d s$ for a section s of $\mathcal{F}$ over $D_+(f)$, and we simply let β_f send m/f^d to s.

It is straightforward to verify that the definitions of β_f and β_g for two elements of degree one agree on the overlaps $D(fg)$ and so glue together to the desired map β.

Injectivity of (16.9): Suppose that m/f^d maps to zero via the map β_f in (16.9), which means that $m \in \Gamma(X, \mathcal{F}(d))$ is a section such that $m|_{D_+(f)} f^{-d} = 0$. Then clearly $m|_{D_+(f)} = 0$, and we want to infer from this that $f^n m = 0$ for some $n \in \mathbb{N}$ (note that m is a global section of $\mathcal{F}(d)$).

Now, the distinguished open sets U_i cover X, and $D_+(f) \cap U_i = D(f_i)$. Because $m|_{D_+(f_i)} = 0$, we get that $f_i^{n_i} m|_{U_i} = 0$ for some $n_i \in \mathbb{N}$ (Exercise 15.2.3), and using the greater n_i, we may assume that the n_i's are equal, to n say. Locality then yields that $f^n m = 0$, and we are done.

Surjectivity of (16.9). Let $t \in \Gamma(D_+(f), \mathcal{F})$ and consider the restrictions $t_i = t|_{D(f_i)}$. Since U_i is affine for each i we know from Exercise 15.2.3 that some $f_i^n t_i$ extends to a section u_i in $\Gamma(U_i, \mathcal{F})$ (and as before we may choose an n that works for all i). In view of the isomorphism $\mathcal{F}|_{U_i}(n) \simeq x_i^n \mathcal{F}|_{U_i}$, we find

$$f^n t_i = x_i^n f_i^n t_i = m_i \in \Gamma(U_i, \mathcal{F}(n)).$$

A potential problem is that the m_i's might not necessarily agree on $U_i \cap U_j$, hindering them to be glued together. However, it holds that

$$m_i|_{D(f_i)} = t|_{D(f_i)} f^n|_{D(f_i)},$$

so at least $m_i = m_j$ on $U_i \cap U_j \cap D_+(f)$. Now, $U_i \cap U_j$ is also affine (because X is separated), and $U_i \cap U_j \cap D_+(f)$ is a distinguished open subset of $U_i \cap U_j$, so arguing as in the injectivity part shows that there is a large integer $l > 0$ such that

$$f^l(m_i|_{U_i \cap U_j} - m_j|_{U_i \cap U_j}) = 0$$

in $\Gamma(U_i \cap U_j, \mathcal{F}(n+l))$. It then follows that $f^{n+l}t_i$ can be glued to a section $m \in \Gamma(X, \mathcal{F}(n+l)$. By construction, this section has the property that it restricts to $tf^{n+l}|_{D_+(f)}$ over $D_+(f)$. Hence m/f^{n+l} maps to t via the map in (16.9). □

We have now have the two functors

$$\sim\colon \mathsf{GrMod}_R \to \mathsf{QCoh}_X$$
$$\Gamma_*\colon \mathsf{QCoh}_X \to \mathsf{GrMod}_R$$

Since $\beta\colon \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is an isomorphism, it follows that the tilde functor is essentially surjective; that is, every quasi-coherent sheaf on X is the tilde of a graded module. However, unlike the affine case, the functors do not give mutual inverses. The functor $\sim$ is not faithful as the tilde of any module M with support in $V(R_+)$ is the zero sheaf. By Lemma 16.6 however, this is the only source of ambiguity.

Putting everything together, we find

Theorem 16.23. Let R be a graded ring, finitely generated in degree one over R_0 and let $X = \operatorname{Proj} R$. Then the functors

$$\sim\colon \mathsf{GrMod}_R \to \mathsf{QCoh}_X$$

and

$$\Gamma_*\colon \mathsf{QCoh}_X \to \mathsf{GrMod}_R$$

satisfy $\widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}$ for all $\mathcal{F} \in \mathsf{QCoh}_X$.
It holds that $\widetilde{M} = 0$ for a graded R-module M if and only if M is supported in $V(R_+)$.

The finite type case

For finitely generated graded modules, the converse of claim (ii) in Lemma **??** holds true and gives another criterion for when two modules have isomorphic tildes. Recall that to each graded module M and each integer d we associated the graded module $M_{>d} = \bigoplus_{i>d} M_d$, and we introduce an equivalence relation on the graded R-modules by declaring M and N to be equivalent when $M_{>d} \simeq N_{>d}$ for some $d \in \mathbb{Z}$.

Theorem 16.24. Assume R be generated over R_0 by finitely many elements of degree one. Let M and N be two finitely generated graded modules.

(i) Then $\widetilde{M} \simeq \widetilde{N}$ if and only if $M_{>d} \simeq N_{>d}$ for some d;

(ii) Moreover, $\mathcal{F}$ is of finite type if and only if it is the tilde of a finitely generated R-module.

Proof Proof of (i): One way is just (ii) of Lemma 16.6. Attacking the other implication, we consider the two each sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \Gamma_*\widetilde{M} & \longrightarrow & K & \longrightarrow & 0 \\ & & & & \downarrow\simeq & & & & \\ 0 & \longrightarrow & N & \longrightarrow & \Gamma_*\widetilde{N} & \longrightarrow & L & \longrightarrow & 0. \end{array} \tag{16.10}$$

Using the assumed isomorphism $\Gamma_*\widetilde{M} \simeq \Gamma_*\widetilde{N}$, we may identify the two Γ_*'s and consider the intersections $M \cap N \subset M$ and $M \cap N \subset N$. Since the support of K and L both are contained in $V(R_+)$, the same holds for $C = M/M \cap N$ and $D = N/M \cap N$, and as M and N both are finitely generated, it follows that $C_d = D_d = 0$ for $d >>0$. Consequently $M_{>d} = (M \cap N)_{>d} = N_{>d}$.

Proof of (ii): If M is finitely generated, there is a surjection $\bigoplus_i R(-d_i) \to M$ of graded R-modules, which induces a surjection $\bigoplus_i \mathcal{O}_X(-d_i) \to \widetilde{M}$ of $\mathcal{O}_X$-modules. This shows that $\widetilde{M}$ is of finite type since the direct sum restricts to free $\mathcal{O}_X$-module on a sufficiently fine open cover.

Assume then that $\mathcal{F}$ is of finite type. It is quasi-coherent, so by 16.22 it equals the tilde $\widetilde{M}$ of a (not necessarily finite) R-module. That $\mathcal{F}$ is of finite type means that there is an open affine cover $\{U_i\}$ of X (we may assume are distinguished open sets $D_+(f_i)$) such that for each i there is a surjection $\mathcal{O}_X^{r_i}|_{U_i} \to \widetilde{M}|_{U_i}$. Let e_{ij} denote the images in $\widetilde{M}(U_i)$ of the standard basis vectors of $\mathcal{O}_X^{r_i}|_{U_i}$. Bearing the equality $\widetilde{M}(U_i) = (M_{f_i})_0$ in mind, we may write $e_{ij} = m_{ij}/f^{\nu_{ij}}$ with $m_{ij} \in M$ homogeneous of degree $\mu_{ij} = \deg m_{ij} = \nu_{ij} \deg f_{ij}$. This yields a map

$$\Phi\colon \bigoplus_{i,j} R(-\mu_{ij}) \to M$$

whose tilde is surjective by construction since over U_i the image contains the relevant e_{ij}'s. The cokernel thus has zero tilde, and so $\widetilde{\operatorname{Im}\Phi} = \widetilde{M}$, but by construction $\operatorname{Im}\Phi$ is finitely generated. □

Exercise 16.4.1. Let $\mathbb{P}^1_k = \operatorname{Proj} k[u_0, u_1]$ and $t = u_0/u_1$. Consider the closed subscheme $Z \subset \mathbb{P}^1_k$ which is supported at $(0, 1)$ and which is locally given as $\operatorname{Spec} k[t]/t^n \subset D_+(u_0) = \operatorname{Spec} k[t]$. Describe the R module $\Gamma_*\mathcal{O}_Z$ and the canonical map $R = \Gamma_*\mathcal{O}_{P^1_k} \to \Gamma_*\mathcal{O}_Z$.

16.5 Closed subschemes of projective space

Having discussed what quasi-coherent sheaves are on projective spectra, we will now use this to study closed subschemes. We saw earlier that given a graded ideal $I \subset R$ we could associate a closed subscheme $V(I) \subset \operatorname{Proj} R$ and a closed immersion $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$. On the other hand, we also saw above that many graded modules M could give rise to the same quasi-coherent sheaf $\widetilde{M}$. This is also the case for graded ideals, as we shall see, but luckily we are again able to completely identify which ideals give rise to the same closed subscheme.

In the discussion it will be convenient to introduce the *saturation* of an ideal. The upshot will be that this will serve as the 'largest' ideal corresponding to a given subscheme. We fix an ideal $B \subset R$ (the case to have in mind is the irrelevant ideal $B = R_+$). Then for a graded ideal $I \subset R$, we define the *saturation* of I with respect to an ideal B as the ideal

$$I : B^\infty := \bigcup_{i \geqslant 0} I : B^i = \{r \in R | B^n r \in I \text{ for some } n \geqslant 0\}.$$

We say that I is B-saturated if $I = I : B^\infty$ and more concisely, *saturated* if it is R_+-saturated. We will here denote $I : (R_+)^\infty$ by $\overline{I}$. It is not hard to check that $\overline{I}$ is homogeneous if I is.

Example 16.25. In $R = k[x_0, x_1]$, the (x_0, x_1)-saturation of (x_0^2, x_0x_1) is the ideal (x_0). Note that both (x_0) and (x_0^2, x_0x_1) define the same subscheme of $\mathbb{P}^1_k$, but in some sense the latter ideal is inferior, since it has a component in the irrelevant ideal (x_0, x_1). This example is typical; the saturation is a process which throws away components of I supported in the irrelevant ideal.

Proposition 16.26. Let A be a ring and let $R = A[x_0, \dots, x_n]$.

(i) If Y is a closed subscheme of $\mathbb{P}^n_A = \operatorname{Proj} R$ defined by an ideal sheaf $\mathcal{I}$, then the ideal

$$I = \Gamma_*(\mathcal{I}) \subset R$$

is a homogeneous saturated ideal. In this setting, Y corresponds to the subscheme $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$.

(ii) Two ideals I, J defined the same subscheme if and only if they have the same saturation.

In particular, there is a 1-1 correspondence between closed subschemes $i : Y \to \mathbb{P}^n_A$ and saturated homogeneous ideals $I \subset R$.

Proof (i) Let $i : Y \to \mathbb{P}^n_A$ be a closed subscheme of $\mathbb{P}^n_A = \operatorname{Proj} R$ and let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n_A}$ denote the ideal sheaf of Y. Using the fact that global sections is left-exact, we have $\Gamma_*(\mathcal{I}) \subset \Gamma_*(\mathcal{O}_{\mathbb{P}^n_A}) = R$. $I = \Gamma_*(\mathcal{I})$ is naturally a graded R-module, so I is a homogeneous ideal of R.

Let us show that I is saturated, i.e., that $\overline{I} = \Gamma_*(\mathcal{J})$. "$\subseteq$": Suppose $f \in R_d$ satisfies $f \cdot B^n \in I$ for some $n > 0$. So in particular, $f \cdot x_i^n \in I_{d+n} = \Gamma(\mathbb{P}^n_A, \mathcal{I}(n + d))$ for some $n > 0$. Over $D_+(x_i)$, the tensor product $(f \cdot x_i^n)|_{D_+(x_i)} \otimes x_i^{-n}$ defines a section of $\Gamma(D_+(x_i), \mathcal{I}(d)) = (I_{x_i})_d$ via the canonical isomorphism $\mathcal{I}(n + d) \otimes \mathcal{O}(-n) = \mathcal{I}(d)$. It is clear that $(fx_i^n)|_{D_+(x_i)} \otimes x_i^{-n}$ and $(fx_j^n)|_{D_+(x_j)} \otimes x_j^{-n}$ restrict to the same section of $\mathcal{I}(d)$ over $D_+(x_i x_j)$ (they are both induced by the element f). Hence they glue to a section $s \in \Gamma(X, \mathcal{I}(d)) = I_d$. Finally, we must have $f = s$, because both restrict to the same sections over each $D_+(x_i)$.

"$\supseteq$": Let $f \in \Gamma(\mathbb{P}^n_A, \mathcal{I}(d))$. Then for each $i = 0, \ldots, n$, we have $f/1 \in (I_{x_i})_d$, i.e., there exists $n_i \geqslant 0$ and $g_i \in I_{n_i+q}$ such that $f/1 = g_i/x_i^{n_i}$, or in other words, $x_i^{n_i} f \in I$. Taking $n = \max n_i$ we see that $fx_i^n \in I$ for all i, so that $f \in \overline{I}$

Now both the subscheme Y and the closed subscheme $j : \operatorname{Proj}(R/I) \to \mathbb{P}^n_A$ are defined by the same ideal sheaf $\mathcal{I}$. Indeed, the first is by definition of I, and the latter because $\widetilde{I} = \mathcal{I}$ by Proposition 16.22. Hence the two subschemes are equal.

(ii) If I, J define the same subscheme, they have the same ideal sheaf $\mathcal{I}$ and so $\overline{I} = \Gamma_*(X, \mathcal{I}) = \overline{J}$. □

Example 16.27. Let k be a field and let $R = k[u, v]$. Moreover introduce the graded ring $S = R^{(n)} = k[u^n, u^{n-1}v, \ldots, v^n]$. We have a graded surjection

$$\phi : k[x_0, \ldots, x_n] \to S$$

given by $x_i \mapsto u^i v^{n-i}$ for $i = 0, \ldots, n$. The ideal $I = \operatorname{Ker} \phi$ is generated by the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}.$$

Thus we have an embedding of $\mathbb{P}^1_k = \operatorname{Proj} S$ into $\mathbb{P}^n$ with image $V(I)$. The image is called a *rational normal curve of degree* n. Note that for $n = 2$, the image of $\mathbb{P}^1_k \to \mathbb{P}^2_k$ is the conic given by $x_1^2 = x_0 x_2$.

Exercise 16.5.1. Check that the saturation $\overline{I}$ is homogeneous if I is.

16.6 Sheaves on $\mathbb{P}^n$

In this section, we write $\mathbb{P}^n_k = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$ with the standard grading.

We recall the following fundamental theorem in commutative algebra:

Theorem 16.28 (Hilbert's syzygy theorem). Let k be a field and let $R = k[x_0, \ldots, x_n]$. Then if M is a finitely generated graded R-module, then there is a finite free resolution (that is, an exact sequence)

$$0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_j = \bigoplus_{i=1}^{b_k} R(-d_{ij})$ is a free graded R-module.

F_i is called the i-th *syzygy module* of the resolution.

If we apply the $\sim$-functor here we obtain an exact sequence of sheaves on $\mathbb{P}^n_k$

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \widetilde{M} \longrightarrow 0$$

where $\mathcal{E}_j = \bigoplus_{i=1}^{b_k} \mathcal{O}_{\mathbb{P}^n_k}(-d_{ij})$ is a direct sum of sheaves of the form $\mathcal{O}(d)$.

Thus any coherent sheaf admit a projective resolution with direct sums of invertible sheaves.

This shows very clearly why the invertible sheaves $\mathcal{O}(d)$ are so important: They are the building blocks of all coherent sheaves on $\mathbb{P}^n$.

Here are a few important special cases:

Example 16.29 (Hypersurfaces). Let $F \in R$ denote an homogeneous polynomial of degree $d > 0$. F determines a projective hypersurface $X = V(F)$, which has dimension $n-1$. $i : X \to \mathbb{P}^n_k$ denote the closed embedding.

Let us consider the sheaf $i_*\mathcal{O}_X$ on $\mathbb{P}^n_k$. We start with the following sequence on $\mathbb{P}^n_k$:

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

Note that we have an isomorphism $R(-d) \to I(X)$ given by multiplication by F. Note the shift in degrees here: The constant '1' gets sent to F, which should have degree d on both sides. Thus the above sequence is simply the tilde of the sequence

$$0 \longrightarrow R(-d) \longrightarrow R \longrightarrow R/(F) \longrightarrow 0.$$

and the ideal sheaf sequence takes the following form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

Example 16.30 (Complete intersections). Let F, G be two homogeneous polynomials without common factors of degrees d, e respectively. Let $I = (F, G)$ and $X = V(I) \subset \mathbb{P}^n_k$. X is called a 'complete intersection' – it is the intersection of the two hypersurfaces $V(F)$ and $V(G)$. To study X, we use the exact sequence

$$0 \longrightarrow R(-d-e) \xrightarrow{\alpha} R(-d) \oplus R(-e) \xrightarrow{\beta} I \longrightarrow 0.$$

The maps here are defined by $\alpha(h) = (-hG, hF)$ and $\beta(h_1, h_2) = h_1F + h_2G$. These maps preserve the grading.

To prove exactness, we start by noting that α is injective (since R is an integral domain) and β is surjective (by the definition of I). Then if $(h_1, h_2) \in \operatorname{Ker}\beta$, we have $h_1F = -h_2G$, which, as F, G are coprime, means that there is an element h so that $h_1 = -hG, h_2 = hF$.

Applying $\sim$, we obtain the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \longrightarrow \mathcal{I}_X \longrightarrow 0.$$

Example 16.31 (The twisted cubic curve). Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will consider the *twisted cubic curve* $C = V(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

i.e., $I = (q_0, q_1, q_2) = (x_1^2 - x_0x_2, x_0x_3 - x_1x_2, -x_2^2 + x_1x_3)$.

Consider the map of R-modules $R^3 \to I$ sending $e_i \mapsto q_i$. This is clearly surjective, since the q_i generate I. Let us consider the kernel of this map, that is, the module of relations of the form $a_0q_0 + a_1q_1 + a_2q_2 = 0$ for $a_i \in R$. There are two obvious relations of this form, i.e., the ones we get from expanding the determinants of the two matrices

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

(So first matrix gives $x_0q_2 - x_1q_1 + x_2q_2 = 0$ for instance). These give a map $R^2 \xrightarrow{m\cdot} R^3$, where M is the matrix above. This map is injective, and it turns out that there is an exact sequence of R-modules

$$0 \longrightarrow R^2 \xrightarrow{A} R^3 \longrightarrow I \longrightarrow 0.$$

Again, if we want to be completely precise, we should consider these as *graded* modules, so we must shift the degrees according to the degrees of the maps above

$$0 \longrightarrow R(-3)^2 \xrightarrow{A} R(-2)^3 \longrightarrow I \longrightarrow 0.$$

This gives the resolution of the ideal I of C. Then applying $\sim$, and using the fact that $\mathcal{I} = \widetilde{I}$, we get a resolution of the ideal sheaf of C:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-3)^2 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3_k}(-2)^3 \longrightarrow \mathcal{I} \longrightarrow 0.$$

We will see later in Chapter **??** how to use sequences like this to extract geometric information about C.

16.7 Morphisms to projective space

Given a scheme X it is natural to ask when there is a morphism to a projective space

$$f : X \to \mathbb{P}^n,$$

or when there is a closed immersion $X \hookrightarrow \mathbb{P}^n$. Given such a morphism, we get geometric information about X using this map, e.g., by studying the fibers $f^{-1}(y)$; pulling back sheaves from $\mathbb{P}^n$; or describing the equations of the image.

The corresponding question for $\mathbb{A}^n$ has already been answered. Morphisms $X \to \mathbb{A}^n$ are in one-to-one correspondence with elements of $\Gamma(X, \mathcal{O}_X)^n$, i.e., an n-tuple of regular functions on X.

Even for projective space itself, there is not so much information in the space of global sections of the structure sheaf. However, we do have something canonical associated to $\mathbb{P}^n$, namely the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$. Given a morphism $f : X \to \mathbb{P}^n$, we get an invertible sheaf $L = f^*\mathcal{O}(1)$ on X. We even get $n + 1$ distinguished global sections $s_i = f^*x_i$ by pulling back the sections $x_0, \ldots, x_n$ of $\mathcal{O}(1)$.

Note that there is no point of $\mathbb{P}^n$ where the x_i simultaneously vanish. More precisely, for every $y \in \mathbb{P}^n$, the stalk $\mathcal{O}_{\mathbb{P}^n}(1)_y$ is generated by the germ of one of the x_i. So by the properties of the pullback, we see that the same statement holds for L and the sections s_i on X. We say that L is *globally generated* by the sections s_i.

The main result in this section is that there is a way to reverse this process. In other words, from a given invertible sheaf L and $n + 1$ global sections $s_i \in \Gamma(X, L)$ with the above property, we can uniquely reconstruct a morphism $f : X \to \mathbb{P}^n$ so that $f^*\mathcal{O}_{\mathbb{P}^n}(1) = L$ and $f^*x_i = s_i$. Thus $(L, s_0, \ldots, s_n)$ is the exactly the data we are after.

Theorem 16.32. Let X be a scheme over a ring A, and let L be an invertible sheaf on X with global sections $s_0, \ldots, s_n \in \Gamma(X, L)$ which generate L. Then there is a *unique* morphism

$$f : X \to \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$$

so that $f^*x_i = s_i$ for $i = 0, \ldots, n$.

First an easy lemma:

Lemma 16.33. Let X be a scheme and let L be an invertible sheaf on X. If $s \in \Gamma(X, L)$ is a global section, then there is an isomorphism

$$\phi : \mathcal{O}_X|_{X_s} \to L|_{X_s}$$

which sends 1 to s.

Proof We define ϕ over an open set $U \subset X_s$, by sending $1 \in \mathcal{O}_X(U)$ to $s \in L(U)$, which is a map of $\mathcal{O}_X$-modules. This is an isomorphism if and only if it is an isomorphism locally, so we may reduce to the case where $X = \operatorname{Spec} A$ and $L = \mathcal{O}_X$. In that case $X_s = D(s) = \operatorname{Spec} A_s$, and $s \in A$ is a unit in A_s, so multiplication by s is an isomorphism $A_s \to A_s$. □

Proof of the theorem We first prove uniqueness. Let $f : X \to \mathbb{P}^n_A$ be a morphism, and consider the pulled back sections $s_i = f^*x_i$ for $i = 0, \ldots, n$. Write for simplicity $X_i = X_{s_i}$ for each i. From Proposition 19.34 we have $f^{-1}(D_+(x_i)) = X_i$ for each i, so X is covered by the $n + 1$ subsets X_i. We can regard the morphism as glued together from the morphisms $f_i : X_i \to D_+(x_i) = \operatorname{Spec} (R_{x_i})_0$, where $R = A[x_0, \ldots, x_m]$. This in turn corresponds to a morphism of A-algebras

$$f^\sharp : (R_{x_i})_0 \to \Gamma(X_i, \mathcal{O}_X).$$

Note that x_i generates $\mathcal{O}(1)$ on $D_+(x_i)$ and $x_j = \frac{x_j}{x_i} x_i$ in $R_{(x_i)}$ for $j = 0, \ldots, n$. Similarly, pulling back via $f^\sharp$ gives

$$s_j = f_i^*(x_j) = f_i^\sharp \left(\frac{x_j}{x_i} x_i \right) = f_i^\sharp \left(\frac{x_j}{x_i} \right) s_i$$

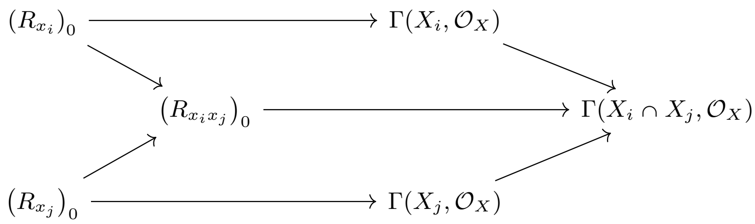
(Here we interpret the fraction $\frac{x_j}{x_i}$ as a section of $\Gamma(D_+(x_i), \mathcal{O}_{\mathbb{P}^n_A})$.) It follows that from each morphism $f : X \to \mathbb{P}^n_A$, we get $n + 1$ distinguished sections $s_0, \ldots, s_n$, from which we can determine the morphisms f_i. Hence f is uniquely determined from the data $(L, s_0, \ldots, s_n)$.

To prove existence, we suppose that we are given $n + 1$ sections $s_0, \ldots, s_n$ of a globally generated invertible sheaf L, we will construct a morphism to $\mathbb{P}^n_A$, such that s_i is the pullback of x_i. As in the above example, we define this morphism on an open cover. Let $X_i = X_{s_i} = \{x \in X | s_i(x) \neq 0\}$. Since the s_i globally generate L, it follows from Lemma 16.33 that the X_i provide a local trivializing cover of L: namely there is an isomorphism

$\psi_i : \mathcal{O}_X|_{X_i} \to L|_{X_i}$ which sends 1 to the section s_i. In particular, if we restrict the global section s_j to X_i, we have $s_j = r_{ij} s_i$ for some $r_{ij} \in \Gamma(X_i, \mathcal{O}_X)$. We denote this section r_{ij} by $\frac{s_j}{s_i}$. These define a map of A-algebras

$$\begin{aligned} (R_{x_i})_0 &\to \Gamma(X_i, \mathcal{O}_{X_i}) \\ \frac{x_j}{x_i} &\mapsto \frac{s_j}{s_i} \end{aligned} \tag{16.11}$$

By the correspondence between ring homomorphisms and maps into affine schemes, we obtain a morphism of schemes $f_i : X_i \to D_+(x_i)$. On $X_i \cap X_k$, the map sends $\frac{x_j}{x_k} = \frac{x_j/x_i}{x_k/x_i}$ to $\frac{s_j}{s_k} = \frac{s_j/s_i}{s_k/s_i}$. In other words, the following diagram commutes:



That means that the morphisms glue to a morphism $f : X \to \mathbb{P}^n$. It is clear that $f^*\mathcal{O}(1) \simeq L$ and that the x_i pull back to the s_i, since this is true over the principal opens $D_+(x_i)$. □

Abusing notation, we will refer to a morphism $\phi : X \to \mathbb{P}^n_A$ as given by the data $(L, s_0, \ldots, s_n)$ and write

$$\begin{aligned} X &\to \mathbb{P}^n_A \\ x &\mapsto (s_0(x) : \cdots : s_n(x)) \end{aligned}$$

One should still keep in mind that the sections s_i are sections of L, not regular functions. In fact, from the above proof, we see that it is the ratios s_j/s_i which can be interpretated as regular functions, locally on $X_i = \{x \in X \mid s_i(x) \neq 0\}$.

We also see that two sets of data $(L, s_0, \ldots, s_n)$, $(L, t_0, \ldots, t_n)$ give rise to the same morphism $f : X \to \mathbb{P}^n_A$ if and only there is a section $\lambda \in \mathcal{O}_X^\times(X)$ so that $t_i = \lambda s_i$ for each i. Thus morphisms $f : X \to \mathbb{P}^n_A$ are in bijective correspondence with the data $(L, s_0, \ldots, s_n)$ modulo this equivalence relation.

Given a scheme X with $s_0, \ldots, s_n$ of a line bundle L, there is a maximal open subset U such that the sections generate L for all points in U, namely $U = \bigcup_{i=0}^n X_i$. Not assuming that the s_i globally generate L, we still get a morphism $\phi : U \to \mathbb{P}^n_A$. In other words, ϕ defines a *rational map* $\phi : X \dashrightarrow \mathbb{P}^n_A$, which is a morphism when restricted to U.

Example 16.34. Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[s, t]$ and $L = \mathcal{O}_{\mathbb{P}^1_k}(2)$. Then L is globally generated by s^2, st, t^2 and the corresponding morphism

$$\begin{aligned} \phi : X &\to \mathbb{P}^2_k \\ (s : t) &\mapsto (s^2 : st : t^2) \end{aligned}$$

has image $V(x_0 x_2 - x_1^2)$ which is a smooth conic.

Example 16.35 (Cuspidal cubic)**.** Let $X = \mathbb{A}^1_k$ and $L = \mathcal{O}_X$. Then, $\Gamma(X, L) = k[t]$ is infinite dimensional over k. Choosing the three sections $1, t^2, t^3$, we get a map of schemes

$$\begin{aligned} X &\to \mathbb{P}^2_k \\ t &\mapsto (1 : t^2 : t^3) \end{aligned}$$

whose image in $\mathbb{P}^2$ is the cuspidal cubic minus the point at infinity.

Example 16.36 ($\mathbb{P}^n$ as a quotient space)**.** Let $X = \mathbb{A}^{n+1}_k$, and $L = \mathcal{O}_X$. Then, $\Gamma(X, L) = k[x_0, \ldots, x_n]$. If we take the sections $x_0, \ldots, x_n$, then they generate L outside $V(x_0, \ldots, x_n)$. Hence we get a morphism of schemes

$$\begin{aligned} \mathbb{A}^{n+1}_k - V(x_0, \ldots, x_n) &\to \mathbb{P}^n_k \\ (x_0, \ldots, x_n) &\mapsto (x_0 : \cdots : x_n) \end{aligned}$$

which is exactly the 'quotient space' description of $\mathbb{P}^n$ from Exercise **??**.

Example 16.37 (Projection from a point)**.** Consider the projective space $X = \mathbb{P}^n_A$ and sections $x_1, \ldots, x_n$ of $\mathcal{O}(1)$, then these sections generate $\mathcal{O}(1)$ outside the point p corresponding to $I = (x_1, \ldots, x_n)$ (that is, the closed point $p = (1 : 0 : \cdots : 0)$). The induced morphism $\mathbb{P}^n_A - V(I) \to \mathbb{P}^{n-1}_A$ is the *projection from p*.

Example 16.38 (Cremona transformation)**.** Consider the projective space $X = \mathbb{P}^2_A$ and sections x_0, x_1, x_2 of $\mathcal{O}(1)$, then the sections x_0x_1, x_0x_2, x_1x_2 generate $\mathcal{O}(2)$ outside $V(x_0x_1, x_0x_2, x_1x_2)$ corresponding to the three points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$. The induced rational map $\mathbb{P}^2_A \dashrightarrow \mathbb{P}^2_A$ is the *Cremona transformation*.

Example 16.39 (The Veronese surface)**.** Consider $X = \mathbb{P}^2$, and $L = \mathcal{O}_{\mathbb{P}^2}(2)$. If x_0, x_1, x_2 are projective coordinates on X, then the quadratic monomials

$$x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$$

form a basis for $H^0(X, L)$, and generate L at every point. The corresponding map $\phi : X \to \mathbb{P}^5$ is in fact a closed immersion; the image is the Veronese surface. It is a classical fact that the image is defined by the 2×2 minors of the matrix

$$\begin{pmatrix} u_0 & u_1 & u_2 \\ u_1 & u_3 & u_4 \\ u_2 & u_4 & u_5 \end{pmatrix}$$

Example 16.40 (The quadric surface)**.** Let us consider again the case $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Keeping the notation from Section 18.10, we have two divisors, $L_1 = (0 : 1) \times \mathbb{P}^1$, $L_2 = \mathbb{P}^1 \times (0 : 1)$. Note that each L_i is globally generated (being the pullback of a base point free divisor on $\mathbb{P}^1$). The corresponding map is of course the i-th projection map $p_i : Q \to \mathbb{P}^1$.

If x_0, x_1 is a basis for $\Gamma(X, L_1)$, and y_0, y_1 is a basis for $\Gamma(X, L_2)$, we find that $\Gamma(X, L_1 + L_2)$ is spanned by the sections

$$s_0 = x_0y_0, s_1 = x_0y_1, s_2 = x_1y_0, s_3 = x_1y_1$$

Moreover, these sections generate $\mathcal{O}_Q(L_1 + L_2)$ everywhere, and so we get a map

$$Q \to \mathbb{P}^3$$

This is of course nothing but the Segre embedding; note the quadratic relation between the four sections $s_0 s_3 - s_1 s_2 = 0$.

16.8 Application: Automorphisms of $\mathbb{P}^n_k$

If k is a field, then any invertible $(n+1) \times (n+1)$ matrix A with entires in k acts on $k[x_0, \ldots, x_n]$ and thus gives rise to a linear automorphism $\mathbb{P}^n_k \to \mathbb{P}^n_k$. Moreover, two matrices A and A' determine the same automorphism if and only if $m = \lambda m'$ for some non-zero scalar $\lambda \in k^*$. So we are led to consider the *projective linear group*

$$PGL_n(k) = GL_n(k)/k^*$$

We will now prove that all automorphisms of $\mathbb{P}^n_k$ are given by linear transformations.

Theorem 16.41. $Aut_k(\mathbb{P}^n) = PGL_n(k)$.

Proof The above shows that there is an injective map from the righthand side to the left. To show the reverse inclusion, let $\phi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ be any automorphism. Then we get an induced map

$$\phi^* : \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(\mathbb{P}^n)$$

which must also be an isomorphism. Since $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, we must have either $\phi^* \mathcal{O}_{\mathbb{P}^n}(1) = O_{\mathbb{P}^n}(1)$ or $\phi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}(-1)$. The latter case is impossible, since $\phi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ has a lot of global sections, whereas $\mathcal{O}_{\mathbb{P}^n}(-1)$ has none. So $\phi^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^n}(1)$. In particular, taking global sections ϕ^* gives a map

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)),$$

which is a isomorphism of k-vector spaces. However, we may choose $\{x_0, \ldots, x_n\}$ as a basis for $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and so in this basis ϕ^* gives rise to an invertible $(n+1) \times (n+1)$-matrix m. By construction m induces the same linear transformation $\mathbb{P}^n_k \to \mathbb{P}^n_k$ as ϕ, and so ϕ comes from an element of $PGL_n(k)$. □

16.9 Exercises

Exercise 16.9.1. The aim of this exericse is to investigate the functor of points of projective space $\mathbb{P}^n$. We will associate to a scheme T, its set of data $(L, s_0, \ldots, s_n)$ where L is an invertible sheaf L, with an $(n+1)$-tuple of sections $s_0, \ldots, s_n$ that locally generate L everywhere. We declare $(L, s_0, \ldots, s_n) \sim (M, t_0, \ldots, t_n)$ if there is an isomorphism $f : L \to M$ so that $f^*(s_i) = \lambda \cdot t_i$ for some $\lambda \in \mathcal{O}_T^\times(T)$.

a) Show that $\sim$ is an equivalence relation.
b) Consider the assignment

$$F(T) = \left\{ (L, s_0, \ldots, s_n) \middle| s_0, \ldots, s_n \in \Gamma(T, L) \text{ generate } L \text{ everywhere} \right\} / \sim$$

Show that F is a functor.

c) Show that there is a natural transformation

$$\Phi(X) : \operatorname{Hom}(T, \mathbb{P}^n) \to F(T)$$

sending a morphism $f : T \to \mathbb{P}^n$ to the equivalence class of the data

$$(L, s_0, \dots, s_n) = (f^*\mathcal{O}(1), f^*x_0, \dots, f^*x_n)$$

d) Construct an inverse to Φ and deduce that F is represented by $\mathbb{P}^n$.

e) Show that elements of $F(\operatorname{Spec} k)$ are in correspondence with $(n+1)$-tuples $(a_0, \dots, a_n) \in k^{n+1}$, so that not all a_i are zero. Thus we recover the usual description of the k-points of projective space as '1-dimensional subspaces of k^{n+1}'.

f) Show that the previous exercise also holds for a local ring.

g) Show that for a ring R, the set $F(\operatorname{Spec} R)$ is in bijection with the set of rank 1 summands of R^{n+1}, i.e., modules of rank 1 such that $M \oplus E \simeq R^{n+1}$ for some module E. This is the right generalization of a "line in k^n" for general rings.

Exercise 16.9.2. This is a continuation of Exercise 16.9.1. We will consider the product $X = \mathbb{P}^m \times \mathbb{P}^n$ and give a new interpretation of the Segre embedding $X \to \mathbb{P}^{(m+1)(n+1)-1}$ in terms of the functor of points.

a) Let T be a scheme and let $(L, s_0, \dots, s_m)$ and $(M, t_0, \dots, t_n)$ be elements of $h_{\mathbb{P}^m}(T)$ and $h_{\mathbb{P}^n}(T)$ respectively. Show that the $(m+1)(n+1)$ tensor products $u_{ij} = pr_1^* s_i \otimes pr_2^* t_j$ generate $pr_1^* L \otimes pr_2^* M$ on $T \times T$.

b) Show that

$$(pr_1^* L \otimes pr_2^* M, u_{00}, \dots, u_{mn}) \tag{16.12}$$

defines an element of $h_{\mathbb{P}^{(m+1)(n+1)-1}}(T)$, and that this defines a contravariant functor from $\mathsf{Sch} \to \mathsf{Sets}$.

c) Deduce that there is a morphism $\phi : X \to \mathbb{P}^{(m+1)(n+1)-1}$.

d) Show that ϕ is an embedding. HINT: Show that the morphism ϕ has the property that $\phi^{-1}(D_+(u_{ij})) = D_+(x_0) \times D_+(y_0)$, and show that ϕ restricts to an embedding on distinguished subsets.

Exercise 16.9.3. This is a continuation of Exercise 16.9.1. We will consider the projective space $\mathbb{P}^n$ and give a new interpretation of the Veronese embedding $X \to \mathbb{P}^N$ in terms of the functor of points.

a) Let T be a scheme and let $(L, s_0, \dots, s_n)$ be an element of $h_{\mathbb{P}^m}(T)$. Show that for each $d \geqslant 1$, the $N = \binom{n+d}{d}$ monomials

$$s_0^{\otimes e_0} \otimes s_1^{\otimes e_1} \otimes \cdots \otimes s_n^{\otimes e_n} \tag{16.13}$$

for $e_0 + \cdots + e_n = d$, generate $L^{\otimes d}$.

b) Show that $L^{\otimes}$ together with the N sections in (16.13) defines an element in $h_{\mathbb{P}^{N-1}}(T)$ and that this defines a contravariant functor from $\mathsf{Sch} \to \mathsf{Sets}$.

c) Deduce that there is a morphism $\phi : X \to \mathbb{P}^{N-1}$.

d) Show that ϕ is an embedding. HINT: Consider distinguished open sets.

17

First steps in sheaf cohomology

One of the main challenges when working with sheaves is that surjective maps of sheaves do not always induce surjections on global sections. Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

one has a sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \tag{17.1}$$

which is exact at each stage except on the right, but the right-most map may fail to be surjective. In many situations in algebraic geometry, knowing that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is surjective is of fundamental importance. For instance, if $U \subset X$ is an open subscheme, it is useful to know when a regular function defined on U extends to a regular function on all of X.

Cohomology groups can be seen as a partial response to this behavior of Γ, and in good situations, they allow us to say something about the missing cokernel. More precisely, the sequence (17.1), induces a *long exact sequence of cohomology groups*

$$\begin{array}{l} 0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \\ \quad \longrightarrow H^1(X, \mathcal{F}') \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}'') \\ \quad \longrightarrow H^2(X, \mathcal{F}') \longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}'') \longrightarrow \cdots \end{array}$$

So the failure of surjectivity of the above is controlled by the group $H^1(X, \mathcal{F}')$ and the other groups in the sequence.

In addition to problems such as lifting, cohomology groups allow us to define many geometric invariants of $\mathcal{F}$ and X. These in turn allow us to distinguish schemes, that is, if two schemes have different cohomology groups they can not be isomorphic.

Cohomology groups can be defined in a completely general setting, for any topological space and a (pre)sheaf on it. There are several ways to define them. The modern approach uses the theory of derived functors. This is in most respects the 'right way' to define the groups in general, but going through the whole machinery of derived functors and homological algebra would take us too far astray. We therefore begin with taking a more down-to-earth approach using *Cech cohomology* which is better suited for computations.

17.1 Some homological algebra

Recall that a *complex of abelian groups* $A^\bullet$ is a sequence of groups A^i together with maps between them

$$\cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for each i. A *map of complexes* $A^\bullet \xrightarrow{f} B^\bullet$ is a collection of maps of groups $f_p : A^p \to B^p$ making the following diagram commutative:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} & & \\ \cdots & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & \cdots \end{array}$$

In this way, we can talk about kernels, images, cokernels, exact sequences of complexes, etc.

We say that an element $\sigma \in A^p$ is a *cocycle* if it lies in the kernel of the map d^p i.e., $d^p\sigma = 0$. A *coboundary* is an element in the image of d^{p-1}, i.e. $\sigma = d^{p-1}\tau$ for some $\tau \in A^{p-1}$. Since $d^p(d^{p-1}a) = 0$ for all a, we have

$$\operatorname{Im} d^{p-1} \subseteq \operatorname{Ker} d^p,$$

and so all coboundaries are cocycles. The *cohomology groups* of the complex $A^\bullet$ are set up to measure the difference between these two notions. We define the *p-th cohomology group* as the quotient group

$$H^pA^\bullet = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

One thinks of $H^pA^\bullet$ as a group that measures the failure of the complex $A^\bullet$ of being exact at stage p: $A^\bullet$ is exact if and only if $H^pA^\bullet = 0$ for every p.

The following result is fundamental in the theory of cohomology groups:

Proposition 17.1. Suppose that $0 \to F^\bullet \xrightarrow{f} G^\bullet \xrightarrow{g} H^\bullet \to 0$ is an exact sequence of complexes. Then there is a *long exact sequence* of cohomology groups

$$\cdots \longrightarrow H^pF^\bullet \longrightarrow H^pG^\bullet \longrightarrow H^pH^\bullet \longrightarrow H^{p+1}F^\bullet \longrightarrow H^{p+1}G^\bullet \longrightarrow H^{p+1}H^\bullet \longrightarrow \cdots$$

Proof For each $p \in \mathbb{Z}$, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^p & \xrightarrow{f_p} & G^p & \xrightarrow{g_p} & H^p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_F^p} & & \downarrow{\scriptstyle d_G^p} & & \downarrow{\scriptstyle d_H^p} & & \\ 0 & \longrightarrow & F^{p+1} & \xrightarrow{f_{p+1}} & G^{p+1} & \xrightarrow{g_{p+1}} & H^{p+1} & \longrightarrow & 0 \end{array}$$

where the rows are exact by assumption. By the Snake lemma, we obtain a sequence

$$0 \longrightarrow \operatorname{Ker} d_F^p \xrightarrow{f_p} \operatorname{Ker} d_G^p \xrightarrow{g_p} \operatorname{Ker} d_H^p \xrightarrow{\delta}$$
$$F^{p+1}/\operatorname{Im} d_F^p \xrightarrow{f_{p+1}} G^{p+1}/\operatorname{Im} d_G^p \xrightarrow{g_{p+1}} H^p/\operatorname{Im} d_H^{p+1} \to 0.$$

Consider now the diagram

$$\begin{array}{ccccccccc}
& & F^p/\operatorname{Im} d_F^{p-1} & \xrightarrow{f_p} & G^p/\operatorname{Im} d_G^{p-1} & \xrightarrow{g_p} & H^p/\operatorname{Im} d_H^p & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle d_F^p} & & \downarrow{\scriptstyle d_G^p} & & \downarrow{\scriptstyle d_H^p} & & \\
0 & \longrightarrow & \operatorname{Ker} d_F^{p+1} & \xrightarrow{f_{p+1}} & \operatorname{Ker} d_G^{p+1} & \xrightarrow{g_{p+1}} & \operatorname{Ker} d_H^{p+1} & &
\end{array}$$

where the rows are exact by the above. For the maps in this diagram, $H^pF^\bullet = \operatorname{Ker} d^p$ and $H^{p+1}F^\bullet = \operatorname{Coker} d^p$ etc. Hence applying the Snake lemma one more time, we get the desired exact sequence. □

A map of complexes $f : C^\bullet \to D^\bullet$ is a *chain map* if $f \circ d_C = d_D \circ f$. Such a map induces a well-defined map between cohomology groups

$$f : H^i(C^\bullet) \to H^i(D^\bullet)$$

A *chain homotopy* between two chain maps $f, g : C^\bullet \to C^\bullet$ is a collection of maps $h : C^n \to D^{n-1}$ such that

$$f - g = d_D \circ h + h \circ d_C$$

If f and g are related by a chain homotopy, they induce the *same* map $H^i(C^\bullet) \to H^i(D^\bullet)$. This follows by

$$[f(c) - g(c)] = [d_D(h(c))] = 0$$

for all c with $d_C c = 0$.

Example 17.2. To show that $H^i(C^\bullet) = 0$ (e.g., that $C^\bullet$ is exact), it is enough find a chain homotopy between the identity map and the zero map. Concretely, the maps $h : C^{p+1} \to C^p$ should satisfy

$$(d^p \circ h + h \circ d^{p+1})(c) = 0 \tag{17.2}$$

for every $c \in C^{p+1}$.

17.1.1 Complexes of sheaves

The definitions and arguments of the previous subsection apply much more generally (in fact to any abelian category). In particular, we make the following sheaf analogue. A *complex of sheaves* $\mathcal{F}^\bullet$ is a sequence of sheaves with maps between them

$$\cdots \xrightarrow{d^{i-2}} \mathcal{F}_{i-1} \xrightarrow{d^{i-1}} \mathcal{F}_i \xrightarrow{d^i} \mathcal{F}_{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for each i. Given such a complex, we define the *cohomology sheaves* $H^p\mathcal{F}^\bullet$ as $\operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$. As above, a short exact sequence of complexes of sheaves gives rise to a long exact sequence of cohomology sheaves.

17.2 Cech cohomology

Let X be a topological space. For simplicity, we will assume that X admits an open cover $\mathcal{U}$ consisting of finitely many open sets $U_1, \ldots, U_r$. We will index the intersections

$$U_I = U_{i_0} \cap \cdots \cap U_{i_p}$$

using strictly increasing sequences of positive integers $I = (i_0 < i_1 < \cdots < i_p)$.

For a sheaf $\mathcal{F}$ on X, we have the sheaf exact sequence (3.2)

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i<j} \mathcal{F}(U_i \cap U_j). \tag{17.3}$$

The *Cech complex* is essentially the continuation of this sequence; it is a complex obtained by adjoining all the groups $\mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_r})$ over all possible intersections $U_{i_1} \cap \cdots \cap U_{i_r}$.

Definition 17.3. For a sheaf $\mathcal{F}$ on X, we define the *Cech complex* $C^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ (with respect to the open covering $\mathcal{U}$) as

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{d^2} \ldots$$

where

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

and the *coboundary maps* $d^p \colon C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ by

$$(d^p\sigma)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots\hat{i_j},\ldots,i_{p+1}}|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}$$

where $i_0, \ldots \hat{i_j}, \ldots, i_{p+1}$ means $i_0, \ldots, i_{p+1}$ with the index i_j omitted.

Note that since the cover is assumed to be finite, say having r elements, $C^p(\mathcal{U}, \mathcal{F}) = 0$ for every $p \geqslant r$, simply because empty products are zero. So the Cech complex is a finite complex.

Example 17.4. The two first groups in the Cech complex are given by

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{i_0} \mathcal{F}(U_{i_0}) \quad \text{and} \quad C^1(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1}).$$

An element $\sigma \in C^0(\mathcal{U}, \mathcal{F})$ is an r-tuple of sections $\sigma = (\sigma_1, \ldots, \sigma_r)$, where $\sigma_i \in \mathcal{F}(U_i)$ for each i. Likewise, an element $\sigma = (\sigma_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$ is a collection of sections $\sigma_{ij} \in \mathcal{F}(U_i \cap U_j)$, one for each pair $i < j$.

The coboundary map $d^0\colon C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F})$ sends an element $\sigma = (\sigma_i)$, to the element $d^0\sigma \in C^1(\mathcal{U}, F)$ whose ij-th component is equal to

$$(d^0\sigma)_{ij} = \sigma_j - \sigma_i\big|_{U_{ij}} \tag{17.4}$$

The coboundary map $d^1\colon C^1(\mathcal{U},\mathcal{F}) \to C^2(\mathcal{U},\mathcal{F})$ sends $\sigma = (\sigma_{ij})$, to the element with ijk-th component equal to

$$(d^1\sigma)_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}\big|_{U_{ijk}} \tag{17.5}$$

Substituting (17.4) into (17.5), there are many cancellations, and we see that $d^1 \circ d^0 = 0$. The same happens also in higher degrees:

Lemma 17.5. For every p, we have

$$d^{p+1} \circ d^p = 0.$$

Proof For an increasing sequence $I = (i_0 < \cdots < i_{p+1})$, we have

$$\begin{aligned}
\left(d^{p+1}d^p s\right)_I &= \sum_{j=0}^{p+2}(-1)^j (d^p\sigma)_{I-\{i_j\}} \\
&= \sum_{j=0}^{p+2}(-1)^j \sum_{k=0}^{j-1}(-1)^k \sigma_{I-\{i_j\}-\{i_k\}} \\
&+ \sum_{j=0}^{p+2}(-1)^j \sum_{k=j+1}^{p+2}(-1)^k \sigma_{I-\{i_i\}-\{i_k\}} \\
&= \sum_{j<k}(-1)^{k+j}\sigma_{I-\{i_j\}-\{i_k\}} \\
&- \sum_{j>k}(-1)^{k+j}\sigma_{I-\{i_i\}-\{i_k\}} = 0.
\end{aligned}$$

□

Therefore, the Cech complex is indeed a complex of abelian groups. The *Cech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined to be the cohomology of this complex:

Definition 17.6. The p-th *Cech cohomology* of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as

$$H^p(\mathcal{U},\mathcal{F}) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

The Cech cohomology groups depend on the open cover $\mathcal{U}$, but not on the choice of the ordering of the open sets U_i. Given two orderings, there is an isomorphism of the two associated Cech complexes given by multiplication by ± 1 on each C^p, so in particular, the cohomology groups are the same.

A sheaf homomorphism $\mathcal{F} \to \mathcal{G}$ induces maps $C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{G})$ (it does so component-wise), and a straightforward computation shows that the induced maps commute with the coboundary maps, and hence they pass to the cohomology. So we obtain functors $H^p(\mathcal{U}, -)$ from sheaves to abelian groups.

Example 17.7. The group $H^0(\mathcal{U}, \mathcal{F})$ is the kernel of the map $d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$, which is simply the usual map

$$\prod_i \mathcal{F}(U_i) \to \prod_{i<j} \mathcal{F}(U_i \cap U_j).$$

This kernel is equal to $\mathcal{F}(X)$ by the sheaf axioms; so $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Example 17.8 (H^1 and lifting of sections)**.** The most interesting cohomology group is arguably $H^1(\mathcal{U}, \mathcal{F})$. It is the group of elements (σ_{ij}) such that $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$ modulo the elements of the form $\sigma_{ij} = \tau_j - \tau_i$ (restricted to $U_i \cap U_j$). As mentioned in the introduction, this group is closely related to the lifting of sections, as we now explain.

Suppose that we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

Suppose we want to try to lift a section $c \in \mathcal{C}(X)$ to a section of $\mathcal{B}(X)$. Since the sequence is exact, we can at least find lifts locally, i.e. there is an open covering $\mathcal{U} = \{\mathcal{U}_i\}$ and sections $b_i \in \mathcal{B}(U_i)$ that map to $c|_{U_i}$ over each U_i. Now we ask if we can assemble the b_i to a section $b \in \mathcal{B}(X)$. For this to be the case, we must have $b_j|_{U_{ij}} - b_i|_{U_{ij}} = 0$. In any case,

$$\sigma = \left(b_j|_{U_{ij}} - b_i|_{U_{ij}}\right)$$

defines an element of $C^1(\mathcal{U}, \mathcal{A})$ (because b_i and b_j map to the same element in $\mathcal{C}(U_{ij})$). Furthermore, $d\sigma = 0$, because

$$(d\sigma)_{ijk} = (b_k - b_j) - (b_k - b_i) + (b_j - b_i) = 0$$

(all terms restricted to U_{ijk}). When is σ zero in $H^1(\mathcal{U}, \mathcal{A})$? This occurs if and only if there is an element $a = (a_i) \in C^0(\mathcal{U}, \mathcal{A})$ such that

$$b_j|_{U_{ij}} - b_i|_{U_{ij}} = a_j|_{U_{ij}} - a_i|_{U_{ij}},$$

which is equivalent to saying that the elements $b_i - a_i \in \mathcal{B}(U_i)$ agree over the overlaps U_{ij}, or in other words, that they glue together to a section $b \in \mathcal{B}(X)$. Note that since $a_i \in \mathcal{A}(U_i)$, the image of $b_i - a_i$ is the same as that of b_i, i.e. b maps to c.

In summary, the section $c \in \mathcal{C}(X)$ can be lifted if and only if the associated element in $H^1(\mathcal{U}, \mathcal{A})$ equals 0. If the latter group is zero, any section of $\mathcal{C}(X)$ lifts.

In Example 17.10 we will see a concrete example of a section which does not lift.

17.3 Examples

Example 17.9 (The projective line)**.** Consider the projective line $\mathbb{P}^1 = \mathbb{P}^1_k$ over a field k. It is covered by the two standard affines $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$ with intersection

$U_0 \cap U_1 = \operatorname{Spec} k[t, t^{-1}]$. For the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$, the Cech-complex takes the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(U_0) \times \mathcal{O}_{\mathbb{P}^1}(U_1) & \xrightarrow{d^0} & \mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1) & \longrightarrow & 0 \\ & & \uparrow \simeq & & \simeq \uparrow & & \\ & & k[t] \times k[t^{-1}] & \xrightarrow[d]{} & k[t, t^{-1}], & & \end{array}$$

where d sends a pair $(p(t), q(t^{-1}))$ to $q(t^{-1}) - p(t)$. We saw in Chapter 7 (during the proof of Proposition 7.1) that $\operatorname{Ker} d = k$. On the other hand, it is clear that each element of $k[t, t^{-1}]$ is a sum of a polynomial in t and one in t^{-1}. Hence d is surjective, and we have

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Coker} d = 0.$$

Example 17.10 (The sheaves $\mathcal{O}_{\mathbb{P}^1}(m)$)**.** Continuing the above example, let us compute the Cech cohomology groups of $\mathcal{O}_{\mathbb{P}^1}(m)$. We use the same affine cover, and the Cech complex still takes the form

$$0 \longrightarrow k[t] \times k[t^{-1}] \xrightarrow{d} k[t, t^{-1}] \longrightarrow 0,$$

but the coboundary map d is different; there is a multiplication by t^m in one of the restrictions, so the coboundary map is now given by

$$d(p(t), q(t^{-1})) = t^m q(t^{-1}) - p(t).$$

(see Section 7.2). As we computed in the proof of Proposition 7.2, the kernel of d is $(m+1)$-dimensional if $m \geqslant 0$, and $\operatorname{Ker} d = 0$ otherwise.

The computation of $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1})$ is slightly more subtle. Consider first the case when $m \geqslant 0$. As before, it is easy to see that any polynomial in $k[t, t^{-1}]$ can be written in the form $t^m q(t^{-1}) - p(t)$. In fact, this also works for $m = -1$; indeed, one has $t^{-k} = t^{-1} \cdot t^{-k+1} - 0$ and $t^k = t^{-1} \cdot 0 - t^k$. Hence $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(m)) = 0$ for $m \geqslant -1$. For $m \leqslant -2$ however, no linear combination of the monomials

$$t^{-1}, t^{-2}, \ldots, t^{m+1}$$

lies in the image, but combinations of all the others do. It follows that $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(m))$ is a k-vector space of dimension $-m-1$.

Example 17.11. Let $Z \subset \mathbb{P}^1_k$ be the subscheme associated to two closed points p, q in $\mathbb{P}^1$. We saw in Example XXX that the ideal sheaf sequence takes the form

$$0 \to \mathcal{O}_{\mathbb{P}^1_k}(-2) \to \mathcal{O}_{\mathbb{P}^1_k} \to i_*\mathcal{O}_Z \to 0$$

Consider the element $(0, 1) \in k \oplus k$, which defines a section of $i_*\mathcal{O}_Z(\mathbb{P}^1_k) = k \oplus k$. One can ask whether this section lifts to a global section s of $\mathcal{O}_{\mathbb{P}^1_k}$. This is not possible, because $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$; any regular function on $\mathbb{P}^1_k$ is constant so it can not take the value 0 at one point and 1 at another.

This failure of ability to lift is of course explained by the cohomology group $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2))$ which is 1-dimensional. Since $H^1(\mathcal{O}_{\mathbb{P}^1_k}) = 0$, one can think of the elements of this group as the group of elements of $k \oplus k$ modulo those that lift to $\mathcal{O}_{\mathbb{P}^1_k}$. Here it is clear that an element $(a, b) \in k \oplus k$ lifts if and only if $a = b$. In fact, in this example, the connecting map

$$\delta : H^0(\mathbb{P}^1, i_*\mathcal{O}_Z) \to H^1(\mathbb{P}^1_k, \mathcal{O}(-2))$$

can be identified with the map $k \oplus k \to k$ sending (a, b) to $a - b$.

Example 17.12 (The cuspidal cubic)**.** The curve $X = \operatorname{Proj} k[x_0, x_1, x_2]/(x_2^3 - x_0 x_1^2)$ admits an open cover $\mathcal{U}$ with two open sets, $U_0 = D_+(x_0)$ and $U_1 = D_+(x_1)$. We have

$$\mathcal{O}_X(U_0) = k[x_0^{-1}x_1, x_0^{-1}x_2]/((x_0^{-1}x_2)^3 - (x_0^{-1}x_1)^2) \tag{17.6}$$
$$\mathcal{O}_X(U_1) = k[x_1^{-1}x_0, x_1^{-1}x_2]/((x_1^{-1}x_2)^3 - (x_1^{-1}x_0)) = k[x_1^{-1}x_2] \tag{17.7}$$
$$\mathcal{O}_X(U_{01}) = k[x_1^{-1}x_2, x_2^{-1}x_1].$$

where we have used the defining equation to identify $x_1^{-1}x_0 = (x_1^{-1}x_2)^3$ and $x_2^{-1}x_0 = (x_1^{-1}x_2)^2$. The coboundary d^1 sends $p(x_0^{-1}x_1, x_0^{-1}x_2)$ and $q(x_1^{-1}x_2)$ to

$$q(x_1^{-1}x_2) - p\left(x_2^{-3}x_1^3, x_2^{-2}x_1^2\right).$$

From these expressions we can obtain any monomial $x_2^{-a}x_1^a$ except $x_2^{-1}x_1$. Therefore,

$$H^1(\mathcal{U}, \mathcal{O}_X) = \operatorname{Coker} d^1 = k \cdot x_2^{-1}x_1 \simeq k.c$$

Example 17.13. Let $\mathcal{U}$ be a finite open cover such that one of the members is the whole space X. In this case, the higher cohomology groups of any sheaf are all zero; that is

$$H^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p \geqslant 1$. To see this, suppose for simplicity that $U_0 = X$ (where $0 \in I$ denotes the smallest element), and define the map $h\colon C^{p+1}(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{F})$ by

$$h(\sigma)_{j_0,\ldots,j_p} = \begin{cases} \sigma_{0,j_0,\ldots,j_p} & \text{if } j_0 \neq 0; \\ 0 & \text{if } j_0 = 0. \end{cases}$$

Then if $i_0 \neq 0$, we have

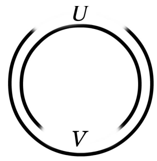
$$\begin{aligned} (dh + hd)(\sigma)_{i_0,\ldots,i_p} &= \sum\nolimits_{j=0}^p (-1)^j h(\sigma)_{i_0,\ldots,\hat{i}_j,\ldots,i_p} + d(\sigma)_{0,i_0,\ldots,i_p} \\ &= \sum\nolimits_{j=0}^p (-1)^j \sigma_{0,i_0,\ldots,\hat{i}_j,\ldots,i_p} + \sigma_{i_0,\ldots,i_p} + \sum\nolimits_{j=0}^p (-1)^{j+1} \sigma_{0,i_0,\ldots,\hat{i}_j,\ldots,i_p} \\ &= \sigma_{i_0,\ldots,i_p}. \end{aligned}$$

Likewise, if $i_0 = 0$, we have

$$\begin{aligned} (dh + hd)(\sigma)_{0,i_1,\ldots,i_p} &= \sum\nolimits_{j=0}^p (-1)^j h(\sigma)_{0,i_1,\ldots,\hat{i}_j,\ldots,i_p} + 0 \\ &= \sigma_{0,i_1,\ldots,i_p}. \end{aligned}$$

Hence h is a homotopy between the identity map on $C^{p+1}(\mathcal{U}, \mathcal{F})$ and the zero map, and the cohomology group $H^{p+1}(\mathcal{U}, \mathcal{F})$ is zero by Example 17.2

Example 17.14 (The unit circle)**.** Here is an example from topology. Consider the unit circle $X = S^1$ (with the Euclidean topology), and equip it with a standard covering $\mathcal{U} = \{U, V\}$ consisting of two intervals intersecting in two intervals as shown in the figure. Let $\mathcal{F} = \mathbb{Z}_X$ be the constant sheaf on $\mathbb{Z}$.



Here we have

$$C^0(\mathcal{U},\mathcal{F}) = \mathbb{Z}_X(U) \times \mathbb{Z}_X(V) \simeq \mathbb{Z}\times\mathbb{Z} \qquad C^1(\mathcal{U},\mathbb{Z}) = \mathbb{Z}_X(U\cap V) \simeq \mathbb{Z}\times\mathbb{Z}.$$

The map $d^0\colon C^0(\mathcal{U},\mathbb{Z}_X) \to C^1(\mathcal{U},\mathbb{Z}_X)$ is the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$d^0(a,b) = (b-a, b-a).$$

Hence

$$H^0(\mathcal{U},\mathbb{Z}_X) = \operatorname{Ker} d^0 = \mathbb{Z}(1,1) \simeq \mathbb{Z},$$

and

$$H^1(\mathcal{U},\mathbb{Z}_X) = \operatorname{Coker} d^0 = \mathbb{Z}^2/\mathbb{Z}(1,1) \simeq \mathbb{Z}.$$

Readers familiar with algebraic topology may recognize that this gives the same answer as singular cohomology. In fact, it is a general fact that the cohomology groups $H^p(\mathcal{U},\mathbb{Z})$ agree with the usual singular cohomology groups $H^p_{\text{sing}}(X,\mathbb{Z})$ for any topological space homotopty equivalent to a CW complex, provided that the open sets in the covering $\mathcal{U}$ are contractible **?**.

Example 17.15 (Constant sheaves on irreducible spaces)**.** In contrast to the above examples, we will show that constant sheaves are not so interesting in algebraic geometry, as we would like to study spaces which are *irreducible* as topological spaces. Then any open set is connected and the constant sheaves A_X are actually constant taking the value A on any open set U.

We claim that for any group A and finite covering $\mathcal{U}$ of X, it holds that

$$H^p(\mathcal{U},A_X) = 0 \qquad \text{for all } p \geqslant 1.$$

The Cech complex takes the form

$$\prod_i A \to \prod_{i<j} A \to \prod_{i<j<k} A \to \cdots \tag{17.8}$$

Note that this complex does not depend on X nor on the covering $\mathcal{U}$; only the index set I plays a role. We can thus use a cover consisting of $(n+1)$ opens, all equal to X, and the higher cohomology groups vanish by Example 17.13.

Exercise 17.3.1. Generalize Example 17.12 to show that the curve $V(x_2^d - x_0x_1^{d-1}) \subset \mathbb{P}^2_k$ has an $H^1(X,\mathcal{O}_X)$ of dimension $\frac{1}{2}(d-1)(d-2)$.

Exercise 17.3.2. Let $X = S^1$ and let $\mathcal{U}$ be the covering of X with three pairwise intersecting open intervals with empty intersection. Show that the Cech complex is of the form

$$\mathbb{Z}^3 \xrightarrow{d^0} \mathbb{Z}^3 \to 0.$$

Compute the map d^0 and use it to verify again that $H^i(\mathcal{U},\mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 1$ as above.

17.4 Cech cohomology on schemes

As the previous examples illustrate, the cohomology groups $H^p(\mathcal{U}, \mathcal{F})$ can be computed if we have adequate information on the sections of $\mathcal{F}$ over the open sets in the finite cover $\mathcal{U}$. In that case, the maps in the Cech complex are completely explicit, and computing their kernels and images involves only basic operations which can be done quite algorithmically.

On the other hand, the definition of the cohomology groups is unsatisfactory for a few reasons. First of all, the groups $H^p(\mathcal{U}, \mathcal{F})$ depend on the open cover $\mathcal{U}$, whereas we want something canonical that only depends on $\mathcal{F}$. More importantly, it is not clear that the definition above really captures the desired information about $\mathcal{F}$. For instance, $\mathcal{U}$ could consist of the single open set X, and so $H^i(\mathcal{U}, \mathcal{F}) = 0$ for all $i \geqslant 1$!

In the context of schemes, the most natural thing is to consider an open covering $\mathcal{U}$ consisting of affine open sets. We will show that in good situations, i.e. if X is Noetherian and separated and the sheaf $\mathcal{F}$ is quasi-coherent, the group $H^i(\mathcal{U}, \mathcal{F})$ will in fact turn out to be independent of the covering.

Theorem 17.16 (Main properties of Cech cohomology). Let X be a Noetherian scheme, and let $\mathcal{U} = \{U_i\}$ be a finite affine cover such all intersections $U_{i_1} \cap \cdots \cap U_{i_p}$ are affine. Then

(i) The Cech cohomology groups are functors $H^i(\mathcal{U}, -)\colon \mathsf{AbSh}_X \to \mathsf{Ab}$;
(ii) $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$;
(iii) Short exact sequences of quasi-coherent sheaves induce long exact sequences of cohomology

$$\cdots \to H^p(\mathcal{U}, \mathcal{F}') \to H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F}'') \to H^{p+1}(\mathcal{U}, \mathcal{F}') \to \cdots$$

(iv) If $\mathcal{V} = \{V_i\}$ is another affine cover with all intersections $V_{i_1} \cap \cdots \cap V_{i_p}$ affine, then there is a natural isomorphism

$$H^p(\mathcal{U}, \mathcal{F}) = H^p(\mathcal{V}, \mathcal{F})$$

for every p and every quasi-coherent sheaf $\mathcal{F}$.
(v) If X has dimension n, then $H^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > n$ and all quasi-coherent $\mathcal{F}$.

Any Noetherian scheme admits an open cover as in the Theorem. Note in particular that the condition on the intersections is automatically satisfied if X is separated.

Definition 17.17. With the assumptions of the above theorem, we write $H^p(X, \mathcal{F})$ for the group $H^p(\mathcal{U}, \mathcal{F})$.

We have already proved the first two of these properties. (In this case, we do not need to assume that the intersections are affine, nor that the cover is finite.) The other items will require a little more work.

17.4.1 The long exact sequence

Proving item (ii) is not so difficult. Consider a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

In Proposition **??** we proved that whenever the $U = \operatorname{Spec} A$ is an open affine in X, the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0 \tag{17.9}$$

is exact. This means that if an affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ has the property that each intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ is affine, as taking products do not disturb exactness, there is an exact sequence

$$0 \longrightarrow C^p(\mathcal{U}, \mathcal{F}') \longrightarrow C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{F}'') \longrightarrow 0,$$

and consequently the sequence of *Cech complexes*

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}') \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}'') \longrightarrow 0$$

is also exact. Thus we are in position to apply Lemma 17.1 to obtain a long exact sequence of Cech cohomology groups

$$\cdots \longrightarrow H^i(\mathcal{U}, \mathcal{F}') \longrightarrow H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}, \mathcal{F}'') \longrightarrow \cdots .$$

17.5 Cohomology of sheaves on affine schemes

The following result is fundamental in the study of sheaf cohomology groups. It is the first example of a 'vanishing theorem' for cohomology. Recalling that cohomology groups were defined to measure the 'failure' of certain desirable statements (e.g. restriction maps being surjective), we are in general happy if cohomology groups are zero.

Theorem 17.18. Let $X = \operatorname{Spec} A$ and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for any affine cover $\mathcal{U}$ of X,

$$H^p(\mathcal{U}, \mathcal{F}) = 0 \quad \text{for all } p > 0.$$

Proof We know the theorem to hold in the 'trivial case' when one of the U_i, say, U_0 is equal to X (see Example 17.13). In general, we reduce to the trivial case as follows. Let $U_i = \operatorname{Spec} A_i$ be the affines in $\mathcal{U}$. As X is affine, it is separated, so all intersections U_I are also affine. We want to show that the complex of A-modules

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \tag{17.10}$$

is exact. The exactness of this sequence can be checked by localizing at each prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Assume without loss of generality that $\mathfrak{p} \in U_0$. Then since $\mathcal{F}$ is quasi-coherent, the localization of (17.10) at $\mathfrak{p}$ coincides with the localization of the Cech complex of $C^\bullet(\mathcal{U} \cap U_0, \mathcal{F})$ at $\mathfrak{p}$, and the latter is exact by the 'trivial case' by Example 17.13. □

Example XXX showed that even $\mathbb{P}^1_k$ admits sheaves with non-vanishing higher cohomology. Here is another example:

Example 17.19 (The affine line with two origins)**.** Consider the ‘affine line with two origins’ X from Example 7.3 on page 95. It is covered by two affine subsets $X_1 = \operatorname{Spec} k[u]$ and $X_2 = \operatorname{Spec} k[u]$ and these are glued together along their common open set $X_{12} = D(u) = \operatorname{Spec} k[u, u^{-1}]$ with the identity as gluing map. The Cech complex for this covering looks like

$$0 \longrightarrow k[u] \times k[u] \xrightarrow{d^1} k[u, u^{-1}] \xrightarrow{d^2} 0$$

where $d^1(p(u), q(u)) = q(u) - p(u)$, and is nothing but the standard sequence that appeared in the example, and as we checked in there, it holds that $\mathcal{O}_X(X) = \operatorname{Ker} d^1 = k[u]$.

More strikingly, $H^1(X, \mathcal{O}_X)$, i.e. the cokernel of the map $k[u] \oplus k[u] \to k[u, u^{-1}]$ is rather big. It equals $k[u, u^{-1}]/k[u] = \bigoplus_{i>0} k u^{-i}$, so that $H^1(X, \mathcal{O}_X)$ is not finite-dimensional as a vector space over k. This gives another proof that X is not isomorphic to an affine scheme.

Exercise 17.5.1. Let $X = \mathbb{A}^n_k - \{0\}$ be the complement of the origin.

a) Compute $H^i(X, \mathcal{O}_X)$ for all i.
b) Give a new proof that X is not an affine scheme for $n \geqslant 2$.

17.6 Independence of the cover

Let us embark on the proof of item (iv). Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ and $\mathcal{V} = \{V_1, \ldots, V_s\}$ be two finite affine covers, and form the following group of sections over all the mixed intersections:

$$C^{n,m} = \prod_{|I|=n, |J|=m} \Gamma(U_I \cap V_J, \mathcal{F})$$

Note that for n fixed

$$C^{n,\bullet} \simeq \prod_{|I|=n} C(U_I \cap V_J, \mathcal{F}|_{U_I})$$

is the Cech complex of $\mathcal{F}|_{U_I}$ with respect to the cover $V_j \cap U_I$. Likewise,

$$C^{\bullet,m} \simeq \prod_{|J|=m} C(V_j \cap U_i, \mathcal{F}|_{U_I})$$

is the Cech complex of $\mathcal{F}|_{V_J}$ with respect to the cover $U_I \cap V_J$

One says that $C^{n,m}$ forms a *bi-complex*. It has two coboundary maps, one written d in the ‘right’ direction, and one in the ‘upwards’ direction, δ. (See the figure below).

The key point is that the intersections $U_I \cap V_J$ are affine. That means that all the higher cohomology groups in each direction vanish, i.e., the complexes $C^{n,\bullet}$ and $C^{\bullet,m}$ are exact in degrees $\geqslant 1$.

In degree 0, the cohomology groups are

$$H^0(C^{n,\bullet}) = \prod_{|I|=n} \Gamma(U_I, \mathcal{F}) = C^n(\mathcal{U}, \mathcal{F})$$

and

$$H^0(C^{\bullet,m}) = \prod_{|J|=m} \Gamma(V_J, \mathcal{F}) = C^m(\mathcal{V}, \mathcal{F})$$

Now item (iv) is a formal consequence of the following homological algebra fact:

Lemma 17.20 (Zig-zag lemma). Let $C^{n,m}$ be a bi-complex with $H^i(C^{n,\bullet}) = H^i(C^{\bullet,m}) = 0$ for all $m, n \geqslant 1$. Then $A^n = H^0(C^{n,\bullet})$ and $B^m = H^0(C^{\bullet,m})$ are complexes and there is a canonical isomorphism between their cohomology:

$$H^i(A^\bullet) = H^i(B^\bullet).$$

Proof We augment the bi-complex above by adding A^i and B^j, to get the picture below.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\delta\uparrow & & \delta\uparrow & & \uparrow & & \uparrow & & \\
B^2 & \longrightarrow & C^{0,2} & \xrightarrow{d} & C^{1,2} & \xrightarrow{d} & C^{2,2} & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
B^1 & \longrightarrow & C^{0,1} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & C^{2,1} & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
B^0 & \longrightarrow & C^{0,0} & \xrightarrow{d} & C^{1,0} & \xrightarrow{d} & C^{2,0} & \xrightarrow{d} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & A^0 & \xrightarrow{d} & A^1 & \xrightarrow{d} & A^2 & \xrightarrow{d} & \cdots
\end{array}$$

Now all rows and columns are exact. The cleanest way to proceed is to show that $H^i(A^\bullet)$ and $H^i(B^\bullet)$ are both isomorphic to a third group, namely the group $H^i(C)$ defined as the quotient of the group of 'zig-zags'

$$\left\{(c^{i,0}, c^{i-1,1}, \dots, c^{0,i}) \mid d(c^{i,0}) = \delta(c^{i-1,1}), \dots, d(c^{1,i-1}) = \delta(c^{0,i})\right\}$$

by the subgroup generated by the 'coboundary zig-zags'

$$d(c^{i-1,0}) + \delta(c^{i-2,1}), d(c^{i-2,1}) + \delta(c^{i-3,2}), \dots, d(c^{1,i-1}) + \delta(c^{0,i-1})$$

For each i there is a map $\alpha_i : H^i(C) \to H^i(A^\bullet)$ sending $(c^{i,0}, \dots, c^{0,i})$ to the image of $c^{i,0}$ in $H^i(A^\bullet)$. Likewise, there is a map $\beta_i : H^i(C) \to H^i(B^\bullet)$.

One shows that α_i is an isomorphism by a diagram chase in the above diagram. For instance, to show that α_2 is surjective, pick $a \in H^2(A^\bullet)$ and an element $c^{2,0} \in C^{2,0}$ that maps to it. Then one builds a the rest of the sequence $(c^{2,0}, c^{1,1}, c^{2,0})$ by chasing up the anti-diagonal of the diagram, as follows:

$$\begin{array}{ccccccc}
\exists c^{0,2} & \longmapsto & \delta(c^{1,1}) & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \exists c^{1,1} & \longmapsto & \delta(c^{2,0}) & \longrightarrow & 0 \\
 & & & & \uparrow & & \\
 & & & & c^{2,0} & \longrightarrow & 0
\end{array}$$

That is, $c^{2,0}$ maps to $\delta(c^{2,0}) \in C^{2,1}$ which in turn is mapped to 0 by both d and δ. Therefore, there exist a $c^{1,1} \in C^{1,1}$ that maps to it. Now we consider $\delta(c^{1,1})$ which must map to zero in $C^{2,2}$ by commutativity, and hence by exactness lifts to an element in $c^{0,2}$. The resulting element defines an element in $H^2(C)$ which clearly maps to a. The argument for injectivity is similar; we leave the details for the interested reader. Of course, once one has established this, one can do the same thing for β_i, so we get the desired isomorphism. □

In particular, we get independence of $H^p(\mathcal{U}, \mathcal{F})$ for *any* open affine covering on a Noetherian separated scheme.

Exercise 17.6.1. Complete the details of the proof of the 'Zig-zag Lemma'.

17.7 Cohomology and dimension

The next result is another 'vanishing theorem' for cohomology groups. It is a general result, due to Grothendieck, that the cohomology groups vanish above the dimension of X, at least for spaces X that are Noetherian and the dimension is interpreted as the Krull dimension.

Theorem 17.21. Let X be a Noetherian topological space and let $\mathcal{F}$ be a sheaf. Then

$$H^p(X, \mathcal{F}) = 0 \text{ for all } p > \dim X$$

A proof valid in the general case may be found in (**?**, Theorem 4.5.12), but we contend ourself with proving it in the special case when X is a quasi-projective scheme over a ring A. We begin with an easy lemma:

Lemma 17.22. Let X be a topological space and let $Z \subset X$ be a closed subset. Then for any sheaf $\mathcal{F}$ on Z and any p,

$$H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F}).$$

Proof Observe that each open cover $\{U_i\}$ of X induces an open cover $\{U_i \cap Z\}$ of Z, and all open covers of Z arise like this. The lemma then follows from the basic fact that for each open subset $U \subset X$ it holds that $\Gamma(U, i_*\mathcal{F}) = \Gamma(Z \cap U, \mathcal{F})$, so the two cohomolgy groups arise from the same Cech complexes. □

Theorem 17.23. Let X be a quasi-projective scheme of finite type over a ring A of dimension n. Then X admits an open cover $\mathcal{U}$ consisting of at most $n+1$ affine open subsets. In particular, for any quasi-coherent sheaf on X,

$$H^p(X, \mathcal{F}) = 0 \text{ for } p > n.$$

Proof We may write $X = \overline{X} - W$ where $\overline{X}, W \subseteq \mathbb{P}^r_A$ are closed subschemes, and we may assume that no irreducible component of $\overline{X}$ is contained in W (simply by discarding such components). Using induction on the dimension, we will prove that X is covered by $n+1$ open affines induced from open affines in $\mathbb{P}^r_A$.

Consider the irreducible decomposition $\overline{X} = \bigcup_i Y_i$, where the Y_i are integral and observe that by prime avoidance $I_W \nsubseteq \bigcup I_{Y_i}$ where $I_T \subseteq A[x_0, \ldots, x_N]$ denotes the homogeneous ideal of a set $T \subseteq \mathbb{P}^N_A$. Pick a homogenous polynomial f such that $f \in I_W - (\bigcup_i I_{Y_i})$, and let $H = V(f)$. Then we infer that the set $\mathbb{P}^r_A - H = D_+(f)$ is affine and hence so is $\overline{X} - H$, being a closed subscheme of an affine scheme.

By construction $\overline{X} - H \subseteq \overline{X} - W = X$ and $H \nsupseteq Y_i$ for any i by the choice of f. Therefore $\dim(Y_i \cap H) < \dim Y_i$ so we may use induction on the dimension to cover $Y \cap H$ by fewer that n open affines, all induced from the ambient projective space, which together with $D(f)$ gives a covering of X with $n+1$ open affine subsets. This shows the first claim.

For the second, note that in a Cech complex built on a covering consisting of at most $n+1$ affines open subsets, terms $C^p(X, \mathcal{F})$ with $p > n$ will vanish, from which follows that $0 = H^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ for each $\mathcal{F}$ and each $p > n$. □

17.8 Cohomology of sheaves on projective space

In Examples 17.9 and 17.10 we computed the sheaf cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(m)$. For $d \geqslant 0$, we found that $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m))$ could be identified with the space of homogeneous polynomials of degree d, and $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m)) = 0$. On the other hand, for $d \leqslant -2$, $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m)) = 0$, while $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m))$ was non-zero.

We will now carry out a more general computation of the cohomology groups of $\mathcal{O}_{\mathbb{P}^n_A}(m)$ for any projective space $\mathbb{P}^n_A$ over a ring A. The strategy is however the same, we have a distinguished cover via the open sets $D_+(x_i)$, and we use Cech complex associated to this cover to compute the cohomology.

For this cover, the groups in the Cech complex are

$$C^0(\mathcal{U}, \mathcal{O}(m)) = \prod_i (A[x_0, \ldots, x_n]_{x_i})_m \tag{17.11}$$

$$C^1(\mathcal{U}, \mathcal{O}(m)) = \prod_{i<j} \left(A[x_0, \ldots, x_n]_{x_i x_j}\right)_m \tag{17.12}$$

$$\vdots \tag{17.13}$$

$$C^n(\mathcal{U}, \mathcal{O}(m)) = (A[x_0, \ldots, x_n]_{x_0 x_1 \ldots x_n})_m$$

and the Cech complex takes the form

$$\prod_i (A[x_0,\ldots,x_n]_{x_i})_m \xrightarrow{d^0} \prod_{i<j} \big(A[x_0,\ldots,x_n]_{x_ix_j}\big)_m \xrightarrow{d^1} \prod_{i<j<k} \big(A[x_0,\ldots,x_n]_{x_ix_jx_k}\big)_m \xrightarrow{d^2} \cdots \tag{17.14}$$

where the maps are as usual composed of alternating sums of localization maps.

In particular, we recover the following isomorphism:

$$\begin{aligned} H^0(\mathbb{P}^n_A, \mathcal{O}(m)) &= \operatorname{Ker} d^0 \\ &= A[x_0,\ldots,x_n]_m. \end{aligned}$$

To compute the higher cohomology groups, we need a careful analysis of the complex (17.14).

Theorem 17.24 (Cohomology of $\mathbb{P}^n$). Let $\mathbb{P}^n_A = \operatorname{Proj} A[x_0,\ldots,x_n]$ where A is a ring.

(i) For each $m \in \mathbb{Z}$,

$$H^0(\mathbb{P}^n_A, \mathcal{O}(m)) = A[x_0,\ldots,x_n]_m.$$

(ii) For all $0 < p < n$ and all $m \in \mathbb{Z}$,

$$H^p(\mathbb{P}^n_A, \mathcal{O}(m)) = 0.$$

(iii) For each $m \in \mathbb{Z}$, we have

$$H^n(\mathbb{P}^n_A, \mathcal{O}(m)) = \big(x_0^{-1}\cdots x_n^{-1} A[x_0,\ldots,x_n]_{x_0\cdots x_n}\big)_m$$

In particular, there is a canonical isomorphism

$$H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) = A.$$

Proof Since $\mathbb{P}^n_A$ is separated, we may compute the cohomology groups using the Cech-complex associated with the standard covering $\mathcal{U} = \{U_i\}$ where $U_i = D_+(x_i) = \operatorname{Spec}(R_{x_i})_0$.

(ii): Suppose $0 < p < n$. We need to check that the Cech complex is exact at $C^p(\mathcal{U}, \mathcal{O}_X(m))$. The main idea is to use the multigrading on the polynomial ring $A[x_0,\ldots,x_n]$ and its localizations, defined by saying that a Laurent monomial $x^e = x_0^{e_0}\cdots x_n^{e_n}$ has multidegree $e = (e_0,\ldots,e_n) \in \mathbb{Z}^{n+1}$. Thus, for instance,

$$A[x_0,\ldots,x_n]_{x_0\cdots x_n} = \bigoplus_{e\in\mathbb{Z}^{n+1}} Ax^e$$

as A-modules. The terms in the Cech complex (17.14) admit a similar decomposition:

$$C^p(\mathcal{U}, \mathcal{O}(m)) = \bigoplus_{e\in\mathbb{Z}^{n+1}} C^p(\mathcal{U}, \mathcal{O}(m))_e \tag{17.15}$$

The differentials in the complex, being alternating sums of localization maps, are also compatible with this decomposition. Therefore it suffices to check that $C^p(\mathcal{U}, \mathcal{O}(m))_e$ is exact for each e.

Note that each $C^p(\mathcal{U}, \mathcal{O}(m))_e$ is a product of finitely many copies of A. In fact, when all the entries of e are non-negative, $C^p(\mathcal{U}, \mathcal{O}(m))_e = \prod_{i_0<\cdots<i_p} Ax^e$, and, forgetting the

monomial x^e, the complex $C^p(\mathcal{U}, \mathcal{O}(m))_e$ takes the form

$$\dots \longrightarrow \prod_{i_0<\dots<i_{p-1}} A \longrightarrow \prod_{i_0<\dots<i_p} A \longrightarrow \prod_{i_0<\dots<i_{p+1}} A \longrightarrow \dots \tag{17.16}$$

We have already seen that this is exact: it is the Cech complex $C^p(\mathcal{V}, \mathcal{O}_{\operatorname{Spec} A})$ for the structure sheaf on $X = \operatorname{Spec} A$, associated with the "trivial covering" with $(n+1)$ copies of $V_i = \operatorname{Spec} A$ as the open sets in the cover.

When some of the entries of e are negative, we still have

$$C^p(\mathcal{U}, \mathcal{O}(m))_e = \prod_{\substack{i_0<\dots<i_p \text{ such that} \\ e_j \geqslant 0 \text{ for all } j \notin \{i_0,\dots,i_p\}}} Ax^e$$

Consequently, $C^p(\mathcal{U}, \mathcal{O}(m))_e$ forms a subcomplex of the complex (17.16). In fact, have the following commutative diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C^{p-1}(\mathcal{U}, \mathcal{O}(m))_e & \longrightarrow & C^p(\mathcal{U}, \mathcal{O}(m))_e & \longrightarrow & C^{p+1}(\mathcal{U}, \mathcal{O}(m))_e & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \prod_{i_0<\dots<i_{p-1}} A & \longrightarrow & \prod_{i_0<\dots<i_p} A & \longrightarrow & \prod_{i_0<\dots<i_{p+1}} A & \longrightarrow & \dots \end{array}$$

When $p < n$, the top row is even a direct summand of the bottom row complex (the projection maps give sections to the inclusions). Having shown that the bottom sequence is exact, we conclude that the top row is exact as well.

To prove (iii), observe that $C^n(\mathcal{U}, \mathcal{O}(m)) = (A[x_0, \dots, x_n]_{x_0 \cdots x_n})_m$ is a free graded A-module spanned by monomials of the form $x_0^{e_0} \cdots x_n^{e_n}$ with multidegrees $(e_0, \dots, e_n) \in \mathbb{Z}^{n+1}$ with $\sum e_i = m$. The image of d^{n-1} is spanned by such monomials where at least one e_i is non-negative. Hence

$$\begin{aligned} H^n(X, \mathcal{O}(m)) &= \operatorname{Coker} d^{n-1} \\ &= A \left\{ x_0^{e_0} \cdots x_n^{e_n} \;\middle|\; e_i < 0 \text{ for every } i \text{ and } \sum a_i = m \right\} \\ &= \left(x_0^{-1} \cdots x_n^{-1} A[x_0, \dots, x_n]_{x_0 \cdots x_n} \right)_m \end{aligned}$$

In degree $m = -n-1$ there is only one such monomial, namely $x_0^{-1} \cdots x_n^{-1}$, so

$$H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) = A \cdot x_0^{-1} \cdots x_n^{-1}. \qquad \square$$

The proof of Theorem 17.24 also gives the following duality between H^0 and H^n on $\mathbb{P}^n$. Consider the following pairing of A-modules given by

$$(\,,\,) : A[x_0, \dots, x_n] \times \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \dots, x_n^{-1}] \right) \to A, \tag{17.17}$$

sending (p, q) to the coefficient of $x_0^{-1} \dots x_n^{-1}$ in the product pq. Note that this is A-linear in each factor. In terms of the standard monomial basis, we have

$$(x_0^{d_0} \dots x_n^{d_n}, x_0^{e_0} \dots x_n^{e_n}) = \begin{cases} 1 & \text{if } d_i + e_i = -1 \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

This pairing is perfect[1] and allows us to canonically identify

$$\big(x_0^{-1}\cdots x_n^{-1}A[x_0^{-1},\ldots,x_n^{-1}]\big)_d = \operatorname{Hom}_A(A[x_0,\ldots,x_n]_{-n-1-m},A).$$

This allows us to regard the n-th cohomology group $H^n(\mathbb{P}^n_A,\mathcal{O}(d))$ as the *dual* of a corresponding H^0:

Corollary 17.25 (Serre duality for $\mathbb{P}^n$). For each $d\in\mathbb{Z}$, there is a canonical isomorphism

$$H^n(\mathbb{P}^n_A,\mathcal{O}(m)) = \operatorname{Hom}_A(H^0(\mathbb{P}^n,\mathcal{O}(-m-n-1)),A). \tag{17.18}$$

When $A=k$ is a field, the dimensions of the cohomology groups are easily computed:

Corollary 17.26. Let k be a field. Then for $m\geqslant 0$

$$\dim_k H^0(\mathbb{P}^n_k,\mathcal{O}_{\mathbb{P}^n_k}(m)) = \binom{m+n}{n} \tag{17.19}$$

$$\dim_k H^n(\mathbb{P}^n_k,\mathcal{O}_{\mathbb{P}^n_k}(-m)) = \binom{m-1}{n}.$$

All other cohomology groups are 0.

17.9 Cohomology groups of coherent sheaves on projective schemes

By the results of the previous section, the cohomology groups of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ over a field k are always finite-dimensional k-vector spaces. This is part of a more general result, saying that on projective schemes of finite type over a ring, the cohomology groups of coherent sheaves are always of finite type. Note that this is definitely not the case for affine schemes: Even the H^0 of the structure sheaf on $\mathbb{A}^1_k$ is infinite dimensional, as it equals $k[t]$.

Theorem 17.27 (Serre). Let $X\subset\mathbb{P}^n_A$ be a projective scheme of finite type over a ring A and let $\mathcal{F}$ be a coherent sheaf on X.

(i) Then the cohomology groups $H^i(X,\mathcal{F})$ are finitely generated A-modules for each i.

(ii) There exists an $n_0>0$ such that

$$H^i(X,\mathcal{F}(n))=0.$$

for all $n\geqslant n_0$ and $i>0$.

Proof Let $i: X\to\mathbb{P}^n_A$ denote the closed embedding and consider the sheaf $i_*\mathcal{F}$. Since i is affine, the sheaf $i_*\mathcal{F}$ is again coherent (Lemma XXX) and $H^i(X,\mathcal{F})=H^i(\mathbb{P}^n_A,i_*\mathcal{F})$, so we immediately reduce to the case $X=\mathbb{P}^n$.

[1] Recall that a bilinear map $M\times N\to A$ is a *perfect pairing* if the induced map $M\mapsto\operatorname{Hom}_A(N,A)$ is an isomorphism.

Recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_A$ is of the form $\widetilde{M}$ for some finitely graded module M over $R = A[x_0, \dots, x_n]$.

Note that both parts of the theorem are trivially satisfied for $i > \dim \mathbb{P}^n_A = n + \dim A$, because $H^i(\mathbb{P}^n_A, \mathcal{F}) = 0$ in that range. The proof will take this as the base case and proceed by downwards induction on i.

(i): As M is finitely genenerated, we may pick a graded surjection $\bigoplus_i R(-a_i) \to M$ for M. Let K be the kernel, so that we have an exact sequence of finitely generated graded R-modules

$$0 \to K \to \bigoplus_i R(-a_i) \to M \to 0$$

Applying tilde, we have a sequence of coherent sheaves

$$0 \to \mathcal{K} \to \bigoplus_i \mathcal{O}_{\mathbb{P}^n_A}(-a_i) \to \mathcal{F} \to 0$$

If we take the long exact sequence of cohomology, we get

$$\cdots \to H^i(\mathbb{P}^n_A, \mathcal{K}) \to \bigoplus_i H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-a_i)) \to H^i(\mathbb{P}^n_A, \mathcal{F}) \to H^{i+1}(\mathbb{P}^n_A, \mathcal{K}) \to \dots$$

By induction on i, the group $H^{i+1}(\mathbb{P}^n_A, \mathcal{K})$ is a finitely generated A-module, as is $\bigoplus_i H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-a_i))$ by Theorem 17.24. $H^i(\mathbb{P}^n_A, \mathcal{F})$ is therefore squeezed between two finitely generated A-modules, so by exactness, it is itself finitely generated.

(ii): Twist the above sequence by $\mathcal{O}_{\mathbb{P}^n_A}(m)$ and take the long exact sequence in cohomology to get

$$H^i(X, \bigoplus_i (m - a_i)) \to H^i(X, \mathcal{F}(m)) \to H^{i+1}(X, \mathcal{K}(m))$$

Again, by downward induction on i, and the fact that $H^i(X, \mathcal{O}_{\mathbb{P}^n_A}(m - a_i)) = 0$ for $i > 0$ and $m > a_i$, we find that $H^i(X, \mathcal{F}(m)) = 0$. □

The Euler characteristic

If X is a projective scheme of finite type over a field k and $\mathcal{F}$ is a coherent sheaf on X, Serre's theorem tells us that the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces. In particular, we can ask about their dimensions. It turns out that the alternating sum of these dimensions has very good formal properties, so we make the following definition:

Definition 17.28. Let X be a projective scheme of finite type over a field k. We define the *Euler characteristic* of $\mathcal{F}$ as

$$\chi(\mathcal{F}) = \sum_{k \geqslant 0} (-1)^k \dim_k H^k(X, \mathcal{F}).$$

Note that the sum is well-defined, as there are only finitely many non-zero cohomology groups appearing on the right hand side.

Proposition 17.29. The Euler characteristic χ is additive on exact sequences, i.e., if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof This follows because if $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of k-vector spaces, then $\sum_i (-1)^i \dim_k V = 0$. Applying this to the long exact sequence in cohomology gives the claim. □

Example 17.30. Let $X = \mathbb{P}^n_k$ and $\mathcal{F} = \mathcal{O}(d)$ for $d \geqslant 0$. Then $\dim_k H^0(\mathbb{P}^n_k, \mathcal{F}) = \binom{n+d}{n}$ and all of the higher cohomology groups are zero. In the case when $d < 0$, only $H^n(X, \mathcal{F})$ can be non-zero, and the rank is given by $\binom{n+d}{n}$, where we use the extended binomial coefficient

$$\binom{x}{d} = x(x-1)\cdots(x-d+1)/d!$$

for any $x \in \mathbb{R}$. In particular,

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{d}$$

is a polynomial in d of degree n, which agrees with $\dim H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ for all $d \geqslant 0$.

The example shows that for a direct sum $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \ldots \mathcal{O}_{\mathbb{P}^n}(a_r)$, the Euler characteristic $\chi(\mathcal{E}(m))$ is a polynomial in m. Even more generally, we can take any coherent sheaf $\mathcal{F}$ and a free resolution of it:

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where the $\mathcal{E}_i$ are direct sums of invertible sheaves of the form $\mathcal{O}(d)$. If we tensor this sequence by $\mathcal{O}_{\mathbb{P}^n_A}(m)$, we get[2]

$$0 \to \mathcal{E}_n(m) \to \cdots \to \mathcal{E}_1(m) \to \mathcal{E}_0(m) \to \mathcal{F}(m) \to 0$$

Note that each of the terms $\chi(\mathcal{E}_i(m))$ is a polynomial in m. Then since the Euler characteristic is additive on exact sequences, also $\chi(\mathcal{F}(m))$ is a polynomial in m. Moreover, again by Serre's theorem, we have $H^i(X, \mathcal{F}(m)) = 0$ for $m \gg 0$ and $i > 0$, and so $\chi(\mathcal{F}(m)) = H^0(\mathcal{F}(m))$ for m large.

If we start with a coherent sheaf $\mathcal{F}$ on a $X \subset \mathbb{P}^n_k$, and apply the previous discussion to $i_*\mathcal{F}$ on $\mathbb{P}^n_k$, we have proved the following:

Corollary 17.31. Let $X \subset \mathbb{P}^n_k$ be a projective scheme of finite type over k and let $\mathcal{O}(1)$ be the Serre twisting sheaf. Then the function

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m))$$

is a polynomial in m, and for large m, $P_{\mathcal{F}}(m) = H^0(X, \mathcal{F}(m))$.

[2] Recall that tensoring by a locally free sheaf preserves exactness.

This polynomial is called the *Hilbert polynomial* of $\mathcal{F}$. While $\chi(\mathcal{F})$ is an intrinsic invariant of $\mathcal{F}$, the Hilbert polynomial is not, as it depends on the choice of embedding $X \subset \mathbb{P}^n_k$.

When $\mathcal{F} = \widetilde{M}$ for a graded module M, $\mathbb{P}_{\mathcal{F}}(m)$ coincides with the usual Hilbert polynomial of M as defined in commutative algebra.

17.10 Extended example: Plane curves

Let $X = V(f) \subset \mathbb{P}^2_k$ be a plane curve, defined by an homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. Let us compute the groups of the structure sheaf $H^i(X, \mathcal{O}_X)$. We have the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

where the ideal sheaf $\mathcal{I}_X$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X$. By Section 16.29, $\mathcal{O}_{\mathbb{P}^2}(-X) \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$, and the sequence can be rewritten as

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0 \tag{17.20}$$

From the short exact sequence, we get the long exact sequence as follows:

$$\begin{aligned} 0 \to H^0(\mathbb{P}^2, \mathcal{O}(-d)) &\to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^0(X, \mathcal{O}_X) \\ \to H^1(\mathbb{P}^2, \mathcal{O}(-d)) &\to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^1(X, \mathcal{O}_X) \\ \to H^2(\mathbb{P}^2, \mathcal{O}(-d)) &\to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0. \end{aligned}$$

Using the results on cohomology of line bundles on $\mathbb{P}^2$, we deduce the equality $H^0(X, \mathcal{O}_X) \simeq k$ and hence

$$H^1(X, \mathcal{O}_X) \simeq k^{(d-1)(d-2)/2}.$$

The dimension of the cohomology group on the left is the *genus* of the curve X (it will be introduced properly in Chapter **??**). So the above can be rephrased as saying *the genus of a plane curve of degree* d *is* $(d-1)(d-2)/2$.

Tensoring the sequence (17.20) by $\mathcal{O}_{\mathbb{P}^2}(m)$, we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow i_*\mathcal{O}_X(m) \longrightarrow 0$$

and the long exact sequence gives that the Hilbert polynomial of $\mathcal{O}_X$ equals

$$P(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = dm - \frac{d^2-3d}{2}.$$

Example 17.32. A plane curve of degree 1, i.e., a projective line, is isomorphic to $\mathbb{P}^1_k$, hence the genus is zero.

A plane curve of degree 2, i.e., a projective conic, also has genus 0. In case the case k is algebraically closed this is clear, because then it is isomorphic to $\mathbb{P}^1_k$.

A plane curve of degree 3, i.e., an elliptic curve, has genus 1. It follows for instance that the curve

$$X = V(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}_k^2$$

is not isomorphic to $\mathbb{P}_k^1$.

17.11 Example: The twisted cubic in $\mathbb{P}^3$

Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will continue Example 16.31 and consider the twisted cubic curve $X = V(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Let us compute the group $H^1(X, \mathcal{O}_X)$. Of course we know what the answer should be, because $X \simeq \mathbb{P}^1$, and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.

Now, to compute $H^1(X, \mathcal{O}_X)$ on X, it is convenient to relate it to a cohomology group on $\mathbb{P}^3$. We have $H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$ where $i : X \to \mathbb{P}^3$ is the inclusion. The sheaf $i_*\mathcal{O}_X$ fits into the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow i_*\mathcal{O}_X \to 0$$

where $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^3$. Applying the long exact sequence in cohomology, we get

$$\cdots \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(\mathbb{P}^3, i_*\mathcal{O}_X) \longrightarrow H^2(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow \cdots$$

By our description of sheaf cohomology on $\mathbb{P}^3$, $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$, which implies that $H^1(X, \mathcal{O}_X) = H^2(\mathbb{P}^3, \mathcal{I})$. We can compute the latter cohomology group using the exact sequence of Example 16.31:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{I} \to 0.$$

Now, taking the long exact sequence we get

$$\cdots \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}(-3)^2) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \longrightarrow H^2(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}(-3)^2) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \longrightarrow H^3(\mathbb{P}^3, \mathcal{I}).$$

Here $H^2(\mathbb{P}^3, \mathcal{O}(-2)) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(-3)) = 0$ by our prevbased computations. Hence by exactness, we find $H^2(\mathbb{P}^3, \mathcal{I}) = 0$. It follows that $H^1(X, \mathcal{O}_X) = 0$ also, as expected.

Exercise 17.11.1. Prove Lemma 17.22 in more detail.

Exercise 17.11.2. Using the sequences above, show that

- $H^0(\mathbb{P}^3, \mathcal{I}(2)) = k^3$ (find a basis!)
- $H^1(\mathbb{P}^3, \mathcal{I}(m)) = 0$ for all $m \in \mathbb{Z}$.
- $H^2(\mathbb{P}^3, \mathcal{I}(-1)) = k$.

17.12 Extended example: Hyperelliptic curves

Let us recall the hyperelliptic curves defined in Chapter 7.

Let k be a field. For an integer $g \geqslant 1$, we consider the scheme X glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} \text{ and } B = \frac{k[u,v]}{(-v^2 + a_{2g+1}u + \cdots + a_1 u^{2g+1})}.$$

and before, we glue $D(x) \subset U$ to $D(u) \subset V$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

Let us compute the Cech cohomology groups of $\mathcal{O}_X$ with respect to the affine covering $\mathcal{U} = \{U, V\}$ above. Viewing the ring A as a $k[x]$-module, we can write

$$\frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} = k[x] \oplus k[x]y$$

and similarly $B \simeq k[u] \oplus k[u]v$ as a $k[u]$-module.

As $\mathcal{U}$ has only two elements, the Cech complex of $\mathcal{O}_X$ has only two terms, $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ and $\mathcal{O}_X(U \cap V)$ and the differential between them,

$$d^0\colon\ (k[x] \oplus k[x]y) \oplus \big(k[x^{-1}] \oplus k[x^{-1}]x^{-g-1}y\big) \to k[x^{\pm 1}] \oplus k[x^{\pm 1}]y,$$

is given by by the assignment

$$\begin{aligned} d^0(p(x) + q(x)y, r(x^{-1}) + s(x^{-1})x^{-g-1}y) \\ = p(x) - r(x^{-1}) + (q(x) - s(x^{-1})x^{-g-1})y. \end{aligned}$$

Comparing monomials $x^m y^n$ on each side, we deduce that

$$H^0(X, \mathcal{O}_X) = \operatorname{Ker} d^0 = k$$

and

$$H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^0 = k\{yx^{-1}, yx^{-2}, \ldots, yx^{-g}\} \simeq k^g.$$

In particular, $\dim_k H^1(X, \mathcal{O}_X) = g$. The latter invariant is usually referred to as the *arithmetic genus* of a curve; we have shown that the hyperelliptic curve X has arithmtic genus g.

For $g = 2$, we get a particularly interesting curve – an irreducible projective curve which cannot be embedded in $\mathbb{P}^2$. Indeed, we showed that for any irreducible curve in $\mathbb{P}^2$ of degree d and the corresponding arithmetic genus equals $\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$. However, there is no integer solution to $\frac{1}{2}(d-1)(d-2) = 2$. This implies the following:

Proposition 17.33. There exist non-singular projective curves which cannot be embedded in $\mathbb{P}^2$.

Note that we still haven't proved that X is projective. Actually, it is not hard to see that X can be embedded into the *weighted* projective space $\mathbb{P}(1,1,g+1) = \operatorname{Proj} k[x_0, x_1, w]$ given by the equation

$$w^2 = a_{2g+1}x_0^{2g+1}x_1 + \cdots + a_1 x_0 x_1^{2g+1}. \tag{17.21}$$

Note that this makes sense if w has degree $g+1$, but it does not define a subscheme of $\mathbb{P}^2$.

17.13 Bezout's theorem

Let k be an algebraically closed field. Let C and D be two curves in $\mathbb{P}^2_k$ of degrees d and e respectively. We assume here that C and D have no common component, so that Z is a 0-dimensional subscheme.

Let us compute the cohomology group $H^0(Z, \mathcal{O}_Z)$. If we assume $Z = \{x_1, \ldots, x_r\}$ is contained in $D(x_0) \simeq k[x,y]$ (which we may arrange by a linear coordinate change), then

$$\mathcal{O}_Z(Z) = \bigoplus_{i=1}^r \mathcal{O}_{Z,x_i} = \bigoplus_{i=1}^r \left(\frac{k[x,y]}{(f,g)}\right)_{\mathfrak{m}_{x_i}} \tag{17.22}$$

where f, g are the dehomogenized equations for C and D. In other words, $\dim_k H^0(Z, \mathcal{O}_Z)$ is the sum of the *multiplicities* at the points x_i:

$$\dim H^0(Z, \mathcal{O}_Z) = \sum_{i=1}^r \dim_k \left(\frac{k[x,y]}{(f,g)}\right)_{\mathfrak{m}_{x_i}}.$$

On the other hand, we can compute $H^0(Z, \mathcal{O}_Z)$ using the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0,$$

and we deduce that $\dim_k H^0(Z, \mathcal{O}_Z) = \dim_k H^1(\mathbb{P}^2, \mathcal{I}_Z) - 1$, and

$$H^2(\mathbb{P}^2, \mathcal{I}) = H^1(\mathbb{P}^2, i_*\mathcal{O}_Z) = 0$$

because Z has dimension 0. We proceed to study the cohomology group $H^1(\mathbb{P}^2, \mathcal{I}_Z)$. Recall the exact sequence from Section 16.30,

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_Z \to 0.$$

Taking the long exact sequence of cohomology we obtain

$$0 \to H^1(\mathbb{P}^2, \mathcal{I}) \to H^2(\mathbb{P}^2, \mathcal{O}(-d-e) \to H^2(\mathbb{P}^2, \mathcal{O}(-d)) \oplus H^0(\mathbb{P}^2, \mathcal{O}(-e)) \to 0.$$

From which we get the pleasant conclusion that

$$\dim_k H^0(Z, \mathcal{O}_Z) = \dim_k H^1(\mathbb{P}^2_k, \mathcal{I}_Z) + 1 \tag{17.23}$$
$$= \dim_k H^2(\mathcal{O}(-d-e) - \dim_k H^2(\mathcal{O}(-d)) - \dim_k H^2(\mathcal{O}(-e)) \tag{17.24}$$
$$= \binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2} + 1 \tag{17.25}$$
$$= de$$

In other words, we have proved Bezout's theorem for $\mathbb{P}^2_k$:

$$\sum_{i=1}^{r} \dim_k \left(\frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}} = de$$

17.14 Extended example: Non-split locally free sheaves

A locally free sheaf is said to be *split* if it is isomorphic to a direct sum of invertible sheaves. We have seen several examples of locally free sheaves that are not free, even on affine schemes, but a priori it is not so clear whether these are direct sums of invertible sheaves. In this section we will study the sheaf $\mathcal{E}$ from Section **??** and show that it is indeed non-split.

The sheaf $\mathcal{E}$ is the locally free sheaf of rank n on $\mathbb{P}^n_k$ sitting in the exact sequence (**??**)

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}^{n+1}_{\mathbb{P}^n_k} \to \mathcal{E} \to 0.$$

Suppose that $\mathcal{E}$ is not split, i.e., $\mathcal{E}$ is not isomorphic to a direct sum of invertible sheaves. Since $\operatorname{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$ is generated by the class of $\mathcal{O}(1)$, this would mean that $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_k}(a_n)$ for some integers $a_1, \ldots a_n \in \mathbb{Z}$.

Recall that for $n \geqslant 2$, we have $H^{n-1}(\mathbb{P}^n_k, \mathcal{O}(m)) = 0$ for any $m \in \mathbb{Z}$. So if we could show that $H^{n-1}(\mathbb{P}^n_k, \mathcal{E}) \neq 0$, we would have a contradiction. Actually, it is the case that $H^{n-1}(\mathbb{P}^n_k, \mathcal{E}) = 0$, but we can instead consider $\mathcal{F} = \mathcal{E}(-n)$, which fits into the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \to \mathcal{O}_{\mathbb{P}^n_k}(-n)^{n+1} \to \mathcal{F} \to 0.$$

Taking the long exact sequence in cohomology, we get

$$\cdots \to H^{n-1}(\mathcal{O}^{n+1}_{\mathbb{P}^n_k}) \to H^{n-1}(\mathcal{F}) \xrightarrow{\delta} H^n(\mathcal{O}_{\mathbb{P}^n_k}(-n-1)) \to H^n(\mathcal{O}^{n+1}_{\mathbb{P}^n_k}) \to \cdots$$

Here the two outer cohomology groups are zero, by Theorem 17.24. Hence, by exactness, we find that $H^{n-1}(\mathbb{P}^n_k, \mathcal{F}(-1)) \simeq H^0(\mathbb{P}^n_k \mathcal{O}_{\mathbb{P}^n_k}) = k$. This implies that $\mathcal{F} = \mathcal{E}(-n)$, and hence $\mathcal{E}$ cannot be a sum of invertible sheaves, and we are done.

The above gives an example of a non-split locally free sheaf of rank n. However, coming up with examples of non-split sheaves of *low rank* on projective space is a famously difficult problem. In fact, a famous conjecture of Hartshorne says that any rank 2 vector bundle on $\mathbb{P}^n$ for $n \geqslant 5$ is split.

Exercise 17.14.1. Let $X \subset \mathbb{P}^5$ denote a quadric hypersurface (i.e., $X = V(q)$ for a homogeneous degree 2 polynomial). Recall the exact sequence 19.4

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1)^4 \to \mathcal{O}^4_{\mathbb{P}^5} \to i_*\mathcal{E} \to 0$$

where $\mathcal{E}$ is a locally free sheaf of rank 2.

Use this exact sequence to show that $\mathcal{E}$ is not split.

Exercise 17.14.2. Let $n > 0$ be an integer and consider the integral projective scheme $X = \operatorname{Proj}(R)$, where R is the ring

$$R = k[x, y, z, w]/(x^2, xy, y^2, u^n x - v^n y).$$

a) Show that X is irreducible, non-reduced, and of dimension 1.
b) Compute $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$.

18

Divisors and linear systems

18.1 Weil divisors

Let X be a normal integral Noetherian scheme scheme. A *prime divisor* is an integral subscheme $Z \subset X$ of codimension 1. A *Weil divisor* is a finite formal sum

$$D = \sum_i n_i Z_i$$

where $n_i \in \mathbb{Z}$ and the Z_i's are prime divisors. The group of Weil divisors will be denoted by $\mathrm{Div}(X)$.

We say that D is *effective* if all $n_i \geqslant 0$ and call $\bigcup_i Z_i$ the *support* of D. This makes $\mathrm{Div}(X)$ into a partially ordered group: Given two Weil divisors $D = \sum_Z n_Z Z$ and $D' = \sum m_Z Z$, we say that $D \geqslant D'$ if $D - D'$ is effective, or equivalently, that $n_Z \geqslant m_Z$ for all prime divisors Z.

Example 18.1. On $X = \mathbb{P}^1_k$, the prime divisors are simply the closed points. Here are some examples of Weil divisors:

$$D_1 = 3 \cdot (1:0) - 5 \cdot (0:1), \qquad D_2 = (1:1) + 5 \cdot (0:1)$$
$$D_1 + D_2 = 3 \cdot (1:0) + (1:1).$$

The reader may wonder why we include the assumption that X is normal. Certainly, the above definition can be made for any scheme, but the concept of a Weil divisor is not particularly useful without this assumption. There are two main reasons reasons for why normality is a natural assumption:

(i) There is a well-behaved notion of 'multiplicity'. This in turn leads to the notion of the divisor associated to a rational function.
(ii) The fact that a pure codimension 1 subscheme is determined by its underlying irreducible components and their multiplicities (Proposition 18.4)

We will explain these two points in the next section.

18.2 Local rings on normal schemes

Let X be a normal integral Noetherian scheme with fraction field $K = k(X)$. By definition, X is normal if all the local rings $\mathcal{O}_{X,x}$ are integrally closed in their fraction field, namely K. One of the most important features of such a scheme is that it is *regular in codimension one*. This means that if ξ is a codimension 1 point, the local ring $A = \mathcal{O}_{X,\xi}$ is a regular local ring of dimension 1. This means that A is also a discrete valuation ring (see Appendix XXX). In particular, any nonzero ideal in A is of the form $\mathfrak{m}^n$, where $\mathfrak{m}$ is the maximal ideal of A.

This algebraic fact has the following geometric implication. Let $Z \subset X$ be a prime divisor of X with generic point ξ. If Y is any codimension 1 subscheme, defined by a coherent ideal sheaf $\mathcal{I}$, then ideal sheaf $\mathcal{I}\mathcal{O}_{X,\xi}$ equals a power $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,\xi}$. We call n the *multiplicity* of Y at ξ or the *multiplicity* of Y in Z and shall write $n_Z(Y)$ for it. Certainly it can happen that $n_Z(Y) = 0$, but this happens only when Z is a not a component of Y. In this way we may associate the Weil divisor

$$[Y] = \sum_Z n_Z(Y)Z,$$

where the summation runs over all prime divisors Z in X. Since X is Noetherian, Y has only finitely many irreducible components, and so the sum is finite.

Example 18.2. On $X = \mathbb{P}^1_k$, the subscheme Y defined by the ideal $I = (x_0^2(x_0 + x_1))$ is of codimension 1, and it has two irreducible components: one as associated to $I_1 = (x_0^2)$ and one to $I_2 = (x_0 + x_1)$. Note that I_1 is supported at the origin $(0 : 1) \in D_+(x_1) = \operatorname{Spec} k[t]$ with $t = x_1/x_0$. The local ring at $(0 : 1)$ equals $\mathcal{O}_{\mathbb{P}^1_k,(0:1)} = k[t]_{(t)}$ and the ideal sheaf $\tilde{I}$ localizes to the ideal $(t^2(t + 1)) = (t)^2$ (since $t + 1$ is unit in this ring). Hence Y has multiplicity 2 at $(0 : 1)$ and

$$[Y] = 2(0 : 1) + (1 : -1).$$

On $\mathbb{P}^1_k$, the subscheme is in fact uniquely determined by the divisor; any codimension 1 subscheme is given by a homogeneous principal ideal

$$I = ((a_1x_0 - b_1x_1)^{n_1} \cdots (a_rx_0 - b_rx_1)^{n_r}) \subset k[x_0, x_1],$$

so knowing the closed points $(a_1 : b_1), \ldots, (a_r : b_r)$ and the exponents n_i determines the ideal I uniquely.

Example 18.3. Consider the nodal cubic curve $X \subset \mathbb{A}^2_k$ given by the equation $y^2 = x^2(x + 1)$. Cutting X with the x-axis we get the subscheme $\operatorname{Spec} k[x, y]/(x^2, y)$, whereas cutting it with the y-axis yields $k[x, y]/(x, y^2)$. The two subschemes are both supported at the origin, with multiplicity 2, but they are different.

The issue in the last example dissears if we make the assumption that X is normal. This is the first reason why normal schemes are desirable; that there is one-to-one correspondence between subschemes of pure codimension one and effective Weil divisors. (Recall that a subscheme is of *pure codimension one* if all its irreducible components are of codimension one.)

Proposition 18.4. Let X be a normal integral Noetherian scheme. Let Y and Y' be two subschemes of pure codimension one. Then $Y = Y'$ if and only if they define the same Weil divisor.

Proof 'Being equal' is a local property for closed subschemes: If $\{U_i\}$ is an open cover and $Z \cap U_i = Z' \cap U$ for all i, it holds that $Z = Z'$. We may therefore assume that X is affine say $X = \operatorname{Spec} A$, where A is a Noetherian normal domain. The point is that if $\mathfrak{p}$ is a height one prime in A, the only $\mathfrak{p}$-primary ideals are the symbolic powers $\mathfrak{p}^{(\nu)}$ for $\nu \geqslant 0$ (see for instance [??]). Don't get scared by these seemingly occult powers, their important feature is that $\mathfrak{p}^{(v)}$ is the only $\mathfrak{p}$-primary ideal which defines a subscheme whose multiplicity at $\mathfrak{p}$ is n.

That a closed subscheme $Y = V(\mathfrak{a})$ of X is of pure codimension one, means that all the associated primes of $\mathfrak{a}$ are of height one; and in view of the discussion above, the primary decomposition of $\mathfrak{a}$ takes the form $\mathfrak{a} = \mathfrak{p}_1^{(\nu_1)} \cap \cdots \cap \mathfrak{p}_r^{(\nu_r)}$, where the ν_i's are exactly the non-zero multiplicities $n_Z(Y)$ with Z a prime divisor. So two ideals with the same multiplicities are equal. □

Orders of vanishing

A second useful fact about the local ring $A = \mathcal{O}_{X,\xi}$ at a codimension one-point ξ is that it is a discrete valuation ring, so there is an assocated valuation map $v : K \to \mathbb{Z} \cup \{\infty\}$ such that $A = v^{-1}(\mathbb{Z}_{\geqslant 0} \cup \{\infty\})$. More explicitly, we can define v as follows. Given a nonzero $a \in A$, we can write it as $a = u \cdot t^m$, where t is the generator for the maximal ideal $\mathfrak{m} \subset A$, and define the valuation of a at ξ to be $v(r) = m$. Alternatively, can define

$$v(r) = \operatorname{length}(A/a)$$

for $a \in A$. (The latter definition extends to singular curves as well.) Finally, we extend this to elements of $K^\times$ by setting $v(ab^{-1}) = v(a) - v(b)$. This gives a map

$$v : K^\times \to \mathbb{Z}$$

We finally extend this to include 0 as well, by defining $v(0) = \infty$.

If $Z \subset X$ is an integral subscheme of codimension 1, and ξ is the generic point, we define the *order of vanishing* of a rational function $f \in K$ along Z to be the number

$$\operatorname{ord}_Z(f) = v(f)$$

where v is the valuation of $\mathcal{O}_{X,\eta}$.

If $X = \operatorname{Spec} A$ is affine, the group of Weil divisors $\operatorname{Div}(\operatorname{Spec} A)$, being the free abelian group on the prime divisors, has a basis consisting of the divisors $V(\mathfrak{p})$ for $\mathfrak{p}$ running through the primes of height one. Then each $A_\mathfrak{p}$ is a discrete valuation ring, so $f \in K$ has order of vanishing along $V(\mathfrak{p})$ if and only if $f \in A_\mathfrak{p}^\times$. In light of Hartog's theorem, which tells us that $\bigcap_\mathfrak{p} A_\mathfrak{p}^\times = A^\times$, we conclude that $\operatorname{ord}_{V(\mathfrak{p})}(f) = 0$ if and only if $f \in A^\times$.

Lemma 18.5. Let A be a Noetherian normal domain, and $f \in K(A)$. Then

- $\operatorname{ord}_{V(\mathfrak{p})} f \geqslant 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $f \in A$; and
- $\operatorname{ord}_{V(\mathfrak{p})} f = 0$ if and only if $f \in A^\times$.

This leads to the definition of the divisor of a non-zero rational function:

Definition 18.6 (Principal divisors)**.** If $f \in K^\times$, we define its corresponding Weil divisor as

$$\operatorname{div}(f) = \sum_Z \operatorname{ord}_Z(f) Z, \tag{18.1}$$

where the sum runs over all prime divisors of codimension one.
Divisors of the form $\operatorname{div}(f)$ are called *principal divisors*, and together with 0 they form a subgroup of $\operatorname{Div}(X)$.

There are only finitely many nonzero terms in the sum (18.1) thanks to the following lemma:

Lemma 18.7. Suppose that X is a normal integral Noetherian scheme scheme with fraction field K and let $f \in K^\times$. Then $\operatorname{ord}_Z(f) = 0$ for all but finitely many prime divisors Z.

Proof We first reduce to the case when X is affine. Let $U = \operatorname{Spec} A$ be any open affine subset such that $f|_U \in \Gamma(U, \mathcal{O}_X)$. Since X is Noetherian and integral, the complement $W = X - U$ is a closed subset of X of codimension at least one, which has finitely many irreducible components; in particular, only finitely many prime divisors Z are supported in W. So we reduce to the affine case $X = \operatorname{Spec} A$ and $f \in \Gamma(X, \mathcal{O}_X) = A$, by ignoring these finitely many components. Then $\operatorname{ord}_Z(f) \geqslant 0$ automatically, and $\operatorname{ord}_Z(f) > 0$ if and only if Z is contained in $V(f)$; and since $V(f)$ has only finitely many irreducible components of codimension one, we are done. □

The proof of the lemma above shows where we make use of some of the finiteness assumptions on our schemes. Unfortunately, there is no getting around it, as Example 18.13 below shows.

Example 18.8. On $X = \operatorname{Spec} \mathbb{Z}$, a Weil divisor is an expression of the form

$$D = n_1 V(p_1) + \cdots + n_1 V(p_r)$$

where the p_i are prime numbers. In the function field $K = \mathcal{Q}$ of $\operatorname{Spec} \mathbb{Z}$, the 'rational function' $f = p_1^{n_1} \cdots p_r^{n_r}$ satisfies $\operatorname{div} f = D$. Thus any divisor is principal on $X = \operatorname{Spec} \mathbb{Z}$.

Example 18.9. Let k be algebraically closed, and consider $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$. Let $K = k(t)$. Here prime divisors in X correspond to closed points $[a] \in \mathbb{A}^1_k$ associated to maximal ideals $(t-a)$ in $k[t]$. Consider the rational function

$$f = t^2(t-1)(t+1)^{-1} \in K.$$

Then f is invertible in all the local rings $\mathcal{O}_{X,[a]}$ except when $a = 0, \pm 1$. When $a = 1$, $t-1$ defines the uniformizer of $\mathcal{O}_{X,[1]} = k[t]_{t-1}$, and we can write f as $(t-1)\cdot$ (unit). Hence the order of vanishing is 1 at the point $\xi = [1]$. Similarly, in $\mathcal{O}_{X,[0]}$, f has the form $t^2(unit)$, and in $\mathcal{O}_{X,[-1]}$, $f = (t+1)^{-1}(unit)$. Thus all the non-zero orders of vanishing are

$$\operatorname{ord}_{[0]}(f) = 2, \quad \operatorname{ord}_{[1]}(f) = 1, \quad \operatorname{ord}_{[-1]}(f) = -1.$$

and the divisor of f is $2[0] + [1] - [-1]$.

In fact, when k is algebraically closed, every Weil divisor on $\mathbb{A}^1_k$ is principal. Indeed, if the divisor is $D = \sum_{i=1}^n n_i[\alpha_i]$ where $n_i \in \mathbb{Z}$ and $\alpha_i \in k$, then the rational function

$$f = \prod_{i=1}^n (t - \alpha_i)^{n_i}$$

has $\operatorname{div}(f) = D$. We will prove a generalization in Corollary 18.19.

Example 18.10. Consider the projective line $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, whose function field is $K = k(t)$ where $t = x_1/x_0$. Consider the function $f = t^2(t-1)^{-1} \in K$. To find the divisor of f, we treat the two affine charts $D_+(x_0)$ and $D_+(x_1)$ separately:

On $U = D_+(x_0) = \operatorname{Spec} k[t]$, f defines an element in $\mathcal{O}_{X,p}$ for every $p \neq [1]$, and it is invertible for every $p \neq [0], [1]$. At the point $p = [0]$, the local ring equals $\mathcal{O}_{X,p} = k[t]_{(t)}$, and since $t - 1$ is invertible here, we have

$$f = t^2(t-1)^{-1} = t^2(\text{unit})$$

Thus f has order of vanishing 2. Similarly, we find $\operatorname{ord}_{[1]}(f) = -1$.

On the open chart $U = D_+(x_1) = \operatorname{Spec} k[u]$, where $u = x_0/x_1 = t^{-1}$, we may write $f = u^{-2}(u^{-1} - 1) = (u - u^2)^{-1}$. The only non-zero valuations are: $\operatorname{ord}_{[0]} = -1$ and $\operatorname{ord}_{[1]} = -1$. Note that the point $[1] \in D_+(x_1)$ is the point $(1:1)$ which we found also in $D_+(x_0)$ above. It follows that the divisor of f is given by

$$\operatorname{div}(f) = 2(1:0) - (1:0) - (1:1).$$

Example 18.11. One may consider the function from Example 18.9 as a rational function on $\mathbb{P}^1_k$. In addition to the prime divisors $(1:a)$ lying on $U_0 = \operatorname{Spec} k[t]$, we have the point $(0:1)$ at infinity. The function f will have pole of order two at $(0:1)$, so that

$$\operatorname{div} f = 2(1:0) + (1:1) - (-1:1) - 2(0:1).$$

Indeed, near $(1:0)$ we may use $s = t^{-1}$ as a parameter, and expressed in t, the function f becomes $f = s^{-2}(s^{-1} - 1)(s^{-1} + 1)^{-1} = s^{-2}(1-s)(1+s)^{-1}$, which vanishes to the order two at $s = 0$.

Example 18.12. Let X be the curve $V(x^3 - y^3 + y) \subset \mathbb{A}^2_k$. Then x, y and y/x^2 define rational functions on X. Note that x and y in fact belong to $\mathcal{O}_X(X)$, thus they have no negative orders of vanishing anywhere. Let us find the points $p \in X$ where $\operatorname{ord}_p(x) > 0$.

The function x vanishes exactly at the points in $V(x, x^3 - y^3 + y) \subset \mathbb{A}^2$, i.e., the points $(0,0)$, $(0,1)$, $(0,-1)$. The local ring at the origin $(0,0)$ is isomorphic to

$$\mathcal{O}_{X,(0,0)} = \big(k[x,y]/(x^3 - y^3 + y)\big)$$

In this ring we have $x^3 - y(y^2 - 1) = 0$, and so $y = x^3(unit)$. Hence x is the uniformizing parameter. In particular, $\operatorname{ord}_{(0,0)} x = 1$. Similar computations show that

$$\operatorname{div} x = (0,0) + (0,-1) + (0,1)$$

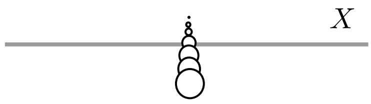
As for y, this can only have non-zero orders of vanishing at the points in $V(x^3 - y^3, y) = V(x, y)$, i.e., at the origin $(0,0)$. We just computed that $y = x^3(unit)$ here, so

$$\operatorname{div} y = 3(0,0)$$

From this we get that

$$\operatorname{div}(y/x^2) = 3(0,0) - 2\left((0,0) + (0,-1) + (0,1)\right) = (0,0) - 2(0,-1) - 2(0,1)$$

Example 18.13. Imitating the construction of the affine line with two origin, we can construct the *affine line X with infinitely many origins*: this scheme is integral, normal, locally Noetherian with fraction field $k(t)$, but there are infinitely many closed points $p \in X$ for which $\operatorname{ord}_p(t) = 1$.



Exercise 18.2.1. Show that all the local rings $\mathcal{O}_{X,p}$ of the curve X given by $y^2 = x^3 - 1$ in $\mathbb{A}^2_k$ are discrete valuation rings, and hence X is a normal variety. We assume that k is algebraically closed and of characteristic different from three and two. More precisely, if (a, b) is a point on X show that $x - a$ is a parameter if $b \neq 0$ and that y is one when $b = 0$. HINT: It holds true that $y^2 - b^2 = x^3 - a^3$.

Exercise 18.2.2. Let $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3 - x)$. Compute the divisors of the rational functions x, y and x^2/y.

18.3 The class group

One of the fundamental invariants of a scheme (or of a ring) is the class group (and its close relative, the Picard group). The term 'class group' comes from algebraic number theory and its origins can be traced back to Kummer's work on Fermat's last theorem. Algebraic number theory was, and to some extent still is, largely devoted to the determination of class groups of the ring of integers in algebraic number fields. One may view the class group as the group giving the obstructions for a divisor being the divisor of a rational function.

Definition 18.14 (The class group)**.** We define the *class group of X* as the group of Weil divisors modulo the principal divisors, i.e.,

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\langle \operatorname{div} f \mid f \in K^\times \rangle$$

Two Weil divisors D and D' are said to be *linearly equivalent* (written $D \sim D'$) if they have the same image in $\operatorname{Cl}(X)$, or equivalently, that $D - D'$ is principal.

When A is a ring, we usually write $\operatorname{Cl}(A)$ for the class group $\operatorname{Cl}(\operatorname{Spec} A)$. This group generalizes the ideal class group $\operatorname{Cl}(A)$ of algebraic number theory, which is usually studied when A is a Dedekind domain. This ring is designed to measure how far A is from being a unique factorization domain.

Example 18.15. It follows from Example **??** that any Weil divisor on $\mathbb{A}^1_k$ is principal, hence $\operatorname{Cl}(\mathbb{A}^1_k) = 0$.

Example 18.16. Similarly, it follows from Example 18.8 that $\operatorname{Cl}(\operatorname{Spec} \mathbb{Z}) = 0$.

Example 18.17. On the other hand, $\mathbb{Z}[\sqrt{-5}]$ is not an UFD, because $2 \cdot 3 = (1 - \sqrt{-5})(1 + \sqrt{-5}))$, and in fact $\mathrm{Cl}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2\mathbb{Z}$.

When $X = \operatorname{Spec} A$ is affine the divisor $\operatorname{div}(f)$ of a rational function f is given as

$$\operatorname{div}(f) = \sum \operatorname{ord}_{\mathfrak{p}}(f) V(\mathfrak{p})$$

Hence $\operatorname{div} f = 0$ if and only if $\operatorname{ord}_{\mathfrak{p}}(f) = 0$ for all height one primes, or, by Lemma 18.5 if and only if $f \in A^{\times}$. Thus the kernel of the map $\operatorname{div}\colon K^{\times} \to \operatorname{Div}(\operatorname{Spec} A)$ equals $A^{\times}$, and the cokernel is by definition the class group $\mathrm{Cl}(A)$. Hence we have the exact sequence

$$0 \to A^{\times} \to K^{\times} \xrightarrow{\operatorname{div}} \operatorname{Div}(\operatorname{Spec} A) \to \mathrm{Cl}(A) \to 0. \qquad (18.2)$$

Proposition 18.18. Let A be a normal Noetherian integral domain and let $X = \operatorname{Spec} A$. Then the following are equivalent:
(i) $\mathrm{Cl}(X) = 0$;
(ii) Every height one prime ideal in A is principal;
(iii) A is a unique factorization domain.

Proof The equivalence of (ii) and (iii) is just Theorem **??**, so the task is to show that statement (i) is equivalent to one of the two other statements; we shall show the equivalence (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (i): If $Z \subset X$ is a prime divisor in $\operatorname{Spec} A$, then $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$ of height one. By assumption, we therefore have $Z = V(f)$ for an element $f \in A$, that is, $Z = \operatorname{div}(f)$, and so $\mathrm{Cl}(X) = 0$.

(i)$\Rightarrow$ (ii): Assume that $\mathrm{Cl}(X) = 0$. Let $\mathfrak{p}$ be a prime of height one, and let $Z = V(\mathfrak{p}) \subset X$. By assumption, there is an $f \in K^{\times}$ such that $\operatorname{div}(f) = Z$. We want to show that in fact $f \in A$ and that $\mathfrak{p} = (f)$. But the first point follows from the exact sequence (18.2), since $\operatorname{ord}_{\mathfrak{q}}(f) = 0$ for $\mathfrak{q} \neq \mathfrak{p}$ and $\operatorname{ord}_{\mathfrak{p}}(f) = 1$, and so f lies in $\{a \in K^{\times} \mid \operatorname{ord}_{\mathfrak{p}}(a) \geqslant 0\} = A$.

Finally, to prove that f generates $\mathfrak{p}$, consider any element $g \in \mathfrak{p}$. Then $\operatorname{ord}_{\mathfrak{p}}(g) \geqslant 1$ and $\operatorname{ord}_{\mathfrak{q}}(g) \geqslant 0$ for all $\mathfrak{q} \neq \mathfrak{p}$. It follows that $\operatorname{ord}_{\mathfrak{q}}(g/f) = \operatorname{ord}_{\mathfrak{q}}(g) - \operatorname{ord}_{\mathfrak{q}}(f) \geqslant 0$ for *all* prime ideals $\mathfrak{q} \in \operatorname{Spec} A$. Hence $g/f \in A_{\mathfrak{q}}$ for all primes $\mathfrak{q}$ of height one, and hence $g/f \in A$, by Hartog's theorem (Theorem **??**). It follows that $g \in fA$, and so $\mathfrak{p} = (f)A$ is principal. ☐

In particular, since $A = k[x_1, \ldots, x_n]$ is a unique factorization domain, we get

Corollary 18.19. The class group of affine n-space over a field is trivial, i.e.,

$$\mathrm{Cl}(\mathbb{A}^n_k) = 0.$$

Example 18.20. Let A be a discrete valuation ring and let $X = \operatorname{Spec} A$. If $x \in X$ denotes the closed point, we have $\operatorname{Div}(X) = \mathbb{Z} \cdot x$. Any Weil divisor on X is principal; if t is a generator for the maximal ideal of A, then $\operatorname{div}(t^n) = n \cdot x$. Hence $\mathrm{Cl}(X) = 0$, in accordance with Proposition 18.18.

Example 18.21. Consider the ideal $\mathfrak{p} = (2, 1 + \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$. One easily checks that $\mathfrak{p}$

is a prime ideal, so that $Y = V(\mathfrak{p})$ is a prime divisor in $\operatorname{Spec} \mathbb{Z}[\sqrt{-5}]$. A small computation shows that the square $\mathfrak{p}^2$ is principal and generated by 2. Thus $2Y = \operatorname{div} 2$, and the class of Y in $\operatorname{Cl}(\mathbb{Z}[\sqrt{-5}])$ is two-torsion. Using the norm, one sees that Y is not principal, so its class is a genuine two-torsion element in $\operatorname{Cl} \mathbb{Z}[\sqrt{-5}]$. For a continuation of this example see page 331.

Exercise 18.3.1. Consider the curve $y^2 = x^3 - 1$ in $\mathbb{A}^2_k$ where k is algebraically closed of characteristic different from two and three. If $(a, b) \in X$ we let $\sigma(a, p) = (a, -b)$, which also lies in X.

a) Show that for any $P \in X$, it holds that $-P \sim \sigma(P)$;
b) Show that if P, Q and R are three collinear points on X, then $P + Q + R \sim 0$;
c) Show that any Weil divisor on X is linearly equivalent to a prime divisor.

18.4 Projective space

Write $\mathbb{P}^n_k = \operatorname{Proj} R$, with $R = k[x_0, \ldots, x_n]$. Prime divisors on $\mathbb{P}^n_k$ are given by homogeneous height one prime ideals in R, that is, ideals $\mathfrak{p} = (g)$ where g is a nonzero homogeneous irreducible polynomial. The generator g is unique up to a scalar, so its degree is well defined. We can use this to define the *degree* of a divisor, by taking the sum of degrees of the corresponding polynomials. More precisely, if $D = \sum_i n_i V(g_i)$, we let

$$\deg D = \sum_i n_i \deg g_i.$$

This gives an additive map $\deg\colon \operatorname{Div} \mathbb{P}^n_k \to \mathbb{Z}$.

Now, a rational function $f \in K(\mathbb{P}^n_k)$ is the quotient of two homogeneous polynomials of the same degree. Factoring the numerator and the denominator, we can write it as $f = \prod_i f_i^{n_i}$ where the f_i are different irreducible polynomials in R and the exponents $n_i \in \mathbb{Z}$, and $\sum_i n_i(\deg f_i) = 0$, because f is homogeneous of degree zero. Let us first show that

Lemma 18.22. $\operatorname{div}(f) = \sum n_i[V(f_i)]$.

Proof If $Z \subset \mathbb{P}^n_k$ is a prime divisor, let $\xi \in Z$ be the generic point. Since Z has codimension one, it holds that $Z = V(g)$ for some irreducible polynomial g of some degree d. For any other polynomial h of degree d, the quotient g/h is a generator of the maximal ideal $\mathfrak{m}_\xi \mathcal{O}_{X,\xi}$. We may write $f = (g/h)^r f'$ and f' a unit in $\mathcal{O}_{Z,\xi}$. Clearly $r = n_i$ if f_i divides g (and 0 if no f_i divides g) and f' a rational function which does not contain g in its numerator nor in its denominator. In any case, we get that $\operatorname{div}(f) = \sum n_i[V(f_i)]$. □

In view of the equality $\deg \operatorname{div}(f) = \sum n_i \deg f_i = 0$, the degree map descends to a group homomorphism

$$\deg : \operatorname{Cl}(\mathbb{P}^n) \to \mathbb{Z}$$

We claim that it is an isomorphism. It is clearly surjective since the degree of any hyperplane, for instance $V(x_0)$, equals one. Now, any $Z = \sum n_i[V(f_i)]$ in the kernel of deg, must have $\sum n_i \deg f_i = 0$. Consequently, the element $f = \prod_i f_i^{n_i}$, is homogeneous of degree zero and defines an element of K. By the lemma above, we have $Z = \operatorname{div}(f)$. Hence Z is a principal divisor, and $\deg$ is injective.

We have thus shown:

Proposition 18.23. The degree map gives an isomorphism $\operatorname{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$.

18.5 The sheaf associated to a Weil divisor

Let $D = \sum n_Z Z$ be a Weil divisor on a Noetherian normal and integral scheme X. We would like to form a sheaf, denoted $\mathcal{O}_X(D)$, consisting of the rational functions with 'poles at worst along D'. There are several ways of expressing this, the simplest is to require of f that $\operatorname{ord}_Z(f) \geqslant -n_Z$ for all Z, so that pole order of f along Z in magnitude is bounded by n_Z. Another way is to say that $\operatorname{div}(f) + D$ is an effective Weil divisor; i.e. that $\operatorname{div} f + D \geqslant 0$. Heuristically, one may say that the pole part of $\operatorname{div} f$ is cancelled out by D. Concretely, we define the sheaf $\mathcal{O}_X(D)$ as follows:

Definition 18.24 (The sheaf of a Weil divisor)**.** Let X be a normal integral Noetherian scheme with function field K, and let D be a Weil divisor on X. We define the sheaf $\mathcal{O}_X(D)$ by letting

$$\mathcal{O}_X(D)(U) = \{\, f \in K \mid \operatorname{ord}_Z(f) \geqslant -n_Z \text{ for all } Z \text{ with } \xi_Z \in U \,\}$$

for each open subset $U \subset X$.

The condition $\xi_Z \in U$ simply means that U meets Z; i.e. that $U \cap Z$ is a dense open subset of Z. The condition in the bracket is relaxed when applied to a smaller subset $U' \subset U$, and so we can define the restriction map $\mathcal{O}_X(D)(U) \subset \mathcal{O}_X(D)(U')$ simply by the inclusion.

You should check that this makes $\mathcal{O}_X(D)$ into a sheaf. As such, it is a subsheaf of the constant sheaf $\mathcal{K}$ on the function field K. Moreover, if $a \in \mathcal{O}_X(U)$ is a regular function U, it holds that $\operatorname{ord}_Y(af) = \operatorname{ord}_Y(a) + \operatorname{ord}_Y(f) \geqslant -n_Z$ for all Z and all $f \in \mathcal{O}_X(D)(U)$, and this makes $\mathcal{O}_X(D)$ into an $\mathcal{O}_X$-module. Even more is true, it will be quasi-coherent:

Proposition 18.25. The sheaf $\mathcal{O}_X(D)$ is quasi-coherent.

Proof Let $U = \operatorname{Spec} A \subset X$ be an open affine subset. We claim that for $f \in A$, there is a canonical isomorphism

$$\Gamma(U, \mathcal{O}_X(D))_f = \Gamma(D(f), \mathcal{O}_X(D)).$$

It follows that $\mathcal{O}_X(D)$ restricts to the tilde of the A-module $M = \Gamma(U, \mathcal{O}_X(D))$ on U, and so it is quasi-coherent.

There is always an injective map $\Gamma(\operatorname{Spec} A, \mathcal{O}_X(D))_f \to \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X(D))$. Conversely, take $s \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X(D))$, so that $\operatorname{div} s + D \geqslant 0$ on $\operatorname{Spec} A_f$. This implies that $\operatorname{div} s + D \geqslant 0$ can fail only over $V(f) \subset \operatorname{Spec} A$. But then there is some $n > 0$ so that

$$(\operatorname{div} f^n s + D) \geqslant 0$$

over $\operatorname{Spec} A$. Then s is the image of $(f^n s)/f^n$ on the left-hand side, so the map above is surjective. $\square$

Example 18.26. Let X be the projective line $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ over k and consider the divisor $D = V(x_1) = (1 : 0)$. We have the standard covering of $\mathbb{P}^1_k$ by the distinguished open sets $U_0 = \operatorname{Spec} k[x_1/x_0] = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[x_0/x_1] = \operatorname{Spec} k[s]$ (so $s = t^{-1}$ on $U_0 \cap U_1$). Let us find the global sections of $\mathcal{O}_X(D)$.

Note that the point $(1 : 0)$ does not lie in $U_1 = D_+(x_1)$, and this means that a rational function $f \in K$ such that $\operatorname{div}(f) + D$ is effective on U_1, must be regular on U_1; that is

$$\Gamma(U_1, \mathcal{O}_X(D)) = k[s].$$

Over the open set U_0, we are looking at elements $f \in k(t)$ having order of vanishing at least -1 at $t = 0$. This implies that

$$\Gamma(U_0, \mathcal{O}_X(D)) = \{\, f \mid f = \alpha t^{-1} + p(t) \,\}$$

where $\alpha \in k$ and $p(t)$ a polynomial.

Now, by the usual sheaf sequence, we may think of the elements in $\Gamma(X, \mathcal{O}_X(D))$ as pairs (f, g) with f and g sections of $\mathcal{O}_X(D)$ over U_0 and U_1 respectively, so that $f = g$ on $U_0 \cap U_1$. Here $g = g(s)$ is a polynomial in s, and

$$f(t) = p(t) + \alpha t^{-1} = p(s^{-1}) + \alpha s.$$

If $f = g$ in $k[t, t^{-1}]$, it is clear that p must be a constant. This implies that

$$\Gamma(X, \mathcal{O}_X(D)) = k \oplus k\, t^{-1}.$$

In fact, we will see in a bit that $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$.

Example 18.27. Let A be a discrete valuation ring with uniformizer t, and let $X = \operatorname{Spec} A$. If $x = (t) \in X$ denotes the closed point, we have $\mathcal{O}_X(nx) = t^{-n} \cdot \mathcal{O}_X$. If $f \in K$, then $\operatorname{div} f = \operatorname{ord}_x f \cdot x$. Moreover,

$$\mathcal{O}_X(\operatorname{div} f) = t^{-\operatorname{ord}_x f} \mathcal{O}_X.$$

Lemma 18.28. Let V be a normal integral Noetherian scheme and let D be a Weil divisor. Then D is a principal divisor if and only if $\mathcal{O}_V(D) \simeq \mathcal{O}_V$. Furthermore, if $f \in K$, we have

$$\mathcal{O}_V(\operatorname{div} f) = f^{-1} \cdot \mathcal{O}_V \subset \mathscr{K}$$

as subsheaves of the constant sheaf $\mathscr{K}$.

Proof Suppose first that $D = \operatorname{div} f$ is principal. Let $U \subset V$ be an open subset. Then $\mathcal{O}_U(D)(U)$ consists of the rational functions $g \in K$ such that

$$0 \leqslant \operatorname{div} g + \operatorname{div} f = \operatorname{div}(fg).$$

In other words, $r = f \cdot g$ is a rational function on V with non-negative order of vanishing at every prime divisor $Z \subset V$. Thus $r \in \mathcal{O}_V(U)$ is a regular section by Hartog's theorem. Thus $g = rf^{-1}$ belongs to the right hand side. The opposite containment is clear.

Conversely, suppose there is an isomorphism $\phi : \mathcal{O}_V \to \mathcal{O}_V(D)$. Define $f \in K$ so that

f^{-1} is a generator of $\mathcal{O}_X(D)$ as an $\mathcal{O}_X$-submodule of $\mathscr{K}$, i.e., $\mathcal{O}_X(D) = f^{-1}\mathcal{O}_X \subset \mathscr{K}$. We claim that $D = \operatorname{div} f$. To see this, it suffices to note that if $Z \subset V$ is a prime divisor, then $\operatorname{ord}_Z f$ equals the coefficient of Z in D. This follows from Example 18.27, by taking the stalk of $\mathcal{O}_X(D)$ at the generic point of Z. □

Proposition 18.29. Let X be a normal integral Noetherian scheme and let D be a Weil divisor on X. Then the following are equivalent:

(i) $\mathcal{O}_X(D)$ is locally free (that is, invertible).

(ii) D is locally principal, that is, there is an open covering U_i and rational functions f_i such that

$$D|_{U_i} = \operatorname{div} f_i.$$

Proof First of all, if $\mathcal{O}_X(D)$ is locally free, then it must have rank 1. This is because over an open set $V \subset X - \operatorname{Supp}(D)$, the group $\mathcal{O}_X(D)(V)$ consists of the rational functions such that $\operatorname{ord}_Z(f) \geqslant 0$, for every prime divisor $Z \subset X$, i.e., $f \in \mathcal{O}_X(V)$. That is, $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ over an open set, so it has rank 1.

(i) ⇒ (ii). If $\mathcal{O}_X(D)$ is an invertible subsheaf of $\mathscr{K}$, we can define $f_i \in K$ by picking local generators so that $\mathcal{O}_X(D)(U_i) = f_i^{-1}\mathcal{O}_X(U_i) \subset K$. Then it follows that $D|_{U_i} = \operatorname{div} f_i$ by Lemma 18.28.

(ii) ⇒ (i). If $D|_{U_i}$ is principal, then Lemma 18.28 shows that $\mathcal{O}_X(D)|_{U_i} \simeq \mathcal{O}_X|_{U_i}$, so $\mathcal{O}_X(D)$ is invertible. □

Definition 18.30. Let X denote a normal integral scheme. We say that a Weil divisor D is *Cartier* if it satisfies one of the conditions in Proposition 18.29.

There are differing opinions on what a Cartier divisor 'is', e.g., whether it is a Weil divisor of a special form or whether it is a section of a certain sheaf (see Exercise 18.6.1). In any case, the prototype example of a Cartier divisor is the Weil divisor associated to the section of an invertible sheaf (see Section 18.7).

There are two main reasons for introducing them: (i) Cartier divisors are conveniently described in terms of rational functions on an affine covering (Definition 18.31); and (ii) Cartier divisors have much better formal properties (e.g., with respect to morphisms). We will come back to the latter point in Section 18.9.

Definition 18.31 (Cartier data)**.** Let X be an integral scheme with function field K. We define a set of *Cartier data* by the following data:

(i) an open covering $\{U_i\}_{i\in I}$ of X;

(ii) elements $f_i \in K$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$ for every i, j.

We consider two defining data $\{(U_i, f_i)\}_{i\in I}$ and $\{(V_j, g_j)\}_{i\in I}$ to be equivalent if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathcal{O}_X^\times)$ for all i, j.

Proposition 18.29 shows that any Cartier divisor is specified by a set of Cartier data, and conversely. Moreover, given an affine covering as in (iii) of the Theorem, the invertible sheaf

$\mathcal{O}_X(D)$ has the following description:

$$\mathcal{O}_X(D)|_{U_i} = f_i^{-1}\mathcal{O}_X \subset \mathscr{K}.$$

Two different data (U_i, f_i) and (V_j, g_j) for the same divisor D give rise to the *same* invertible sheaf. This is because over $U_i \cap V_j$, we have $f_i = d_{ij}g_j$ for some units $d_{ij} \in \mathcal{O}_X(U_i \cap V_j)^\times$. This means that $f_i^{-1}\mathcal{O}_{U_i \cap V_j} = g_i^{-1}\mathcal{O}_{U_i \cap V_j}$, and so the sheaf is uniquely determined as a subsheaf of $\mathscr{K}_X$.

The set of Cartier divisors naturally forms a subgroup, denoted $\mathrm{CaDiv}(X)$, of $\mathrm{Div}(X)$. In terms of the Cartier data, the identity is given by $(X, 1)$. Given D and E represented by the data $\{(U_i, f_i)\}_{i \in I}$ and $\{(V_i, g_i)\}_{i \in J}$, then the Cartier data for $D + E$ is given by

$$\{(U_i \cap V_j, f_i g_j)\}_{i,j}$$

Moreover, the inverse $-D$ will be defined as $\{(U_i, f_i^{-1})\}_{i \in I}$. Note also that (U_i, f_i) is a principal divisor if and only if it is equal to (X, f) for some $f \in K^\times$.

The subgroup of Cartier divisors may certainly be smaller than $\mathrm{Div}(X)$, but as we will see shortly, any Weil divisor is Cartier whenever X has mild singularities (Theorem XXX).

Example 18.32. Consider the projective n-space $\mathbb{P}^n_k$ over a field k. Write $\mathbb{P}^n_k = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$.

Any homogeneous polynomial of degree d, $F(x_0, \ldots, x_n) \in R_d$ defines a closed subscheme of $\mathbb{P}^n_k$ of codimension 1. The corresponding Weil divisor D is Cartier. Concretely, we can write down the Cartier data with respect to the standard covering $U_i = D_+(x_i)$ of $\mathbb{P}^n_k$. Note that $F(x/x_i) = F(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$ defines a non-zero regular function on U_i, and the collection

$$(U_i, F(x/x_i))$$

forms a Cartier divisor D on X. Indeed, on the overlap $U_i \cap U_j$, we have the relation

$$F(x/x_i) = (x_j/x_i)^d F(x/x_j)$$

and x_j/x_i is a regular and invertible function on $U_i \cap U_j$. The corresponding invertible $\mathcal{O}_X(D)$ sheaf is isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$. Two homogeneous polynomials F, G of the same degree d give linearly equivalent divisors, because the quotient $F(x)/G(x)$ is a global rational function on $\mathbb{P}^n_k$.

Proposition 18.33. Let X be a normal integral Noetherian scheme scheme and let D and D' be two Cartier divisors. Then:
(i) $\mathcal{O}_X(D + D') \simeq \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$
(ii) $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ if and only if D and D' are linearly equivalent.

Proof We may pick a common covering U_i so that both D and D' are both represented by data (U_i, f_i), (U_i, f_i'). Then $D + D'$ is determined by the Cartier data $(U_i, f_i f_i')$. Locally, over U_i the sheaf $\mathcal{O}_X(D + D')$ is defined as the subsheaf of $\mathscr{K}$ given by $(f_i f_i')^{-1}\mathcal{O}_{U_i} = f_i^{-1} f_i'^{-1}\mathcal{O}_{U_i}$. The tensor product is locally(!) given as $f_i^{-1}\mathcal{O}_{U_i} \otimes f_i'^{-1}\mathcal{O}_{U_i}$, which is clearly isomorphic to $f_i^{-1} f_i'^{-1}\mathcal{O}_{U_i}$ via the map $af_i^{-1} \otimes bf_i'^{-1} \mapsto abf_i^{-1}f_j'^{-1}$.

For the second claim, it suffices (by point (i)) to show that $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ if and only if D is a principal Cartier divisor. But this is a consequence of Lemma 18.28. □

18.6 Divisors and invertible sheaves

By the item (i) and (ii) in Proposition 18.33, we see that the natural map

$$\rho : \operatorname{CaDiv}(X) \to \operatorname{Pic}(X),$$

which sends D to the class of $\mathcal{O}_X(D)$ in $\operatorname{Pic}(X)$ is additive and has the subgroup of principal divisors as its kernel. This means that the induced map $\rho : \operatorname{CaCl}(X) \to \operatorname{Pic}(X)$ is injective. It is also surjective: given any invertible sheaf L on X, we can take a rational section $s \in L(V)$, so that

$$L \simeq \mathcal{O}_X(\operatorname{div} s)$$

We have therefore shown

Corollary 18.34. Let X be a normal integral Noetherian scheme. Then ρ induces an isomorphism

$$\operatorname{CaDiv}(X) \simeq \operatorname{Pic}(X).$$

Having defined Weil and Cartier divisors, it is natural to ask when the two coincide. We will soon see (rather simple) examples where they do not. However, when X has 'mild singularities', any Weil divisor is in fact Cartier.

Proposition 18.35. Let X be an normal integral Noetherian scheme. Then the following are equivalent:

(i) $\operatorname{CaDiv}(X) = \operatorname{Div}(X)$

(ii) X is factorial (all the local rings $\mathcal{O}_{X,x}$ are UFDs).

Proof Note that any Weil divisor is a linear combination of prime divisors, so (i) holds if and only if any such divisor is Cartier. But here the equivalence follows from Theorem 18.47. □

So if X is factorial, every Weil divisor comes from a Cartier divisor, and vica versa. The intuition is that this holds whenever X has 'mild' singularities. For instance, *regular* local Noetherian rings (i.e., $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$) are also UFDs ([Atiyah-MacDonald Ch. 7] or [Stacks 0AG0]). So in particular, the above applies to the main examples of interest:

Corollary 18.36. On a non-singular variety X, then the map $\iota : \operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ is an isomorphism. Moreover, this induces natural isomorphisms between the groups of

(i) Weil divisors (up to linear equivalence)

(ii) Cartier divisors (up to linear equivalence)

(iii) Invertible sheaves (up to isomorphism)

From our previous computation of $\mathrm{Cl}(\mathbb{A}_k^n)$, we get the following theorem:

Theorem 18.37. Let k be a field. Then $\mathrm{Pic}(\mathbb{A}_k^n) = \mathrm{Cl}(\mathbb{A}_k^n) = \mathrm{CaCl}(\mathbb{A}_k^n) = 0$.

We previously computed that $\mathrm{Cl}(\mathbb{P}_k^n) = \mathbb{Z}$, so Corollary 18.36 gives the following:

Corollary 18.38. On $\mathbb{P}_k^n$ any invertible sheaf is isomorphic to some $\mathcal{O}_{\mathbb{P}_k^n}(m)$.

Exercise 18.6.1. Let X be an integral normal scheme and let $\mathscr{K}_X$ denote the constant sheaf on $K = k(X)$. Note that the sheaf $\mathcal{O}_X^\times$ of invertible sections of $\mathcal{O}_X$ embeds as a subsheaf of $\mathscr{K}_X$. Show that a Cartier divisor is the same thing as a global section of the sheaf $\mathscr{K}_X^\times/\mathcal{O}_X^\times$.

Exercise 18.6.2. Check that the inverse of a Cartier divisors and the sum of two are well defined; that is, that all cocycle conditions are fulfilled and that the inverse, respectively the sum, is independent of choices of representatives.

18.7 The Weil divisor associated to a section of an invertible sheaf

We have now come to one of the most important classes of Weil divisors; those assocated to sections of an invertible sheaf. The construction pararells the definition of a principal divisor.

Let L be an invertible sheaf on X and let s be a global section of L. Let U_i be an open cover of X such that L is trivial on each U_i. This means that there are isomorphisms

$$\phi_i : L|_{U_i} \to \mathcal{O}_X|_{U_i}.$$

Let $f_i \in \mathcal{O}_X(U_i)$ be the image of $s|_{U_i}$ via ϕ_i. Since f_i is regular on U_i, we may regard it as an element of the function field $K = k(X)$. We define

$$\mathrm{ord}_Z(s) = \mathrm{ord}_Z(f_i).$$

The number on the right does in fact not depend on the index i. This is because on $U_i \cap U_j$, the two rational functions f_i and f_j are related by $f_j = c_{ji} f_i$ where $c_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ is an invertible section. (See section XXX). Thus if the prime divisor Z belongs to both U_i and U_j, we get that $\mathrm{ord}_Z(f_j) = \mathrm{ord}_Z(f_j)$.

Summing up over all prime divisors, we get a Weil divisor associated to s:

$$\mathrm{div}(s) = \sum_Z \mathrm{ord}_Z(s) Z. \tag{18.3}$$

Some choices have been made on the way, but of course they don't matter, moreover the sum in (18.3) is finite:

Lemma 18.39. The divisor $\mathrm{div}(s)$ is independent of the choice of the open sets U and V and the trivialization ϕ. The sum in (18.3) is finite.

Proof Suppose we are given two trivializations $\phi_\alpha : L|_{U_\alpha} \to \mathcal{O}_{U_\alpha}$ and $\psi : L|_{U_\beta} \to \mathcal{O}_{U_\beta}$. Then over each $U \subset U_\alpha \cap U_\beta$, we have $\phi(s|_U) = c \cdot \psi(s|_U)$ where c is unit in $\mathcal{O}_X(U)$. As $\mathrm{ord}_Z(c) = 0$, we get that $\mathrm{ord}_Z(\phi(s)) = \mathrm{ord}_Z(\psi(s))$.

For the last statment, let U be any affine open set over which L is trivial. There is only finitely many components of the complement U; and replacing $s|_U$ by the image under a trivialization $L|_U \simeq \mathcal{O}_U$, we are back to Lemma 18.7, and so we are done. $\square$

Example 18.40. On $X = \mathbb{P}^1_k$, the monomial $x_0^3 x_1^2$ defines a global section s of the invertible sheaf $L = \mathcal{O}_{\mathbb{P}^1}(5)$. Over $U_0 = D_+(x_0)$, we have trivialization

$$\phi_0 : k[\widetilde{x_1/x_0}]x_0^5 \to k[\widetilde{x_1/x_0}]$$

given by multiplication by x_0^{-5}. Thus $s|_{U_0}$ is transported to the rational function $t^2 = x_1^2/x_0^2$ on U_0, which has order of vanishing two at $(1:0)$. Similarly, the order of vanishing of s at $(0:1)$ is equal to 3. Thus the divisor of s is equal to

$$\operatorname{div} s = 2(1:0) + 3(0:1).$$

Geometrically, the divisor $\operatorname{div}(s)$ equals the Weil divisor $[Y]$ where $Y = V(s)$ is the zero scheme of s. To give some more details, we recall from Section 19.7 that the subscheme $V(s)$ is constructed from the coherent ideal sheaf

$$\mathcal{I} = \operatorname{Im}(s^\vee : L^\vee \to \mathcal{O}_X) \subset \mathcal{O}_X.$$

where $s^\vee$ is the map dual to the map $\mathcal{O}_X \to L$ given by multiplication by s.

Note that the map $s^\vee$ is injective (the restriction of s to each U_i is non-zero and any non-zero map between invertible sheaves on an integral scheme is injective). This implies that the ideal sheaf $\mathcal{I}$ is in fact isomorphic to $L^\vee$. Letting f_i denote the rational functions associated to s, we note that $s^\vee|_{U_i}$ in fact corresponds to the map $\mathcal{O}_{U_i} \to \mathcal{O}_{U_i}$ which sends 1 to f_i. The image of $s^\vee$ over $\mathcal{I}(U_i)$ is therefore the ideal generated by f_i. Thus the Cartier divisor D determined by the f_i is exactly the zero scheme $V(s)$.

In particular, if Z is a prime divisor, we see that the multiplicity of Y along Z is exactly the order of vanishing $\operatorname{ord}_Z(s)$. Thus

$$[Y] = \sum_Z \operatorname{ord}_Z(s) Z.$$

Weil divisors associated to rational sections

It is important to notice that the above constuction can in fact be carried out for a section s of L defined over any subset $V \subset X$. We call such a section a *rational section*. Indeed, if $s \in L(V)$, the trivializations of L still give rational functions f_i (working over the open sets $U_i \cap V$) and we have well-defined orders of vanishing $\operatorname{ord}_Z(s)$ for any prime divisor $Z \subset X$.

The above construction, when $V = X$, always produced Weil divisors which are *effective*; this may no longer be the case when s is only a rational section. Here is a typical example:

Example 18.41. Continuing the example of $X = \mathbb{P}^1_k$, consider the quotient $s = \frac{x_0^3}{x_1}$ which defines a section of $L = \mathcal{O}_{\mathbb{P}^1_k}(2)$ over $D_+(x_1)$, hence a rational section on X. Let us compute the divisor associated to s: Let $t = \frac{x_0}{x_1}$ be the coordinate on $U = D_+(x_1) = \operatorname{Spec} k[t]$.

$$\mathcal{O}_X(2)(U) = k[x_0/x_1]x_1^2 = k[t]x_1^2.$$

So the rational function $f = \phi(s)$ is given by $\frac{x_0^3}{x_1^3} = t^3$ which has non-zero order of vanishing only at the point $t = 0 \in U$, where we have $\operatorname{ord}_t(f) = 3$. To compute $\operatorname{div}(s)$, we must also consider the point outside $D_+(x_1)$. On $U = D_+(x_0)$, we use the coordinate $u = \frac{x_1}{x_0}$, and we have

$$\mathcal{O}_X(2)(U) = k[u]x_0^2.$$

So the rational function $\phi(s)$ is given by $f = \frac{x_0}{x_1} = u^{-1}$. This has order of vanishing $\operatorname{ord}_t(f) = -1$ at $t = 0$ (and $\operatorname{ord}_Z(f) = 0$ at all other points). Hence we obtain

$$\operatorname{div}(s) = 3(0:1) - (1:0).$$

We finish by yet another characterization of when a Weil divisor is Cartier:

Proposition 18.42. Let X be a normal integral Noetherian scheme and let D be a Weil divisor on X. Then D is Cartier if and only if $D = \operatorname{div}(s)$ for some rational section s of an invertible sheaf.

Proof Most of the work here has been done already. First of all, any divisor $D = \operatorname{div}(s)$ of a rational section is clearly locally principal.

Conversely, if D is Cartier, we can consider the invertible sheaf $L = \mathcal{O}_X(D)$, which admits a distinguished rational section s_D of the sheaf $\mathcal{O}_X(D)$. Namely, the element '$1 \in K$' defines an element of $\Gamma(V, \mathcal{O}_X(V))$ over the open set $V = X - \operatorname{Supp}(D)$. We then have

$$\operatorname{div} s_D = D.$$

□

18.8 Subschemes of codimension one and effective divisors

One of the benefits of using zero schemes of sections of invertible sheaves is that they can be defined on any scheme. In this section, we do not assume that X is normal, integral or Noetherian.

The main theme of this section is to study closed subschemes of codimension one. We are particularly interested in the subschemes which are locally defined by a single equation which is a nonzerodivisor, i.e. subschemes whose ideal sheaf is an invertible sheaf. We will see that this is often the case, at least when the ambient scheme X has mild singularities. However, we shall also see simple examples of subschemes which are not possible to define locally by just one equation (see for instance Example 18.10).

For now, let us note the following characterisation of such subschemes:

Proposition 18.43. Let X be a scheme and let $D \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}$. Then the following are equivalent:

(i) $\mathcal{I}$ is an invertible sheaf;

(ii) For every $x \in X$, the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ is principal and generated by a nonzerodivisor;

(iii) There is an open covering U_i of X and nonzerodivisors $f_i \in \mathcal{O}_X(U_i)$ such that f_i generates $\mathcal{I}(U_i)$;

(iv) For every $x \in X$, there is an open affine neighbourhood $U = \operatorname{Spec} A$ of x such that $U \cap D = \operatorname{Spec} A/(f)$ where $f \in A$ is a nonzerodivisor.

(v) D is the zero scheme of a global section s of an invertible sheaf L.

When X is normal integral Noetherian, this is equivalent to

(vi) $[D]$ is an effective Cartier divisor.

Proof Clearly the first three conditions are equivalent; this is just a restatement of what it means to be locally free of rank one. Let us show the equivalence (i) ⇔ (iv).

We begin with the implication (i) ⇒ (iv). Let $x \in X$ and pick an open affine set $V = \operatorname{Spec} A \subset X$ so that $\mathcal{I}_D|_V \simeq \mathcal{O}_V$. This means that there is an element $f \in \mathcal{I}_D(V) \subset A = \mathcal{O}_X(V)$ which is an A-basis for $\mathcal{I}_D|_V$, and in particular, f must be a nonzerodivisor. Moreover, $D \cap V$ is the subscheme of $\operatorname{Spec} A$ defined by f, so that $D \cap V = \operatorname{Spec} A/(f)$.

For the implication (iv) ⇒ (i) we need to show that $\mathcal{I}$ is invertible near every point $x \in X$. Pick an open affine $U = \operatorname{Spec} A$ neighbourhood of x so that $D \cap U = \operatorname{Spec} A/(f)$, for some nonzerodivisor f. Then $\mathcal{I}|_U \simeq \widetilde{(f)} \simeq \widetilde{A} \simeq \mathcal{O}_U$, which means that $\mathcal{I}$ is invertible.

(iii) ⇒ (vi). Let us now assume that X is normal. Note that the sections f_i in (iii) form a set of Cartier data. This is because on $U_i \cap U_j$, f_i and f_j generate the same, principal ideal, so they must be related by a unit $c_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$.

Write $[D] = \sum n_Z Z$ for the Weil divisor associated to D. On each U_i, the ideal sheaf of D is generated by f_i, so the multiplicity n_Z of D is exactly $\operatorname{ord}_Z(f)$. It follows that

$$[D]|_{U_i} = \operatorname{div} f_i$$

for each i. Thus $[D]$ is an effective Cartier divisor.

(vi) ⇒ (iii). Conversely, suppose that $[D]$ is Cartier, and of the form $\operatorname{div} f_i$ over each U_i in an open covering. By the Cartier data conditions, the f_i define an ideal sheaf $\mathcal{J}$ of $\mathcal{O}_X$. The corresponding subscheme Y, is supported on D with the same multiplicities on each component, so $Y = D$. Thus $\mathcal{I} = \mathcal{J}$ by Proposition 18.4, so $\mathcal{I}$ is locally generated by the f_i.

(vi) ⇒ (v). If D is an effective Cartier divisor, it is the zero set of the distinguished section s_D of $\mathcal{O}_X(D)$. (v) ⇒ (iii). If D is the zero set of $s \in \Gamma(X, L)$, then the ideal sheaf $\mathcal{I}$ is isomorphic to $L^\vee$, which is invertible. □

If D is an effective Cartier divisor, then D is determined by the ideal sheaf $\mathscr{I} = \mathcal{O}_X(-D)$ which is locally generated by the elements f_i. Informally, we say that the f_i's are *local equations* of D. As before, these equations are not unique; two sets of local equations (U_i, f_i), (V_j, g_j) give the same subscheme if and only if they define the same Cartier data.

If $i : D \to X$ denotes the inclusion, the ideal sheaf sequence (**??**) takes the form

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0$$

Example 18.44. Let $X = \mathbb{A}^n_k$ over a field k and let G be a polynomial. Then $D = V(G)$ is an effective Cartier divisor. It is specified by the obvious data $(\mathbb{A}^n, G)$.

Example 18.45. Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ over a field k and let P be the point $(1:0)$. Using the standard covering $D_+(x_0)$ and $D_+(x_1)$, we see that P is the effective Cartier divisor determined by the data

$$(D_+(x_0), x_1 x_0^{-1}) \text{ and } D_+(x_1, 1).$$

Note that on the intersection $D_+(x_0) \cap D_+(x_1)$ the function $x_1 x_0^{-1}$ is invertible, so the data yields an effective Cartier divisor.

On the open set $D_+(x_0) = \operatorname{Spec} k[x_1 x_0^{-1}] = \mathbb{A}^1_k$ the ideal is generated by $x_1 x_0^{-1}$ which defines the point P, and on $D_+(x_1)$ the local equation is 1 which is without zeros, so the devisor defined is exactly P.

We might also consider the data $(D_+(x_0), (x_1 x_0^{-1})^n)$ and $(D_+(x_1), 1)$. In the distinguished open set $D_+(x_0) = \operatorname{Spec} k[x_1 x_0^{-1}]$ it gives the ideal $((x_1 x_0^{-1})^n)$ which defines a subscheme supported at P and of length n, and in $D_+(x_1)$ the ideal will be the unit ideal, whose zero set is empty. We denote the corresponding divisor by nP.

Proposition 18.46. Let X be a normal integral Noetherian scheme. Then:

(i) Each effective Cartier divisor $D \subset X$ is the zero set $V(s)$ of some regular section $s \in \Gamma(X, L)$ of some invertible sheaf L;

(ii) Two regular sections $s, t \in \Gamma(X, L)$ give rise to the same divisor if and only if $t = \lambda s$ for some unit $\lambda \in \mathcal{O}_X^\times(X)$.

Proof We only need to prove the last statement. Suppose that s and t define the same ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$, so that we have isomorphisms

$$L^\vee \xrightarrow{s^\vee} \mathcal{I} \xrightarrow{(t^\vee)^{-1}} L^\vee.$$

Note by Proposition **??** on page ?? it holds that $\mathscr{H}om_{\mathcal{O}_X}(L^\vee, L^\vee) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X$ so that $\operatorname{Hom}_{\mathcal{O}_X}(L^\vee, L^\vee) = \Gamma(X, \mathcal{O}_X)$. Hence each isomorphism $L^\vee \to L^\vee$ is given by multiplication by some element in $\mathcal{O}_X^\times(X)$. Thus s and t differ only by a unit. □

Theorem 18.47. Let X be a Noetherian integral scheme. Then the following two statements are equivalent:

(i) Every integral subscheme of codimension one is an effective Cartier divisor;

(ii) X is locally factorial (that is, the local rings $\mathcal{O}_{X,x}$ are all UFD's).

Proof Both conditions can be checked locally, so we may assume that $X = \operatorname{Spec} A$ is affine. Let $D = \operatorname{Spec}(A/\mathfrak{q})$ be an integral subscheme. Saying that D has codimension one is equivalent to saying that $\mathfrak{q}$ is a prime ideal of height one. If $\operatorname{Spec} A$ is factorial, then each $A_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$ is a UFD and $\mathfrak{q}A_\mathfrak{p}$ is principal according to Theorem **??**. One may extend the generator to a generator for $\mathfrak{q}$ over a neighbourhood of $\mathfrak{p}$, and thence by Proposition 18.43, D will be an effective Cartier devisor.

For the converse, assume (i). Note first that for each $\mathfrak{p} \in \operatorname{Spec} A$ every prime ideal in $A_\mathfrak{p}$ is of the form $\mathfrak{q}A_\mathfrak{p}$ for a prime $\mathfrak{q}$ in A, and when $\mathfrak{q}A_\mathfrak{p}$ is of height one, $\mathfrak{q}$ is also of height one (there is a one-to-one correspondence between primes in A lying in $\mathfrak{p}$ and primes in $A_\mathfrak{p}$). Consequently, if $\mathfrak{q}A_\mathfrak{p}$ is of height one, $\mathfrak{q}$ is locally principal by (i), which means that $\mathfrak{q}A_\mathfrak{p}$ is principal. □

Definition 18.48. The set of effective divisors D' linearly equivalent to D is denoted by $|D|$. This is called the *complete linear system* of D.

The name 'linear system' comes from the special case when X is a projective variety X over a field k (thus X is integral, separated of finite type over k). In this case, we have $\Gamma(X, \mathcal{O}_X)^\times = k^\times$, and the previous discussion shows that the linear system $|D|$ is given by

$$\begin{aligned} |D| &= \{D' | D' \geqslant 0 \text{ and } D' \sim D\} && (18.4)\\ &= \left(\Gamma(X, \mathcal{O}_X(D)) - 0\right)/k^\times && (18.5)\\ &= \mathbb{P}\Gamma(X, \mathcal{O}_X(D)) \end{aligned}$$

When X is projective over k, the groups $\Gamma(X, \mathcal{O}_X(D))$ are finite dimensional as k-vector spaces (we will prove this fact in Chapter **??**), so the set of effective divisors D' linearly equivalent to D is (as a set) a projective space $\mathbb{P}^n_k$.

Definition 18.49. A *linear system of divisors* is a linear subspace of a complete linear system $|D|$.

Example 18.50. Consider the case $X = \mathbb{P}^n_k$ and $D = dH$, where H is the hyperplane divisor (so H is a Cartier divisor with $\mathcal{O}_X(H) \simeq \mathcal{O}_X(1)$). In this case the linear system of D associated to $\mathcal{O}_X(dH)$ is given by the set of homogeneous polynomials of degree d modulo scalars, i.e.,

$$|D| = \left\{ \sum_{i_0+\cdots+i_n=d} a_{i_0,\ldots,i_n} x_0^{i_0} \cdots x_n^{i_n} \right\} / k^\times \simeq \mathbb{P}^N(k)$$

where $N = \binom{n+d}{d} - 1$. The points of this projective space correspond to degree d hypersurfaces, and the coefficients $a_{i_0,\ldots,i_n}$ give homogeneous coordinates on it.

Exercise 18.8.1. Given data $\{(U_i, f_i)\}$ as in **??**. Assume that there are units $c_{ij} \in \mathcal{O}_X(U_i \cap U_j)$ with $f_j = c_{ji} f_i$ which satisfy the cocycle condition. Show that the data then defines a sheaf of invertible ideals.

Exercise 18.8.2. Check that the ideal sheaf $\mathcal{I}_{nP}$ of the divisor nP in Example 18.45 is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n)$.

Exercise 18.8.3. Describe Cartier data that defines the hyperplane $V(x_i)$ in $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$.

18.9 Pullbacks of divisors

Given a morphism $\phi : X \to Y$ and a Weil divisor D on Y, we can ask whether we can define a Weil divisor on X supported on $\phi^{-1}D$. In general, this is not possible. Consider for instance, the case where ϕ is the inclusion of a closed subscheme, say $Y = \mathbb{P}^2_k$ and $f : X \to Y$ is the inclusion of a line $X = V(x_0)$. Then of course $D = X$ defines a Weil divisor on Y, but there is no reasonable definition of $\phi^{-1}D$ that defines a codimension one subscheme of X.

Example 18.51. Let $X = V(x_0)$ be a line in $Y = \mathbb{P}^2$ and let $f : X \to Y$ be the inclusion. We may consider $D = X$ as a Weil divisor on Y. But then

There is a situation where we can always define the pullback of a divisor D. This is when $\phi : X \to Y$ is a dominant morphism and D is a Cartier divisor. In that case, there is a covering U_i such that $D|_{U_i}$ is given by $\operatorname{div} f_i$ over U_i. The fact that f is dominant, means that there is an induced map on function fields $\phi^\sharp : k(Y) \to k(X)$. We can therefore define a divisor $\phi^* D$ by

$$\phi^* D = \sum_Z \operatorname{ord}_Z(\phi^\sharp f_i) Z$$

This is well-defined, because over each intersection $U_i \cap U_j$.

Example 18.52. $\operatorname{Cl}(\mathbb{P}^2)$ is generated by the class of line $L \subset \mathbb{P}^2$, e.g. $L = V(x_0)$, and any two lines L, L' are linearly equivalent.

Example 18.53. Consider the curve X as in Figure 18.1, given by

$$X = V(y^2 z - x^3 - z^3) \subset \mathbb{P}^2.$$

For a line $L = V(y)$ on $\mathbb{P}^2_k$, let $L|_X$ denote the restriction of L to X (i.e., the Weil divisor $L \cap X$ on X which is of codimension 1 as X is integral). Moreover, for another line $L' = V(z)$, the two restrictions $L|_X$ and $L'|_X$ are linearly equivalent divisors on X, since $L|_X - L'|_X = \operatorname{div}(\frac{x}{z}|_X)$. This argument applies for any two lines L, L' in $\mathbb{P}^2$, so we get many relations between divisors on X. The figure below shows one example where $L|_X = P + Q + R$ and $L' = 2S + T$.

18.10 Examples

A useful exact sequence

Given a Noetherian, normal and integral X and an open subset U, the restriction of a prime divisor on X is a prime divisor on U, so it is natural to ask how the two class groups are related. The answer is given by the theorem below.

Before stating the result, let us make the restriction map a bit more precise. Consider a prime divisor Z in X. If $Z \cap U \neq \emptyset$, it is dense in Z, and so the generic point of Z lies in U. Since $\operatorname{Div} X$ is free abelian on the prime divisors, this allows us to define a restriction map $\operatorname{Div} X \to \operatorname{Div} U$ by

$$Z \mapsto \begin{cases} Z \cap U & \text{if } Z \cap U \neq \emptyset, \\ 0 & \text{else .} \end{cases}$$

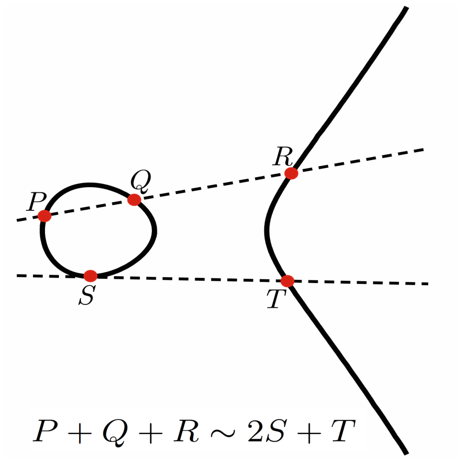


Figure 18.1 Two linearly equivalent divisors on a plane cubic

Moreover, if f is a rational function on X, the restriction $f|_U$ is a rational function on U, and it holds that $\operatorname{ord}_{Y\cap U}(f|_Y) = \operatorname{ord}_Y(f)$ (the two valuation rings are equal), and consequently the divisor $\operatorname{div}(f)$ restricts to the divisor $\operatorname{div}(f|_U)$. The restriction map passes to the quotient and yields a map $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$.

Theorem 18.54. Let X be a normal integral Noetherian scheme. Let $W \subset X$ be a closed subscheme and let $U = X - W$. If $Z_1, \dots, Z_r$ are the prime divisors corresponding to the codimension one components of W, there is an exact sequence

$$\bigoplus_{i=1}^{r} \mathbb{Z}Z_i \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0, \tag{18.6}$$

where the map $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ is defined by $[Z] \mapsto [Z \cap U]$.

Proof If Z is a prime divisor on U, the closure in X is a prime divisor in X, so the map is surjective, and we just need to check exactness in the middle.

Suppose Z is a prime divisor which is principal on U. Then $Z|_U = \operatorname{div}(f)$ for some $f \in k(U) = K = k(X)$. Now $D = \operatorname{div}(f)$ is a divisor on X such that $D|_U = \operatorname{div}(f)|_U$. Hence $D - Z$ is a Weil divisor supported in $X - U$, and hence it must be a linear combination of the Z_i's. Thus $D - Z$ is in the image of the left-most map, and we are done. □

As a special case, we see that removing a codimension two subset does not change the group of Weil divisors. So for instance $\operatorname{Cl}(\mathbb{A}^2 - 0) = \operatorname{Cl}(\mathbb{A}^2)$.

Example 18.55. Consider the projective line $\mathbb{P}^1_k$ over a field k, and let P be a point. We have the exact sequence

$$\mathbb{Z}[P] \to \operatorname{Cl}(\mathbb{P}^1) \to \operatorname{Cl}(\mathbb{A}^1) \to 0.$$

We saw that $\operatorname{Cl}(\mathbb{A}^1) = 0$, so the map $\mathbb{Z} \to \operatorname{Cl}(\mathbb{P}^1)$ is surjective. It is also injective: If $[nP] = 0$ in $\operatorname{Cl}(\mathbb{P}^1)$ for some n, then $nP = \operatorname{div}(f)$ for some $f \in k(\mathbb{P}^1)$. Consider the open set $U = \mathbb{P}^1 - P \simeq \mathbb{A}^1$. Then $nP|_U = 0$, so we must have $\operatorname{div}(f)|_{\mathbb{A}^1} = 0$. Thus f has neither zeros, nor poles, and so $f \in \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}^\times) = k^\times$. Hence f is constant, and so $n = 0$. This gives another proof of $\operatorname{Cl}(\mathbb{P}^1) = \mathbb{Z}$.

Exercise 18.10.1. Let $\mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2]$. An irreducible homogeneous polynomial f of degree $d \geqslant 1$ determines a prime divisor $D = V(f)$. Consider the open set $U = \mathbb{P}_k^2 - D$. Show that the above exact sequence above takes the form

$$0 \to \mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \to \operatorname{Cl}(U) \to 0.$$

Deduce that $\operatorname{Cl}(U) = \mathbb{Z}/d\mathbb{Z}$.

The smooth quadric surface

Let k be an field, and let $Q = \mathbb{P}_k^1 \times \mathbb{P}_k^1$. Recall that Q embeds as a quadric surface in $\mathbb{P}_k^3$ via the Segre embedding. So we can view Q both as a fiber product $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric $V(xy - zw) \subset \mathbb{P}^3$.

Since Q is a product of two $\mathbb{P}^1$s there are natural ways of constructing Weil divisors on Q from those on $\mathbb{P}^1$. For instance, we can let

$$L_1 = (0:1) \times \mathbb{P}^1 \subset Q,$$

which is a prime divisor on Q corresponding to the 'vertical fiber' of Q. Similarly, $L_2 = \mathbb{P}^1 \times (0:1)$ is a Weil divisor on Q. From these we obtain an exact sequence

$$\mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \to \operatorname{Cl}(Q) \to \operatorname{Cl}(Q - L_1 - L_2) \to 0$$

Here $Q - L_1 - L_2 = U_{11} = \operatorname{Spec} k[x^{-1}, y^{-1}]$. The latter is isomorphic to $\mathbb{A}_k^2$, so $\operatorname{Cl}(Q - L_1 - L_2) = 0$. This shows that $\operatorname{Cl}(Q)$ is generated by the classes of L_1 and L_2. We claim that the first map is also injective, so that in fact that

$$\operatorname{Cl}(Q) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2.$$

If the map is not injective, there must be a relation $aL_1 - bL_2 \sim 0$, or equivalently,

$$\mathcal{O}_Q(aL_1) \simeq \mathcal{O}_Q(bL_2) \tag{18.7}$$

for some integers $a, b \in \mathbb{Z}$. We will show that this is not the case, by showing

(i) $\mathcal{O}_Q(L_1)|_{L_1} \simeq \mathcal{O}_{\mathbb{P}^1}$;
(ii) $\mathcal{O}_Q(L_2)|_{L_1} \simeq \mathcal{O}(1)_{\mathbb{P}^1}$

Then restricting (18.7) to L_1, we get $b = 0$, and hence also $a = 0$, by switching the roles of L_1 and L_2.

To prove i): Note that $L_1 \simeq L_1'$ where $L_1 = (1:0) \times \mathbb{P}^1$. This follows because we can consider the divisor of the rational function $x \in k(Q) = k(x, y)$:

$$\operatorname{div} x = (0:1) - (1:0) = L_1 - L_1'$$

Then note that $\mathcal{O}_Q(L_1')|_U \simeq \mathcal{O}_U$ over the open set $U = Q - L_1'$. However L_1 is contained in U, so the isomorphism i) follows.

Q is covered by four affine subsets

$$\begin{aligned} U_{00} &= \operatorname{Spec} k[x,y] \quad U_{10} = \operatorname{Spec} k[x^{-1},y] \\ U_{01} &= \operatorname{Spec} k[x,y^{-1}]\; U_{11} = \operatorname{Spec} k[x^{-1},y^{-1}] \end{aligned} \tag{18.8}$$

Consider $\mathbb{P}^1_k = W_0 \cup W_1$, where $W_0 = \operatorname{Spec} k[t], W_1 = \operatorname{Spec} k[t^{-1}]$. The first projection $p_1 : Q \to \mathbb{P}^1_k$ is induced by the ring maps

$$\begin{aligned} k[x] \to k[x,y] \quad k[x^{-1}] \to k[x^{-1},y] \\ k[x] \to k[x,y^{-1}];\; k[x^{-1}] \to k[x^{-1},y^{-1}]; \end{aligned} \tag{18.9}$$

Let $p = (0:1)$ be the Weil divisor on $\mathbb{P}^1$. The Cartier data of p is given by $(W_0,t),(W_1,1)$, so that $\mathcal{O}_{\mathbb{P}^1}(p) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. The pullback $D = p_1^*(p)$ is a Cartier divisor on Q, corresponding to the Weil divisor $(0:1)\times\mathbb{P}^1$. The corresponding Cartier data is given by

$$\begin{aligned} &(U_{00},x),\,(U_{10},1) \\ &(U_{01},x),\,(U_{11},1) \end{aligned} \tag{18.10}$$

Let $L_1 = (0:1)\times\mathbb{P}^1_k$ and $L_2 = \mathbb{P}^1_k \times (0:1)$. Consider the restriction of D to L_2. L_2 is covered by the two open subsets $V_0 = U_{00}\cap L_2 = \operatorname{Spec} k[x,y]/y = \operatorname{Spec} k[x], V_1 = U_{10}\cap L_2 = \operatorname{Spec} k[x^{-1},y]/(y) = \operatorname{Spec} k[x^{-1}]$. In terms of these opens, the restriction $D|_{L_2}$ has Cartier data

$$(V_0,x),(V_1,1)$$

obtained by restricting the data above. In particular, identifying $L_2 \simeq \mathbb{P}^1$, we see that $\mathcal{O}_Q(D)|_{L_1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. In particular, since $\operatorname{Cl}(\mathbb{P}^1) = \mathbb{Z}$, no multiple nD is equivalent to 0 in $\operatorname{Cl}(Q)$: if that were the case, we would have $\mathcal{O}_Q(nD)\simeq\mathcal{O}_Q$, and hence $\mathcal{O}_Q(nD)|_{L_2} \simeq \mathcal{O}_Q|_{L_2} \simeq \mathcal{O}_{\mathbb{P}^1}$, a contradiction.

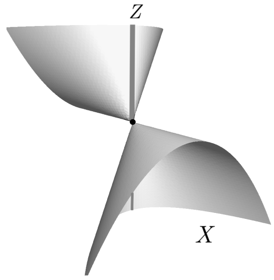
This completes the proof that

$$\operatorname{Cl}(Q) \simeq \mathbb{Z}L_1 \oplus \mathbb{Z}L_2.$$

If D is a divisor on Q, $D \sim aL_1 + bL_2$ and we call (a,b) the 'type' of D. A divisor of type $(1,0)$ or $(0,1)$ is a line on the quadric surface $Q \subset \mathbb{P}^3$. We have $i^*\mathcal{O}_{\mathbb{P}^3}(1) \simeq \mathcal{O}_Q(L_1+L_2)$, so a $(1,1)$-divisor is represented by a hyperplane section of Q (a conic). A prime divisor of type $(1,2)$ or $(2,1)$ is a *twisted cubic curve*.

The quadric cone

Let $X = \operatorname{Spec} R$ where $R = k[x,y,z]/(xy-z^2)$, and k has characteristic $\neq 2$. Let $Z = V(y,z)$ be the closed subscheme corresponding to the line $\{y=z=0\}$. Note that $Z \simeq \operatorname{Spec} k[x,y,z]/(xy-z^2,y,z) = \operatorname{Spec} k[x]$, so it is integral of codimension 1.



A singular quadric surface

Note that $X - Z = X - V(y) = D(y)$, and the latter equals

$$\operatorname{Spec} k[x, y, y^{-1}, z]/(xy - z^2) = \operatorname{Spec} k[y, y^{-1}][t, u]/(t - u^2) = \operatorname{Spec} k[y, y^{-1}, u]$$

which is the spectrum of a UFD. It follows that $\mathrm{Cl}(X - Z) = 0$. Recall now the sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - Z) \to 0$$

where the first map sends 1 to $[Z]$. Hence $\mathrm{Cl}(X)$ is generated by $[Z]$.

We first show that $2Z = 0$ in $\mathrm{Cl}(X)$. This is because we can consider the divisor of y. The rational function y is invertible in every stalk $\mathcal{O}_{X,p}$ except when $p \in V(y)$. Moreover, by the defining equation $xy = z^2$, we see that the divisor of y can only be non-zero along Z. The valuation at the generic point η of Z is 2: The local ring equals

$$\mathcal{O}_{X,\eta} = (k[x, y, z]/(xy - z^2))_{(y,z)}$$

and since x is invertible here, we see that $y \in (z^2)$ and that z is the uniformizer.

Now we show that Z is not a principal divisor. It suffices to prove that this is not principal in $\operatorname{Spec} \mathcal{O}_{X,p}$ where $p \in X$ is the singular point of X. The local ring here equals

$$\mathcal{O}_{X,p} = (k[x, y, z]/(xy - z^2))_{(x,y,z)}$$

In this ring $\mathfrak{p} = (x, z)$ is a height 1 prime ideal, but it is not principal: Let $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideal. Note that $x, y \in \mathfrak{m}$, since x, y are not units. Moreover, it is clear that the vector space $\mathfrak{m}/\mathfrak{m}^2$ (which is the Zariski cotangent space at x) is 3-dimensional, spanned by $\{\overline{x}, \overline{y}, \overline{z}\}$. Then $\overline{x}, \overline{y}$ gives a 2-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$. Hence, since $\overline{x}$ and $\overline{y}$ are linearly independent in this quotient, there couldn't be an non-constant element $f \in \mathcal{O}_{X,x}$ for which $x = af, y = bf$. This means that $[Z] \neq 0$ in $\mathrm{Cl}(X)$ and hence

$$\mathrm{Cl}(X) = \mathbb{Z}/2.$$

Note that the open subscheme $X - (0, 0, 0)$ *is* factorial. Hence removing a codimension 2 subset has an effect on $\mathrm{CaCl}(X)$. Recall however, that the class group of Weil divisors $\mathrm{Cl}(X)$ stays unchanged under removing a codimension 2 subset.

Projective quadric cone

Let $X = \operatorname{Proj} R$ where $R = k[x, y, z, w]/(xy - z^2)$. Let $H = V(w)$ be the hyperplane determined by w. We have

$$0 \to \mathbb{Z}H \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - H) \to 0$$

(Here H is a divisor corresponding to the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$, hence it is non-torsion in $\mathrm{Cl}(X)$, so the first map is injective). $X - H$ is isomorphic to the affine quadric cone from before, hence $\mathrm{Cl}(X - H) = \mathbb{Z}/2$. Using this sequence, we see that $\mathrm{Cl}(X) = \mathbb{Z}$, generated by a Weil divisor D such that $H = 2D$. More precisely, D is the divisor $V(x, z)$ which is supported on a line on X.

The Weil divisor D is not Cartier; being Cartier is a local condition, so this follows from the example of the affine quadric cone above. Here is an alternative way to see it: If $D = V(x, z)$ is Cartier, the sheaf $L = \mathcal{O}_X(D)$ is invertible, and hence so is its restrtction to the line $\ell = V(x, z) \simeq \mathbb{P}^1_k$. The Picard group of $\mathbb{P}^1_k$ is $\mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^1}(1)$, so we have $L|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$. On the other hand, we know that the divisor $H = 2D$ is Cartier and in fact $\mathcal{O}_X(H) \simeq \mathcal{O}_{\mathbb{P}^3}(1)|_X$ (the local generator is given by w). Restricting further to ℓ, we obtain $\mathcal{O}_{\mathbb{P}^3}(1)|_\ell \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ (as the divisor of w is just one point on ℓ). But these two observations imply that $2a = 1$, a contradiction. Hence D is not Cartier.

Quadric hypersurfaces in higher dimension

Here is an application of the 'useful exact sequence' (18.6).

Let $A = k[x_1, \ldots, x_n, y, z]/(x_1^2 + \cdots + x_m^2 - yz)$. We will prove that A is a UFD for $m \geqslant 3$. A is a domain, since the defining ideal is prime. Apply Nagata's lemma with the element y:

$$A_y = k[x_1, \ldots, x_n, y, y^{-1}, z]/(y^{-1}(x_1^2 + \cdots + x_m^2) - z) \simeq k[x_1, \ldots, x_n, y, y^{-1}, z]$$

which is a UFD. We show that y is prime: Taking the quotient we get

$$A/y = k[x_1, \ldots, x_n, x]/(x_1^2 + \cdots + x_m^2)$$

which is an integral domain, because $x_1^2 + \cdots + x_m^2$ is irreducible (for $m \geqslant 3$).

Note that for $m = 2$, we get the quadric cone, which we have seen is not a UFD.

Applying a change of variables, we find the following description of the class groups of quadrics in any dimension:

Proposition 18.56. Let k be a field containing $\sqrt{-1}$ and let $X = V(x_0^2 + \cdots + x_m^2) \subset \mathbb{A}^{n+1}_k = \operatorname{Spec} k[x_0, \ldots, x_n]$.

(i) $m = 2$, $\mathrm{Cl}(X) = \mathbb{Z}/2$
(ii) $m = 3$, $\mathrm{Cl}(X) = \mathbb{Z}$
(iii) $m \geqslant 4$, $\mathrm{Cl}(X) = 0$

There is also the following statement for projective quadrics:

Proposition 18.57. Let $X = V(x_0^2 + \cdots + x_m^2) \subset \mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$.

(i) $m = 2$, $\mathrm{Cl}(X) = \mathbb{Z}$;
(ii) $m = 3$, $\mathrm{Cl}(X) = \mathbb{Z}^2$;
(iii) $m \geqslant 4$, $\mathrm{Cl}(X) = \mathbb{Z}$.

18.11 Exercises

Exercise 18.11.1. Show that for the weighted projective space $\mathbb{P} = \mathbb{P}(1,1,d)$ we have $\mathrm{Cl}(\mathbb{P}) = \mathbb{Z}D$ and $\mathrm{CaCl}(\mathbb{P}) = \mathbb{Z}H$ where $H = dD$.

Exercise 18.11.2. The same reasoning as for $\mathbb{P}^1_k$ can be applied to the affine line X with two origins. Compute $\mathrm{Pic}(X)$ for this example.

Exercise 18.11.3. The aim of this exercise is to prove the following statement, known as "Nagata's Lemma": Let A be a noetherian integral domain, and let $x \in A - 0$. Suppose that (x) is prime, and that A_x is a UFD. Then A is a UFD.

a) Show that A_x is normal.
b) Show that A is normal. HINT: If $t \in K(A)$ is integral over A, then $t \in A_x$.
c) Show that there is an exact sequence

$$\mathbb{Z}D \to \mathrm{Cl}(\operatorname{Spec} A) \to \mathrm{Cl}(\operatorname{Spec} A_x) = 0 \to 0$$

d) Use the above sequence to show that $\mathrm{Cl}(A) = 0$, and conclude that A is a UFD.

An example from number theory

We turn to an example from number theory and pick up the thread from Example 18.21. There we claimed that the class group of the quadratic extension $A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[x]/(x^2+5)$ is equal to $\mathbb{Z}/2\mathbb{Z}$. We also verified that the class of $Y = V(2, 1+\sqrt{-5}))$ was a non-trivial two-torsion element. Here we complete the claim and show that the class of Y generates $\mathrm{Cl}(A)$.

Since A is a Dedekind ring, the class group is generated by prime divisors, so we will be through when we show that $V(\mathfrak{p})$ is equivalent to Y for each prime ideal in A that is not principal. The only non-principal prime ideals in A are those of the form $(p, a \pm \sqrt{-5})$ where $p \in \mathbb{Z}$ is a prime and a is an integer that satisfy a congruence $a^2 + 5 \equiv 0 \mod p$, and altering a by a multiple of p, we may assume that $0 \leqslant a < p$.

The proof goes by induction on p, and to lubricate the induction, we shall prove a somehow more general statement. Note that the lemma with $n = p$ yields what we want; that the class of every non-principal prime divisor equals Y.

Lemma 18.58. For each ideal $\mathfrak{a} = (n, a+\sqrt{-5})$ for any integers n and a satisfying $n \geqslant 2$ and $a^2 + 5 \equiv 0 \mod n$, there are non-zero elements f and g in A so that

$$(f)(n, a \pm \sqrt{-5}) = (2, 1+\sqrt{-5})^{\epsilon}(g)$$

where either $\epsilon = 0$ or $\epsilon = 1$.

The two signs in the statement reflects the two possible choices of the square root, and it will suffice to do the case $a + \sqrt{-5}$; it is however crucial that $(2, 1+\sqrt{-5}) = (2, 1-\sqrt{-5})$.

Proof The most significant portion of the proof is the induction part which reduces the proof to the case that $n \leqslant 5$ (which subsequently is done case by case):

So we assume $n > 5$ and proceed by induction on n; we write $a^2 + 5 = bn$, and compute

$$(b)(n, a + \sqrt{-5}) = (bn, b(a + \sqrt{-5})) = \tag{18.11}$$

$$= (a^2 + 5, b(a + \sqrt{-5})) = (a - \sqrt{-5}, b)(a + \sqrt{-5}). \tag{18.12}$$

Now, $bn = a^2 + 5 < n^2 + 5$, so as $n > 5$, it follows that $b < n$ and clearly $a^2 + 5 \equiv 0 \mod b$. By induction, it follows that for approriate elements f' and g from A, we have the equality

$$(f') \cdot (b, a + \sqrt{-5}) = (2, 1 + \sqrt{-5})^{\epsilon} \cdot (g),$$

and so muliplying xxx through by f' we arrive at

$$(bf')(n, a + \sqrt{-5}) = (2, 1 + \sqrt{-5})^{\epsilon} \cdot (g)(a + \sqrt{-5}).$$

It remains to treat the special cases with $n \leqslant 5$. Again, write $a^2 + 5 = bn$ with $0 \leqslant a < n$. When $n = 5$, it holds that $a^2 = 5(b - 1)$, and this implies that $b = 1$ and $a = 0$. One easily verifies that all ideals $(n, \sqrt{-5})$ are principal (either generated by 1 or by $\sqrt{-5}$). That $n = 4$ is impossible since no square is congruent $-1 \mod 4$. Finally, if $n = 3$ and $a = 1$, we have

$$(2)(3, 1 + \sqrt{-5}) = (6, 2(1 + \sqrt{-5})) = (1 - \sqrt{-5}, 2)(1 + \sqrt{-5}),$$

and if $a = 2$, we note that $(3, 2 + \sqrt{-5}) = (3, 1 - \sqrt{-5})$. We are left with the case $n = 2$ and $a = 1$, which is exactly what we want. □

Exercise 18.11.4. Let d be a square free integer and assume that $d \not\equiv 1 \mod 4$ so that $\mathbb{Z}[\sqrt{d}]$ is a Dedekind ring. Show that the class group of $\mathbb{Z}[\sqrt{d}]$ is finite. HINT: Concider the indction portion of the proof above.

19

Locally free sheaves

The most important examples of quasi-coherent sheaves are the locally free sheaves. As the name suggests, these are sheaves which are locally isomorphic to a direct sum of copies of the structure sheaf of the scheme. Because of this 'freeness' property, these sheaves are in many respects the nicest examples of sheaves on a scheme and the easiest to work with. They are also the algebraic counterpart to the vector bundles in topology.

19.1 Basic properties

Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{E}$ is called *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if there exists a *trivializing cover*; that is, an open cover $\{U_i\}$ of X such that the restriction $\mathcal{E}|_{U_i}$ is free for each i. In view of the criterion in Exercise xxxx every locally free $\mathcal{O}_X$-module is quasi-coherent. A locally free $\mathcal{O}_X$-modules which is globally free; that is, one which is isomorphic to a free $\mathcal{O}_X$-module, is often said to be *trivial*.

The *rank* $r_x(\mathcal{E})$ of $\mathcal{E}$ at a point $x \in U_i$ is the number of copies of $\mathcal{O}_{U_i}$ needed to express $\mathcal{E}|_{U_i}$ as a free $\mathcal{O}_X$-module. This may be finite or infinite, but we shall almost exclusively concern ourself with the case of finite rank. The local rank is obviously constant throughout each U_i, so the sets $\{\, x \in X \mid r_x(\mathcal{E}) = r \,\}$ are all open as r varies, and consequently they are also all closed. It follows that the rank is constant along each connected component of X. If the rank is constant all over, say equal to r, we say that $\mathcal{E}$ is of rank r and write $r(\mathcal{E})$ for it. In particular, this is the case when X is connected. A locally free sheaf of rank one is called an *invertible sheaf*.

Example 19.1. It is easy to give examples of locally free sheaves with varying rank. If X is disconnected with connected components U and V, we are free to define $\mathcal{E}$ by letting $\mathcal{E}|_U = \mathcal{O}_U^n$ and $\mathcal{E}|_V = \mathcal{O}_V^m$ with $n, m \in \mathbb{N}$ arbitrary.

Example 19.2. On the projective line $X = \mathbb{P}^1_A$ one has the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(m)$ constructed on page 94. These were made by gluing together trivial sheaves of rank one, so they are locally free of rank one. Most of them are non-trivial; we showed that $\mathcal{O}_{\mathbb{P}^1_A}(m)$ is not isomorphic to $\mathcal{O}_{\mathbb{P}^1_A}$ when $m \neq 0$.

Example 19.3. There are many ways of constructing new locally free sheaves from given ones. For instance, if $\mathcal{E}$ and $\mathcal{F}$ are locally free, their direct sum will be locally free as well. Indeed, if $\{U_i\}$ is a trivializing cover for $\mathcal{E}$ and $\{V_i\}$ one for $\mathcal{F}$, the cover $\{U_i \cap V_j\}$ will be trivializing for $\mathcal{E} \oplus \mathcal{F}$.

In particular, if $m_1, \ldots, m_r$ are integers, the sum $\bigoplus_i \mathcal{O}_{\mathbb{P}^1_A}(m_i)$ will be locally free.

In general, a locally free sheaf $\mathcal{E}$ of finite rank r is obtained by gluing together copies of the trivial sheaves. More precisely, if $\mathcal{E}$ is locally free of rank r, there is by definition an open cover $\{U_i\}$ trivializing $\mathcal{E}$; that is, there are isomorphisms of $\mathcal{O}_{U_i}$-modules

$$\phi_i \colon \mathcal{O}_{U_i}^r \xrightarrow{\simeq} \mathcal{E}|_{U_i}. \tag{19.1}$$

Over the intersections $U_{ij} = U_i \cap U_j$ the maps $\tau_{ji} = \phi_j^{-1} \circ \phi_i$ are well defined, and they give isomorphisms

$$\tau_{ji} \colon \mathcal{O}_{U_{ij}}^r \xrightarrow{\simeq} \mathcal{O}_{U_{ij}}^r,$$

which restricted to triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ satisfy the cocycle condition

$$\tau_{ki} = \tau_{kj} \circ \tau_{ji}. \tag{19.2}$$

Indeed, we have $\phi_k^{-1} \circ \phi_i = (\phi_k^{-1} \circ \phi_j) \circ (\phi_j^{-1} \circ \phi_i)$.

Conversely, we know from the Gluing lemma for sheaves that given isomorphisms τ_{ji} as above, satisfying the cocycle condition (19.2) on the triple overlaps, the sheaves $\mathcal{O}_{U_{ij}}^r$ may be glued together to a sheaf $\mathcal{E}$, which by definition is locally free of rank r.

Note that there are many ambiguities in this process, both the trivializing cover and the bases for the trivial sheaves are chosen. So it is not an ideal way of classifying locally free sheaves. The same sheaf may be constructed in an infinite number of ways, and it is generally very hard to decide when two constructions yields isomorphic sheaves.

Note that any isomorphism of $\mathcal{O}_{U_{ij}}$-modules $\mathcal{O}_{U_{ij}}^r \to \mathcal{O}_{U_{ij}}^r$ is given by some $r \times r$-matrix with entries in $\mathcal{O}_X(U_{ij})$. Thus, we will sometimes specify $\mathcal{E}$ by giving the gluing maps τ_{ji} as matrices satisfying the cocycle condition (19.2).

Stalks

If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, the stalk $\mathcal{E}_x$ is clearly a free $\mathcal{O}_{X,x}$-module for every $x \in X$. Having stalks that are free modules over the stalks of the structure sheaf, is *a priori* a weaker property than being locally free, and in fact, in a general the two properties are not equivalent. However, under mild finiteness conditions, they are equivalent for finitely presented sheaves:

Lemma 19.4. A finitely presented quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ on a scheme X having the property that $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module for every $x \in X$, is locally free.

Proof One way is trvial. So assume that the stalks $\mathcal{E}_x$ are free. Pick a point $x \in X$ and a enxtend a basis for $\mathcal{E}_x$ to sections $\sigma_1, \ldots, \sigma_r$ of $\mathcal{E}$ over an open U. This gives

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathcal{O}_U^r \xrightarrow{\phi} \mathcal{E} \longrightarrow \mathscr{C} \longrightarrow 0$$

where ϕ sends the i-th basis vector to σ_i. The cokernel $\mathscr{C}$ is finitely generated so its support is closed, and it is a proper subset because $\mathscr{C}_x = 0$. Shrinking U, we may thus assume that $\mathscr{C} = 0$. Hence $\mathscr{K}_x \otimes k(x) = 0$ since $\mathcal{E}_x$ is free, and $\phi \otimes \mathrm{id}_{k(x)}$ is an isomorhism. The kernel $\mathscr{K}$ is of finite type since $\mathcal{E}$ is of finite presentation, and its support is therefore closed. It does not contain x, and so has a complement which is a non-empty open neigbourhood V about x where $\mathscr{K}$ restricts to zero. Hence $\phi|_V$ is an isomorpism. □

The hypothesis that X be locally Noetherian may be relaxed; what is needed is that sheaves of finite type are finitely presented, which certainly is the case over locally Noetherian schemes. Examples of schemes for which the lemma fails are exotic and rather involved (we give one below, Example 19.16). A simple example that the coherence hypothesis is necessary appears already on the spectrum of a DVR, a continuation of Example 14.23 on page 224.

Example 19.5. Let A be DVR with fraction field K, and let x and η be respectively the closed and the open point of $X = \operatorname{Spec} A$. Let $\mathcal{E}$ be the $\mathcal{O}_X$-module with $\Gamma(X, \mathcal{E}) = A$ and $\Gamma(\{\eta\}, \mathcal{E}) = K$, and with the restriction map being the zero map. Then $\mathcal{E}$ is an $\mathcal{O}_X$-module with exactly the same stalks as the structure sheaf $\mathcal{O}_X$, but it is not locally free (in fact, it is not even quasi-coherent).

19.2 Examples

Example 19.6 (The tangent bundle of the 2-sphere)**.** Consider $X = \operatorname{Spec} A$ where we put $A = \mathbb{R}[x, y, z]/(x^2+y^2+z^2-1)$, and consider the A-module homomorphism $\phi\colon A^3 \to A$ given by multiplication by the vector $V = (x, y, z)$. Then $M = \operatorname{Ker} \phi$ gives rise to a quasi-coherent sheaf $\mathcal{T} = \widetilde{M}$. Any element in the kernel corresponds to a vector of polynomials $(p, q, r) \in A^3$ so that

$$xp + yq + zr = 0$$

On $U = D(x)$ we may divide by x, and solve for p, so (p, q, r) is uniquely determined by the elements q, r. Conversely, given any pair q, r of elements in A, we may define the element $(-x^{-1}(yq + zr), q, r)$ which lies in M_x. This implies that $M_x \simeq A^2$. A similar argument works for y and z, showing that $\mathcal{T}$ is locally free of rank 2.

It is a non-trivial fact that M is not free, i.e. not isomorphic to A^2. Every element of A^3 gives a vector field on the sphere S^2. For instance, $(x, y, z) \in A^3$ defines the vector field normal to the sphere which points out from the origin to the point (x, y, z). Any element of M therefore gives a tangent vector to S^2. If M were free, elements of a basis would be non-vanishing vector fields on S^2, which is impossible (from topological reasons).

Example 19.7. Let k be a field and let $R = k[u_0, \ldots, u_n]$. Let further $\mathbb{A}^{n+1} = \operatorname{Spec} R$ and $U = \mathbb{A}^{n+1} - \{0\}$. Consider the eaxct sequence of R-modules

$$0 \longrightarrow R \xrightarrow{\ \phi\ } R^{n+1} \longrightarrow M \longrightarrow 0$$

where the map ϕ sends a plolynmial p to $\sum_i px_ie_i$ where e_i is the i-th standard basis vector. We contend that the restriction $\mathcal{E} = \widetilde{M}|_U$ is a locally free sheaf of rank n. Taking tildes and restricting to U we obatin the sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_U^{n+1} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{19.3}$$

of sheaves on U. Over the distinguished open set $D(x_i)$, this sequence splits since the map π

that sends $\sum_j a_j e_j$ to $a_i x_i^{-1}$ is section of ϕ. Consequently $\mathcal{E}_{D(x_i)} \simeq \mathcal{O}^n_{D(x_i)}$. The sheaf $\mathcal{E}$ is not free, and we will come back to that later.

Example 19.8. Let $\mathcal{E}$ a be finitely presented quasi-coherent sheaf on an integral scheme X and assume that the dimension $\dim_{k(x)} \mathcal{E}_x \otimes_{\mathcal{O}_{x,x}} k(x)$ is the same for all points $x \in X$. Then $\mathcal{E}$ will be locally free. Citing Lemma 19.4 it suffices to show that if M is a finitely presented module over a local domain A with fraction field K and residue class field k and M satisfying $\dim_{k(x)} M \otimes_A k(x) = \dim M \otimes_A K$, then M is free. This is a standard application of Nakayama's lemma: lifting a basis for $M \otimes_A k(x)$ to elements in M, we find an exat sequence

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow A^r \xrightarrow{\ \phi\ } M \longrightarrow 0$$

where ϕ is surjective after Nakayama's lemma. Tensoring the sequence with K, yields $\operatorname{Ker}\phi \otimes_A K = 0$ since $M \otimes_A K$ is a vector space over K of rank $r = \dim_{k(x)} M \otimes_A k(x)$. It follows that K is torsion module, but being contained in a free module over an integral domain, it is torsion free, so $K = 0$.

Example 19.9. If X is an integral scheme and $\phi\colon \mathcal{E} \to \mathcal{F}$ is map of locally free sheaves. Assume that $\dim_{k(x)} \phi_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is constant for $x \in X$. Then $\operatorname{Coker}\phi$ is locally free.

Exercise 19.2.1. Let $\mathcal{E}$ be of finite presentation on an integral scheme X. Show that there is an open sense subset $U \subset X$ where $\mathcal{E}|_U$ is free. Assume that the closed points are dense in X, and that $\dim_{k(x)} E_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is constant for all closed points $x \in X$. Show that $\mathcal{E}$ is locally free. Note that this applies to sheaves on varieties.

Exercise 19.2.2. The most general version of the algebraic statement in Example 19.7 is the following. Let A be a reduced ring and M a finitely generated A-module. Assume that $\dim_{k(\imath)} K \otimes_A k(\mathfrak{p})$ is constant for all maximal and all minimal prime ideals in A. Then M is free. Show this.

.

Example 19.10 (The four-dimensional quadric hypersurface)**.** Let k be a field and let $R = k[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}]$. Consider the matrix

$$M = \begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} \\ -p_{01} & 0 & p_{12} & p_{13} \\ -p_{02} & -p_{12} & 0 & p_{23} \\ -p_{03} & -p_{13} & -p_{23} & 0 \end{pmatrix}.$$

Let us consider the closed subschemes in $\mathbb{P}^5 = \operatorname{Proj} R$ defined by the conditions that this matrix has a rank less than a given bound. Note that M has rank at most 3 precisely when the determinant vanishes. In fact, this matrix M has the special property that the determinant is a square: one computes that $\det M = q^2$ where

$$q = p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$$

It is a fact that the rank of an antisymmetric matrix is even, so the $g3 \times 3$-minors of M are all identically zero. The locus of points where M has rank at most 2 is given by the ideal generated by the 2×2-minors, which by direct calculation has radical equal to the irrelevant ideal R_+. Consider the exact sequence

$$0 \to R(-1)^4 \xrightarrow{M} R^4 \to \operatorname{Coker} M \to 0.$$

Applying the tilde-functor we obtain an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1)^4 \to \mathcal{O}_{\mathbb{P}^5}^4 \to \mathcal{F} \to 0 \tag{19.4}$$

where $\mathcal{F} = \widetilde{\operatorname{Coker} M}$, and by Exercise 19.2.1 the sheaf $\mathcal{F}$ is locally free of rank two.

Consider the quadric hypersurface $X = V(q)$ and let $\iota\colon X \to \mathbb{P}^5$ denote the inclusion. Applying, ι^* we arrive at an exact sequence of sheaves on X

$$\mathcal{O}_X(-1)^4 \to \mathcal{O}_X^4 \to \mathcal{E} \to 0$$

where $\mathcal{E} = i^*\mathcal{F}$ (recall that i^* is only right-exact). Now the discussion above shows that $\mathcal{E}$ is locally free of rank 2 (as it has rank 2 at all closed points).

19.3 Locally free sheaves on affine schemes

On an affine scheme $X = \operatorname{Spec} A$ every quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ is isomorphic to $\widetilde{M}$ for some A-module M. Thus a natural question is which A-modules give rise to locally free sheaves. The main result of this section is that $\mathcal{E}$ is locally free of finite rank if and only if M is finitely generated and projective.

We recall a few basic facts about projective modules (for a more extensive treatment see the appendix). An A-module M is called *projective* if it is a direct summand in a free module; that is, if there is another module N so that $M \oplus N \simeq A^I$. A module M being projective can further be characterized by saying that the functor $N \mapsto \operatorname{Hom}_A(M, N)$ is exact. Clearly free modules have this property, but examples of projective modules which are not free abound. However, over local rings the two notions are equivalent for finitely generated modules:

Lemma 19.11. A finitely generated projective module M over a local ring A is free.

Proof This is a standard application of Nakayama's lemma. Let $k = A/\mathfrak{m}$ denote the residue field, and consider the module $M \otimes_A k = M/\mathfrak{m}M$, which is a finite dimensional vector space over k. Lifting the elements of a basis to elements m_i of M, we obtain a map $\phi\colon A^r \to M$ that sends the standard basis elements e_i to m_i. Then $\phi \otimes \mathrm{id}_k$ is an isomorphism, so by Nakayama's lemma ϕ is surjective, and we have a short exact sequence

$$0 \to \operatorname{Ker} \phi \to A^r \xrightarrow{\phi} M \to 0.$$

As M is a projective module, this sequence splits. This shows that $\operatorname{Ker} \phi$ is finitely generated, and that the sequence stays exact when tensorized by k. Again, since $\phi \otimes \mathrm{id}_k$ is an isomorphism, it holds that $K \otimes_A k = 0$, and hence that $K = 0$, once more by Nakayama's lemma. It follows that $M \simeq A^r$, and is M free. □

Being projective is a local property for finitely presented modules:

Lemma 19.12. A finitely presented module is projective if and only if $M_{\mathfrak{p}}$ is projective for all $\mathfrak{p} \in \operatorname{Spec} A$.

Proof That being projective is preserved under localization is clear. For the converse, the salient point is that for finitely presented modules forming hom's commutes with localization; that is, for all $\mathfrak{p} \in \operatorname{Spec} A$ it holds that $\operatorname{Hom}_A(M,N)_{\mathfrak{p}} = \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ when M is finitely presented. Coupling this with the standard facts that localisation is an exact operation and being zero is a local property, the lemma follows. □

Proposition 19.13. Let $X = \operatorname{Spec} A$ where A is Noetherian, and let $\mathcal{F} = \widetilde{M}$ be a coherent sheaf. The following are equivalent:

(i) $\mathcal{F}$ is locally free;
(ii) $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module for all $x \in X$;
(iii) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec} A$;
(iv) M is a *projective* A-module.

Proof That (i) and (ii) We have already seen that (i) and (ii) are equivalent (Lemma 19.4), and that (ii) and (iii) are equivalent follows by definition of $\widetilde{M}$. Finally, (iii) and (iv) are equivalent by the two preceding lemmas. □

Example 19.14. This is a minimalistic example of a projective module that is not free (see also Example 19.1 above). Let $A = \mathbb{Z}/2 \times \mathbb{Z}/2$ and consider the ideal $I = \mathbb{Z}/2 \times (0)$. Then I is a projective A-module, since if $J = (0) \times \mathbb{Z}/2$, we have $I \oplus J \simeq A$. However, I is not free because any free A-module must have at least four elements!

The sheaf $\widetilde{I}$ on $\operatorname{Spec} A$ is thus locally free, but not free. Note that $\operatorname{Spec} A$ is the disjoint union of two copies of $\operatorname{Spec} \mathbb{Z}/2$, and $\widetilde{I}$ restricts to the structure sheaf on one of these and to the zero sheaf on the other.

Example 19.15. A less trivial example arises in number theory. We consider $A = \mathbb{Z}[i\sqrt{5}]$ and the ideal $\mathfrak{a} = (2, 1 + i\sqrt{5})$. Then a direct computation shows that $\mathfrak{a} \oplus \mathfrak{a} \simeq A \oplus A$, so $\mathfrak{a}$ is projective. However, $\mathfrak{a}$ is an ideal in A, so it is free if and only if it is principal. We therefore conclude that it is not free.

Example 19.16 (An exotic example)**.** Let $R = \prod_{i=1}^{\infty} \mathbb{F}_2$ be a direct product of countably many copies of the field $\mathbb{F}_2$ with two elements, and let $I = \bigoplus_{i=1}^{\infty} \mathbb{F}_2$. Then I is an ideal and R/I is locally free in the sense that $(R/I)_{\mathfrak{p}}$ is free for all primes $\mathfrak{p}$, but R/I is not projective.

Elements of R are sequences $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ where $\alpha_i \in \{0, 1\}$, and I consists of the α's with only finitely many non-zero components. Let e_i denote the sequence with a 1 in slot i and a 0 in all the others. Then I is generated by the e_i's. Moreover, no non-zero element in R is killed by all the e_i's, and this shows that R/I cannot be a summand in a free module.

For the other part of the claim, let $\mathfrak{p}$ be a prime ideal in R. If $I \not\subset \mathfrak{p}$, it holds that $(R/I)_{\mathfrak{p}} = 0$, hence is free. So assume that $I \subset \mathfrak{p}$. For each element $\alpha \in I$, it holds that $\alpha_i = 0$ when $i \geqslant n$ for some n. Let β be the sequence with $\beta_i = 0$ for $i < n$ and $\beta_i = 1$ for $i \geqslant n$. Then β kills α. But $1 = \sum_{i<n} e_i + \beta$ and as $\sum_{i<n} e_i \in I \subset \mathfrak{p}$, it follows that $\beta \notin \mathfrak{p}$.

Hence α maps to 0 in $I_{\mathfrak{p}}$, and as this holds for all $\alpha \in I$, it ensues that $I_{\mathfrak{p}} = 0$. Consequently $(R/I)_{\mathfrak{p}} = R_{\mathfrak{p}}$ is free.

Exercise 19.3.1. Let $X = \operatorname{Spec} A$, where $A = \prod_{i=0}^{\infty} \mathbb{Z}$. Show one may give $M = \mathbb{Z}$ an A-module which is projective, but not free.

Exercise 19.3.2 (Torsion sheaves)**.** Let X be an integral scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Define for each open set $U \subset X$, a subgroup $T(U) \subset \mathcal{F}(U)$ consisting of all the elements $m \in \mathcal{F}(U)$ such that the germ m_x is torsion in $\mathcal{F}_x$ for all $x \in X$, i.e., $a_x \cdot m_x = 0$ for some non-zero $a_x \in \mathcal{O}_{X,x}$.

a) Show that T is a subsheaf of $\mathcal{F}$. Also, show that T is quasi-coherent. T is called the *torsion subsheaf of* $\mathcal{F}$; another notation for it is $\mathcal{F}_{\text{tors}}$.
b) Let $\mathscr{K}$ denote the constant sheaf on $K = k(X)$. Define a map of sheaves
$$\nu : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{K}.$$
Show that $T = \operatorname{Ker} \nu$.
c) A sheaf is called *torsion free if* $\mathcal{F}_{\text{tors}} = 0$. Show that the quotient $\mathcal{F}/T$ is always torsion free, i.e., $(\mathcal{F}/T)_{\text{tors}} = 0$.
d) Show that any locally free sheaf is torsion free.

19.4 Properties of locally free sheaves

From the previous proposition, local properties of coherent locally free sheaves are obtained from corresponding properties of coherent projective modules. And by using sufficiently fine affine covers, one may even (at least, when maps are globally defined) reduce to the case of free modules.

In Example 19.3 we saw that the direct sum of two locally free sheaves is locally free. I the same manner, numerous of the standard operation in commutative algebra when performed on locally free sheaves, yield locally free sheaves. The ensuing formulas are indispensable when working with locally free sheaves, and we summarize some in Proposition 19.17below.

So let $\mathcal{E}$ and $\mathcal{F}$ be two locally free sheaves of finite rank. As in Example19.3 there are covers of X that trivialize both of them and let $\{U_i\}$ be one, and refining it if necessary, we may assume that the U_i's are affine. When $U = \operatorname{Spec} A$ is an open affine subscheme, E and F will denote A-modules such that $\mathcal{E}|_U = \widetilde{E}$ and $\mathcal{F}|_U = \widetilde{F}$.

On $U = \operatorname{Spec} A$ the tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ restricts to
$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}|_U = (E \otimes_A F)^{\sim}$$
which obviously is free of finite rank when E and F are, and the rank will be $r(E)r(F)$.

Let us take a closer look at the hom-sheaves. Each A-module M has a *dual* module $M^{\vee} = \operatorname{Hom}_A(M, A)$. When M is a free module of finite rank, the dual $M^{\vee}$ will likewise be free, and it will have the same rank as M: given a basis $\{e_i\}$ for M, the maps $\delta_i \colon M \to A$ with $e_i \mapsto 1$ and $e_j \mapsto 0$ when $i \neq j$ form a basis for $M^{\vee}$, called the *dual basis*. For any module$M$ there is a canonical evaluation map $M \to (M^{\vee})^{\vee}$ defined by $m \mapsto (\phi \mapsto \phi(m))$, and when M is free, it is straightforward to verify it is an isomorphism.

Given another module N, there is a canonical map

$$M^\vee \otimes_A N \to \operatorname{Hom}_A(M, N),$$

which is given by the assignment $\phi \otimes n \mapsto (m \mapsto \phi(m)n)$. When M and N are free of finite rank it will be an isomorphism; indeed, when $N = A$, this is obviously true, and both sides are additive in N.

Returning to the locally free sheaves on X, we let $\mathcal{E}$ and $\mathcal{F}$ be two of finite rank. The Hom-sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is then well-behaved; it is quasi-coherent and restricts to affine open sets $U = \operatorname{Spec} A$ as expected:

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})|_U = \operatorname{Hom}_A(E, F)\tilde{}.$$

The canonical maps in (19.4) are defined over each open affine and glue together to a global map

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}).$$

When both $\mathcal{E}$ and $\mathcal{F}$ are locally free of finite rank, it will be an isomorphism.

The next proposition summarises some of the basic properties of locally free sheaves of finite rank.

Proposition 19.17. Let X be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be two locally free $\mathcal{O}_X$-modules of finite rank.

(i) The direct sum $\mathcal{E} \oplus \mathcal{F}$ is locally free of rank $r_x(\mathcal{E}) + r_x(\mathcal{F})$;
(ii) The tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free of rank $r_x(\mathcal{E}) \cdot r_x(\mathcal{F})$;
(iii) The dual sheaf $\mathcal{E}^\vee$ is locally free of rank $r_x(\mathcal{E})$, and the canonical evaluation map $(\mathcal{E}^\vee)^\vee \to \mathcal{E}$ is an isomorphism;
(iv) The canonical map $\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is an isomorphism; and rank of $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ equals $r_x(\mathcal{E})r_x(\mathcal{F})$.

Example 19.18. Suppose $\mathcal{E}$ is locally free of rank r. Let U_i be a trivializing cover, and let τ_{ji} denote the gluing functions for $\mathcal{E}$. As before, we interpret τ_{ji} as an $r \times r$ matrix with entries in $\mathcal{O}_X(U_i \cap U_j)$. Then $\mathcal{E}^\vee$ is obtained by the gluing matrices $\nu_{ji} = (\tau_{ji}^t)^{-1}$.

Example 19.19. Suppose $\mathcal{E}$ and $\mathcal{F}$ are locally free of ranks r and s respectively. After refining, we may assume that they admit the same trivializing cover. Suppose that the gluing functions are given by τ_{ji} and ν_{ji} respectively. Then $\mathcal{E} \oplus \mathcal{F}$ is obtained by gluing together the different $\mathcal{O}_{U_i}^r \oplus \mathcal{O}_{U_i}^s$ with help of the matrices

$$\Phi_{ji} = \begin{pmatrix} \tau_{ji} & 0 \\ 0 & \nu_{ji} \end{pmatrix}$$

Thus, for instance, the sheaf $\mathcal{O}_{\mathbb{P}^1_A} \oplus \mathcal{O}_{\mathbb{P}^1_A}(-1)$ on the projective line is obtained using the gluing matrix

$$\tau_{01} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$

over $U_0 \cap U_1 = \operatorname{Spec} A[u, u^{-1}]$ with $U_0 = \operatorname{Spec} A[u]$ and $U_1 = \operatorname{Spec} A[u^{-1}]$.

When it comes to tensor products the gluing functions of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ will of course be the tensor products $\tau_{ij} \otimes \nu_{ij}$, whose matrices will be the Kronecker products of the matrices of the τ_{ij}'s and ν_{ij}'s. In the particular case that $\mathcal{F}$ is of rank one the gluing maps are just multiplication by invertible sections g_{ij} in $\mathcal{O}_X(U_{ij})$, and the gluing maps of $\mathcal{E} \otimes \mathcal{F}$ will be $g_{ij}\tau_{ij}$.

19.5 Locally free sheaves on $\mathbb{P}^1$

In 1955, Grothendieck wrote his paper "Sur la classification des fibres holomorphes sur la sphere de Riemann", showing that any locally free sheaf on the projective over a field splits as a sum of invertible sheaves:

Theorem 19.20. Let $X = \mathbb{P}^1_k$ and let $\mathcal{E}$ be a locally free sheaf of rank r. Then there are integers $a_1, \ldots, a_r$ such that

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r). \tag{19.5}$$

Grothendieck's proof was sheaf-theoretic, but in fact this is a rather elementary result which has been rediscovered and reproved several times. For instance, Grothendieck was not aware of the following result, due to Dedkind–Weber from 1882. Dedekind, Weber. Theorie der algebraischen Funktionen einer Veränderlichen', Crelle's Journal, 1882

Lemma 19.21 (Dedekind–Weber). Let k be a field and let $A \in GL_r(k[x, x^{-1}])$. Then there exist matrices $B \in GL_r(k[x])$ and $C \in GL_r(k[x^{-1}])$ such that

$$BAC = \begin{pmatrix} x^{a_1} & & 0 \\ & \ddots & \\ 0 & & x^{a_r} \end{pmatrix}. \tag{19.6}$$

This lemma is completely elementary, and can be proved by induction on r with only basic row-operations on matrices.

In any case, Theorem 19.20 follows immediately from the description of quasi-coherent sheaves on $\mathbb{P}^1_k$ from Example 15.11. In the notation of that example, we have $M_0 = k[x]^r$, $M_1 = k[x^{-1}]^r$ and $\tau : k[x^{\pm 1}]^r \to k[x^{\pm 1}]^r$. The lemma above implies that after changing bases, the map τ is given by a diagonal matrix 19.6. Hence $\mathcal{E}$ splits as (19.5).

Exercise 19.5.1. Prove Lemma 19.21 for $r = 2$.

19.6 Pushforwards and pullbacks

A word of warning: the pushforward of a locally free sheaf is not locally free in general. For instance, if $\iota\colon \operatorname{Spec} k \to \mathbb{A}^1_k$ is the inclusion of a closed point p in $\mathbb{A}^1_k$, $\mathcal{F} = i_*\mathcal{O}_{\operatorname{Spec} k}$ has stalk $k(p)$ at p but zero stalks everywhere else, so $\mathcal{F}$ is not locally free. For pullbacks, however, we have the following:

Proposition 19.22. Let $f\colon X \to Y$ be a morphism of schemes. If $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then $f^*\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of the same rank as $\mathcal{E}$.

Proof Let $U \subset Y$ be an open over which $\mathcal{E}$ is trivial; that is $\mathcal{E}|_U \simeq \mathcal{O}_U^r$. Then, since $f^*\mathcal{O}_Y = \mathcal{O}_X$ for any morphism and f^* is an additive functor, we see that $f^*\mathcal{E}|_{f^{-1}U} \simeq \mathcal{O}_{f^{-1}U}^r$. Hence $f^*\mathcal{E}$ is locally free. □

Note that if τ_{ij} are the gluing matrices for $\mathcal{E}$ over U_{ij} corresponding to a trivializing cover $\{U_i\}$, the cover $f^{-1}U_i$ will be trivialising for $f^*\mathcal{E}$ and the gluing matrices are just the pullbacks of those of $\mathcal{E}$; i.e. the images under the maps $f^\sharp|_{U_{ij}}\colon \mathcal{O}_Y(U_{ij}) \to f_*\mathcal{O}_X(U_{ij})$.

Example 19.23. Let k be a field and $\mathbb{P}^1_k = \operatorname{Proj} k[u_0, u_1]$. Fix a natural number n and consider the map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $u_i \mapsto u_i^n$ introduced i Example 9.23 on page 142. We contend that $f^*\mathcal{O}_{\mathbb{P}^1_k}(m) = \mathcal{O}_{\mathbb{P}^1_k}(nm)$.

With $u = u_1u_0^{-1}$ the projective line $\mathbb{P}^1_k$ is as usual covered by the open sets $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$, and the transition function

$$\tau_m\colon \mathcal{O}_{U_1}|_{U_0\cap U_1} \to \mathcal{O}_{U_0}|_{U_0\cap U_1}$$

of the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k}(m)$ acts as multiplication by u^m, as explained in Section 7.2.

The map f maps each of the open sets U_i into itself, and the action of $f^\sharp$ on u is $u \mapsto u^n$. It follows that the pullback of the transition function τ_m is just multiplication by u^{nm}:

$$\tau_{nm}\colon \mathcal{O}_{U_1}|_{U_0\cap U_1} \to \mathcal{O}_{U_0}|_{U_0\cap U_1}.$$

Hence $f^*\mathcal{O}_{\mathbb{P}^1_k}(m) = \mathcal{O}_{\mathbb{P}^1_k}(nm)$.

Example 19.24. Letting $n = 2$ in the previous exercise we obtain the 'squaring-morphism'

$$f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$$

We claim that the pushforward $f_*\mathcal{O}_{\mathbb{P}^1_k}$ is locally free of rank two. We shall use two copies $\operatorname{Proj} k[t_0, t]$ and $\operatorname{Proj} k[u_0, u_1]$ and the map is induced by the assignments $u_i \mapsto t_i^2$. We let $u = u_1u_0^{-1}$ and $t = t_0t_1^{-1}$.

Over the local chart $U_0 = \operatorname{Spec} k[u]$ the map f is induced by $k[u] \mapsto k[t]$ with $u \mapsto t^2$, and over the chart $U_1 = \operatorname{Spec} k[u^{-1}]$ it is given by the map $k[u^{-1}] \to k[t^{-1}]$ such that $u^{-1} \mapsto t^{-2}$.

It follows that the restriction $f_*\mathcal{O}_{\mathbb{P}^1}|_{U_0}$ to U_0 equals the tilde of $k[t]$ as a $k[u]$-module, which clearly is free with basis 1 and t; indeed, one has $k[t] = k[t^2] \oplus k[t]t = k[u] \oplus k[u]t$. In a symmetric way, on the chart $U_1 = \operatorname{Spec} k[u^{-1}]$ the pushforward $f_*\mathcal{O}_{\mathbb{P}^1}$ restricts to the tilde of the module $k[u^{-1}] \oplus k[u^{-1}]t^{-1}$. Hence $f_*\mathcal{O}_{\mathbb{P}^1}$ is locally free of rank 2.

In fact, one can readily check that there is an isomorphism $f_*\mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$ where $\mathcal{O}_{\mathbb{P}^1_k}(-1)$ Indeed, over $U_0 \cap U_1$ we have the equality

$$k[t, t^{-1}] = k[t^2, t^{-2}] \oplus k[t^2, t^{-2}]t = k[t^2, t^{-2}] \oplus k[t^2, t^{-2}]t^{-1}$$

or in other words

$$k[t, t^{-1}] = k[u, u^{-1}] \oplus k[u, u^{-1}] = k[u, u] \oplus k[u, u]t^{-1}$$

Now, $p(u, u^{-1})t^{-1} = p(u, u^{-1})u^{-1}t$ so when the equality $k[t^2, t^{-2}]t^{-1} = k[t^2, t^{-2}]t$ is translated into a gluing function $k[u, u^{-1}] \to k[u, u^{-1}]$, it becomes multiplication by u^{-1}; that is, the corresponding sheaf is $\mathcal{O}_{\mathbb{P}^1_k}(-1)$.

Example 19.25. Let Y be an integral scheme with the property that the local ring $\mathcal{O}_{Y,y}$ at each closed point y is a DVR (one would call Y a regular curve). Let X be a Noetherian scheme and $f\colon X \to Y$ a finite morphism and assume that all components of X (including embedded ones) dominate Y. Then $f_*\mathcal{O}_X$ is locally free of finite rank.

After localization this boils down to the case that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ with A a DVR. Each component $V(\mathfrak{p})$ of X dominates Y so $\mathfrak{p}$ pulls back to the zero ideal in A (item (iii) of Proposition **??** on page ??), and since the union of the primes $\mathfrak{p} \subset B$ corresponding to the components of X, equals the set of zero divisors, we infer that B is a torsion free A-module, and it is finitely generated by hypothesis. The claim then follows by the general property of DVR's that finitely generated torsion free modules are free.

Exercise 19.6.1. With the setup of Example 19.23, show that $f_*\mathcal{O}_{\mathbb{P}^1_k}$ is locally free of rank n, and in fact that the more precise formula

$$f_*\mathcal{O}_{\mathbb{P}^1_k} = \bigoplus_{0 \leqslant i \leqslant n-1} \mathcal{O}_{\mathbb{P}^1_k}(-i).$$

holds true.

Exercise 19.6.2 (The projection formula)**.** Let $f\colon X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{E}$ a locally free sheaf of finite rank. Show that there is a natural isomorphism of $\mathcal{O}_Y$-modules

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \simeq f_*(\mathcal{F}) \otimes \mathcal{E}.$$

19.7 Zero sets of sections

Let $\mathcal{E}$ be a locally free sheaf on a scheme X and let $x \in X$ be a point. We will call the *fibre of $\mathcal{E}$ at x* the $k(x)$-vector space $\mathcal{E}(x)$

$$\mathcal{E}(x) = \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x \simeq \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

If $U \subset X$ is an open subset containing x and $s \in \Gamma(U, \mathcal{E})$ is a section of $\mathcal{E}$ over U, we shall denote by $s(x)$ the image of the germ $s_x \in \mathcal{E}_x$ in the fibre $\mathcal{E}(x)$. This is in close analogy with what we called the 'value' of a regular function in Chapter 2.

Definition 19.26. Let $\mathcal{E}$ be an locally free sheaf on the scheme X, and suppose $s \in \Gamma(X, \mathcal{E})$ is a global section. We define the *zero set of s* by

$$V(s) = \{x \in X \mid s(x) = 0\}.$$

Also, we define the open set X_s by

$$X_s = \{\, x \in X \mid s(x) \neq 0 \,\}.$$

Equivalently, X_s is the set of points x where $s \notin \mathfrak{m}_x\mathcal{E}_x$.

The set $V(s)$ is indeed a closed subset of X: the sheaf $\mathcal{E}$ is locally free, so every point has an open affine neighbourhood U such that $\mathcal{E}|_U \simeq \mathcal{O}_X|_U$, and we may safely assume that $\mathcal{E} = \mathcal{O}_X^r$ with $X = \operatorname{Spec} A$. This brings us back to the 'function case': the section s is an element in A^r, and $V(s)$ coincides with the usual closed set. It follows that $X_s = X - V(s)$ is also open in X.

Proposition 19.27. Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{E}$ be a locally free sheaf on Y. Then

$$f^{-1}(V(s)) = V(f^*s) \quad \text{ and } f^{-1}(X_s) = X_{f^*s}.$$

Proof For each of these statements, we may reduce to the case $X = \operatorname{Spec} B; Y = \operatorname{Spec} A$ and $L = \mathcal{O}_Y^r$. In that case (i) follows from the fact that $f^{-1}(V(a)) = V(\phi(a))$ for $a \in A$, which we have seen several times before. □

The set $V(s)$ just defined is a priori just a closed subset of X, but we can put a canonical scheme structure on it as follows. We may view a global section $s \in \Gamma(X, \mathcal{E}) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{E})$, as a map of $\mathcal{O}_X$-modules $s : \mathcal{O}_X \to \mathcal{E}$. Applying $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$, we get a map

$$s^\vee : \mathcal{E}^\vee \to \mathcal{O}_X \tag{19.7}$$

The image of $s^\vee$ is a quasi-coherent ideal sheaf of $\mathcal{O}_X$. We define the *subscheme of zeroes of* s to be the closed subscheme $Z(s)$ of X.

Example 19.28. Let $X = \operatorname{Spec} A$, and $\mathcal{E} = \mathcal{O}_X^r$. Then a section $s \in \Gamma(X, \mathcal{E})$ is given by an r-tuple $(f_1, \ldots, f_r) \in A^r$ of elements in A. The map $s^\vee$ is simply the tilde of the map $A^r \to A$, that sends the i-th basis vector e_i to f_i. Therefore, $Z(s)$ is simply the usual subscheme given by the ideal $I = (f_1, \ldots, f_r)$. Locally, any subscheme $Z(s)$ looks like this example.

Exercise 19.7.1. Show that the subscheme $Z(s)$ satisfies the following universal property: A morphism $f : T \to X$ satisfies $f^*s = 0$ if and only if it factors through $Z(s)$. (Hint: Understand the subscheme on each open affine $\operatorname{Spec} A \subset X$ first. Reduce to the case $\mathcal{E} = \mathcal{O}_X$.)

19.8 Globally generated sheaves

Definition 19.29. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *globally generated* (or *generated by global sections*) if there is a family of sections $s_i \in \mathcal{F}(X)$, $i \in I$, such that the germs of s_i generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module for each $x \in X$.

Equivalently, $\mathcal{F}$ is globally generated if there is a surjection

$$\mathcal{O}_X^I \to \mathcal{F} \to 0$$

for some index set I. In particular, any quotient of a globally generated sheaf is also globally generated.

Let us consider a few examples:

Example 19.30. On an affine scheme any quasi-coherent sheaf is globally generated. Indeed, if $X = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{M}$, for some A-module M, then picking any presentation $A^I \to M \to 0$ for M and applying tilde shows that $\mathcal{F}$ is globally generated.

Example 19.31. Let R be a graded ring generated in degree 1 and set $X = \operatorname{Proj} R$. Then $\mathcal{F} = \mathcal{O}(1)$ is globally generated. Indeed, the only way $\mathcal{F}$ could fail to be globally generated is that there is a point $x \in X$ for which all sections $s \in \Gamma(X, \mathcal{O}(1)) = R_1$ simultaneously vanish. However, by assumption R_1 generates the irrelevant ideal, so this is impossible.

On the other hand, if R is not generated in degree 1, then it can happen that the sheaf $\mathcal{O}(1)$ has no global sections at all. This happens for instance for the weighted projective space $\mathbb{P}(2,3,4) = \operatorname{Proj} k[x_2, x_3, x_4]$ (with $\deg x_i = i$). The sheaf $\mathcal{O}(-1)$ is likewise not typically globally generated (unless, say, X is a point).

Example 19.32. For a closed subscheme $Y \subset X$, the structure sheaf $i_*\mathcal{O}_Y$ is globally generated (generated by the section '1'). On the other hand the corresponding ideal sheaf $\mathcal{I}$ is typically not globally generated. For instance, if Y a closed point in $\mathbb{P}^1_k$, then $\mathcal{I}_Y \simeq \mathcal{O}(-1)$, which has no global sections.

Example 19.33. The locally free sheaves from Section **??** are both globally generated. For instance, the sheaf $\mathcal{E}$ from (**??**) admits a surjection $\mathcal{O}^{n+1} \to \mathcal{E} \to 0$.

Proposition 19.34. Let $f : X \to Y$ be a morphism of schemes and let L be an invertible sheaf on Y. Then if L is generated by global sections $s_0, \ldots, s_n$, then f^*L is generated by the sections $t_0 = f^*s_0, \ldots, t_n = f^*s_n$, and X is covered by the open sets $X_{t_0}, \ldots, X_{t_n}$.

Proof For each of these statements, we may reduce to the case $X = \operatorname{Spec} B$; $Y = \operatorname{Spec} A$ and $L = \mathcal{O}_Y$. In that case (i) follows from the fact that $f^{-1}(V(a)) = V(\phi(a))$, which we have seen several times before.

For (ii), we note that hypothesis gives that the sections $s_0, \ldots, s_n$ are elements in A that generate the unit ideal. But then clearly the same holds for the pullbacks $\phi(s_0), \ldots, \phi(s_n)$. □

Example 19.35. For the pushforward, $f_*\mathcal{F}$ may fail to be globally generated even when $\mathcal{F}$ is the structure sheaf. For example, if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is the 'squaring map', i.e., the map induced by $k[u^2, v^2] \subset k[u, v]$, then $f_*\mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The latter sheaf is not globally generated, since it has $\mathcal{O}(-1)$ as a quotient.

20

Differentials

So far we have defined schemes and surveyed a few of their basic properties (e.g. how to study sheaves on them). In this chapter, we introduce tangent spaces and *Kähler differentials*, which allow us in some sense to do calculus on schemes. This in turn will allow us to define the most important sheaves in algebraic geometry, namely, the cotangent sheaf, the tangent sheaf, and the sheaves of n-forms.

Differentials appear prominently throughout many areas of mathematics, multivariable analysis, manifolds and differential geometry to mention a few. In algebraic geometry they are introduced algebraically using their formal properties and are usually referred to as Kähler differentials after the German mathematician Erich Kähler (1906–2000).

20.1 Derivations and Kähler differentials

We will work over a base ring A, and B will be an A-algebra. We will also need a B-module M. The geometric picture to have in mind is that $A = k$, where k is a field, and $X = \operatorname{Spec} B \to \operatorname{Spec} k$ is the structure morphism.

Definition 20.1. An *A-derivation* (from B with values in M) is an A-linear map $D\colon B \to M$ satisfying the product rule, also called the Leibniz rule:

$$D(bb') = bD(b') + b'D(b).$$

Given that the product rule holds, D is A-linear if and only if it vanishes on all elements of the form $a \cdot 1$ with $a \in A$; if D is A-linear, we have $D(a \cdot 1) = aD(1) = 0$ since $D(1) = 0$, which follows from the product rue applied to $1^2 = 1$. If D vanishes on A, the product rule gives $D(ab) = aD(b) + bD(a) = aD(b)$. We may therefore think of the elements in B of the form $a \cdot 1$ as 'constants'; note however, that a derivation also can vanish on other elements in B (a silly example is the zero map, which is a derivation). For a more constructive example see Example 20.8 below).

Example 20.2. The map of the polynomial ring $B = k[x]$ to itself which is given by $P(t) \mapsto P'(t)$, is a k-derivation. More generally, the partial differential operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$, as well as their k-linear combinations, are k-derivations on the polynomial ring $k[x_1, \ldots, x_n]$.

A straightforward induction shows that the good old rules from calculus have analogues in the abstract situation: it holds true that $D(b^n) = nb^{n-1}D(b)$ and, in case b is invertible in B, that $D(1/b) = -D(b)/b^2$. Moreover, if $P(t)$ is a polynomial in $A[t]$, one has the chain rule $D(P(b)) = P'(t)D(b)$, where $P'(t)$ is the formal derivative defined as $P'(t) = \sum_i ia_i t^{i-1}$ when $P(t) = \sum_i a_i t^i$.

The set of A-derivations $D\colon B \to M$ is denoted by $\mathrm{Der}_A(B, M)$. This set inherits a B-module structure from M, and it is as such naturally a submodule of $\mathrm{Hom}_A(B, M)$. This gives rise to a covariant functor $\mathrm{Der}_A(B, -)$ from Mod_B to itself. More precisely, if $\phi\colon M \to M'$ is a B-linear map, we can map a derivation $D \in \mathrm{Der}_A(B, M)$ to $\phi \circ D\colon B \to M'$, which is in turn an A-derivation of B with values in M'.

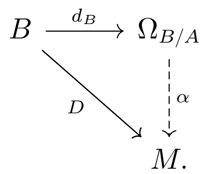
The set of derivations $\mathrm{Der}_A(B, M)$ is also functorial in the base ring A and the A-algebra B; in both cases it is contravariant. If $A \to A'$ is a ring homomorphism, any A'-derivation $B \to M$ is in turn an A-derivation. We therefore obtain an inclusion $\mathrm{Der}_{A'}(B, M) \subset \mathrm{Der}_A(B, M)$.

20.1.1 The module of Kähler differentials

The covariant functor $\mathrm{Der}_A(B, -)$ on the category of B-modules is representable. This simply means that there exists a distinguished B-module $\Omega_{B/A}$ and an isomorphism of functors

$$\mathrm{Der}_A(B, -) \simeq \mathrm{Hom}_B(\Omega_{B/A}, -). \tag{20.1}$$

In more down-to-earth terms, this condition is equivalent to there being a *universal derivation*[1] $d_B\colon B \to \Omega_{B/A}$ that has the following property: For any A-derivation $D\colon B \to M$ there exists a unique B-module homomorphism $\alpha\colon \Omega_{B/A} \to M$ such that $D = \alpha \circ d_B$. In terms of diagrams, we have



To see directly why such a module exists, we can construct it via generators and relations. For each element $b \in B$ introduce a symbol db and consider the free B-module $G = \bigoplus_{b \in B} B db$ they generate. Inside G we have the submodule H generated by the expressions of the form

$$d(b + b') - db - db', \text{ or } \quad d(bb') - bdb' - b'db, \text{ or } \quad da$$

for $b, b' \in B$ and $a \in A$. We then define $\Omega_{B/A} = G/H$, and the map $d_B\colon B \to \Omega_{B/A}$ is given by $d_B(b) = db$. It is well-defined as a group homomorphism since any $\mathbb{Z}$-linear relation among the db's maps to zero in G/H by the imposed additive constraint, and it is a derivation because all relations $d(bb') = bdb' + b'db$ are forced to hold in G/H. Finally, it will be A-linear because $da = 0$ in G/H.

It is not hard to see that this module indeed satisfies the universal property above: given an A-derivation $D\colon B \to M$, we define the B-homomorphism $\alpha\colon \Omega_{B/A} \to M$ by $\alpha(db) = D(b)$ (which is well-defined precisely because D is a derivation!).

[1] The ring A is an essential part of the structure,but for the sake of a practical notation is not shown; when it is necessary to emphasize the base ring, the notation will be $d_{B/A}$

Definition 20.3. The elements of the module $\Omega_{B/A}$ are called the *Kähler differentials*, or simply *differentials* of B over A.

Example 20.4 (Change of constants)**.** To any homomorphism of rings $\rho\colon A \to A'$ corresponds the natural inclusion $\mathrm{Der}_{A'}(B, M) \subset \mathrm{Der}_A(B, M)$, which via the isomorphism (20.1) induces a surjective B-linear map

$$\beta\colon \Omega_{B/A} \to \Omega_{B/A'}.$$

It is just the B-linear map that arises from $d_{B/A'} \circ \rho$ by the universal property of $d_{B/A}$. In terms of the generating sets in the construction above, the map β simply sends db to db; note that $da' \mapsto 0$ for all $a' \in B$ coming from A'.

Proposition 20.5 (Polynomial rings). Let A be any ring and let $B = A[x_1, \ldots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $dx_1, \ldots, dx_n$ and the universal derivation is given by

$$d_B f = \sum (\partial f/\partial x_i) dx_i.$$

Proof The universal property follows from the general chain rule: for any A-derivation $D\colon B \to M$ into a B-module M, the formula

$$D(f) = \sum_i (\partial f/\partial x_i) D(x_i). \tag{20.2}$$

holds true. Indeed, an easy induction, using the product rule, shows it to be true when f is a monomial, and then A-linearity finishes he story. The B-linear map $\alpha\colon \bigoplus_i B dx_i \to M$ which sends each basis element dx_i to $D(x_i)$, will be the wanted factoring map; by the general chain rule (20.2), it satisfies the equality $D = \alpha \circ d_B$. □

20.2 Examples

Here are some more explicit calculations of $\Omega_{B/A}$:

Example 20.6 (Localization)**.** If $B = S^{-1}A$ is a localization of A, then $\Omega_{B/A} = 0$. Indeed, take $b \in B$, and choose $s \in S$ so that $sb \in A$. Then $s d_B b = d_B(sb) = 0$, which implies that $d_B b = 0$ since s is invertible in B.

Example 20.7 (Surjections)**.** Generally, if $\phi\colon A \to B$ is surjective, then $\Omega_{B/A} = 0$, because if $b = \phi(a)$, then $d_B b = a \cdot d_B(1) = 0$ in $\Omega_{B/A}$.

Example 20.8 (Separable field extensions)**.** Let $K = k(a)$ be a separable field extension and let $P(t)$ be the minimal polynomial of a. For any k-derivation $D\colon K \to K$ it holds that $0 = D(0) = D(P(a)) = P'(a)D(a)$. Hence $D(a) = 0$ since $P'(a) \neq 0$ the element a being separable over k. The product rule implies that $D(a^n) = na^{n-1}D(a) = 0$ for each natural number n, and since the powers a^n generate K as a vector space over k, it follows that $D = 0$.

Example 20.9 (Inseparable field extensions)**.** Contrary to the separable ones, inseparable extensions have non-trivial derivations. Let us consider the simplest case when K is obtained by adjoining a p-th root to a field k of characteristic p; that is, $K = k(b)$ with $b^p = a$, where $a \in k$ is not a p-th power. The minimal polynomial of b is $P(t) = t^p - a$, and $K = k[t]/(t^p - a)$. The point is that $P'(t) = pt^{p-1} = 0$, so for each $c \in K$ the k-linear map $k[t] \to K$ given by $Q(t) \mapsto Q'(t)c$ vanishes on $P(t)$ and descends to a k-linear map $D_c \colon K \to K$. Leibniz' rules immediately yields that D_c is a derivation, and as $D_c(b) = c$, the derivation D_c does not vanish. We conclude that $\mathrm{Der}_k(K, K) \simeq K$ and that $\Omega_{K/k} \simeq K$ as well; in fact, D_b serves as a universal derivation.

Example 20.10 (The differentials of a tensor product)**.** Let B and C be two A-algebras. Then the map

$$d \colon B \otimes_A C \to (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A})$$

given as $b \otimes c \mapsto b \otimes d_C c + d_B b \otimes c$ on decomposable tensors and extended by bilinearity is a universal A derivation. We compute

$$\begin{aligned} d(b'b \otimes c'c) &= bb' \otimes (c'd_C c + c d_C c') + (b'd_B b + b d_B b') \otimes cc' = \\ &= b' \otimes c' \cdot (b \otimes d_C c + d_B b \otimes c) + b \otimes c \cdot (b' \otimes d_C c' + d_B b' \otimes c') \end{aligned}$$

and δ is a derivation, and which is universal in view of the formula

$$\gamma(db \otimes c + b' \otimes dc') = 1 \otimes c' \cdot \alpha(b' \otimes 1) + b' \otimes 1 \cdot \beta(1 \otimes c'),$$

which defines the required map $\gamma \colon (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A}) \to M$. Here $\alpha \colon \Omega_{B/A} \to M$ and $\beta \colon \Omega_{C/A} \to M$ are the linear maps corresponding to the derivations $D|_{B \otimes 1} \colon B \to M$ and $D|_{1 \otimes C} \colon C \to M$ and $D \colon B \otimes_A C \to M$ is a given A-derivation.

20.3 Properties of Kähler differentials

There are a few useful ways for computing modules of differentials when changing rings.

20.3.1 Base change

The Kähler differentials behave well with respect to base change:

Proposition 20.11. Let A be a ring and B be an A-algebra, and let A' be another A-algebra. Define $B' = B \otimes_A A'$. Then there is a canonical isomorphism

$$\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B'$$

Proof The universal derivation $d_B \colon B \to \Omega_{B/A}$ induces an A'-linear map

$$d' = d_B \otimes \mathrm{id}_{A'} \colon B' \to \Omega_{B/A} \otimes_A A' = \Omega_{B/A} \otimes_B B'$$

which clearly is a derivation. This will be the required universal derivation of $\Omega_{B'/A'}$, and so the claim follows: let $\iota \colon B \to B' = B \otimes_A A'$ be the canonical map $b \mapsto b \otimes 1$. Given an A'-derivation $D \colon B' \to M$ into a B'-module, the map $D \circ \iota \colon B \to M$ will be an

A-derivation, and consequently it will factor as $\alpha \circ d_B$ for a B-linear map $\alpha\colon \Omega_{B/A} \to M$. The map $\alpha \otimes \mathrm{id}_{A'}\colon \Omega_{B/A} \otimes_A A' \to M \otimes_A A' = M$ then yields the desired factorization of D. □

20.3.2 Two exact sequences

Let A be a ring and let $\rho\colon B \to C$ be a homomorphism of A-algebras. There is natural homomorphism of C-modules

$$\rho_*\colon \Omega_{B/A} \otimes_B C \to \Omega_{C/A}$$

defined by $\rho_*(d_B b \otimes c) = c d_C \rho(b)$. The dual of ρ_* corresponds, under the identification (20.1), to the map $\mathrm{Der}_A(C, N) \to \mathrm{Der}_A(B, N)$ that sends a derivation $D\colon C \to N$ to $D \circ \rho$. (Note that $\mathrm{Hom}_B(\Omega_{A/B}, N) = \mathrm{Hom}_C(\Omega_{A/B} \otimes_B C, N)$ since N is a C-module.)

Moreover, there is a canonical 'change-of-constants-map'

$$\beta\colon \Omega_{C/A} \longrightarrow \Omega_{C/B}$$

as explained in Example 20.4 above.

The next propositions describes the kernel of this 'change-of-constants-map', and as one would suspect, it is generated by the elements shaped like db where $b \in C$ comes from B:

Proposition 20.12. The following sequence of C-modules is exact

$$\Omega_{B/A} \otimes_B C \xrightarrow{\rho_*} \Omega_{C/A} \xrightarrow{\beta} \Omega_{C/B} \longrightarrow 0$$

Proof That $\beta \circ \rho_* = 0$ is clear. Checking exactness amounts to showing that for any C-module N, the dual sequence

$$0 \to \mathrm{Hom}_C(\Omega_{C/B}, N) \to \mathrm{Hom}_C(\Omega_{C/A}, N) \to \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N)$$

is exact, and, as the map β is surjective (Example 20.4), only exactness in the middle is an issue. Note that $\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N) = \mathrm{Hom}_B(\Omega_{B/A}, N)$, so the in view of the constituting isomorphisms (20.1), the sequence can be written as

$$0 \longrightarrow \mathrm{Der}_B(C, N) \longrightarrow \mathrm{Der}_A(C, N) \longrightarrow \mathrm{Der}_A(B, N).$$

The map on the left merely considers a B-derivation to be an A-derivation, whereas the one on the right sends $D\colon C \to N$ to the composition $D \circ \rho$. Saying that D is mapped to zero in $\mathrm{Der}_A(B, N)$, is saying that it vanishes on all elements b in C coming from B, which is equivalent to saying it is a B-derivation; indeed, it will B-linear by Leibniz rule:

$$D(bx) = bD(x) + xD(b) = bD(x),$$

for $x \in C$ and $b \in C$ coming from B. □

In the next proposition, we establish an exact sequence that relates the differentials of an A-algebra B and those of a quotient $C = B/I$. It involves a map $\delta\colon I/I^2 \to \Omega_{B/A} \otimes_B C$ which sends the class of $f \in I$ mod I^2 to $d_B f \otimes 1$, or more formally, which results from applying the tensor functor $-\otimes_B C$ to the restriction $d_B|_I\colon I \to \Omega_{B/A}$. (Note that $I \otimes_B C = I/I^2$ as $C = B/I$).

Proposition 20.13 (Conormal sequence). Suppose that B is an A-algebra. Let $C = B/I$ for some ideal $I \subset B$ and let $\alpha\colon B \to C = B/I$ be the canonical map. Then there is an exact sequence of C-modules

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha*} \Omega_{C/A} \longrightarrow 0.$$

Proof As in the previous proposition it suffices to check that for each C-module N, the dual sequence

$$0 \longrightarrow \mathrm{Der}_A(C, N) \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Hom}_C(I/I^2, N) = \mathrm{Hom}(I, N)$$

is exact. In view of Proposition 20.12 and Exampe 20.7 the map α_* is surjective, and hence the leftmost map is injective. The rightmost map associates to a derivation $D\colon B \to N$ its restriction to I. (Note that this is indeed a homomorphism of C-modules since $IN = 0$). If $D|_I = 0$, clearly D passes to the quotient and yields a $D'\colon C = B/I \to N$, which is a C-derivation since D is a B-derivation. In other words, D lies in the image of $\mathrm{Der}_A(C, N)$, and the sequence is exact in the middle as well. □

Corollary 20.14. Let A be a ring and let B be a finitely generated A-algebra (or a localization of such). Then $\Omega_{B/A}$ is finitely generated over B.

Proof Write $B = A[x_1, \ldots, x_n]/I$ for some variables $x_1, \ldots, x_n$ and apply Proposition 20.5 on page 348 and the above proposition. □

Exercise 20.3.1 (The diagonal and $\Omega_{B/A}$)**.** Suppose that B is an A-algebra. There is an exact sequence of A-modules

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

where μ is the multiplication map, which acts as $b_1 \otimes b_2 \mapsto b_1 b_2$ on decomposable tensors, and where I is the kernel of μ. Since $B \otimes_A B/I \simeq B$, the module I/I^2 has the structure of a B-module.

a) Show that I is generated by elements of the form $a \otimes 1 - 1 \otimes a$;
b) Show that the two B-module structure on I/I^2 induced from each factor of the tensor product agree; that is, $b \otimes 1 \cdot x = 1 \otimes b \cdot x$ for all $x \in I/I^2$;
c) Show that $d\colon B \to I/I^2$ defined by $db = b \otimes 1 - 1 \otimes b$ is an A-derivation;
d) Show that d is a universal derivation so that $I/I^2 \simeq \Omega_{B/A}$ and $d = d_{B/A}$.

Exercise 20.3.2. Let $A \to B$ be a map of Noetherian rings, $\pi\colon X \to Y$. Assume that $\Omega_{B/A} = 0$. Show that the diagonal Δ is a connected component of $X \times_Y X = \mathrm{Spec}\, B \otimes_A B$.

Assume that $I \subset A$ is finitely generated ideal such that $I^2 = I$. Show that I is a principal ideal generated by an idempotent. HINT: Let $\{x_i\}$ generate I and write $x_i = \sum_j a_{ij} x_j$ with $a_{ij} \in I$. Consider the matrix $\Phi = (\delta_{ij} - a_{ij})$. Show that $\det \Phi$ annihilates I, and hence there is an $e \in I$ so that $(1 - e)I = 0$. Show that $e^2 = e$ and that $I = (e)$.

20.3.3 Kähler differentials and localization

When we later shall globalize the construction of the Kähler differenials, the following two results about their behavior with respect to localizations are important. They both rely on the sequence in Proposition 20.12.

Proposition 20.15. Let $S \subset A$ be a multiplicative subset mapping into the group of units in B. Then 'change-of-constants-map' is an isomorphism

$$\Omega_{B/A} \simeq \Omega_{B/S^{-1}A}.$$

Proof The 'change-of-constants-map' is the map β in the sequence

$$\Omega_{S^{-1}A/A} \otimes_{S^{-1}A} B \longrightarrow \Omega_{B/A} \xrightarrow{\beta} \Omega_{B/S^{-1}A} \longrightarrow 0,$$

and by Example 20.6 we have $\Omega_{S^{-1}A/A} = 0$. □

Proposition 20.16. Suppose S is a multiplicative system in B and let $\iota\colon B \to S^{-1}B$ be the localization map. Then the natural map ι_* yields an isomorphism

$$\iota\colon S^{-1}\Omega_{B/A} \simeq \Omega_{S^{-1}B/A}.$$

Proof Note that $S^{-1}\Omega_{B/A} = \Omega_{B/A} \otimes_B S^{-1}B$, so we are in the context of Proposition 20.12 and may use the exact sequence there. We previously checked that $\Omega_{S^{-1}B/B} = 0$ (Example 20.6) and hence ι_* is surjective. Thus in view of the identity $\mathrm{Hom}_{S^{-1}B}(S^{-1}\Omega_{B/A}, M) = \mathrm{Hom}_B(\Omega_{B/A}, M,)$ which is valid for any $S^{-1}B$-module M, it suffices to see that the map

$$\mathrm{Der}_A(S^{-1}B, M) \longrightarrow \mathrm{Der}_A(B, M)$$

corresponding to ι_* is surjective. This is the case since every $D\colon B \to M$ extends to a derivation $D'\colon S^{-1}B \to M$ by the formula

$$D'(bs^{-1}) = (sDb - bDs)s^{-2}, \tag{20.3}$$

some checking must be done, which is left to the reader. □

Exercise 20.3.3. Check that the expression $D'(bs^{-1})$ in (20.3) does not depend on the choice of representative for bs^{-1} and that the resulting D' is a derivation.

20.4 Exercises

Exercise 20.4.1. Let $B = k[x, y]/(x^2 + y^2)$. Show that if k has characteristic $\neq 2$, then

$$\Omega_{B/k} = (Bdx + Bdy)/(xdx + ydy)$$

If k has characteristic 2, then $\Omega_{B/k}$ is the free B module $Bdx + Bdy$.

Exercise 20.4.2 (Torsion in the Kähler differentials)**.** (This exercise requires some knowledge of Koszul complexes and homological algebra). Let $f \in k[x, y]$ be a polynomial without multiple factors and let $A = k[x, y]/(f)$. Show that the submodule of torsion elements of

$\Omega_{A/k}$ is isomorphic the quotient $\big((f_x, f_y) : f\big)/(f_x, f_y)$ of the transporter ideal $\big((f_x, f_y) : f\big)$ in the polynomial ring $R = k[x, y]$. Show that $X = V(f)$ is regular if and only if $\Omega_{A/k}$ is torsion free. Show that, more precisely, the torsion is of length $\dim_k A/(f_x, f_y)A$. (This number is the sum of a contribution from each singular point, often called *the Tjurina number* of the singular point. The formula for the length is due to Zariski (**?**)).

Exercise 20.4.3. Let $f \in k[x, y]$ be the equation of a non-singular curve. Let $A = k[x, y]/(f)$ and $B = k[x, y]/(f^2)$. Show that $\Omega_{B/k} \simeq Bdx \oplus Bdy$ if k is of characteristic two and that $\Omega_{B/k} \simeq \Omega_{A/k}$ if not.

Exercise 20.4.4 (Transcendental extensions)**.** Let k be a field and $K = k(x_1, \ldots, x_n)$ a purely transcendental field extension. Show that $\Omega_{K/k} \simeq K^n$ with $dx_1, \ldots, dx_n$ as a basis. HINT: Consider $k[x_1, \ldots, x_n]$ and use (**??**), then localize and use 20.16.

Exercise 20.4.5. Assume that $k \subset K$ is a finitely generated field extension.

a) Show that $\dim_K \Omega_{K/k} \geqslant \operatorname{trdeg} K/k$;
b) Show that equality holds if and only if K is separably generated[2] over k.
c) Show that if k is perfect, it holds that $\dim_K \Omega_{K/k} = \operatorname{trdeg} K/k$, hence K is separably generated over k. HINT: Let $P(t) = \sum_i a_i t^i$ be a minimal polynomial in for x, show that $dP = P'(t)dt + \sum_i da_i \cdot t^i \in \Omega_{K[t]/k}$ is non zero.

20.5 The sheaf of differentials

For us, the primary motivation for studying $\Omega_{B/A}$ is that its tilde gives an intrinsic sheaf on $\operatorname{Spec} B$ associated to any morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. We would like to globalize this construction to an arbitrary morphism of schemes $f \colon X \to S$. This will lead us to form the *sheaf of relative differentials* $\Omega_{X/S}$ which will be a quasi-coherent $\mathcal{O}_X$-module.

This sheaf is locally built out of the various $\Omega_{B/A}$ on local affine charts. These are not just arbitrary modules that just happen to glue together to a sheaf; each of them come with the universal property of classifying derivations $D \colon B \to M$. For this reason, we would like to say that the $\Omega_{X/Y}$ should satisfy a similar universal property. We make the following definition:

Definition 20.17. Let $\mathcal{F}$ be a quasi-coherent (?) $\mathcal{O}_X$ module. A morphism $D \colon \mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules is an S*-derivation* if for all open affine subsets $V \subset S$ and $U \subset X$ with $f(U) \subset V$, the map $D|_U$ is an $\mathcal{O}_S(V)$-derivation of $\mathcal{O}_X(U)$ with values in $\mathcal{F}$. The set of all such S-derivations is denoted by $\operatorname{Der}_S(\mathcal{O}_X, \mathcal{F})$.

[2] A field extension $k \subset K$ is separably generated if there is a transcendence basis $x_1, \ldots, x_n$ for K over k so that K is separable over $k(x_1, \ldots, x_n)$. If in addition K is finitely generated over k, the K will be finite over $k(x_1, \ldots, x_n)$.

Definition 20.18. The *sheaf of relative differentials* is a pair $(\Omega_{X/S}, d_{X/S})$ of a quasi coherent (?) $\mathcal{O}_X$-module $\Omega_{X/S}$ and a S-derivation $d_{X/S}\colon \mathcal{O}_X \to \Omega_{X/S}$ that satisfies the following universal property: For each quasi-coherent (?) $\mathcal{O}_X$-module $\mathcal{F}$, and each S-derivation $D\colon \mathcal{O}_X \to \mathcal{F}$ there exists a unique $\mathcal{O}_X$-linear map $\alpha\colon \Omega_{X/S} \to \mathcal{F}$ such that $D = \alpha \circ d_{X/S}$.
When $S = \operatorname{Spec} A$, we sometimes write $\Omega_{X/A}$ for $\Omega_{X/S}$.

In other words, $\Omega_{X/S}$ is a sheaf that represents the functor of S-derivations, in the sense that there is a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, -) \simeq \operatorname{Der}_S(\mathcal{O}_X, -).$$

Exercise 20.5.1. Prove, using the universal property of differentials, that gives that this sheaf is unique up isomorphism, if it exists.

In the affine situation with a morphism $X = \operatorname{Spec} B \to S = \operatorname{Spec} A$ we have the module of Kahler differentials $\Omega_{A/B}$ and the corresponding sheaf $\widetilde{\Omega_{A/B}}$ will serve as the sheaf of relative differential on X; this is just a consequence of $\sim$ being an equivalence of categories Mod_B and QCoh_X. In the general case, gluing the local differential on affine covers works well, and the main theorem of this section says that sheaves of relative Kähler differentials exist unconditionally.

Theorem 20.19. Let $f\colon X \to S$ be a morphism of schemes. Then there is a sheaf of relative differentials $\Omega_{X/S}$, which is a quasi-coherent sheaf on X.
Moreover, $\Omega_{X/S}$ has the property that for each open affine open $V = \operatorname{Spec} A$ and each open affine $U = \operatorname{Spec} B \subset f^{-1}(V)$ it holds that

$$\Omega_{X/S}|_U \simeq \widetilde{\Omega_{B/A}}.$$

Also for each $x \in X$, we have

$$(\Omega_{X/S})_x \simeq \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}.$$

Proof Fix an open subset $V = \operatorname{Spec} A$ of S, and let $U = \operatorname{Spec} B$ be an affine open subset in X so that $f(U) \subset V$. For these two, we define

$$\Omega_{U/V} = \widetilde{\Omega_{B/A}}$$

which is a sheaf on U. We first show that the different $\Omega_{U/V}$ glue together to an $\mathcal{O}_{f^{-1}V}$-module $\Omega_{f^{-1}(V)/V}$ when U runs through an open affine cover of $f^{-1}(V)$. This comes down to showing that if $U' = \operatorname{Spec} B'$ is a distinguished open affine subset of U, then

$$\Omega_{U/V}|_{U'} \simeq \Omega_{U'/V}.$$

But as B' is a localization of B, Proposition 20.16, tells us that ι_* is such an isomorphism with $\iota\colon B \to B' \to$ the localization map. These maps depend functorially on the inclusions, so the gluing conditions are trivially fulfilled.

Then we show that the sheaves $\Omega_{f^{-1}V/V}$ for all affine opens $V \subseteq S$ glue to a $\mathcal{O}_X$-module

$\Omega_{X/S}$. This amounts to showing that for each distinguished open $V' = \operatorname{Spec} A' \subseteq V$, and all open $U = \operatorname{Spec} B$ of $f^{-1}(V')$, we have

$$\Omega_{U/V} = \Omega_{U/V'}$$

But this follows from Proposition 20.15, as A' is a localization of A in a single element (which maps to an invertible element in B).

This means that we get an $\mathcal{O}_X$-module $\Omega_{X/S}$. Let us check that it satisfies the above universal property. So we need to define the universal derivation $d_{X/S}\colon \mathcal{O}_X \to \Omega_{X/S}$.

Let $V = \operatorname{Spec} A \subseteq S$ and $U = \operatorname{Spec} B \subseteq X$ be an affine open subset such that $f(U) \subseteq V$. Define $d_{X/S}(U) = d_{B/A}$. By the gluing construction above, this map does not depend on the chosen affine open V, and it can be checked that the assignment is compatible with restriction maps. Hence this gives a morphism of sheaves $d_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$, which by construction is an S-derivation.

To check that this is universal, we again work locally. Let $d\colon \mathcal{O}_X \to \mathcal{F}$ be an S-derivation, where $\mathcal{F}$ is an $\mathcal{O}_X$-module. Let $U = \operatorname{Spec} A \subseteq S$ and $V = \operatorname{Spec} B \subseteq X$ so that $f(U) \subseteq V$. By the universal property of $\Omega_{B/A}$, we get an A-derivation $D(V)\colon B \to \mathcal{F}(V)$, and hence a unique B-linear map $\alpha(V)\colon \Omega_{X/S}(V) = \Omega_{B/A} \to \mathcal{F}(V)$ such that $D(V) = \alpha(V) \circ d_{X/V}(V)$. One has to check that these maps are compatible with restriction maps (use the universal property of $\Omega_{B/A}$), but after that, we obtain a unique $\mathcal{O}_X$-linear map $\alpha : \Omega_{X/S} \to \mathcal{F}$ so that $D = \alpha \circ d_{X/S}$. □

Note that the sheaf $\Omega_{X|S}$ is always quasi-coherent (it is by definition locally of the form $\widetilde{M}$ for some module). Moreover, when X is of finite type over a field, $\Omega_{B/k}$ is finitely generated, and so $\Omega_{X|k}$ is even coherent.

Example 20.20. Let A be a ring and let $X = \mathbb{A}^n_S = \operatorname{Spec} A[x_1, \ldots, x_n]$ be affine n-space over $S = \operatorname{Spec} A$. Then $\Omega_{X/S} \simeq \mathcal{O}_X^n$ is the free $\mathcal{O}_X$-module generated by $dx_1, \ldots, dx_n$.

If X is a separated scheme over S then one could also define $\Omega_{X/S}$ as follows. Let $\Delta : X \to X \times_S X$ be the diagonal morphism and let $\mathcal{I}_\Delta$ be the ideal sheaf of the image of Δ. Then $\Omega_{X/S} = \Delta^*(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)$. This does in fact give the same sheaf as above, since these two definitions coincide when X and S are both affine (Exercise 20.3.1). This definition gives a slick way of defining the sheaf $\Omega_{X/S}$, but it is not very suited for computations.

The properties of the Kähler differentials $\Omega_{B/A}$ translate into the following results for $\Omega_{X/Y}$:

Proposition 20.21 (Base change). Let $f\colon X \to S$ be a morphism of schemes and let S' be a S-scheme. Let $X' = X \times_S S'$ and let $p\colon X' \to X$ be the projection. Then

$$\Omega_{X'/S'} \simeq p^*\Omega_{X/S}$$

Proposition 20.22. Let X, Y, and Z be schemes along with maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then there is an exact sequence of $\mathcal{O}_X$-modules

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0. \tag{20.4}$$

Proposition 20.23 (Conormal sequence). Let Y be a closed subscheme of a scheme X over S. Let $\mathcal{I}_Y$ be the ideal sheaf of Y on X. Then there is an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_{X/S} \otimes \mathcal{O}_Y \to \Omega_{Y/S} \to 0. \tag{20.5}$$

20.6 The Euler sequence and differentials of $\mathbb{P}^n_A$

We have seen that the sheaf of differentials on affine space $\mathbb{A}^n$ is trivial, that is, $\Omega_{\mathbb{A}^n_k} \simeq \mathcal{O}^n_{\mathbb{A}^n}$. In this section we will give a concrete description of the cotangent bundle of projective space, suitable for explicit computations.

Euler's theorem states that if f is a rational function of degree d, it holds that $\sum x_i f_{x_i} = df$, or, in particular, when f is of degree zero, one has $\sum_i x_i f_{x_i} = 0$. Now, the functions on open sets in projective space are all rational functions of degree zero, and so Euler tells us that their differentials all live in the kernel of the map

$$\bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(-1)dx_i \to \mathcal{O}_{\mathbb{P}^n}$$

that sends $\sum_i f_i dx_i$ to $\sum_i x_i f_i$. This gives a strong heuristic argument for the next theorem:

Theorem 20.24. Then there is an exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n_A/A} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow 0.$$

Proof Choosing coordinates on $\mathbb{P}^n_A$ we have $\mathbb{P}^n_A = \operatorname{Proj} R$ where R is the graded A-algebra $R = A[x_0, \dots, x_n]$. We introduce a graded R-module M by the exact sequence

$$0 \longrightarrow M \longrightarrow \textstyle\bigoplus_i R(-1)dx_i \xrightarrow{\eta} R$$

where η is the 'Euler map' $\sum_i f_i dx_i \mapsto \sum_i f_i x_i$. It is homogenous of degree zero when we give each dx_i degree one. Note that $\operatorname{Coker} \eta = R/(x_0, \dots, x_n)$, so that when 'tilded' the sequence becomes

$$0 \longrightarrow \widetilde{M} \longrightarrow \textstyle\bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-1)dx_i \xrightarrow{\tilde{\eta}} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

We will use the covering of $\operatorname{Proj} R$ by the standard open affines $D_+(x_i)$ each equal to $\operatorname{Spec}(R_{x_i})_0$, where $(R_x)_0$ is the degree zero piece of the localization R_{x_i} (equipped with natural grading). The overlaps of the standard opens are the distinguished open sets $D_+(x_i x_j) = \operatorname{Spec}(R_{x_i x_j})_0$.

The universal derivation $d_R\colon R \to \Omega_{R/A} = \bigoplus_j R dx_j$ extends to a derivation

$$d_{R_{x_i}} : R_{x_i} \to \Omega_{R_{x_i}/A} = \bigoplus_j R_{x_i} dx_j$$

by the usual rule for the derivative of a fraction, and it preserves degrees when each dx_j is given degree one; that is $(R_{x_i} dx_i)_\nu = (R_{x_i})_{\nu-1} dx_i$. Taking the degree zero part, yields a derivation

$$(R_{x_i})_0 \to \bigoplus_j (R_{x_i}(-1))_0 dx_j;$$

that is, when exposed to tilde, a derivation

$$\mathcal{O}_{D_+(x_i)} \to \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-1)|_{D_+(x_i)} dx_j.$$

Since these derivations for different i originate from the *same* global derivation d_R, they are forced to agree on the overlaps, and glue together to a derivation

$$\mathcal{O}_{\mathbb{P}^n} \to \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-1) dx_i.$$

It takes values in $\widetilde{M}$, and by universality there is a map $\Omega_{\mathbb{P}^n/A} \to \widetilde{M}$. The rest of the proof consists of checking that this is an isomorphism, which is a local issue. Both $\Omega_{\mathbb{P}^n/A}$ and $\widetilde{M}$ are locally free of rank n, so it suffices to see that α is surjective.

On the open set $D_+(x_i)$ the sheaf $\Omega_{\mathbb{P}^n/A} = \Omega_{D_+(x_i)/A}$ originates from the module $\Omega_{(R_{x_i})_0/A}$, which has a basis formed by the $d(x_j/x_i)$ for $j \neq i$, and one checks without much resistance that the map α sends $d(x_i/x_j)$ to $(x_j dx_i - x_i dx_j)/x_i^2$. But the kernel of the Euler map η is generated by the elements $x_i dx_j - x_j d_{x_i}$, and so we are through. □

Since $\Omega_{\mathbb{P}^n_A}$ injects into $\mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1}$ (which has no global sections), we get:

Corollary 20.25. $\Gamma(\mathbb{P}^n_A, \Omega_{\mathbb{P}^n_A}) = 0$

Exercise 20.6.1. Show that the kernel of η is generated by $n(n-1)/2$ expressions $x_i dx_j - x_j dx_i$.

20.7 Relation with the Zariski tangent space

The tangent space to a differentiable manifold at a point is defined at the space of 'point derivations' as the point, i.e. derivations from the ring of C^∞-germs near the point to $\mathbb{R}$. The analogue to this for a scheme X over a field k would be the space of derivations $\mathrm{Der}_k(\mathcal{O}_{X,x}, k(x))$, where $k(x)$ is the residue class field at x, and in view of the fundamental relation (20.5) and the equality

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, k(x)) = \mathrm{Hom}_{k(x)}(\Omega_{X/k} \otimes k(x), k(x)),$$

the cotangent space; i.e. the dual of the tangent space, will be $\Omega_{X/k} \otimes_{\mathcal{O}_X} k(x)$.

Another candidate is, however, the Zariski tangent space $\mathrm{Hom}_{k(x)}(\mathfrak{m}/\mathfrak{m}^2, k(x))$, In contrast to the 'point derivations', the Zariski tangent space is not a relative notion, it does not depend on the subfield k, and can be defined for any local ring. The Zariski cotangent space will simply be the dual space $\mathfrak{m}/\mathfrak{m}^2$.

These two possible tangent spaces give rise to two different notions, *regularity* and *smoothness*, which both in some sense mimic the property of being a manifold. Fortunately, in several cases the two are equivalent; one such situation is in described in the following proposition:

Proposition 20.26. Suppose $(B, \mathfrak{m})$ is a local ring with residue field $K = B/\mathfrak{m}$ and assume that B contains a field k. If the extension $k \subset K$ is finite and separable, then the map from the conormal sequence

$$\delta\colon \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{B/k} \otimes_B K$$

is an isomorphism.

Proof The conormal sequence with $A = k$ and $C = K$ takes the following shape:

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B K \longrightarrow \Omega_{K/k} \longrightarrow 0,$$

and according to Example 20.8 on page 348 it holds that $\Omega_{K/k} = 0$, so δ is surjective.

The map δ sends $x \in \mathfrak{m}$ to dx. We shall exhibit an inverse $\psi\colon \Omega_{B|k} \otimes_B K \to \mathfrak{m}/\mathfrak{m}^2$ to δ. Constructing such a map is equivalent to constructing a map of B-modules $\Omega_{B|k} \to \mathfrak{m}/\mathfrak{m}^2$, or equivalently, a derivation $D\colon B \to \mathfrak{m}/\mathfrak{m}^2$.

The derivation $D\colon B \to \mathfrak{m}/\mathfrak{m}^2$ will be the composition $D \circ \pi$ of the canonical 'reduction-mod-$\mathfrak{m}^2$-map' $\pi\colon B \to B/\mathfrak{m}^2$ and a derivation $D_0\colon B/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$. To construct the latter, we cite the lemma below that the k-algebra $B/\mathfrak{m}^2$ splits as a direct sum $B/\mathfrak{m}^2 = K \oplus \mathfrak{m}/\mathfrak{m}^2$, and simply let D_0 be the projection onto $\mathfrak{m}/\mathfrak{m}^2$; that is

$$D_0(a + x) = x,$$

where $\alpha \in K$ and $x \in \mathfrak{m}/\mathfrak{m}^2$. The reduction map π being an algebra homomorphism, it suffices to see that D_0 is a k-derivation. To this end, we compute:

$$\begin{aligned} D_0((a + x)(a' + x')) &= D_0(aa + (ax' + a'x) + x'x) \\ &= D_0(aa) + D_0(ax' + a'x) + D_0(x'x) = ax' + a'x, \end{aligned}$$

and we get the same answer when we expand

$$(a' + x')D_0(a + x) + (a + x)D_0(a' + x')$$

since $xx' = 0$. Hence D_0 is a derivation, and we get the desired inverse. It is indeed an inverse to the map δ, since via the identification $\mathrm{Der}_A(B, M) = \mathrm{Hom}_B(\Omega_{B/A}, A)$, it sends dx to x. □

Lemma 20.27. Let B be a local ring with maximal ideal I that satisfies $I^2 = 0$. Assume that B contains a field k and that the extension $k \subset K = B/I$ is finite and separable. Then B contains a subring isomorphic to K; so that $B = K \oplus I$.

Proof Since K is finite and separable over k, it is primitive. So let $K = k(x)$ and let P being the minimal polynomial of x over k. It is separable, so $P'(x) \neq 0$. We shall lift x

to an element $y \in B/\mathfrak{m}^2$ such that that $P(y) = 0$ (meaningful as $k \subset B/\mathfrak{m}^2$ and P has coefficients in k). Then the the subring $k(y)$ maps isomorphically onto K.

Chose any lifting z of x. Then $P(z) = \epsilon \in I$. For any $\alpha \in I$ Taylor's formula yields

$$P(z + \alpha) = P(z) + P'(z)\alpha$$

as $\alpha^2 = 0$. Now $P'(x)$ is a unit in B/I, and as units reduce to units (Lemma 20.28 below)and hence $y = z + \alpha$ is such that $P(y) = 0$. □

Recall that a Noetherian local ring B is called *regular* if the Krull dimension equals the embedding dimension; or with $\mathfrak{m}$ the maximal ideal and $K = B/\mathfrak{m}$, it holds that $\dim_K \mathfrak{m}/\mathfrak{m}^2 = \dim B$.

Lemma 20.28. Let $\pi\colon B \to A$ be a surjective ring homomorphism with kernel I. Assume that $I^2 = 0$. Then every element in B that maps to a unit in A is invertible, and there is an exact sequence of groups

$$1 \longrightarrow 1 + I \longrightarrow B^* \stackrel{\pi}{\longrightarrow} A^* \longrightarrow 1\ .$$

Proof All elements in $1 + I$ are units, since if $x^2 = 0$, it holds that $(1 + x)(1 - x) = 1$. Assume that $\pi(x)y = 1$ and let $z \in B$ be so that $\pi(z) = y^{-1}$ Then $xz \in 1 + I$ and is therefore invertible, so *a fortiori* x is invertible. □

Exercise 20.7.1. Show that if K is a finitely generated extension of k with a separating basis, there is a field $K' \subset B$ mapping isomorphically to K. HINT: First treat the case that $K = k(x)$ with x a variable; then use induction on the cardinality of a separating basis.

Corollary 20.29. With notation as in Proposition 20.26 but additionally with B being Noetherian, the ring B is a regular local ring if and only if

$$\dim B = \dim_k \Omega_{B/K} \otimes_B K.$$

The separability condition in Proposition 20.26 is certainly necessary, this is already the case for fields: fields are regular local rings of dimension zero, and for a inseparable field extension $k \subset K$ the module of differentials $\Omega_{K/k}$ is never zero; for instance, if $K = k(x)$ with $x^p = a$, it holds that $\Omega_{K/k} = K$.

Smooth varieties

We give a definition for smoothness of varieties. In general schemes can have components of different dimension, so we if $x \in X$ is a point, we let $\dim_x X$ be the Krull dimension of a sufficiently small affine neighbourhood of x; if x is a closed point it coincides with $\dim \mathcal{O}_{X,x}$.

Definition 20.30 (Smoothness over fields)**.** Let X be a (separated (?)) scheme of (essential (?)) finite type over a field k and let $x \in X$ be a point. We say that X is *smooth* at x if $\Omega_{X/k}$ is locally free of rank $\dim_x X$ near x. The scheme X is called *smooth* if it is smooth at every closed point.

Theorem 20.31. Let X be a variety (integral separated scheme of finite type) over a perfect field k (e.g. k algebraically closed, finite or of characteristic zero) and let $x \in X$ be a closed point. Then the following are equivalent:

(i) X is smooth at x;
(ii) $(\Omega_{X/k})_x$ is free of rank $\dim X$;
(iii) X is non-singular at x.

Proof (i) $\iff$ (ii) is just the definition of X being smooth together with the fact that a coherent module $\mathcal{F}$ over $\mathcal{O}_X$ is locally free in near x if and only $\mathcal{F}_x$ is free.

(ii) $\implies$ (iii). Assuming that $\Omega_{\mathcal{O}_{X,x}/k}$ is free of rank $n = \dim \mathcal{O}_{X,x}$, we infer, by the above proposition, that $\dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = n$, and so $\mathcal{O}_{X,x}$ is a regular local ring.

(iii) $\implies$ (ii). There are two salient points: The first is that if x is a regular point, the integer $d(y) = \Omega_{X/k} \otimes_{\mathcal{O}_X} k(y)$ takes on its minimal value at x, and the second is that $d(y)$ can only increase upon specialization. The details are as follows: Let K be the function field of X. If the local ring $\mathcal{O}_{X,x}$ is regular, it follows from Proposition 20.26 that $\dim (\Omega_{X/k}) \otimes k(x) = \dim_{k(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim X$. From Exercise 20.4.5 on page 353 follows that $\dim_K \Omega_{K/k} = \dim_K \Omega_{X/k} \otimes K \geqslant \operatorname{trdeg} K/k$. The transcendence degree of the function field of a variety equals $\dim X$, and hence $(\Omega_{X/k})_x$ is a free $\mathcal{O}_{X,x}$-module by the general fact that a finite module over an integral local ring having generic fibre of the larger dimension than the special one, is free (Exercise 20.7.2 below). □

Exercise 20.7.2 (Jumping of fibre dimension upon specialization)**.** Let A be a local integral domain with maximal ideal $\mathfrak{m}$, residue field $k = A/\mathfrak{m}$ and fraction field K. Let M be a finite A-module and assume that $\dim_K M \otimes_A K \geqslant \dim_k M \otimes_A k$. Then M is a free A-module. (See also Proposition **??** on page ?? in CA)

Exercise 20.7.3. Let X be a variety over a perfect field. Show that the function field K of X is separably generated over k and that the smooth (hence regular) closed points of X form an open dense subset. Give a counterexample if k is not perfect.

21

Curves

We have through the course seen several examples of curves. Plane curves with conics and hyper elliptic curves have been favourites, the normal rational curves as examples of curves that are not plane. In this chapter we shall study curves more systematically and from a intrinsic point of view, that is we curves per se and not as subschemes of larger scheme.

So far we have not given a formal definition of a curve; here it comes: a *curve* is a one dimensional variety over a field k. Recall that this means that X apart from being of dimension one, is an integral scheme separated and of finite type over k.

We shall restrict our attention to curves over perfect fields; in addition to all fields of characteristic zero this covers the cases that k is algebraically closed or a finite field.

21.1 The local ring at regular points of a curve

A variety X is smooth at a point x if the $\Omega_{X,x}$ is locally of rank $\dim X$ near x, and over a perfect field this is equivalent to $\mathcal{O}_{X,x}$ being a regular local ring. In other words, it is equivalent to the Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ being of dimension $\dim X$ as a vector space over $k(x)$.

For curves, the important points is that the Noetherian regular local rings of dimension one are precisely the discrete valuation rings; that is the Noetherian local PID's. The ideal structure of these rings is particularly simple, the powers of the maximal ideal are the only non-zero ideals.

Lemma 21.1. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. The following statements are equivalent:

(i) A is a DVR;
(ii) the maximal ideal $\mathfrak{m}$ is principal;
(iii) all ideals are principal and powers of the maximal ideal;
(iv) $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.

At a regular point $x \in X$ the maximal ideal $\mathfrak{m}_x$ is principal, and any generator is called a *local parameter* or a *uniformizing parameter* at x. Each rational function on X can expressed in a unique fashion as $f = \alpha t^\nu$ where ν is an integer and α is unit in $\mathcal{O}_{X,x}$; that is, it is a regular function which does not vanish at x. To every DVR is associated a normalized valuation on the fraction field, which we in the present case denote by ν_x. Note that $\nu_x(f)$ is precisely the integer ν above. One may think about the valuation $v_x(f)$ as the order of f at x, either the order of vanishing, if f is regular at x, or the order of the pole if not.

Our ground field is assumed to be perfect, and the differential criterion for regularity in Theorem 20.31 on page 360 applies:

Proposition 21.2. A curve X over a perfect field is regular at a closed point x if and only if the stalk $(\Omega_{X/k})_x$ is free of rank one.

Example 21.3 (Plane curves)**.** Consider $X = \operatorname{Spec} A$ where $A = k[u, v]/(f)$. In Example **??** we found the following expression for the Kähler differentials of A:

$$\Omega_{A/k} = Adu \oplus Adv/(f_u du + f_v dv),$$

and this is not of rank one (i.e. of rank two) exactly at the points of X where the two partials f_u and f_v vanish. Hence a point $x \in X$ is a smooth point if and only if at least one of the partials does not vanish at x, and X is a regular curve when $V(f, f_u, f_v) = \varnothing$. In terms of ideals this reads $(f, f_u, f_v) = k[u, v]$.

Example 21.4 (A regular but not smooth curve)**.** Over fields that are not perfect, being regular and being smooth are not the same, every smooth curve is regular but regular curves need not be smooth; it might cease being regular after a base extension. For instance, assume that k is of characteristic two and that $\alpha \in k$ is an element that is not a square. Then $\mathfrak{m} = (v, u^2 + \alpha)$ is a maximal ideal in $k[u, v]$ and $x = V(u^2 + \alpha, v)$ a closed point in $\mathbb{A}^2_k$.

The plane affine curve $V(f)$ with $f = v^2 - u(u^2 + \alpha)$ is regular at all points: since $df = (u^2 + \alpha)du$, it is smooth except at $x = V(u^2 + \alpha, v)$, where it, however, is regular. Indeed, f does not belong to $\mathfrak{m}^2 = (v^2, v(u^2 + \alpha), u^4 + \alpha^2)$.

The curve $X' = X \otimes_k k'$ acquires a singular point if k' is an extension of k containing a square root of α, say $\beta^2 = \alpha$. Then f takes the form $f = v^2 - u(u^2 + \beta^2) = v^2 - u(u + \beta)^2$, and X' has a node at $(-\beta, 0)$.

The moral is that regularity is not always invariant under base change.

Exercise 21.1.1. Find the singularities of the curve in $\mathbb{P}^2_k$ whose equation is $x^2y^2 + x^2z^2 + y^2z^2 = 0$.

Another all important feature of one dimensional Noetherian domains is that they are regular precisely when they are normal:

Proposition 21.5. Let A be a one-dimensional Noetherian domain A. Then A is normal if and only if it is regular.

Proof Being normal is a local property, and by definition a Noetherian ring is regular precisely when then all the local rings $A_\mathfrak{p}$ are regular, so the proposition boils down to the local case, which is standard algebra: a one-dimensional local domain is normal if and only if it is a DVR. □

Back in Chapter **??** we constructed the normalization $\overline{X}$ of an integral scheme X (Theorem 13.12) together with a morphism $\pi\colon \overline{X} \to X$. In view of the above proposition, $\overline{X}$ is in fact a *desingularization* of X.

Theorem 21.6. The normalization $\overline{X}$ of a curve X over k, is a non-singular curve. The normalization map $\pi\colon \overline{X} \to X$ is finite and birational. If X is proper over k the same holds for $\overline{X}$.

Proof This is just Theorem 13.13 on page 207. □

21.2 Morphisms between curves

We recall the following three fundamental facts about morphisms of curves

Proposition 21.7. Let X be a variety and Y a curve over k, and let $f\colon X \to Y$ be a morphism. Then either

(i) $f(X)$ is a point in Y; or

(ii) $f(X)$ is open and dense in Y.

In the case (ii), when X is a curve, the extension $k(Y) \subseteq k(X)$ of function fields will be a finite extension. Moreover, when X is proper over k, so is Y, and f is a finite morphism.

Proof The first statement follows from lemma below; indeed, let $\operatorname{Spec} A \subset Y$ and $\operatorname{Spec} B \subset X$ be open affines such that $\operatorname{Spec} B$ maps into $\operatorname{Spec} A$. The image of $\operatorname{Spec} B$ is either a point, in which case the image of X will be that point, or $\operatorname{Spec} B$ dominates $\operatorname{Spec} A$, and its image contains an open subset. The image of f will then be open because subsets of an irreducible curve containing a non-empty open set are open.

Lemma 21.8. Let A and B be a domains and $\phi\colon A \to B$ a ring homomorphism. Assume that A is of Krull dimension one. Then either ϕ is injective or factors by a field. In particular, the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is either dominant or has a closed point as image.

Proof Since B is domain, so is also $\phi(A)$, and $\operatorname{Ker}\phi$ is a prime ideal. Since A is a domain of Krull dimension one, the kernel $\operatorname{Ker}\phi$ is either maximal or zero. □

Assume then that X is a curve and that f is dominant. The two function fields $k(X)$ and $k(Y)$ are both of transcendence degree one, and so $k(X)$ is algebraic over Y, but X is of finite type over Y, since it is of finite type over k, and thus $k(X)$ is a finite extension of $k(Y)$. When f is proper and dominant, it will be surjective, and by general properties of proper maps (xxxx) Y will be proper over k as well. Is X also a curve, every fibre of f over a closed point will be a proper closed set, and so will be finite. Hence f is quasi-finite, and also being proper, it is finite (xxxx). □

This leads to the notion of the degree of a morphism between curves:

Definition 21.9 (The degree of a finite morphism)**.** Let $f: X \to Y$ be a dominant morphism between curves. The degree $[k(X) : k(Y)]$ is called the *degree* of f and is denoted $\deg f$.

Since the degree of field extensions is multiplicative in towers, one has:

Proposition 21.10. If f and g are dominant composable morphisms between curves, the composition $f \circ g$ is dominant and $\deg f \circ g = \deg f \deg g$.

21.2.1 The fibre of a morphism

We shall examine the scheme theoretic fibreFor the basic details about scheme theoretic fibres see Section 10.5 $f^{-1}(y)$ over a closed point $y \in Y$ of a morphism $f: X \to Y$ between two curves in more detail. The most interesting case is when both X and Y are regular curves and the morphism is finite and dominant, and we will confine the analysis to that case. The analysis is local on Y, so we may additionally assume that X and Y are affine; say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, where A and B are reular one-dimensional rings and B is a finite A-algebra. If $x \in X$ the ring $\mathcal{O}_{X,x}$ is a valuation ring and we denote by v_x the corresponding valuation on $k(X)$.

Proposition 21.11. Let $f: \operatorname{Spec} B \to \operatorname{Spec} A$ be a finite morphism where A is a regular one-dimensional ring. If each component of $\operatorname{Spec} B$ dominates $\operatorname{Spec} A$, then B is a locally free A module. In the case that B is integral, the rank of B equals $\deg f$.

Proof The zero divisors of B is the union of the minimal prime ideals $\{\mathfrak{p}_i\}$ in B, and since each component of X dominates Y, it holds that $\mathfrak{p}_i \cap A = 0$. This means that each non-zero element t of A is a non-zero divisor on B. Hence B is a torsion free finite A-module, and as A is a Dedekind ring, it follows (see xxxx)) that B locally free. □

Example 21.12 (Illustartive example)**.** Let $f(t)$ be a polynomial in $k[t]$. The assignment $t \to f(t)$ defines a map $k[t] \to k[t]$ and hence a map $\mathbb{A}^1_k \to \mathbb{A}^1_k$. The scheme theoretic fibre over the closed point $(t-a) \in \mathbb{A}^1_k$ (heuristically speaking over $\alpha \in \mathbb{A}^1(k)$) is the closed subscheme $V(f(t) - \alpha)$. The polynomial $f - \alpha$ factors as

$$f - \alpha = f_1^{\nu_1} \cdot \cdots \cdot f_r^{\nu_r}$$

where the f_i's are irreducible and pairwise distinct. One would like to think about $V(f(t)-\alpha)$ as the solutions of $f(t) - \alpha = 0$, but the roots β_i of the f_i's do not necessarily lie in k, but in extensions $k(\beta_i)$; and of course, each appears with multiplicity ν_i. The Chinese Remainder Theorem gives

$$k[f]/(f(t) - \alpha) = \prod_{1 \leqslant i \leqslant r} k[t]/(f_i^{\nu_i}).$$

And so we get the expression

$$\dim_k k[f]/(f(t)-\alpha) = \deg f = \sum [k(\beta_i : k)]\nu_i.$$

for ‘the number of points in the fibre’; indeed, if all $\beta_i \in k$ and all $\nu_i = 1$, it is equal to the cardinality of the fibre.

Coming back to general situation, with X and Y be regular curves over k and $f\colon X \to Y$ be a finite non-constant morphism, we shall describe the scheme fibre $f^{-1}(y)$ quit similarly as done in the illustrative example above. Let t be a uniformizer at y.

Consider a point $x \in X$ mapping to y. The induced map $f_y^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ gives rise to a field extension $k(y) \subset k(x)$, which is finite since the Nullstellensatz tells us that both $k(x)$ and $k(y)$ are finite extensions of k. The degree $d_x = [k(x) : k(y)]$ is called the *local degree* of f at x_i. In the case that k is algebraically closed, the two fields coincide with k, and the local degree equals one.

The number $e_x = v_x(f^\sharp(t))$ will be called the *ramification index* of f at x. It does not depend on the choice of parameter t, and it holds that $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x^e\mathcal{O}_{X,x}$ and we have the equality

$$e_x = \dim_{k(x)} \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}.$$

We say that f *ramifies* in x when $e_x > 1$.

The scheme theoretic fibre $f^{-1}(y)$ equals $\operatorname{Spec} B/\mathfrak{m}_y B$, and as the domain B is of Krull dimension one and $\mathfrak{m}_y B \neq 0$, the ring $B/\mathfrak{m}_y B$ will be of dimension zero. It is of finite length and decomposes as the product of its localizations:

$$B/\mathfrak{m}_y B = \prod_{f(x)=y} \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}.$$

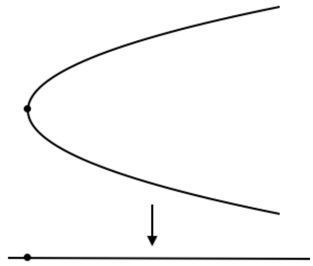
Combining this formula with Proposition 21.11 above one gets:

Proposition 21.13. Let $f\colon X \to Y$ a finite morphism between regular curves over k. For each closed point $y \in Y$, it holds that

$$\deg f = \sum_{f(x)=y} d_x e_x.$$

Example 21.14. Let $A = k[u]$ and let $B = k[u,v]/(u - v^2) \simeq k[v]$ where k is a field whose characteristic is not two. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let $f\colon X \to Y$ be the morphism induced by the inclusion $A \hookrightarrow B$ (thus $u \mapsto v^2$). The morphism f is ramified at the origin $x = (0,0)$, and here the ramification index is two. Indeed, u is a uniformizing parameter of $\mathcal{O}_{Y,y} = k[u]_{(u)}$ at $y = 0$, while v is the uniformizer of $\mathcal{O}_{X,x} = B_{(u,v)} = k[v]_{(v)}$. Then we have $v_y(u) = v_x(v^2) = 2$.

The reader might notice a resemblance between the previous example and Example **??** , where ramification was defined in terms of the relative sheaf of differentials $\Omega_{X/Y}$. In that



example, $\Omega_{X/Y}$ was a torsion sheaf supported on the single point $x = (0,0)$. This correspondence between the two notions of ramification is a general fact (at least in characteristic 0), and we have the useful formula for the ramification indexes of curves:

Proposition 21.15. Let $: X \to Y$ be a morphism between non-singular curves over k, and let $x \in X$ be a closed point. Assume that the ramification index e_x is invertible in k. Then one has

$$e_x = \operatorname{length}(\Omega_{X/Y})_x + 1.$$

Proof From a general perspective, one has the exact sequence

$$f^*\Omega_{Y/k} \xrightarrow{df} \Omega_{Y/k} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \tag{21.1}$$

from Proposition 20.22 on page 356. In our setting Y is a regular curve, so at a point $y \in Y$ the stalk $(\Omega_{Y/k})_y$ is a free $\mathcal{O}_{Y,y}$-module with basis du for u a uniformizer at y. Similarly, at a point $x \in X$ the stalk $(\Omega_{X/k})_x$ is free $\mathcal{O}_{X,x}$-module with basis dv for v a uniformizer at x.

Now $f_y^\sharp \colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ acts as $u \mapsto \alpha v^e$ with α a unit and $e = e_x$, and so the stalk at x of df in (21.1) is determined by the assignment $du \mapsto d\alpha v^e$, and we compute

$$d\alpha v^e = v^e \alpha' dv + e\alpha \cdot v^{e-1} dv = v^{e-1}(v\alpha' + e\alpha)dv.$$

Now, by hypothesis, e is invertible in k so that $\alpha' v + e\alpha$ is a unit in $\mathcal{O}_{X,x}$. Consequently the image of df is the submodule generated by $v^{e-1}dv$; and so the cokernel of df (which equals $(\Omega_{X/Y})_x$) is isomorphic to $\mathcal{O}_{X,x}dv/v^{e-1}\mathcal{O}_{X,x}dv \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x^{e-1}$. □

Example 21.16. Continuing Example 21.14 above, we se that the origin is the only place where f ramifies since $df = 2udu$, and the characteristic of k is supposed to be different from 2. If k is not algebraically closed, the local degree may be two; this happens, for instance, at a point $a \in k$ if a does not have a square root in k.

Extension of maps and The Fundamental Theorem

This section presents two basic result about non-singular curves. The first basically says that any rational map on a non-singular curve into a a projective variety is globally defined. Combining this with the Main Theorem of Birational Geometry (Theorem 12.33 on page 200) and the fact that every curve is birationally equivalent to a non-singular one, we obtain the second, which states that the category of projective non-singular curves over k with dominant maps is equivalent to the category of finitely generated field extensions of k of transcendence degree one.

Later it will turn out that all curve are projective, so in fact the claim applies to the category of non-singular curves proper over k.

Proposition 21.17. Let X be an irreducible curve over k and let $x \in X$ be a closed point where X is regular. Then any morphism $f\colon X - \{x\} \to Y$ to a projective variety Y has a unique extension $\overline{f}\colon X \to Y$.

Proof The salient point of the proof is precisely the same as in the proof of Lemma **??** on page ??. Fixing the notation, we let t be a uniformizer at x, and denote by K the function field of X.

The morphism f yields a K-point $\operatorname{Spec} K \to \mathbb{P}^n_K$ which is described by homogenous coordinates $(a_0 t^{\nu_0} : \cdots : a_n t^{\nu_n})$ where the a_i are units in $\mathcal{O}_{X,x}$ and the ν_i's are integers. After scaling through by $t^{-\min \nu_i}$ we may assume that for each i it holds that $\nu_i \geqslant 0$ and at $\nu_{i_0} = 0$ for at least one i_0.

Now the a_i are non-vanishing sections of $\mathcal{O}_X$ over some some open neighbourhood U of x and after shrinking U if need be, t will also be a section of $\mathcal{O}_X$ over U with x as the sole zero. Hence the $a_i t^{v_i}$ define a map $U \to \mathbb{P}^n_k$.

□

Corollary 21.18. Any rational map between two non-singular projective curves extends to a morphism. In particular, any birational map extends to an isomorphism.

Proof The first statement is just a reformulation of Proposition 21.17.

That two curves X and Y are birationally equivalent, means that there are open subsets $U \subset X$ and $V \subset Y$ and an isomorphism $f\colon U \to V$. Now, both f and f^{-1} extends respectively to morphisms $g\colon X \to Y$ and $h\colon Y \to X$, and since $h \circ g|_U = \mathrm{id}_U$ and $g \circ h|_V = \mathrm{id}_V$, it follows that $h \circ g = \mathrm{id}_X$ and $g \circ h = \mathrm{id}_Y$; indeed, morphisms that agree on an open dense set are equal. □

Theorem 21.19 (Main theorem of non-singular projective curves). There is an equivalence of categories between the following categories:

(i) The category of non-singular projective curves over k and dominant morphisms;

(ii) The category of finitely generated field extensions of k of transcendence degree one and k-algebra homomorphisms.

Proof First, if X and Y are two nonsingular projective curves, any rational map extends to a morphism. This shows, combined with Theorem 12.31 on page 199, that the functor $X \mapsto k(X)$ is fully faithful.

Next we show it is essentially surjective: each finitely generated field K of transcendence degree one over k is of the form $k(X)$ for some nonsingular projective curve X. If K is generated by $a_1, \ldots, a_r$ the k-subalgebra $A = k[a_1, \ldots, a_r]$ will be of dimension one according to Theorem **??** on page ??. The curve $X = \operatorname{Spec} A$ is contained in the affine space $\mathbb{A}_k^r$ in a natural way, and closing it up in $\mathbb{P}_k^r$, yields a projective curve, whose normalization is projective (after xxx) and has K as function field. □

Example 21.20 (Morphisms into $\mathbb{P}_k^1$)**.** For a non-singular curve X over k, there is a natural one-to-one correspondence between non-constant rational functions on X and dominant maps from X to $\mathbb{P}_k^1$. A rational function on X is just a morphism from some open subset to $\mathbb{A}_k^1$ and, this extends to a morphism from the entire X to $\mathbb{P}_k^1$.

To be somehow more explicit, we let $\mathbb{P}_k^1 = \operatorname{Proj} k[t_0, t_1]$ and the affine line $\mathbb{A}_k^1$ in the construction above be $D_+(t_0)$. Let $g \in k(X)^\times$ be given. To lessen the confusion, denote by G the extended map $G\colon X \to \mathbb{P}_k^1$.

Let U_g be the maximal open where g is defined; i.e. $U_g = G^{-1}D_+(t_1)$, then $G^\sharp(t_0/t_1) = g$ in $\mathcal{O}_X(U_g)$. Let $U_{g^{-1}}$ be the maximal open where g^{-1} is defined, then $U_{g^{-1}} = G^{-1}D_+(t_0)$, and it holds that $G^\sharp(t_1/t_0) = 1/g$ in $\mathcal{O}_X(U_{g^{-1}})$.

Coherent sheaves on curves

Recall that an element of an A-module is called a *torsion element* if it is killed by a nonzero-divisor of A, and a module is a *torsion module* if all elements are torsion. On the other hand, a module is *torsion free* if no non-zero element is torsion. The sum of two torsion elements is clearly torsion, so the subset of a module M formed by the torsion elements, is a submodule T. It has the property that M/T is torsion free.

We shall need the following result from algebra:

Proposition 21.21. Let A be a PID. Then any finitely generated torsion free module M is free. In particular, if A is regular of dimension one, every finitely generated torsion free module is locally free.

Proof Observe first that there are non-zero maps $M \to A$. Indeed, the natural map $M \to M \otimes_A K$ that sends m to $m \otimes 1$ is injective since M is torsion free. Then choose a K-linear map $M \otimes_A K \to K$ that does not vanish on M. If $\{m_i\}$ is a finite generating set for M, the images $\phi(m_i)$ may be brought on the form a_i/b with a common denominator. Then $b\phi$ is our map.

We proceed by induction on the rank of M. If the rank is one, M is an ideal in A, and hence is free since A is a PID. If the rank is superior to one, chose a non-zero map $M \to A$. The image is an ideal, hence free of rank one, and M splits as $M = \operatorname{Ker}\phi \oplus \operatorname{Im}\phi$. By induction, $\operatorname{Ker}\phi$ is free, and we are done.

Finally, that A is regular of dimension one, means that all the local rings $A_\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Spec} A$ are DVR's, and in particular, they are PID's. Hence each localization $M_\mathfrak{p}$ is free; in other words, M is locally free. □

Returning to the global situation, any coherent sheaf $\mathcal{F}$ on a scheme X contains a torsion subsheaf $\mathcal{T}$, whose sections over an open set $U \subset X$ equals the subgroup of $\mathcal{F}(U)$ of elements annihilated by some nonzerodivisor of $\mathcal{O}_X(U)$ (see Exercise 19.3.2 on page 339). The quotient $\mathcal{F}/\mathcal{T}$ is torsion free in the sense that on open affine subsets U its section space (which equals $\mathcal{F}(U)/\mathcal{T}(U)$) is a torsion free module over $\mathcal{O}_X(U)$.

When X is a curve, the support of $\mathcal{T}$ is finite, say it consists of the points $p_1, \ldots, p_r$, and $\mathcal{T}$ is the direct sum of its stalks at these points: $\mathcal{T} = \bigoplus_{i=1}^r \mathcal{T}_{p_i}$.

Theorem 21.22. Let X be a non-singular curve and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a decomposition

$$\mathcal{F} = \mathcal{E} \oplus \mathcal{T}$$

where $\mathcal{T} \subset \mathcal{F}$ is the torsion subsheaf and $\mathcal{E}$ is locally free.

Proof The quotient $\mathcal{E} = \mathcal{F}/\mathcal{T}$ is locally free by Proposition 21.21, and are to see that the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{21.2}$$

is split exact. Let U be an affine neighbourhood about p_i. Since $\mathcal{F}/\mathcal{T}|_U$ is the tilde of a projective module, the sequence (21.2) splits when restricted to U. Hence there is a map $\phi_i \colon \mathcal{F}|_U \to \mathcal{T}_{p_i}$ splitting the inclusion $\mathcal{T}_{p_i} \to \mathcal{F}$. This map extended by zero is a map $\phi_i \colon \mathcal{F} \to \mathcal{T}_{p_i}$ that splits off $\mathcal{T}_{p_i}$. The sum $\sum \phi_i$ then splits of the entire torsion subsheaf $\mathcal{T}$. □

The torsion sheaves on X are easily classified, but only for rather few class curves are the locally free sheaves satisfactory understood, but there is a vast literature about them. For instance, back on page 341 we proved Theorem 19.20 which states that every coherent locally free sheaf on $\mathbb{P}^1_k$ decomposes as a direct sum $\bigoplus \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ of line bundles. In fact this property characterises $\mathbb{P}^1_k$ among non-singular curves (even among normal projective varieties).

21.3 Divisors on regular curves

We shall mostly work with regular curves in this section, in which case there is no substantial distinction between Weil and Cartier divisors, every Weil divisor has a a set of Cartier data, and every set of Cartier data yields a Weil divisor. The distinction only shows up in the way a divisor is presented.

The codimension one-subsets of a curve are precisely the closed points, so that a Weil divisor is a finite formal combination

$$D = \sum_{x \in X} n_x x$$

of closed points in X, where the coefficients are integers. Each residue field $k(x)$ is a finite extension of the ground field k whose degree is denoted by $[k(x) : k]$. Note that in case the ground field is algebraically closed, all these degrees equal one. We define the degree of the prime divisor x as $\deg x = [k(x) : k]$, and extending this by linearity, every Weil divisor is given a *degree*, namely the sum:

$$\deg D = \sum [k(x) : k] n_x.$$

As noted above, every Weil divisor on a regular curve has a Cartier representation. To a given Weil divisor $D = \sum_x n_x x$ we may associate Cartier data $\{(U_x, g_x)\}$, indexed by

$\operatorname{Supp} D$, by letting U_x be any open affine neighbourhood of x disjoint from the rest of $\operatorname{Supp} D$, and letting $g_x = t_x^{n_x}$, where t_x is a uniformizing parameter at x. In terms of the Cartier data, the degree is given as

$$\deg D = \sum_{x \in X} [k(x) : k] v_x(g_x),$$

where, as usual, $v_x$is the valuation associated to $\mathcal{O}_{X,x}$.

Each non-zero coherent sheaf of ideals on a regular curve X is invertible (all the local rings are PID's), so with each finite subscheme Z of X is associated an effective Weil divisor (as in **??** on page ??):

$$D_Z = \sum_{x \in X} \operatorname{length}(\mathcal{O}_{Z,x}) x,$$

where the sum is finite because $\operatorname{length}(\mathcal{O}_{Z,x}) = 0$ for x outside the finite set Z. The $\operatorname{length}(\mathcal{O}_{Z,x})$ is the number of terms in composition series, and each subquotent equals $k(x)$, so $[k(x) : k] \operatorname{length}(\mathcal{O}_{Z,x}) = \dim_k \mathcal{O}_{Z,x}$. Summing up over closed points $x \in X$ yields

$$\deg D_Z = \dim_k \mathcal{O}_Z.$$

Recall also that each Weil divisor determines an invertible sheaf $\mathcal{O}_X(D)$, which over an open set U takes the value

$$\mathcal{O}_X(D)(U) = \{ f \in K | (\operatorname{div} f + D)|_U \geqslant 0 \}$$

Then D is effective if and only if $\Gamma(\mathcal{O}_X(D)) \neq 0$. In particular, if $\Gamma(X, \mathcal{O}_X(D))$ has dimension at least 2, there is a second effective divisor $D' = \sum m_i q_i$ such that D and D' are linearly equivalent.

Example 21.23. Consider the prime divisor x on $X = \mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[t]$ corresponding to the maximal ideal (t^2+1) in $\mathbb{R}[t]$. Then $k(x) = \mathcal{O}_{X,x}/(t^2+1)\mathcal{O}_{X,x} = \mathbb{R}[t]/(t^2+1) = \mathbb{C}$ and hence $\deg x = 2$.

Example 21.24. Consider the closed subset $D = V(t^5 - 1)$ of $X = \operatorname{Spec} \mathbb{Q}[t]$. Since $s(t) = t^4 + t^3 + t^2 + t + 1$ is an irreducible polynomial over $\mathbb{Q}$ the ideal $(s(t))$ in $\mathbb{Q}[t]$ is maximal. Thus the set D consists of the two points $p = V(t-1)$ and $q = V(s(t))$, and we may consider D as the divisor $D = p+q$. The residue fields are $k(p) = \mathbb{Q}$ and $k(q) = \mathbb{Q}(\eta)$, where η is a primitive fifth-root of unity. Consequently, the degree of $D = p+q$ is

$$\deg D = [\mathbb{Q} : \mathbb{Q}] + [\mathbb{Q}(\eta) : \mathbb{Q}] = 1 + 4 = 5,$$

which of course fits well with $t^5 - 1$ being of degree 5.

Over the field $k = \mathbb{Q}(\eta)$, the divisor $V(t^5 - 1)$ in $\operatorname{Spec} \mathbb{Q}(\eta)[t]$ splits as the sum of five different points, each with local degree one. Indeed, $t^5 - 1 = \prod_i t - \eta^i$, and letting $p_i = V(t_{\eta^i})$, we find $D = p_0 + \cdots + p_5$.

Example 21.25 (The circle)**.** Consider the circle $X = \operatorname{Spec} \mathbb{R}[u,v]/(u^2+v^2-1)$. The example is about the divisors on X obtained by intersecting X with lines $u + v - a$ where a real; or in the present terminology, the principal divisor $D = \operatorname{div}(u + v - a)$. The support of D equals $V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal $\mathfrak{a} = (u + v - a, u^2 + v^2 - 1)$. It holds that $\mathfrak{a} = (u + v - a, P(v))$ where $P(v) = 2v^2 - 2av + a^2 - 1$, so that

$$k[u, v]/\mathfrak{a} \simeq k[v]/(P(v)).$$

Now there are three cases. Firstly, if $a > \sqrt{2}$, the polynomial $P(v)$ does not have real roots, and $\mathbb{R}[u, v]/\mathfrak{a} = \mathbb{C}$. The divisor D is the prime divisor $x = V(\mathfrak{a})$ and $k(x) = \mathbb{C}$.

Secondly, if $a < \sqrt{2}$, the polynomial $P(v)$ splits as the product of two disinct linear factors l_1 and l_2, and $\mathfrak{a} = \mathfrak{m}_1 \cap \mathfrak{m}_2$ with $\mathfrak{m}_i = (u + v - a, l_i)$. Each ideal $\mathfrak{m}_i$ is maximal, and $\mathbb{R}[u, v]/\mathfrak{m}_i = \mathbb{R}$. The divisor D equals $D = x_1 + x_2$ with $x_i = V(\mathfrak{m}_i)$, and $k(x_i) = \mathbb{R}$.

Finally, when $a = \sqrt{2}$, we find $\mathfrak{a} = (u + v - a, l^2)$ where $l = v - \sqrt{2}/2$. The divisor D becomes $D = 2x$ with $x = (u - \sqrt{2}/2, v - \sqrt{2}/2)$ and $k(x) = \mathbb{R}$.

21.3.1 Pullbacks of divisors

If $f\colon X \to Y$ is a morphism, we can pull back invertible sheaves from Y to X, as well as sections of these. By the correspondence between divisors and invertible sheaves, this gives us a way of pulling back divisors from Y to X. In the context of curves, we can make this a little bit more explicit. We assume that the morphism $f\colon X \to Y$ is finite, it is then surjective, and $f^\sharp$ induces an inclusion of function fields $k(Y) \hookrightarrow k(X)$.

We aim at defining the pull back of Weil divisors and start by just pulling back a point $y \in Y$. This pullback is just the divisor associated to the scheme theoretic fibre $f^{-1}(y)$. To give a detailed description, choswe a local parameter $t \in \mathcal{O}_{Y,y}$ at y and define

$$f^*(y) = \sum_{f(x)=y} v_x(f^\sharp t)x,$$

where as usual v_x is the valuation at x. Changing t by a unit in $\mathcal{O}_{Y,y}$ does not alter the valuation $v_x(f^\sharp t)$ because a unit in $\mathcal{O}_{Y,y}$ stays a unit in $\mathcal{O}_{X,x}$. Extending this by linearity, we obtain a well defined group homomorphism

$$f^*\colon \operatorname{Div} X \to \operatorname{Div} Y.$$

We can also understand this map on the level of Cartier divisors: if D is a Cartier divisor on Y given by the data $\{(U_i, g_i)\}$, where $g_i \in k(Y)^\times$, we can consider the data $\{(f^{-1}U_i, f^\sharp g_i)\}$, which defines a Cartier divisor on X.

Example 21.26 (Principal divisors)**.** The principal divisor $\operatorname{div} g$ of a rational function $g \in k(X)^\times$ equals the pullback $G^*((0:1) - (1:0))$, where $G\colon X \to \mathbb{P}^1_k$ is the extension of g as in Example 21.20 on page 368. With the notation there it holds that

$$\operatorname{div} g = \sum_{x \in U_g} v_x(g) + \sum_{x \in U_{g^{-1}}} v_x(g) = \sum_{x \in U_g} v_x(g) - \sum_{x \in U_{1/g}} v_x(1/g)$$

since $v_x(g) = 0$ for $x \in U_g \cap U_{g^{-1}}$. But this is precisely the pullback $G^*((0:1) - (1:0))$ since $G^\sharp(t_0/t_1) = g$ and $g^\sharp(t_1/t_0) = 1/g$ and and t_0/t_1 and t_1/t_0 are uniformizers at $(0:1)$ and $(1:0)$ respectively.

Lemma 21.27. If $f \colon X \to Y$ is finite and D is a divisor on Y, we have $\deg f^*D = \deg f \cdot \deg D$.

Proof It suffices to treat the case of prime divisors, so let $D = y$. Now, let $\operatorname{Spec} A$ be an open neighbourhood of y and $\operatorname{Spec} B$ the inverse image of $\operatorname{Spec} A$. Then B is a torsion free A-algebra a so is locally free of rank equal to $[k(X) : k(Y)] = \deg f$. For t a uniformizer at y the value $v_x(f^\sharp t)$ is the ramification index of f at x and is written e_x. Moreover,

$$\deg x = [k(x) : k] = [k(x) : k(y)][k(y) : k] = d_x \deg y$$

where d_x is the local degree of f at x. From (21.3.1) and Proposition 21.13 follows that

$$\deg f^*y = \sum_{f(x)=y} v_x(f^\sharp t) \deg x = (\sum_{f(x)=y} d_x e_x) \deg y = \deg f \cdot \deg y.$$

□

Lemma 21.28. For a non-zero $g \in k(X)$ and a morphism $f \colon X \to Y$, we have

$$f^* \operatorname{div} g = \operatorname{div} g \circ f.$$

Proof Let $G \colon X \to \mathbb{P}^1$ be the extension of g then $G \circ f$ is the extension of $g \circ f$ and so according to Example 21.26 above, it holds that

$$f^* \operatorname{div} g = f^*(G^*((0:1) - (1:0)) = (G \circ f)^*((0:1) - (1:0)) = \operatorname{div} g \circ f.$$

□

Corollary 21.29. For a non-zero rational function $g \in k(X)$, we have $\deg \operatorname{div} g = 0$. Hence the degree map descends to a well-defined map

$$\deg \colon \operatorname{Cl}(X) \to \mathbb{Z}.$$

In other words, linearly equivalent divisors have the same degree.

Proof This is clear if g is a constant. If not, g defines a morphism $G \colon X \to \mathbb{P}^1_k$ so that

$$\operatorname{div} g = G^*((1:0) - (0:1)).$$

Thus we are done by the above lemma. □

Example 21.30. Assume that k is a field whose characteristic is different from 2 and 31. and consider the curve $X \subset \mathbb{A}^2_k = \operatorname{Spec} k[u, v]$ given by the equation

$$v^2 = u^3 + u^2 + 1 \tag{21.3}$$

which is a regular curve. Consider the the rational function $g = v + 1$ on X. What is div g? Note that g is regular, so there are no points x for which $v_x(g) < 0$. Rewriting (21.3) as

$$(v-1)(v+1) = u^2(u+1), \tag{21.4}$$

we see that the zeros of $v+1$ are the points $x = (0, -1)$ and $y = (-1, -1)$. Near $x = (0, 1)$ both $(v + 1)$ and $u + 1$ are invertible, and the equality

$$v - 1 = u^2(u + 1)(v + 1)^{-1} \tag{21.5}$$

shows that u is uniformizer there (the maximal ideal $\mathfrak{m}_x$ is generated by $v - 1$ and u). In the same vein, near $y = (-1, -1)$ both $v + 1$ and u are invertibel, and we infer from (21.5) that $u + 1$ is a uniformizer. It follows that

$$\operatorname{div} g = v_x(u^2) + v_y(u + 1) = 2x + y. \tag{21.6}$$

Example 21.31. Consider the curve $Y \subset \mathbb{P}^2_k = \operatorname{Proj} t_0, t_1, t_2$ given by the equation

$$t_2^2 t_0 = t_1^3 + t_1^2 t_0 + t_0^3$$

Note that the curve in the previous example equals $X \cap D(t_0)$, where we use coordinates $u = t_1/t_0, v = t_2/t_0$. Let us compute $\operatorname{div} g$ for the same rational function $g = t_2/t_0 + 1$ as before, but this time on Y. For this, we only need to consider the points where $t_0 = 0$. From the equation, we see that there is a single point in $Y \cap V(t_0)$, namely the point $z = (0 : 0 : 1)$. To compute $v_x(g)$ here, we use the chart $D(t_1)$. Then $Y \cap D(x_2)$ is isomorphic to the plane curve given by the equation

$$u = v^3 + v^2 u + u^3 \tag{21.7}$$

where now $u = t_0/t_2$ and $v = t_1/t_2$. The point z is then the origin $(u, v) = (0, 0)$ in $D_+(t_2)$. Note that $g = u^{-1} + 1$. Rewriting (21.7) as

$$v^3 = u(1 - v^2 u - u^3),$$

we see that v is a uniformzer at z and that $v_x(u) = 3$. Hence we find we also see that $v_z(u) = 3$, and so

$$v_x(g) = v_x(u^{-1} + 1) = v_x(u + 1)/u) = v_x(u + 1) - v_x u = -3$$

Finally, we concude thats

$$\operatorname{div} g = 2(1 : 0 : -1) + (1 : -1 : -1) - 3(0 : 0 : 1). \tag{21.8}$$

Note that, since Y is projective we may use Corollary 21.29 and conclude that $\deg \operatorname{div} g = 0$, which immediately yields 21.8.

21.3.2 Pushforward of divisors

For a morphism of curves $f\colon X \to Y$, one may also define a pushforward map $f_*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ as follows. If $D = \sum_{x \in X} n_x x$, we define

$$f_*(D) = \sum_{x \in X} d_x n_x x,$$

where d_x is the local degree of f at x (as defined on page 365). This defines an element of $\operatorname{Div}(Y)$ and f_* will be a linear map $\operatorname{Div} X \to \operatorname{Div} Y$. In this case it is not so obvious that the map descends to a map between the class groups $\operatorname{Cl}(X)$ and $\operatorname{Cl}(Y)$. However, it turns out that this is indeed the case: for $f \in k(X)^\times$, we have $f_* \operatorname{div}(f) = \operatorname{div} N(f)$ where $N\colon k(X) \to k(Y)$ is the *norm map* between the function fields.

Proposition 21.32. If $f\colon X \to Y$ is a finite morphism between regular curves and $g \in k(X)$ a non-zero rational function. then $f_* \operatorname{div} f = \operatorname{div} N(f)$. In particular, f_* passes to the quotient and gives a homomorphism $f_*\colon \operatorname{Cl}(X) \to \operatorname{Cl}(Y)$.

Quit generally there is norm for any finite field exetension $K \subset L$. Then norm $N(g)$ of an element $g \in L^\times$ is defined as the determinant of the multiplication map $L \to L$ given as $t \mapsto gt$. It is a multiplicative map $N\colon L^\times \to K^\times$. If $B \subset L$ is a subsring and $A = B \cap K$, it holds that $N(g) \in A$ for each $g \in B$.

Lemma 21.33. If A is a DVR and $\phi\colon A^n \to A^n$ is an injective map, then $v(\det \phi) = \operatorname{length}(\operatorname{Coker} \phi)$.

Proof Represent ϕ by a matrix (a_{ij}). After a permutation of rows and columns we may assume that $v(a_{11}) \leqslant v(a_{ij})$ for all other entries a_{ij}. It is then straightforward to perform elementary row and column operations to make the matrix have $a_{1i} = a_{i1} = 0$ for $i \neq 1$. Repeated application of this procedure yields bases for the source and target of ϕ in which ϕ has a diagonal matrix. If the i-th diagonal element is $\alpha_i t^{e_i}$ with α_i a unit, then $\operatorname{Coker} \phi \simeq \bigoplus_i A/(t^{e_i})A$ and $\operatorname{length}(\operatorname{Coker} \phi) = \sum e_i$, which obviously equals $v(\det \phi)$. □

Corollary 21.34. If A is a DVR and B a finite free A-algebra, then $\operatorname{length}_B(B/(b)B) = v_A(N(b))$ for any element $b \in B$.

Proof As to the proof of Proposition 21.32, it will be sufficient to establish that

$$v_y(N(g)) = \sum_{f(x)=y} d_x v_x(g)$$

for all $y \in Y$. We shall apply the Corollary with $A = \mathcal{O}_{Y,y}$ and $B = (f_*\mathcal{O}_X)_y$; the latter is finite and free over A (Proposition 21.21 on page 368).

The ring B is a one-dimensional semi-local ring whose maximal ideals correspond to the points in the fibre $f^{-1}(y)$. Hence $B/(g)B$ is Artinian, and it decomposes as $B/(g)B = \bigoplus \mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}$. We claim that

$$\operatorname{length}_A \mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x} = d_x v_x(g),$$

from which 21.32 follows in view of the Corollary. Indeed, it holds true that $\operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}) = v_x(g)$, which means that $\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}$ has a composition series of length $v_x(g)$, and the subquotients are all isomorphic to $k(x)$; hence $\operatorname{length}_A(\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x}) = [k(x) : k(y)] \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(g)\mathcal{O}_{X,x})$, and the claim follows. □

Exercise 21.3.1 (The projection formula)**.** Let $f\colon X \to Y$ be a finite morphism between regular curves and let D be a divisor on Y. Show that $f_* f^* D = \deg f \cdot D$.

21.4 The canonical divisor

In this section we exclusively work over a perfect field k. A curve over k will then be smooth if and only if it is regular, and the sheaf $\Omega_{X/k}$ of regular differential is then locally free of rank one.

The elements of $\Omega_{k(X)/k}$ are called *rational differential forms* We are going to associate a Weil divisor with each non-zero rational differential form on X. These divisors will all be rationally equivalent, and so we find a well defined divisor class in $\operatorname{Div} X$, which only depends on the curve. It is called the *canonical class* and the divisors belonging to it will be called *canonical divisors*; often denoted by K_X. The canonical class is the most important invariant of the curve.

Since the Kähler differentials localize well (Theorem 20.31), the module $\Omega_{k(X)/k}$ is a one-dimensional $k(X)$-vector space being the stalk of the invertible sheaf $\Omega_{X/k}$. Any local generator η of $\Omega_{X/k}$ at a point $x \in X$, is a generator for $\Omega_{k(X)/k}$ as well, so that each rational differential ω is of the form $\omega = g\eta$ for some $g \in k(X)$; indeed, $\Omega_{k(X)/k} = \Omega_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} k(X)$.

To every rational differential one may associate a Weil divisor $\operatorname{div} \omega$ by the following procedure. For each point $x \in X$ chose a generator η_x for $\Omega_{\mathcal{O}_{X,x}/k}$ and write $\omega = g_x \eta_x$ with $g_x \in k(X)$. Then let

$$\operatorname{div} \omega = \sum_{x \in X} v_x(g_x) x. \tag{21.9}$$

The expression on the right in (21.9) is independent of the choice of local generators, which is clear since two generators η'_x and η_x will be related through an equality $\eta_x = \alpha \eta'_x$ with α a unit in $\mathcal{O}_{X,x}$. Hence $\omega = g_x \eta_x = \alpha g_x \eta'_x$, and $v_x(\alpha g_x) = v_x(g_x)$. Note also that the sum in fact is finite. This ensues from any local generator η_x being a generator for $\Omega_{X/k}$ in some neighbourhood U of x.

That the divisors associated with two rational differentials are linearly equivalent, is clear from the definition in (21.9); indeed, two rational differentials are proportional with a factor from $k(X)$, and for each $x \in X$ it holds that $v_x(hg_x) = v_x(h) + v_x(g_x)$, so that

$$\operatorname{div}(h\omega) = \operatorname{div} h + \operatorname{div} \omega.$$

This leads to:

Definition 21.35 (The canonical class)**.** Let X be a smooth curve over k. The *canonical class* of X in $\operatorname{Cl} X$ is the divisor class of $\operatorname{div} \omega$ for any non-zero $\omega \in \Omega_{k(X)/k}$. Any divisor in the canonical class is called *a canonical divisor* and often will be denoted by K_X.

What we have done so far is valid over any field as long as the curve is smooth. When the ground field is perfect, there is as a good local description of the rational differentials in terms of uniformizers making calculations easier.

Lemma 21.36. Assume that X is smooth at the closed point $x \in X$ and that t is uniformizer at x. Then each element of $\Omega_{k(X)/k}$ is of the form gdt with $g \in k(X)$; in other words, $\Omega_{k(X)/k}$ is of rank one over $k(X)$ with dt as a basis.

Proof The Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ at x is always generated by the class of a uniformizer, and in virtue of Proposition 20.26, if follows that dt generates $\Omega_{\mathcal{O}_{X,x}/k}$ when X is smooth and $k(x)$ is a separable extension of k. □

Example 21.37. When the residue field $k(x)$ is not separable over the ground field k, it happens that $dt = 0$ for a uniformizer t at x. For instance, if k is of characteristic p and $\alpha \in k$ does not have a p-th root in k, the ideal $\mathfrak{m} = (t^p - \alpha)$ in $k[t]$ is maximal. The element $t^p - \alpha$ is a uniformizer in the local ring $k[t]_{\mathfrak{m}}$, but its derivative equals 0.

Be reminded that two Weil divisors D and D' are linearly equivalent precisely when the two associated invertible sheaves are isomorphic.

Proposition 21.38. The invertible sheaf associated to $\operatorname{div} \omega$ equals Ω_X.

Example 21.39. Let $X = \mathbb{P}_k^1$ with the usual covering $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$. The differential form dt is an element of $\Omega_{k(X)/k}$, which generate $\Omega_{X/k}|_{U_0}$. This means that $v_x(dt) = 0$ for every $x \in U_0$. For the remaining point $(1:0)$ at infinity, note that t^{-1} is the uniformizer there, $(1:0)$ corresponding to the origin in U_1. We have $d(t^{-1}) = -t^{-2}dt$; hence $dt = -(t^{-1})^{-2}d(t^{-1})$. This means that $v_{(1:0)}dt = -2$, so that $\operatorname{div} dt = -2(1:0)$.

As a Cartier divisor, the corresponding divisor is given by $(U_0, 1), \quad (U_1, t^2)$ This shows that $\Omega_X \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$.

Example 21.40. Assume that k is of characteristic different from 2. Let $X \subset \mathbb{A}_k$ be the elliptic curve given by the equation

$$v^2 = u^3 - u,$$

and consider the differential $\omega = du$. At a point $p = (a, b)$ where $b \neq 0$, the coordinate u is a uniformizer, and so $du = d(u - a)$ has zero valuation at p. When $b = 0$, the curve has three points: $p_1 = (0, 0)$, $p_2 = (-1, 0)$, and $p_3 = (1, 0)$.

At these points, v will be a uniformizer, and since $2vdv = (3u^2 - 1)du$, it holds that

$$du = 2v/(3u^2 - 1)dv.$$

Hence $v_{p_i}(du) = 1$ for all three. Summing up, we conclude that

$$\operatorname{div} \omega = p_1 + p_2 + p_3.$$

Example 21.41. We consider the projectivization $X \subset \mathbb{P}_k^2$ of the previous example, i.e. the curve whose homogeneous equation is

$$x_1^2 x_2 = x_0^3 - x_0 x_2^2.$$

Consider again the rational differential ω from before; that is, $\omega = d(x_0/x_2)$. We know the behaviour of ω on the distinguished open set $D_+(x_2)$, so what remains to compute the divisor of ω, is the valuation $v_p(\omega)$ for each point in $X \cap V(x_2)$, but this intersection has just one single point $x = (0:1:0)$.

Dehomogenizing the chart $D_+(x_1)$ by setting $u = x_0/x_1$ and $v = x_2/x_1$, the equation of X in $D_+(x_1)$ becomes

$$v = u^3 - uv^2.$$

Since $1 + uv$ is invertible near x, this shows that u is a uniformizer at x and that $v_x(v) = 3$. Our differential ω takes the form $\omega = d(x_0/x_1 \cdot x_1/x_2) = d(u/v) = (udv - vdu)/v^2$. We find

$$dv = 3u^2 - v^2 - 2uvv')dv,$$

which yields

$$\begin{aligned} udv - vdu &= (3u^3 - uv^2 - 2u^2vv' - v)du \\ &= (2u^3 - 2uv^2 - 2u^2vv')du \end{aligned} \tag{21.10}$$

The terms uv^2, $2u^2vv'$ vanish to order at least 5 at x, and the dominating term in (21.10) is $2u^3$, which means that $v_x(\omega) = v_x(u^3) - 2v_x(v) = -3$. We conclude that

$$\operatorname{div} \omega = (0:0:1) + (-1:0:0) + (1:0:0) - 3(0:1:0)$$

The genus of a curve

The genus of a curve belongs to the hall of fame of notions in algebraic geometry, and is even one of the most promient membres. Traces are found in Abels work on the addition formula, where it appeats as a 'mysteroius number', but it would be fair to say that it was Riemann who brought it into daylight. The genus is an invariantwith at least two faces: it is a topological invariant of real surfaces (as the number of handles you attache to the sphere to get the surface) and at the same tme a purely algebraic invariant, which is defined over any field (even for curves over the feld with two elements).

Definition 21.42. Let k be a field and X a curve proper over k. The *arithmetic genus* of X is defined as the number

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X).$$

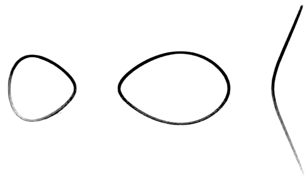
The *geometric genus* of X is defined as

$$p_g(X) = \dim_k H^0(X, \Omega_X).$$

Note that $\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$, so we get

$$p_a(X) = 1 - \chi(\mathcal{O}_X).$$

These numbers are defined using different sheaves, and there is no a priori reason to expect that they should have anything to do with each other. However, we shall see later in the chapter that there is a strong relation between them: $p_a = p_g$ whenever X is non-singular. For the time being we will still refer to the arithmetic genus p_a as the *genus* of X.



Example 21.43. When $X = \mathbb{P}^1$, we have $H^1(\mathbb{P}^1, \mathcal{O}) = 0$ so the arithmetic genus is 0. Likewise, we have that $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, so also $p_g = 0$.

Example 21.44. Let $X \subset \mathbb{P}^2$ be a plane curve, defined by a homogeneous polynomial $f(x_{0,1}, x_2)$ of degree d. In Chapter **??**, we computed that $H^1(X, \mathcal{O}_X) \simeq k^{\binom{d-1}{2}}$. Hence the genus of X is $\frac{(d-1)(d-2)}{2}$.

21.5 Hyperelliptic curves

Let us recall the hyperelliptic curves defined in Chapter 3. For an integer $g \geqslant 1$ we consider the scheme X glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x, y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} \text{ and } B = \frac{k[u, v]}{(-v^2 + a_{2g+1}u + \cdots + a_1 u^{2g+1})}$$

As before, we glue $D(x)$ to $D(u)$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

In Chapter **??** we showed that the genus of X was g and claimed that X was actually projective.

Let us examine the last point in more detail, and give a new projective embedding of X. To do this, we will need to work out the groups $\Gamma(X, \mathcal{O}_X(nP))$ for a point $p \in X$.

Let us for simplicity assume that $a_{2g+1} = 1$. Let p be the unique closed point given by $V(u, v)$ in X. In the local ring at p, we have

$$u = v^2(1 + a_{2g}u + \cdots + a_1 u^{2g})^{-1} = v^2(\text{unit}),$$

and hence v generates $\mathfrak{m}_p$. Hence v is the local parameter. The valuations of v, u, x, y are given by

$$\nu_p(v) = 1, \quad \nu_p(u) = 2, \quad \nu_p(x) = -2, \quad \nu_p(y) = 1 + (g+1)(-2) = -(2g+1)$$

We computed in XXX that $\Gamma(X, \mathcal{O}_X) = k$, which agrees with our expectation that there are no non-constant regular function on a projective curve. Let us consider the case where the rational functions are allowed to have poles at p (and only at p). In other words, we are interested in elements $s \in \Gamma(X, \mathcal{O}_X(p))$. Note that the point p does not lie in U; this means that s is regular there, and hence can be viewed as a *polynomial* in x, y. Now, as $A = k[x] \oplus k[x]y$ as a $k[x]$-module, we can express any element s can be expressed as $f(x) + h(x)y$. We can then calculate

$$\begin{aligned} \nu_p(f(x) + h(x)y) &= \min\{\nu_p(f(x)), \nu_p(h(x))\nu_p(y)\} \\ &= \min\{-2\deg f, -(2\deg h + 2g + 1)\} \end{aligned}$$

Thus, since we assume $g \geqslant 1$, any non-constant rational function with a pole at p must have valuation $\leqslant -2$ there, and hence we have only the constants in $\Gamma(X, \mathcal{O}_X(p)) = k$.

On the other hand for the divisor $2p$ we obtain an extra section, corresponding to the rational function x:

$$\Gamma(X, \mathcal{O}_X(2p)) = k \oplus kx$$

Note that $\mathcal{O}_X(2p)_p = \mathcal{O}_{X,p} \cdot x$. The section $x \in \Gamma(X, \mathcal{O}_X(2p))$ is non-vanishing at p, while the section $1 \in \Gamma(X, \mathcal{O}_X(2p))$ is vanishing at p, since $1 = u \cdot x$ and $u \in \mathfrak{m} \subset \mathcal{O}_p$. Note that the linear series generated by $1, x$ generates $\mathcal{O}_X(2p)$ everywhere, so we get the morphism

$$\begin{aligned} X &\xrightarrow{\varphi} \mathbb{P}^1 \\ (x, y) &\mapsto (1 : x) \end{aligned} \tag{21.11}$$

This morphism is exactly the double cover above. It gets even more interesting if we allow even higher order poles at p. The computation above shows that

$$\Gamma(X, \mathcal{O}(3p)) = \begin{cases} k \oplus kx \oplus ky & \text{if } g = 1 \\ k \oplus kx & \text{if } g > 1 \end{cases}$$

Case $g = 1$. We can show, using the embedding criterion of Chapter **??**, that the sections $x_0 = 1, x_1 = x, x_2 = y$ give an embedding

$$X \rightarrow\hookrightarrow \mathbb{P}^2_k$$

The image is even seen to be a cubic curve: One computes that $\Gamma(X, \mathcal{O}(6p))$ is 6-dimensional, but we have 7 global sections: $1, x, y, x^2, xy, x^3, y^2$. That means that there must be some relation between them of the form - of course it is just the relation in A:

$$y^2 = a_3x^3 + a_2x^2 + a_1x.$$

This gives the following defining equation of X in $\mathbb{P}^2$:

$$x_2^2x_0 = a_3x_1^3 + a_2x_0x_1^2 + a_1x_0^2x_1$$

Case $g = 2$. In this case, the divisor $3p$ does not give a projective embedding. However, the map given by $5p$ gives something interesting: We obtain

$$\Gamma(X, \mathcal{O}_X(5p)) = k \oplus kx \oplus kx^2 \oplus ky$$

These sections generate $\mathcal{O}_X(5p)$, so we obtain a morphism

$$\phi : X \to \mathbb{P}^3$$

given by the sections $u_0 = 1, u_1 = x, u_2 = x^2, u_3 = y$ of $L = \mathcal{O}_X(5p)$. Notice that $u_0u_2 - u_1^2 = 0$, so the image of X lies on a quadric surface. In fact, the image of ϕ is precisely the relations between the sections:

The map ϕ is in this case a closed immersion, showing that X is projective.

22

The Riemann–Roch theorem

When X is a projective curve over a field k, the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces and we define

$$h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$$

Note that in this case, $h^i(X, \mathcal{F}) = 0$ for all $i \geqslant 2$, so we have two cohomology groups $h^0(X, \mathcal{F})$ and $h^1(X, \mathcal{F})$ to work with. We will mostly be interested in the case when $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor D; any invertible sheaf on X is of this form.

Our most basic tool for studying the cohomology groups $H^0(X, \mathcal{O}_X(D))$ is the ideal sheaf sequence of a point $p \in X$, which takes the form

$$0 \longrightarrow \mathcal{O}_X(-p) \longrightarrow \mathcal{O}_X \longrightarrow k(p) \longrightarrow 0 \tag{22.1}$$

where the first map is the inclusion and the second is evaluation at p. Here we have identified the ideal sheaf $\mathfrak{m}_p \subset \mathcal{O}_X$ by the invertible sheaf $\mathcal{O}_X(-p)$, and the sheaf $i_*\mathcal{O}_p$ with the skyscraper sheaf with value $k(p)$ at p. If L is an invertible sheaf, we can tensor (22.1) by L and get

$$0 \longrightarrow L(-p) \longrightarrow L \longrightarrow k(p) \longrightarrow 0 \tag{22.2}$$

where $L(-p)$ is the invertible sheaf of sections of L vanishing at p. (Here we also identify $L \otimes k(p) \simeq k(p)$, because every invertible sheaf over a point is trivial). In particular, taking $L = \mathcal{O}_X(D + p)$ in (22.2) we get the following basic bound:

Lemma 22.1. Let X be a non-singular projective curve, and let D be a divisor on X. Then

- $h^0(X, \mathcal{O}(D + p)) \leqslant h^0(X, \mathcal{O}(D)) + 1$ for each $p \in X$.
- $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D + 1$.

Proof We only need to prove the last part. Also, it suffices to constder the case when $D = \sum n_p p$ is effective (otherwise the left-hand side is 0). In that case, the inequality follows by applying the first inequality $\deg D$ times. □

Recall, that we defined for a sheaf $\mathcal{F}$, the Euler characteristic $\chi(\mathcal{F})$ as the alternating sum of the $h^i(X, \mathcal{F})$. One useful property of $\chi(X, -)$ is that it is additive on short exact sequences: $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. Thus applying χ to (22.2), we get

$$\chi(L(-p)) = \chi(L) - \chi(k(-p)) = \chi(L) - 1.$$

Theorem 22.2 (Easy Riemann–Roch). Let X be a smooth projective curve of genus g and let D be a Cartier divisor on X. Then

$$\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

Proof Let $p \in X$ be a point and consider the sequence (22.2) with $L = \mathcal{O}_X(D + p)$. Then, as we just saw, $\chi(\mathcal{O}_X(D + p)) = \chi(\mathcal{O}_X(D)) + 1$. Also the right-hand side of the equation above increases by 1 by adding p to D (since $\deg(D + p) = \deg D + 1$). This means that the theorem holds for a divisor D if and only if it holds for $D + p$ for any closed point p. So by adding and subtracting points, we can reduce to the case when $D = 0$. But in that case, the left hand side of the formula is by definition $\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g$, which equals the right hand side. □

The formula above is useful because the right hand side is so easy to compute. The number we are really after is the number $h^0(X, \mathcal{O}_X(D))$, since this is the dimension of global sections of $\mathcal{O}_X(D)$. This in turn would help us to study X geometrically, since we could use sections of $\mathcal{O}_X(D)$ to define rational maps $X \dashrightarrow \mathbb{P}^n$. So if we, for some reason, could argue that say, $H^1(X, \mathcal{O}_X(D)) = 0$, we would have a formula for the dimension of the space of global sections of $\mathcal{O}_X(D)$.

In any case, we can certainly say that $h^1(X, \mathcal{O}_X(D)) \geqslant 0$, so we get the following bound on $h^0(X, \mathcal{O}_X(D))$. It is a *lower bound* on $h^0(X, \mathcal{O}_X(D))$, which is often enough in applications.

Corollary 22.3. $h^0(X, \mathcal{O}_X(D)) \geqslant \deg D + 1 - g$.

Example 22.4. A typical feature is that $H^1(X, \mathcal{O}_X(D)) = 0$ provided that the degree $\deg D$ is large enough. This is basically a consequence of Serre's theorem. To give an example, consider again the case where X is a hyperelliptic curve of genus 2, as in XXXX. We have the following table of the various cohomology groups $H^i(X, \mathcal{O}_X(np))$ for the point $p = (u, v)$:

D	0	$1p$	$2p$	$3p$	$4p$	$5p$	$6p$	$7p$
$H^0(X, \mathcal{O}_X(D))$	1	1	2	2	3	4	5	6
$H^1(X, \mathcal{O}_X(D))$	2	1	1	0	0	0	0	0
$\chi(\mathcal{O}_X(D))$	-1	0	1	2	3	4	5	6

and it is not so hard to prove directly using the Cech complex that $H^1(X, \mathcal{O}_X(np)) = 0$ for all $n \geqslant 3$.

Fortunately, there are more general results which tell us when $H^1(X, \mathcal{O}_X(D)) = 0$. This is due to the following fundamental theorem:

Theorem 22.5 (Serre duality). Let X be a smooth projective variety of dimension n and let D be a Cartier divisor on X. Then for each $0 \leqslant p \leqslant n$,

$$\dim_k H^p(X, \mathcal{O}_X(D)) = \dim_k H^{n-p}(X, \mathcal{O}_X(K_X - D))$$

So if X is a curve, we get that $h^1(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(K_X - D))$ and the Riemann–Roch theorem takes the following form:

Theorem 22.6 (Riemann–Roch). Let X be a non-singular projective curve of genus g and let $D \in \mathrm{Div}(X)$ be a divisor. Then

$$h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - g$$

This is a much stronger statement than the Riemann–Roch formula we had before. It is more applicable because the group $H^0(X, \mathcal{O}_X(K_X - D))$ is easier to interpret: it is the space of global sections of the sheaf associated to the divisor $K_X - D$, or equivalently $\Omega_X(-D)$. It is also often easier to argue that there can be no such global sections of this sheaf. For instance, in the case $\deg D > \dim K_X$ then $K_X - D$ cannot be effective: effective divisors $\sum n_i p_i$ have non-negative degree.

So what is this degree of the canonical divisor K_X? From Serre duality, we get that $H^0(X, \mathcal{O}_X(K_X))$ and $H^1(X, \mathcal{O}_X)$ have the same dimension, so the geometric genus and arithmetic genus agree:

$$p_g = p_a = g.$$

Then applying the Riemann–Roch formula to $D = K_X$, we get

$$g - 1 = \dim_k H^0(X, \mathcal{O}_X(K_X)) - \dim_k H^0(X, \mathcal{O}_X(K_X - K_X)) = \deg K + 1 - g$$

and so $\deg K_X = 2g - 2$. We summarize this in the following corollary.

Corollary 22.7. Suppose that D is a Cartier divisor of degree $\leqslant 2g - 1$. Then $H^1(X, \mathcal{O}_X(D)) = 0$, and

$$h^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

Moreover, if $\deg D = 2g - 2$, then $H^1(X, \mathcal{O}_X(D)) \neq 0$ only if $D \sim K_X$.

Example 22.8. Let us verify the Riemann–Roch formula for $X = \mathbb{P}^1$. It suffices to check it for all divisors of the form $D = dP$ where $P \in \mathbb{P}^1$ is a point. In this case, the right-hand-side of the fomula equals $\deg D + 1 - 0 = d + 1$.

If $d \geqslant 0$, we may identify $H^0(X, D)$ with the space of homogenous degree d polynomials in x_0, x_1. Hence $h^0(X, D) = d + 1$. Moreover, $h^1(X, D) = 0$, as we saw in Chapter XXX. If $d < 0$, we have $h^0(X, D) = 0$ and $h^1(X, D) = -d - 1$.

22.1 Serre duality

The aim of the next few sections is to prove the following:

Theorem 22.9 (Serre duality). Let X be a projective curve over an algebraically closed field k. Then there is a coherent sheaf ω_X on X, together with an isomorphism $t : H^1(X, \omega_X) \to k$, such that for any locally free sheaf $\mathcal{E}$ on X, there is a perfect pairing

$$H^0(X, \mathcal{F}) \times H^1(X, \omega_X \otimes \mathcal{E}^\vee) \to H^1(X, \omega_X) \simeq k \tag{22.3}$$

In particular, $H^1(X, \omega_X \otimes \mathcal{E}^\vee) \simeq H^0(X, \mathcal{E})^\vee$.

The sheaf ω_X is called a *dualizing sheaf*. The existence of ω_X is usually not enough for applications or explicit computations. The important point is that, in the smooth case, the dualizing sheaf equals with the cotangent sheaf, which is easier to study (e.g., because there are formulas for the canonical divisor).

Theorem 22.10. If X is a non-singular, projective curve, the dualizing sheaf ω_X is isomorphic to the cotangent sheaf Ω_X.

There are many proofs in the literature of this result **?**, **?**, **?**, **?**. Our proof is quite elementary, in the sense that it requires no derived functors, $\mathcal{E}xt$-sheaves, residues, adeles, etc. The ad hoc approach here is however much less conceptual than the standard proofs, and give essentially no information about the isomorphism $H^1(X, \Omega_X) \simeq k$.

We will prove the two theorems in three steps:

(i) First we prove both theorems for $X = \mathbb{P}^1$. In which $\omega_X = \mathcal{O}_{\mathbb{P}^1}(-2)$ serves as a dualizing sheaf (and we know this coincides with $\Omega_{\mathbb{P}^1}$).

(ii) Then we prove existence of ω_X for a general curve, using a Noether normalization $f : X \to \mathbb{P}^1$. Here the sheaf ω_X is constructed just to satisfy the formal properties of Serre duality.

(iii) We finally prove that $\omega_X \simeq \Omega_X$ by a computation on the self product $X \times_k X$.

The fact that $\mathbb{P}^1$, and hence X, can be a covered by two affine open sets simplifies things a lot. In particular, we have a concrete interpretation of the first cohomology group H^1 of a sheaf, in terms of the Cech complex.

22.2 Proof of Serre duality for $X = \mathbb{P}^1$

Lemma 22.11. Serre duality holds for $\mathbb{P}^1$ with $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{F}$ is an invertible sheaf.

Proof Recall that we may identify $H^0(X, \mathcal{O}_{\mathbb{P}^1}(d))$ with the k-vector space of homogeneous polynomials of degree d and $H^1(X, \mathcal{O}_{\mathbb{P}^1}(-2-d))$ with the k-vector space spanned by

Laurent monomials $x_0^{-u}x_1^{-v}$ with $u + v = d + 2\ u, v \geqslant 1$. The multiplication map

$$x_0^a x_1^b \times x_0^{-u}x_1^{-v} := \begin{cases} x_0^{-1}x_1^{-1} & \text{if } (u, v) = (a-1, b-1) \\ 0 & \text{otherwise} \end{cases}$$

defines a perfect pairing, which induces (22.3) when $\mathcal{F} = \mathcal{O}(d)$ with $d \geqslant 0$. For $d < 0$ all groups are zero. □

22.3 A simple cohomological lemma

In our proof, we will deduce Serre duality on a general curve X by considering a finite (hence affine) morphism $\pi : X \to \mathbb{P}^1$. The following lemma will allow us to transport computations of cohomology of sheaves on X to computations on $\mathbb{P}^1$, at the cost of replacing $\mathcal{F}$ with $\pi_*\mathcal{F}$.

Lemma 22.12. Let $\pi : X \to Y$ be an affine morphism of varieties. Then for each coherent sheaf $\mathcal{F}$ on X, and $i \geqslant 0$, we have a canonical isomorphism

$$H^i(X, \mathcal{F}) = H^i(Y, \pi_*\mathcal{F}).$$

Proof Let $\mathcal{U} = \{U_i\}$ be a finite affine covering of Y such that $H^i(X, \pi_*\mathcal{F})$ is computed by the Cech complex $C^\bullet(U_i, \pi_*\mathcal{F})$. The hypotheses give that X is covered by the affine subsets $\pi^{-1}(U_i)$. The lemma follows simply because the Cech complexes of $\mathcal{F}$ and $\pi_*\mathcal{F}$ with respect to the respective coverings are the same. □

22.4 Curves obtained by gluing two affines

If X is a non-singular projective curve over k, we can pick a Noether normalization $\pi : X \to \mathbb{P}^1$, which is affine, finite and flat.

Recall the standard gluing construction of $\mathbb{P}^1$ as $U \cup U'$ where $U = \operatorname{Spec} A$, and $U' = \operatorname{Spec} A'$, and $A = k[a]$ and $A' = k[a']$. The gluing is defined by the isomorphism $D(a) = \operatorname{Spec} A_a \simeq \operatorname{Spec} A'_{a'} = D(a')$, using the isomorphism $\tau : A_a \to A'_{a'}$ given by $\tau(a) = {a'}^{-1}$.

Because the morphism π is affine, we find that also X can be covered by two affine subsets $\pi^{-1}(U)$, $\pi^{-1}(U')$. We write $V = \operatorname{Spec} B$ and $V' = \operatorname{Spec} B'$ for these subsets. Note that $\pi|_V$ (resp. $\pi|_{V'}$) is induced by a ring map $u : A \to B$ (resp. $u' : A' \to B'$), so that $b = u(a)$ (resp. $b' = u'(a')$). Thus X is obtained by gluing V and V' along $\operatorname{Spec} B_b$ and $\operatorname{Spec} B'_{b'}$ using an isomorphism $\sigma : B_b \to B_{b'}$, which is compatible with π, in the sense that the diagram below commutes:

$$\begin{array}{ccc} B_b & \xrightarrow{\sigma} & B'_{b'} \\ {\scriptstyle u_a}\uparrow & & \uparrow{\scriptstyle u'_{a'}} \\ A_a & \xrightarrow{\tau} & A'_{a'} \end{array} .$$

22.4.1 Gluing sheaves

Given a quasi-coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1$, we get an A-module $N = \Gamma(U, \mathcal{G})$, and an A'-module $N' = \Gamma(U', \mathcal{G})$. On $D(a) = \operatorname{Spec} A_a \simeq \operatorname{Spec} A'_{a'} = D(a')$, these are related by an isomorphism of $A_{a'}$-modules

$$\mu : N'_{a'} \to N_a$$

(where we view N_a as an $A_{a'}$-module using the isomorphism τ). Conversely, by the tilde-construction and the Gluing lemma for sheaves, given modules N, N' and an isomorphism μ as above, we can construct a quasi-coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1$.

Similarly, giving a quasi-coherent sheaf $\mathcal{F}$ on X is equivalent to giving: A B-module M; A B'-module; and an isomorphism of $B_{b'}$-modules

$$\nu : M_{b'} \to M_b$$

$\mathcal{F}$ is coherent if and only if M and M' are finitely generated.

22.5 The dualizing sheaf

We will use the gluing construction of the previous section to define a sheaf ω_X on X, starting from $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. To define it, we need to define two modules on each affine open and check that they glue over the intersection.

The general construction goes as follows. Start with an A-module N and consider the A-module

$$M = \operatorname{Hom}_A(B, N).$$

The crucial point is that M can be viewed as a B-module, via the rule

$$b \cdot \phi(y) := \phi(b \cdot y), \quad y \in B$$

for each A-linear map $\phi : B \to N$. Likewise, for an A'-module N', the A'-module $M' = \operatorname{Hom}_{A'}(B', N')$ can be viewed as a B'-module.

If N and N' arise from a sheaf $\mathcal{G}$ on $\mathbb{P}^1$ in the construction above, there is a natural isomorphism

$$\operatorname{Hom}_{A'_{a'}}(B'_{b'}, N'_{a'}) \to \operatorname{Hom}_{A_a}(B_b, N_a)$$

sending $\phi : B'_{b'} \to N'_{a'}$ to $\mu^{-1} \circ \phi \circ \sigma : B_b \to N_a$. One checks that this is an isomorphism of B_b-modules. Thus from any sheaf $\mathcal{G}$ on $\mathbb{P}^1$, we obtain a sheaf, denoted by $\pi^! \mathcal{G}$, on X. In fact, the map $\mathcal{G} \mapsto \pi^! \mathcal{G}$ defines a *functor* from the category of coherent $\mathcal{O}_{\mathbb{P}^1}$-modules to $\mathcal{O}_X$-modules, but we will not need this fact here.

The crucial ingredient we need is that there is a canonical isomorphism

$$\pi_* \mathscr{H}om_X(\mathcal{F}, \pi^! \mathcal{G}) \simeq \mathscr{H}om_{\mathbb{P}^1}(\pi_* \mathcal{F}, \mathcal{G}). \tag{22.4}$$

We first prove this locally:

Lemma 22.13. For a finitely generated B-module L, there is a natural isomorphism (of A-modules)

$$\operatorname{Hom}_B(L, \operatorname{Hom}_A(B, N)) \to \operatorname{Hom}_A(L, N) \tag{22.5}$$

Proof The map is defined by sending $\phi : L \to \operatorname{Hom}_A(B, N)$ to $\ell \mapsto \phi(\ell)(1)$.

The map (22.5) is clearly an isomorphism for $L = B^{\oplus n}$. To prove it in general, pick a presentation

$$B^{\oplus m} \to B^{\oplus n} \to L \to 0.$$

Applying $\operatorname{Hom}_B(-, Hom_A(B, N))$, we get a diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Hom}_B(L, \operatorname{Hom}_A(B, N)) & \longrightarrow & \operatorname{Hom}_B(B, \operatorname{Hom}_A(B, N))^{\oplus n} & \longrightarrow & \operatorname{Hom}_B(B, \operatorname{Hom}_A(B, N))^{\oplus m} \\ & \downarrow & & \| & & \| \\ 0 \longrightarrow & \operatorname{Hom}_B(L, N) & \longrightarrow & \operatorname{Hom}_B(B, N)^{\oplus n} & \longrightarrow & \operatorname{Hom}_B(B, N)^{\oplus m} \end{array}$$

Then (22.5) is the left-most vertical map, and this is an isomorphism by the 5-Lemma. □

Inspecting the proof of Lemma 22.5, we note that the isomorphism in (22.5) is compatible with localizations. Thus the isomorphisms sheafify, and we get the sheaf isomorphism (22.4).

Definition 22.14. We define the *dualizing sheaf* of X as the sheaf $\omega_X = \pi^! \omega_{\mathbb{P}^1}$.

So far we haven't used the fact that X is non-singular; any projective curve admits a dualizing sheaf ω_X. which is a coherent $\mathcal{O}_X$-module. Note that ω is a coherent sheaf on X (because locally it is constructed by $\operatorname{Hom}_A(B, N)$, which is finitely generated). In the non-singular case, we will prove in Section 22.7 that $\omega_X \simeq \Omega_X$. A first step towards this, is to show that ω_X is invertible.

Proposition 22.15. Let X be a non-singular projective curve. Then ω_X is an invertible sheaf.

Proof Since X is a non-singular curve, ω_X is locally free if and only if it is torsion free. Let $\mathcal{T} = (\omega_X)_{\text{tors}}$ denote the torsion subsheaf and $\mathcal{E}$ is the torsion free part, so that there is an exact sequence

$$0 \to \mathcal{T} \to \omega_X \to \mathcal{E} \to 0$$

Applying π_* to this, we get

$$0 \to \pi_* \mathcal{T} \to \pi_* \omega_X \to \pi_* \mathcal{E}$$

Then applying formula (22.4), shows that $\pi_* \omega_X = \mathscr{H}om(\pi_* \mathcal{O}_X, \omega_{\mathbb{P}^1})$. As π is finite and surjective, $\pi_* \mathcal{O}_X$ is locally free. Thus since $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ is invertible, we find that $\pi_* \omega_X$ is also locally free. Note that $\pi_* \mathcal{T}$ is again a torsion sheaf on $\mathbb{P}^1$. As $\pi_* \mathcal{T}$ maps injectively into a locally free sheaf, we must have $\pi_* \mathcal{T} = 0$. This implies that $\Gamma(X, \mathcal{T}) = \Gamma(\mathbb{P}^1, \pi_* \mathcal{T}) = 0$. On a curve, the only torsion sheaf with no global sections is the zero sheaf, so $\mathcal{T} = 0$ as well.

Finally, to compute the rank of ω_X, we use the fact that the formation of $\pi^!\mathcal{G}$ behaves well with localization. This implies that $\omega_{X,\eta}$ at the generic point $\eta = \operatorname{Spec} k(X)$ coincides with $\operatorname{Hom}_{k(\mathbb{P}^1)}(k(X), k(\mathbb{P}^1))$. The latter is a $k(\mathbb{P}^1)$-vector space of dimension equal to the degree of $k(X) : k(\mathbb{P}^1)$. Hence, as a $k(X)$-vector space it has dimension 1. □

22.6 Proof of Theorem 22.9

From here on, we can finish the proof of Serre duality on X:

$$\begin{aligned}
H^1(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X) &= H^1(\mathbb{P}^1, \pi_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)) && \text{(Lemma 22.12)}\\
&= H^1(\mathbb{P}^1, \pi_* \mathscr{H}om(\mathcal{F}, \omega_X))\\
&= H^1(\mathbb{P}^1, \mathscr{H}om(\pi_*\mathcal{F}, \omega_{\mathbb{P}^1})) && \text{(by (22.4))}\\
&= H^1(\mathbb{P}^1, (\pi_*\mathcal{F})^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1}} \omega_{\mathbb{P}^1})\\
&= H^0(\mathbb{P}^1, \pi_*\mathcal{F})^\vee && \text{(Serre duality on } \mathbb{P}^1)\\
&= H^0(X, \mathcal{F})^\vee. && \text{(Lemma 22.12)}
\end{aligned}$$

22.7 The dualizing sheaf equals the canonical sheaf

The goal of this section is to show that the dualizing sheaf ω_X is isomorphic to the cotangent sheaf Ω_X. Note that both of these sheaves are locally free: the first by Corollary 22.15, and Ω_X because X is smooth.

We will work with the self-product $X \times X$ with the two projections $p, q : X \times X \to X$.

Lemma 22.16 ("Kunneth formula"). Let V and X be varieties over k with V affine. Let $\mathcal{F}$ denote a coherent $\mathcal{O}_V$-module and let $\mathcal{G}$ denote a coherent $\mathcal{O}_X$-module and write $p, q : X \times V \to X$ for the two projections on $V \times X$. Then there is a natural isomorphism

$$H^i(V \times X, p^*\mathcal{F} \otimes q^*\mathcal{G}) = \Gamma(V, \mathcal{F}) \otimes_k H^i(X, \mathcal{G}) \tag{22.6}$$

Proof Let $\mathcal{U} = \{U_i\}$ denote an open affine covering of X so that $C^\bullet(\mathcal{U}, \mathcal{G})$ computes the cohomology group $H^i(X, \mathcal{G})$. Tensoring $C^\bullet(\mathcal{U}, \mathcal{G})$ with the module $M = \Gamma(V, \mathcal{F})$ gives a complex $C^\bullet(\mathcal{U}, \mathcal{G}) \otimes_k M$ which is easily seen to compute the cohomology of both sides of (22.6). □

Consider the diagonal embedding $i : \Delta \to X \times X$. Recall that the normal bundle of Δ in $X \times X$ is isomorphic to the tangent bundle T_X. We thus have an exact sequence

$$0 \to \mathcal{O}_{X \times X} \to \mathcal{O}_{X \times X}(\Delta) \to i_* T_X \to 0. \tag{22.7}$$

We now tensor this sequence by $q^*\omega_X$, to we get a sequence

$$0 \to q^*\omega_X \to q^*\omega_X(\Delta) \to i_*(\omega_X \otimes T_X) \to 0 \tag{22.8}$$

Here we have used the projection formula:

$$i_* T_X \otimes q^*\omega_X = i_*(T_X \otimes i^* q^* \omega_X) = i_*(T_X \otimes \omega_X).$$

If we restrict the sequence (22.8) to the open set $V \times X$, where $V = \operatorname{Spec} R$ is affine, and take the long exact sequence in cohomology, we get

$$\Gamma(V \times X, i_*(\omega_X \otimes T_X)) \to H^1(V \times X, q^*\omega_X) \to H^1(V \times X, q^*\omega_X(\Delta)) \qquad (22.9)$$

By Lemma 22.12), we may identify the first group with $\Gamma(V, \omega_X \otimes T_X)$. By Lemma 22.16, we may identity the second with $\Gamma(V, \mathcal{O}_X) \otimes_k H^1(X, \omega_X) = \Gamma(V, \mathcal{O}_X)$ (we also use the isomorphism $H^1(X, \omega_X) = k$). These identifications are compatible with restriction maps, so we get a map of sheaves $\omega_X \otimes T_X \to \mathcal{O}_X$, or equivalently,

$$\rho : \omega_X \to \Omega_X$$

We claim that ρ is surjective. Since both sheaves are locally free, it must then be an isomorphism because the kernel is locally free of rank 0. Thus we find that $\omega_X \simeq \Omega_X$.

To conclude, it suffices to prove that the group $H^1(V \times X, q^*\omega_X(\Delta))$ in (22.9) vanishes for each affine $V \subset X$. Note that by Lemma 22.12, we have

$$H^1(V \times X, q^*\omega_X(\Delta)) = H^1(X, \omega_X \otimes q_*\mathcal{O}_{V\times X}(\Delta))$$

Note that $q_*\mathcal{O}_{V\times X}(\Delta)$ is locally free. By the duality property of ω_X, we may identify this with

$$H^0(X, \mathscr{H}om_X(q_*\mathcal{O}_{V\times X}(\Delta), \mathcal{O}_X))^\vee$$

By the change-of-rings property of Hom, the latter equals

$$H^0(X, q_*\mathscr{H}om_X(\mathcal{O}_{V\times X}(\Delta), \mathcal{O}_{V\times X}))^\vee = H^0(X, q_*\mathcal{O}_{V\times X}(-\Delta))^\vee$$

Using 22.12 agian, the latter cohomology group equals

$$H^0(V \times X, \mathcal{O}_{V\times X}(-\Delta))^\vee$$

But this last group is indeed zero: sections of $\mathcal{O}_{V\times X}(-\Delta) \simeq \mathcal{I}_\Delta$ correspond to sections of $\mathcal{O}_{V\times X}$ that vanish along the diagonal. However, we have

$$\Gamma(V \times X, \mathcal{O}_{V\times X}) = \Gamma(V, \mathcal{O}_V) \otimes_k \Gamma(X, \mathcal{O}_X) \simeq \Gamma(V, \mathcal{O}_V) \otimes_k k,$$

which implies that any such section can only vanish along a 'vertical' divisor $D \times X$ for $D \subset V$.

22.8 Exercises

Exercise 22.8.1. a) Show that the pushforward of a torsion sheaf is torsion

b) Show that a sheaf $\mathcal{F}$ is torsion iff it is supported on a proper closed subset

c) Show that if $\mathcal{F}$ is a torsion sheaf on a curve X then $H^0(X, \mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$.

23

Applications of the Riemann–Roch theorem

In this chapter we give a few of the (many) consequences of the Riemann–Roch formula. We start by translating the results of Chapter **??** into concrete numerical criteria for a divisor D to be base point free or very ample. Then we use these results to classify all curves of all genus $\leqslant 4$.

An embedding theorem

Let X be a variety over (an algebraically closed) field k with an invertible sheaf L on X. For each point $x \in X$ there is a evaluation map $L \to L(x) = L \otimes k(x)$ whose kernel is $\mathfrak{m}_x L$. One says that L is generated by global sections if the induced map on global sections $H^0(X, L) \to L(x)$ is surjective. Choosing a basis $\{\sigma_i\}$ for $H^0(X, L)$, we get a well defined morphism $f\colon X \to \mathbb{P}^n_k$ with the property that $f^*\mathcal{O}_{\mathbb{P}^n_k}(1) = L$ and, on the level of sections, that $f^*x_i = \sigma_i$. Here the x_i's are homogeneous coordinates on $\mathbb{P}^n_k$; i.e. they form basis for the space of global sections $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$.

For each pair of points x, y from X there is an evaluation map $L \to L(x) \oplus L(y)$, and we say that L *separates points*, if the induced map on global sections $H^0(X, L) \to k(x) \oplus k(y)$ is surjective for all x and y. This means there is a section σ vanishing at x but not at y, so that corresponding hyperplane in $\mathbb{P}^n_k$ contains x but not y. We conclude that f is injective on closed points, hence injective. Note that the condition that L separates points, ensures it is globally generated.

Next step is to control the 'derivative' of f. We say that L *separates tangent directions* if for each closed point $x \in X$, the map $\mathfrak{m}_x L \to \mathfrak{m}_x L/\mathfrak{m}_x^2 L = \mathfrak{m}_x/\mathfrak{m}_x^2 \otimes_k L(x)$ induces a surjection on global sections. This means the elements from the section space $H^0(X, \mathfrak{m}_x L)$ generate $\mathfrak{m}_x/\mathfrak{m}_x^2 \otimes L(x)$. Now, if σ_0 is a section that does not vanish at x, it holds that $L(x) = \sigma_0(x) \cdot k$, so dividing by $\sigma_0(x)$, we obtain a surjective map $H^0(X, \mathfrak{m}_x L) \to \mathfrak{m}_x/\mathfrak{m}_x^2$.

Theorem 23.1. Let X be a variety proper over an algebraically closed field k and L an invertible sheaf on X which is generated by its global sections.

(i) If L separates points, then f is a homeomorphism onto its image, and the image is closed;
(ii) If L also separates tangent directions, then f is a closed embedding.

Proof Since X is assumed to be proper, the map f is closed, and as shown above, it will be injective on closed points. We conclude that it is bijective onto its image, and hence a homeomorphism onto $f(X)$.

For the second statement, we have to show that map of sheaves $f^\sharp \colon \mathcal{O}_{\mathbb{P}^n} \to f_*\mathcal{O}_X$ is surjective. This is a local issue, so chose a point $x \in X$. It suffices to see that the local map $f^\sharp_x \colon \mathcal{O}_{\mathbb{P}^n k, f(x)} \to \mathcal{O}_{X,x}$ is surjective. By an appropriate choice of homogeneous coordinates on $\mathbb{P}^n_k$ we may assume that $f(x) \in D_+(x_0)$; that is , $\sigma_0(x) \neq 0$, and that the maximal ideal $\mathfrak{m}_{f(x)}$ in $\mathcal{O}_{\mathbb{P}^n, f(x)}$ is generated by the $x_i x_0^{-1}$. Then the hypotheses ensure that their images $\sigma_i(x)\sigma_0(x)^{-1}$ in $\mathcal{O}_{X,x}$ generate $\mathfrak{m}_x/\mathfrak{m}_x^2$. The next lemma finishes the proof. □

Lemma 23.2. Let $\phi\colon A \to B$ be a homomorphism of Noetherian local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. Assume that B is a finite A-module, and assume further that the induced map $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. Then ϕ is surjective.

Proof Consider the ideal $\mathfrak{a} = \mathfrak{m}_A B$ in B. Clearly $\mathfrak{a}/\mathfrak{m}_B\mathfrak{a} \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, so Nakayama's lemma yields that $\mathfrak{a} = \mathfrak{m}_B$. Now, B is finite over A and $A/\mathfrak{m}_A = B/\mathfrak{m}_B = B/\mathfrak{m}_A B$, so Nakayama's lemma once more gives that $A \to B$ is surjective. □

As an application Serre duality and of the embedding theorem, let us show the promised result that every proper smooth curve is projective:

Theorem 23.3. Let X be a proper smooth curve over a field k. Then X is projective.

Proof We are to see that $H^1(X, L(-x-y)) = 0$ for each pair of points x, y on X. Its the separetes points and tangent vectors

$$0 \longrightarrow L(-x-y) \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow 0$$

Where $\mathcal{E}$ equals $L(x) \oplus L(y)$ when $x \neq y$ and $\mathcal{E} = L/\mathfrak{m}_x^2 L$ when $x = y$ (Indeed, $\mathcal{O}(-2x)$ is the locally ideal t_x^2).

By Serre duality, this H^1 group is dual to $H^0(X, \omega \otimes L^{-1}(x+y))$; in terems of divisors, if $L = \mathcal{O}_x(D)$, this means that $h^0(K_X + x + y - D) = 0$. But we achieve this by choosing D to be of degree greater than $2g - 2 + 2 = 2g$, indeed, then the degree of $K_X + x + y - D$ will be negative. □

23.1 Very ampleness criteria

Recall the criterion of Theorem **??**, that an invertible sheaf L is very ample if and only if its linear system separates points and tangent vectors. Using Riemann–Roch we can translate that result into a very simple, numerical criterion for very ampleness on a curve:

Theorem 23.4. Let X be a non-singular projective curve and let D be a divisor on X. Then

(i) $|D|$ is base point free if and only if

$$h^0(D-P) = h^0(D) - 1 \qquad \text{for every point } P \in X.$$

(ii) D is very ample if and only if

$$h^0(D-P-Q) = h^0(D) - 2 \qquad \text{for every two points } P, Q \in X$$

(including the case $P = Q$)

(iii) A divisor D is ample iff $\deg D > 0$

Proof (i) We take the cohomology of the following exact sequence

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to k(P) \to 0$$

and get

$$0 \to H^0(X, \mathcal{O}_X(D-P)) \to H^0(X, \mathcal{O}_X(D)) \to k$$

From this sequence, we get $h^0(D) - 1 \leqslant h^0(D-P) \leqslant h^0(D)$.

The right-most map takes a global section of $\mathcal{O}_X(D)$ and evaluates it at P. To prove that $|D|$ is base point free, we must prove that there is a section $s \in \mathcal{O}_X(D)$ which does not vanish at P, or equivalently, that the map $H^0(X, \mathcal{O}_X(D)) \to k$ is surjective. But this happens if and only if $h^0(D-P) = h^0(D) - 1$.

(ii) If the above inequality is satisfied, we see in particular that $|D|$ is base point free. So it determines a morphism $\phi : X \to \mathbb{P}^n$. We can use Theorem **??** that ensure that ϕ is an embedding. That is, we need to check that ϕ separates (a) points and (b) tangent vectors.

For (a), we are assuming that $h^0(D-P-Q) = h^0(D) - 2$, so the divisor $D - P$ is effective and does not have Q as a base point (by (i)). But this means that there is a section of $H^0(X, \mathcal{O}_X(D-P))$ which doesn't vanish at Q. We have $H^0(X, D-P) \subseteq H^0(X, D)$, so we get a section of $\mathcal{O}_X(D)$ which vanishes at P, but not at Q. Hence $|D|$ separates points.

For (b), we need to show that $|D|$ separates tangent vectors, i.e., the elements of $H^0(X, \mathcal{O}_X(D))$ should generate the k-vector space $\mathfrak{m}_P\mathcal{O}_X(D)/\mathfrak{m}_P^2\mathcal{O}_X(D)$ at every point $P \in X$. This condition is equivalent to saying that there is a divisor $D' \in |D|$ with multiplicity 1 at P: Note that $\dim T_P(X) = 1$, $\dim T_P D' = 0$ if P has multiplicity 1 in D' and $\dim T_p(D') = 1$ if P has higher multiplicity. But this is equivalent to P not being a base point of $D - P$. Applying (i) again, we see that $h^0(D - 2P) = h^0(D) - 2$, so we are done.

(iii) By definition, D is ample if mD is very ample for $m \gg 0$. So the result follows by the next result, since any divisor of degree $\geqslant 2g + 1$ is very ample. □

Proposition 23.5. Let X be a non-singular projective curve and let D be a divisor on X. Then

(i) If $\deg D \geqslant 2g$, then D is base point free.
(ii) If $\deg D \geqslant 2g + 1$, then D is very ample.

Proof By Serre duality, $h^1(D) = h^0(K - D) = 0$ because $\deg D > \deg K = 2g - 2$. Similarly, $h^1(D - P) = 0$.

(i) Applying Riemann–Roch, we find that $h^0(D - p) = h^0(D) - 1$ for any $P \in X$, so we are done by the above theorem.

(ii) In this case we also get $h^1(D - P - Q) = 0$, so Riemann–Roch shows that $h^0(D - P - Q) = h^0(D) - 2$, which is the conclusion we want. □

Example 23.6. On $X = \mathbb{P}^1$ a divisor D is base point free if and only if $\deg \geqslant 0$. Moreover, D is very ample if and only if $\deg D > 0$

Example 23.7. If X is a curve of genus 1, a divisor D is base point free if $\deg D \geqslant 2$. We will see later that, if $D = p$ for some point p, we have $h^0(X, \mathcal{O}_X(D)) = 1$, so D can not be base point free (because the generator of $H^0(X, \mathcal{O}_X(D))$ vanishes at p).

A divisor D of degree $\geqslant 3$ is very ample if $\deg D \geqslant 3$.

23.2 Curves on $\mathbb{P}^1 \times \mathbb{P}^1$

Let us consider one central example, namely curves on the quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Recall that $\mathrm{Cl}(Q) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$ where $L_1 = [0 : 1] \times \mathbb{P}^1$ and $L_2 = \mathbb{P}^1 \times [0 : 1]$.

We can use this to prove that Q contains non-singular curves of any genus $g \geqslant 0$. (This is in contrast with the case of $\mathbb{P}^2$, where only genera of the form $\binom{d-1}{2}$ were allowed).

To prove this, consider the divisor $D = aL_1 + bL_2$ where $a, b \geqslant 1$. D is effective, so let $C \in |D|$ be a generic element.

Lemma 23.8. C is non-singular.

Proof D is defined by a bihomogeneous equation

$$\sum_{i+j=a, l+k=b} c_{ij,kl} x_0^i x_1^j y_0^l y_1^k = 0$$

On the open set $D_+(x_0) \cap D_+(y_0) \simeq \mathbb{A}^2 = \operatorname{Spec} k[x, y]$ this is given by

$$\sum_{i+j=a, l+k=b} c_{ij,kl} x^j y^k = 0$$

and it is clear that if the coefficients $c_{ij,kl}$ are general, this is non-singular. By symmetry this also happens in the other charts, so C is non-singular. □

To compute the genus of C, we use the formula $2g - 2 = \deg \Omega_C$. So we need to find Ω_C and compute its degree. This is best computed using the Adjunction formula of Proposition **??**:

$$\begin{aligned} \Omega_C &= \omega_Q \otimes \mathcal{O}_Q(C)|_X && (23.1) \\ &= \mathcal{O}_Q(-2L_1 - 2L_2) \otimes \mathcal{O}(aL_1 + bL_2)|_C && (23.2) \\ &= \mathcal{O}_C((a-2)L_1 + (b-2)L_2) \end{aligned}$$

To compute the degree of this, we consider the degrees of $L_1|_C$ and $L_2|_C$ separately. Note that the degree $\deg L_1|_C$ is invariant under linear equivalence, so we can compute the degree of any $[s : t] \times \mathbb{P}^1$ for a general point $[s : t] \times \mathbb{P}^1$. The point is that as a Weil divisor, $L_1|_X$ is obtained by intersecting $[s : t] \times \mathbb{P}^1$ with X. When $[s : t] \in \mathbb{P}^1$ is a general point, the intersection $X \cap [s : t] \times \mathbb{P}^1$ is a reduced subscheme of X, consisting of b points (as $C \subset Q = \mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of type $aL_1 + bL_2$). Hence $\deg L_1|_C = b$ and $\deg L_2|_C = a$. It follows that

$$2g - 2 = \deg \Omega_C = (a-2)b + (b-2)a = 2ab - 2a - 2b$$

Solving for g gives us the following theorem:

Theorem 23.9. Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Then a generic divisor C in $|aL_1 + bL_2|$ is a smooth projective curve of genus

$$g = (a-1)(b-1).$$

In particular, Q contains non-singular curves of any genus $g \geqslant 0$.

23.3 Curves of genus 0

The results of the previous results are particularly strong when the genus is small. For instance, when $g = 0$, any divisor of positive degree is very ample! We can use this to classify all curves of genus 0. First a simple lemma:

Lemma 23.10. Let X be a non-singular curve. Then $X \simeq \mathbb{P}^1$ if and only if there exists a Cartier divisor D such that $\deg D = 1$ and $h^0(X, \mathcal{O}_X(D)) \geqslant 2$. In this case, the divisor D is even very ample.

Proof Let $g \in H^0(X, \mathcal{O}_X(D))$. Then $D' \sim \operatorname{div} g + D \geqslant 0$, so replacing D by D' we may assume that D is effective. Since $\deg D = 1$, we must have $D = p$ for some point $p \in X$. Now take $f \in H^0(X, \mathcal{O}_X(D)) - k$. As above, f induces a morphism $\phi : X \to \mathbb{P}^1$. This morphism has degree equal to 1, so it is birational, and hence X is isomorphic to $\mathbb{P}^1$. □

Proposition 23.11. A non-singular curve X is isomorphic to $\mathbb{P}^1$ if and only if $\mathrm{Cl}(X) \simeq \mathbb{Z}$.

Proof We have seen that the Picard group of any $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}$ via the degree map $\deg : \operatorname{Pic}(\mathbb{P}^n_k) \to \mathbb{Z}$.

Conversely, suppose X is a curve with $\mathrm{Cl}(X) \simeq \mathbb{Z}$. Let p, q be two distinct points on X. By assumption, p and q are linearly equivalent, so the linear system $|p| = \mathbb{P}H^0(X, \mathcal{O}_X(D))$ is at least 1-dimensional. Then $X \simeq \mathbb{P}^1_k$ by the previous lemma. □

Theorem 23.12. Any curve of genus 0 over an algebraically closed field is isomorphic to $\mathbb{P}^1$.

Proof Let $p \in X$ be a point and consider the divisor $D = p$. If X has genus 0, then

$1 = \deg D > 2g - 2 = -2$, so $H^1(X, \mathcal{O}_X(D)) = 0$. Then Riemann-Roch tells us that

$$\dim H^0(X, \mathcal{O}_X(D)) = 1 + 1 - 0 = 2$$

Hence $X \simeq \mathbb{P}^1_k$ by Lemma 23.10. □

We conclude by yet another characterisation of $\mathbb{P}^1$:

Lemma 23.13. Let C be a non-singular projective curve and D any divisor of degree $d > 0$. Then

$$\dim |D| \leqslant \deg D$$

with equality if and only if $C \simeq \mathbb{P}^1$.

Proof Although one might guess that this lemma follows directly from Riemann-Roch, this does not seem to be the case: Riemann-Roch gives a different sort of relationship between the dimension and degree of a divisor.

We may assume that D is effective, i.e., $D = P_1 + \cdots + P_d$ for some points $P_1, \ldots, P_k \in C$ (possibly equal) (otherwise replace D by some different effective divisor $D' \in |D|$). We induct on d.

First suppose $d = 1$. There is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(P) \to k(P) \to 0.$$

Now $h^0(\mathcal{O}_C) = 1$ and $h^0(k(P)) = 1$ therefore $h^0(\mathcal{O}_C(P)) \leqslant 2$ so $\dim |P| \leqslant 1$. If $\dim |P| = 1$ then $|P|$ has no base points so we obtain a morphism $C \to \mathbb{P}^1$ of degree $\deg P = 1$ which must be an isomorphism, and so $C \simeq \mathbb{P}^1$ is rational.

Next suppose $D = P_1 + \cdots + P_d$. Let $D' = P_1 + \cdots + P_{d-1}$. There is an exact sequence

$$0 \to \mathcal{O}_C(D') \to \mathcal{O}_C(D) \to k(P_d) \to 0.$$

Now $h^0(\mathcal{O}_C(D')) \leqslant d$ by induction and $h^0(k(P_d)) = 1$ so $h^0(\mathcal{O}_C(D)) \leqslant d + 1$, therefore $\dim |D| \leqslant d$ with equality iff $h^0(\mathcal{O}_C(D')) = d$. By induction $h^0(\mathcal{O}_C(D')) = d$ iff C is rational. □

23.3.1 Non-algebraically closed fields

It is of course possible to develop the theory of curves over any field k, not just algebraically closed ones. In this case, there tend to be more divisors around than just the combinations of closed points. For instance, for $X = \mathbb{P}^1_{\mathbb{R}}$, the subscheme $D = V(x^2 + 1)$ is of codimension 1, so it gives a Weil divisor on X. The results of this chapter, including the Riemann–Roch theorem, still holds true, provided the degree of a divisor D is defined in terms of the degree of the field extension over which D is defined. In the above example, we would for instance have $\deg D = [\mathbb{R}(D) : \mathbb{R}] = [\mathbb{C} : \mathbb{R}] = 2$.

In this setting, a curve of genus 0, need not be isomorphic to $\mathbb{P}^1_k$ (although certainly this is true over the algebraic closure: $X \times_k \overline{k} \simeq \mathbb{P}^1_{\overline{k}}$). For instance, the curve $X = V(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_{\mathbb{R}}$ has genus 0, but is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$: This is because $X(\mathbb{R}) = \varnothing$, whereas $\mathbb{P}^1(\mathbb{R}))$ is infinite. A nice and surprising fact, however, is that a curve of genus 0 over a field k

is at least always isomorphic to a projective conic in $\mathbb{P}^2_k$. This is because of the anticanonical divisor: $-K_X$ has degree 2 and defines an embedding $X \mapsto \mathbb{P}^2_k$.

Example 23.14. Let k be any field, and consider the conic $X = V(x_0^2 + x_1^2 - x_2^2) \subset \mathbb{P}^2_k$. This X has a k-rational point $p_0 = (1 : 0 : 1)$. Projecting from p_0, we obtain a rational map $X \dashrightarrow \mathbb{P}^1_k$, which is birational. Hence X is isomorphic to $\mathbb{P}^1_k$.

Example 23.15. The conic $X = V(x_0^2 + x_1^2 - 3x_2^2)$ has many $\mathbb{R}$-points, but no $\mathbb{Q}$-points!

23.4 Curves of genus 1

A plane curve $X \subset \mathbb{P}^2_k$ of degree 3 has genus 1. This follows from our earlier work on the canonical divisor, which showed $\omega_X \simeq \mathcal{O}_{\mathbb{P}^2_k}(d-3)|_X \simeq \mathcal{O}_X$, and so $g = h^0(\omega_X) = h^0(\mathcal{O}_X) = 1$. In this section, we show that in fact *every* curve of genus 1 arises this way:

Theorem 23.16. Any projective curve X of genus 1 can be embedded as a plane cubic curve in $\mathbb{P}^2_k$.

Proof Pick a point $P \in X$ and consider the divisor $D = 3P$. D has degree $3 \geqslant 2g + 1$, so it is very ample. Furthermore, by Riemann–Roch, $h^0(3P) = 3$, so there is a projective embedding $\phi : X \to \mathbb{P}^2_k$. The image $\phi(X)$ is a smooth curve of degree equal to $\deg \phi^*\mathcal{O}_{\mathbb{P}^2}(1) = \deg D = 3$. □

In contrast to the $g = 0$ case however, there are many non-isomorphic genus 1 curves. For instance, in the *Legendre family* of curves in $X_\lambda \subset \mathbb{P}^2$ given by

$$y^2 z = x(x - z)(x - \lambda z)$$

where $\lambda \in k$, each X_λ is isomorphic to at most a finite number of other $X_{\lambda'}$'s.

Actually, these are essentially all the curves of genus 1.

Theorem 23.17. Let k be a field of char $k \neq 2, 3$. Then any genus 1 curve X admits a projective model given by an homogeneous equation

$$x_2^2 x_0 = x_1^3 + a x_1 x_0^2 + b x_0^3$$

for some $a, b \in k$ with $4a^3 + 27b^2 \neq 0$.

23.4.1 Divisors on X

Let X be a curve of genus 1. We will study the divisors on X. To make the discussion a bit more concrete, let $X \subset \mathbb{P}^2$ be the curve given by $y^2 z = x^3 - xz^2$. We claim that there is an exact sequence

$$0 \to X(k) \to \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

This means that the class group $\mathrm{Cl}(X)$ is very big – its elements are in bijection with the k-points of X, of which there might be uncountably many. (In particular, this is another reason why X cannot be isomorphic to $\mathbb{P}^1$.)

If $L \subset \mathbb{P}^2$ is a line, we get a divisor $L|_X$: That is, we take a section $s \in \mathcal{O}_{\mathbb{P}^2}(1)$ defining L and restrict it to X. The divisor of $s \in \mathcal{O}_X(1)$ consists of three points P, Q, R (counted with multiplicity). In particular, since any two lines are linearly equivalent on $\mathbb{P}^2$, we get for every pair of lines L, L' and corresponding triples P, Q, R, a relation

$$P + Q + R \sim P' + Q' + R'$$

(where $\sim$ denotes linear equivalence).

Let us consider the point $O = [0, 1, 0]$ on X. This is a special point on X: it is an *inflection point*, in the sense that there is a line $L = V(z) \subset \mathbb{P}^2$ which has multiplicity three at O, so that L restricts to $3O$ on X. This has the property that any three collinear points P, Q, R in X satisfy

$$P + Q + R \sim 3O$$

We will use these observations to define a group structure on the set of closed points $X(k)$, using the point O as the identity. The group structure will be induced from that in $\mathrm{Cl}(X)$.

Consider the subgroup $\mathrm{Cl}^0(X) \subset \mathrm{Cl}(X)$ consisting of degree 0. This fits into the exact sequence

$$0 \to \mathrm{Cl}^0(X) \to \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

We now define a map

$$\begin{aligned} \xi : X(k) &\to \mathrm{Cl}^0(X) \\ P &\mapsto [P - O] \end{aligned} \tag{23.3}$$

Lemma 23.18. ξ is a bijection.

Proof ξ is injective: $\xi(P) = \xi(Q)$ implies that $P \sim Q$. Then $P = Q$ (otherwise X would be rational, by Proposition 23.11). (Alternatively, it follows because $h^0(X, \mathcal{O}_X(P)) = 1$).

ξ is surjective: Take a divisor $D = \sum n_i P_i \in \mathrm{Div}(X)$ of degree 0. Then $D' = D + O$ has degree 1, so by Riemann–Roch, $H^0(X, \mathcal{O}_C(D'))$ is 1-dimensional. Hence there exists an effective divisor of degree 1 in $|D'|$, which must then be of the form $D' = Q$. But that means that $D + O \sim Q$, or, $D \sim Q - O$, as desired. □

Using this bijection, we can put a group structure on the set $X(k)$:

Theorem 23.19. The set of k-points $X(k)$ on a genus 1 form a group.

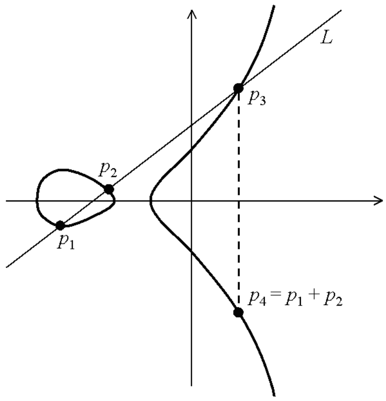
The group law has the following famous geometric interpretation. Given two points $p_1, p_2 \in X$, we draw the line L they span (see the figure below). This intersects X in one more point, say p_3. In the group $\mathrm{Cl}^0(X)$ we have

$$p_1 + p_2 + p_3 = 3O$$

To define the 'sum' $p_1 + p_2$ (which should be a new k-point of X), we then look for a point p_4 so that

$$p_4 - O = (p_1 - O) + (p_2 - O)$$

or in other words, $p_4 + O = p_1 + p_2$. By the above, this becomes $p_4 + O = 3O - p_3$ or, $p_3 + p_4 + O = 3O$. This tells us that we should define p_4 as follows: We draw the line L' from O to p_3 (shown as the dotted line in the figure), and define p_4 to be the third intersection point of L' with X. By construction, we get $(p_1 - O) + (p_2 - O) = (p_4 - O)$ in $\mathrm{Cl}^0(X)$.



Given the equation of X in $\mathbb{P}^2$, and coordinates for the points p_1, p_2, we can of course write down explicit formulas for the coordinates of p_4, and they are rational functions in the coordinates of p_1, p_2. This is almost enough to justify that X is a *group variety*, i.e., it is an algebraic variety equipped with a multiplication map $m : X \times X \to X$ satisfying the usual group axioms, and m is a morphism of algebraic varieties.

23.5 Curves of genus 2

Let X be a non-singular projective curve of genus 2. We saw one example of such a curve earlier in this chapter, namely the curve obtained by gluing two copies of the affine curve $y^2 = p(x)$ where $p(x)$ is a polynomial of degree five. The condition that X is non-singular implies that p has distinct roots.

We already saw in Chapter XX that a genus 2 curve cannot be embedded in the projective plane $\mathbb{P}^2_k$ (since 2 is not a trigonal number). However, we show the following:

Theorem 23.20. Any curve of genus 2 is isomorphic to a hyperelliptic curve

Here, a curve C is said to be *hyperelliptic* if there is a degree 2 map $X \to \mathbb{P}^1$. Equivalently, there is a base point free linear system of degree 2 and dimension 1. Equivalently again, there exists points $P, Q \in X$ so that the invertible sheaf $L = \mathcal{O}_X(P + Q)$ is globally generated and by two global sections.

It is classical notation that a base point free linear system of degree d and dimension r is called a g^r_d. So to say that a curve is hyperelliptic is to say that it has a g^1_2.

Example 23.21. If $g = 0$, then $X \simeq \mathbb{P}^1$. Let $D = 2P$, then $H^0(D) = kx_0^2 + kx_0x_1 + x_1^2$, so $|D| \simeq \mathbb{P}^2$ is identified with the space of quadratic polynomials up to scaling. If we take two quadratic polynomials q_0, q_1 with no common zeroes, we get a base point free linear system $g^1_2 \subset |D|$.

Example 23.22. If $g = 1$ any divisor of degree 2 gives a g^1_2 by Riemann-Roch. Indeed, if D has degree 2 then

$$h^0(D) - h^0(K - D) = 2 + 1 - g = 2$$

and $\deg(K - D) = -2$ so $h^0(D) = 2$ and hence $\dim |D| = 1$. This D is base point free, since $D - p$ has degree 1, and hence since X is not rational, $h^0(D - p) = 1 = h^0(D) - 1$.

Example 23.23. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth divisor of bidegree $(2, g + 1)$. Then $K_X \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0, g-1)$ and X has genus g. Moreover, the projection $p_2 : X \to \mathbb{P}^1$ is finite of degree 2, which shows that X is hyperelliptic.

The projections $p_1, p_2 : Q \to \mathbb{P}^1$ give rise to a degree 2 and a degree $g + 1$ morphism of X to $\mathbb{P}^1$. Thus there exists a 2:1 morphism $f : X \to \mathbb{P}^1$. f corresponds to a base point free linear system on X of degree 2 and dimension 1. Thus X is hyperelliptic.

In this example, $\Omega_X = \mathcal{O}_Q(X)\otimes\omega_Q|_X = \mathcal{O}_Q(2, g+1)\otimes\mathcal{O}_Q(-2, -2) = \mathcal{O}_X(0, g-1)$. The latter invertible sheaf has g independent global sections so X has genus g. Moreover K_X is base point free, but not very ample, since the corresponding morphism $X \to \mathbb{P}^{g-1}$ is not an embedding (it maps X onto a conic).

To prove the theorem, we must produce a degree two map $\phi : X \to \mathbb{P}^1$. We have a natural candidate: the canonical divisor K_X, which has degree $2g - 2 = 2$. We claim that K_X is base point free.

Note that we cannot apply Proposition 23.5 directly to prove this, since the degree is too small. However, we can use Riemann–Roch to check directly that the conditions in Theorem 23.4 apply. That is, we need to show that for every point $P \in X$, we have

$$h^0(X, K_X - P) = h^0(X, K_X) - 1 = 2 - 1 = 1$$

Applying Riemann–Roch to the divisor $D = P$, we also get $h^0(P) - h^0(K_X - P) = 1 + 1 - 2 = 0$. As P is effective, and X is not rational, we have $h^0(P) = 1$, and so also $h^0(X, K_X - P) = 1$, as we want.

23.6 Curves of genus 3

The case of curves of genus 3 is especially interesting. We have seen two examples of curves of genus 3 so far:

Example 23.24. A plane curve $X \subset \mathbb{P}^2$ of degree $d = 4$ has genus $\frac{1}{2}(d-1)(d-2) = 3$.

Notice that

$$\Omega_X = \mathcal{O}_{\mathbb{P}^2}(d-3)|_X = \mathcal{O}_X(1)$$

so Ω_X is very ample, since it is the restriction of the very ample invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. Hence K_X is very ample, and the corresponding morphism is exactly the given embedding $X \hookrightarrow \mathbb{P}^2$.

Example 23.25. The curves in Section 21.5 on page 378 can be chosen to have genus $g = 3$. In this case, X admits a 2:1 map to $\mathbb{P}^1$, and thus X is hyperelliptic.

Example 23.26. A curve X on the quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ of type (2,4) is hyperelliptic. It is a curve of degree 6 and genus 3.

Thus these examples are a bit different. The curves in the first example have very ample canonical divisor K_X (they are 'canonical') whereas the two others to do not ('hyperelliptic'). We show that this distinction is a general phenomenon for curves of genus three:

Proposition 23.27. Let X be a curve of genus 3. Then there are two possibilities:
(i) K_X is very ample. Then X embeds as a plane curve of degree 4.
(ii) K_X is not very ample. Then X is a hyperelliptic curve, and it embeds as a $(2,4)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $K_X \sim 2F$, where $F = L_1|_X$.

We will deduce this from a more general result:

Theorem 23.28. Let X be a curve of genus $\geqslant 2$. Then K is very ample if and only if X is not hyperelliptic.

Proof K is very ample if and only if $h^0(K-P-Q) = h^0(K) - 2 = g-2$ for every $P, Q \in X$. By Riemann–Roch, we compute

$$h^0(P+Q) - h^0(K-P-Q) = 2+1-g = 3-g$$

Hence K is very ample if and only if $h^0(P+Q) = 1$ for every P, Q.

If X is hyperelliptic, then there is a map $\phi : X \to \mathbb{P}^1$, so that $\phi^*([1:0]) = P+Q$ for some points $P, Q \in X$ (possibly equal). Here the linear system $|P+Q|$ is 1-dimensional, so $h^0(X, P+Q) = 2$, and hence K_X is not very ample.

If X is not hyperelliptic, we have $h^0(X, P+Q) = 1$ for any P, Q (otherwise it is $\geqslant 2$, and $P+Q$ induces a map $X \to \mathbb{P}^1$ of degree two), and hence K_X is very ample.

We still need to check the last part of the above theorem, namely that every hyperelliptic curve arises as a curve of type (2,4) on $Q \subset \mathbb{P}^3$.

We proceed as follows. Let $D = P_1 + \cdots + P_4$ denote a generic degree 4 divisor on X (so $P_1, \ldots, P_4$ are general points of X). By Riemann–Roch, we get

$$h^0(D) - h^0(K-D) = 4+1-3 = 2$$

We claim that $h^0(K-D) = 0$, so that $h^0(D) = 2$. Note that $K-D$ has degree $2g-2-4 = 0$, so $K-D$ is a divisor of degree 0. This is effective if and only if $K \sim D$. However, there is a 4-dimensional family of divisors of the form $P_1 + \cdots + P_4$, whereas the space of effective canonical divisors has dimension $\dim |K| = 2$. Hence if the points P_i are chosen generically, $K-D$ will not be effective, and hence the claim holds.

From this, we obtain a linear system $|D|$ of dimension 1. We claim that D is base point free. We need to show that

$$h^0(D-P) = h^0(D) - 1 = \deg D + 1 - 3) - 1 = 1$$

for every point P. Suppose not, and let P be a base point of D. Since $D = P_1 + P_2 + P_3 + P_4$ we may suppose that $P = P_4$.

By Riemann–Roch, we are done if we can show $h^0(K-D+P) = 0$. However, $K-D+P = K-P_1-P_2-P_3$. There is a 3-dimensional space of effective divisors of the form $P_1+P_2+P_3$ for points $P_i \in X$, but only a 2-dimensional linear system of effective canonical divisors $|K|$. Hence $K-D+P$ is not effective.

We therefore have two morphisms from our hyperelliptic curve X; $f : X \to \mathbb{P}^1$ (induced by the g^1_2) and $g : X \to \mathbb{P}^1$ (induced by D). By the universal property of the fibre product, this gives a morphism

$$\phi = (f \times g) : X \to \mathbb{P}^1 \times_k \mathbb{P}^1$$

We claim that this is a closed immersion. Let $F = P + Q \in |g^1_2|$. The map $D + F$ induces the map $F : X \to \mathbb{P}^3$, which coincides with $j \circ \phi$ where $j : \mathbb{P}^1 \times \mathbb{P}^1$ is the Segre embedding. To prove the claim, it suffices to show that F is an embedding, or equivalently that $D + F$ is very ample.

First claim that $K \sim 2F$. Since both of these divisors have degree 4 it suffices to show that $K - 2F$ is effective. Note that in any case $h^0(X, 2F) \geqslant 3$, since if $H^0(X, F) = \langle x, y\rangle$, then x^2, xy, y^2 are linearly independent in $H^0(X, 2F)$ (understand why!). Now applying Riemann–Roch to $D = 2F$, we get

$$h^0(2F) - h^0(K - 2F) = 4 + 1 - 3 = 2$$

so $h^0(K - 2F) \geqslant 1$, and $K \sim 2F$ as we want.

Now, to show that $D + F$ is very ample, we need to show that

$$h^0(X, D + F - p - q) = h^0(D + F) - 2$$

for any pair of points $p, q \in X$. By Riemann–Roch again, we can conclude if we know that $h^0(K - D - F + p + q) = 0$. But since $K \sim 2F$, we have

$$K - D - F + p + q \sim F - D + p + q$$

These are divisors of degree 0, so if this is effective, we must have $D \sim F + p + q$. However, the space of effective divisors of the form $D' + p + q$ with $D' \sim F$ is 3-dimensional (since $|F|$ has dimension 1, and p and q can be chosen freely on X). On the other hand, as we have seen, the space of divisors of the form $D = P_1 + \cdots + P_4$ is of dimension 4, so choosing D generically means that this$F - D + p + q$ is not effective. It follows that $h^1(D - p - q) = h^0(D + F) - 2$ and hence D is very ample. □

23.7 Curves of Genus 4

Recall that curves of genus $g \geqslant 2$ split up into two disjoint classes.

(i) Hyperelliptic curves: X admits a 2:1 to $\mathbb{P}^1$

(ii) Canonical curves: K_X is very ample

Here's an example of a genus 4 curve in $\mathbb{P}^1 \times \mathbb{P}^1$:

Example 23.29. Consider a type $(2, 5)$ curve C on $Q \subset \mathbb{P}^3$. Then C has degree $7 = 2 + 5$ and C is hyperelliptic (because of the degree 2 map coming from projection onto the first fact $p_1 : Q \to \mathbb{P}^1$). A type $(3, 3)$ curve on Q is also of genus 4. It is a degree 6 complete intersection of Q and a cubic surface. Curves of type $(3, 3)$ have at least two g^1_3's.

In fact, using the same strategy as for $g = 3$, one can show that any hyperelliptic curve of genus 4 arises this way.

23.7.1 Classifying curves of genus 4

We start with an abstract curve X of genus 4. We may assume that X is not hyperelliptic (since in that case it embeds as a $(2,5)$-divisor on $\mathbb{P}^1 \times \mathbb{P}^1$). So we assume that K_X is very ample. Therefore we have the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1} = \mathbb{P}^3$. The degree of the embedded curve is $\deg \omega_X = 2g-2 = 6$. Thus we can view X as a degree 6 genus 4 curve in $\mathbb{P}^3$.

What are the equations of X in $\mathbb{P}^3$? To answer this question we use a very powerful technique in curve theory, namely we combine Riemann–Roch with the sheaf cohomology on $\mathbb{P}^n$. Twisting the ideal sheaf sequence of X by $\mathcal{O}_{\mathbb{P}^3}(2)$ and taking cohomology gives the exact sequence

$$0 \to H^0(\mathbb{P}^3, I_X(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(X, \mathcal{O}_X(2)) \to \cdots$$

Note that $\mathcal{O}_{\mathbb{P}^3}(1)|_X = K_X$. Applying Riemann-Roch states to the divisor $D = 2K_X$, we get

$$h^0(\mathcal{O}_X(2)) = \deg 2K_X + 1 - g + h^1(\mathcal{O}_X(D)) = 12 + 1 - 4 + 0 = 9.$$

(Note that $h^1(\mathcal{O}_X(2)) = h^0(K_X - 2K_X) = h^0(-K_X) = 0$ since K_X is effective). Since $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$ it follows that the map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2))$ must have a nontrivial kernel so $h^0(\mathbb{P}^3, I_X(2)) > 0$.

The upshot of this is that we now know that X lies in some surface of degree 2. Since X is integral, this surface cannot be a union of hyperplanes. So X lies on either a singular quadric cone $Q_0 = V(xy - z^2)$ or the nonsingular quadric surface $Q = V(xy - zw)$.

If C lies on Q then it must have a type (a,b) which must satisfy $a + b = 6$ and $(a-1)(b-1) = 4$. The only solution is $a = b = 3$. Since $\mathcal{O}_Q(3,3) \simeq \mathcal{O}_{\mathbb{P}^3}(3)|_Q$, this implies that C is the restriction of a divisor on $\mathbb{P}^3$, that is, $C = Q \cap S$ for a degree 3 surface S.

The other possibility is that C lies on Q_0. Computing as before, we obtain

$$0 \to H^0(\mathcal{O}_X(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_X(3)) \to \cdots$$

As before one sees that $h^0(\mathcal{O}_X(3)) = 15$ and $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$. Thus $h^0(\mathcal{O}_C(3)) \geqslant 5$. Let $q \in H^0(\mathcal{O}_C(2))$ be the defining equation of Q_0. Then $xq, yq, zq, wq \in H^0(\mathcal{O}_C(3))$. But $h^0(\mathcal{O}_C(3)) \geqslant 5$ so there exists an $f \in H^0(\mathcal{O}_C(3))$ so that the global sections xq, yq, zq, wq, f are independent. Thus there is an f not in (q). Since $f \notin (q)$ we see that $S = Z(f) \not\supset Q$ so $C' = S \cap Q$ is a degree 6 not necessarily nonsingular or irreducible curve. Since $C \subset S$ and $C \subset Q$ it follows that $C \subset C'$. Since these are both integral curves of the same degree $\deg C = 6 = \deg C'$, we must have $C = C'$. Thus in the case that C lies on Q_0 we see that C is also a complete intersection $C = Q_0 \cap S$ for some cubic surface S.

This proves the following theorem:

Theorem 23.30. Let X be a non-singular curve of genus 4. Then either

(i) X is hyperelliptic (in which case X embeds as a $(2,5)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1$); or

(ii) $X = Q \cap S$ is the intersection of a quadric surface and a cubic surface in $\mathbb{P}^3$.

24

Further constructions and examples

24.1 Gluing relative schemes

In this section, we explain a general procedure for constructing morphisms of schemes via gluing. The setup is as follows:

Let X be a scheme and suppose that we are given the following data:

a) For each affine subscheme, $U \subset X$ a scheme $Y(U)$ and a morphism $\pi_U : Y(U) \to U$.
b) Whenever U, V are affine with $V \subset U$ there is a morphism $\rho_{VU} : Y(V) \to Y(U)$ such that

$$\begin{array}{ccc} Y(V) & \xrightarrow{\rho_{UV}} & Y(U) \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi_U} \\ V & \hookrightarrow & U \end{array} \tag{24.1}$$

is Cartesian (i.e., induces an isomorphism $Y(V) \simeq \pi_U^{-1}(V)$).
c) If $W \subset V \subset U$ are three affines, then $\rho_{UW} = \rho_{VU} \circ \rho_{WV}$.

Proposition 24.1 (Gluing relative schemes). Given the above data, there exists a scheme $Y(X)$ together with a morphism $\pi_X : Y(X) \to X$, and isomorphisms $\iota_U : \pi^{-1}(U) \to Y(U)$ so that for each $V \subset U$ affine, the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(V) & \hookrightarrow & \pi^{-1}(U) \\ \downarrow{\scriptstyle \iota_V} & & \downarrow{\scriptstyle \iota_U} \\ Y(V) & \xrightarrow{\rho_{UV}} & Y(U) \end{array}$$

As an X-scheme, $Y(X)$ is unique up to isomorphism.

Proof Let $\{U_i\}_{i\in I}$ be an affine cover of X and cover the double intersections U_{ij} with affines U_{ijk}. The schemes $Y(U_i) \times_{U_i} U_{ijk}$ and $Y(U_j) \times_{U_i} U_{ijk}$ are canonically isomorphic (to $Y(U_{ijk})$). Therefore the open subschemes $Y(U_i) \times_{U_i} U_{ij}$ and $Y(U_j) \times_{U_i} U_{ij}$ are isomorphic, and the isomorphisms satsify the cocycle conditions on the triple overlaps. The same holds for the morphisms $\pi_{U_i} : Y(U_i) \to U_i$, so we get the desired morphism $\pi_X : Y(X) \to X$.

$\pi_X : Y(X) \to X$ must be unique up to isomorphism, because it restricts to π_{U_i} over each U_i. $\square$

24.2 Relative Spec

Let X be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$*-algebras*. This means that $\mathcal{A}$ is a quasi-coherent sheaf and for each open set $U \subseteq X$, the group $\mathcal{A}(U)$ is an algebra over the ring $\mathcal{O}_X(U)$.

Let us apply Proposition 24.1 to the case where

$$Y(U) = \operatorname{Spec} \mathcal{A}(U)$$

and $\pi_U : Y(U) \to U$ is the morphism induced by the ring map $\mathcal{O}_X(U) \to \mathcal{A}(U)$. Let us check the that the second condition in the proposition is satisfied. If $V \subset U$ is another affine subset, we have there is a ring map $\mathcal{A}(U) \to \mathcal{A}(V)$ which induces a morphism $Y(V) \to Y(U)$ making the diagram (24.1) commutative. The diagram is actually Cartesian, because $\mathcal{A}(V) = \mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$, as $\mathcal{A}$ is quasi-coherent.

The third condition is also satisfied, because the ring map $\mathcal{A}(U) \to \mathcal{A}(W)$ factors via restriction to V.

It follows that the schemes $Y(U)$ glue together to a scheme, which we denote $\mathbf{Spec}(\mathcal{A})$ which we call the 'relative spectrum of $\mathcal{A}$'. There is a morphism $\pi : \mathbf{Spec}(\mathcal{A}) \to X$ which satisfies

$$\pi_* \mathcal{O}_{\mathbf{Spec}(\mathcal{A})} = \mathcal{A}.$$

The scheme $\mathbf{Spec}(\mathcal{A})$ satisfies the following universal property: For each morphism $h : Z \to X$ with a map of $\mathcal{O}_X$-algebras $\mathcal{A} \to h_* \mathcal{O}_Z$, there should be a unique morphism $f : Z \to \mathbf{Spec}(\mathcal{A})$ such that $h = \pi \circ f$.

Example 24.2. For $\mathcal{A} = \mathcal{O}_X[t_1, \ldots, t_n]$, the relative Spec coincides with the relative affine space $\mathbb{A}^n_X$.

Example 24.3. Let $X = \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ and let $f \in k[x_1, \ldots, x_n]$ be a polynomial. Then

$$\mathcal{A} = \mathcal{O}_X[t]/(t^m - f(x_1, \ldots, x_n))$$

is an $\mathcal{O}_X$-algebra. The relative spec $Y = \operatorname{Spec} \mathcal{A}$ is exactly the double cover of $\mathbb{A}^n$ ramified along f.

Example 24.4. More generally, let X be a normal integral scheme, $D \subset X$ an effective divisor, and let L be an invertible sheaf on X such that $L^{\otimes m} \simeq \mathcal{O}_X(D)$. Let $s \in \mathcal{O}_X(D)$ be the section that defines D; we will view it as a map $s : \mathcal{O}_X \to L^{\otimes m}$. Define the $\mathcal{O}_X$-module

$$\mathcal{A} = \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-m+1}$$

This becomes an $\mathcal{O}_X$-algebra via the multiplication

$$L^{-a} \otimes L^{-b} \simeq L^{-a-b} \otimes \mathcal{O}_X \xrightarrow{id \otimes s} L^{-a-b} \otimes L^m \simeq L^{-a-b+m}.$$

Let $Y = \mathbf{Spec}\mathcal{A}$ with the projection $\pi : Y \to X$. We call Y the ramified cyclic cover of s.

Over an open set U where $L \simeq \mathcal{O}_U$, pick a local generator s. The image $s^m \in \Gamma(U, L^m)$. On such an open, we have $\mathcal{A}|_U \simeq \mathcal{O}_U^m$, which is generated by 1 and f subject to the relation $z^m = f$.

It is not hard to show that Z is regular if and only if X and D are.

Exercise 24.2.1. Check that the scheme $\mathbf{Spec}(\mathcal{A})$ and the morphism π satisfies the above universal property.

Example 24.5 (Closed subschemes)**.** An important special case is when $\mathcal{A} = \mathcal{O}_X/\mathcal{I}$ for some quasi-coherent ideal $\mathcal{I}$. In this case there is a morphism

$$i : \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}) \longrightarrow \operatorname{Spec}(\mathcal{O}_X) = X$$

and $Y = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$ is exactly the closed subscheme associated to $\mathcal{I}$.

Example 24.6 (Vector bundles)**.** Let $\mathcal{E}$ denote a locally free sheaf of rank r. The symmetric algebra

$$\operatorname{Sym}^*(\mathcal{E}) = \mathcal{O}_X \oplus \mathcal{E} \oplus S^2(\mathcal{E}) \oplus \cdots$$

is naturally an algebra over $\mathcal{O}_X$. The corresponding relative Spec is denoted by $V(\mathcal{E})$. The projection $\pi : V(\mathcal{E}) \to X$ is what's known as a *vector bundle*; all the scheme-theoretic fibers are affine spaces of dimension r. More precisely, if $x \in X$, the fiber $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}_X} k(x)$ is isomorphic to $k(x)^r$, and so the scheme theoretic fiber of π over x is isomorphic to the spectrum of

$$\operatorname{Sym}^*(k(x)^r) \simeq k(x)[t_1, \ldots, t_r]$$

24.3 Relative Proj

Let X be a scheme and let $\mathcal{R}$ be a quasi-coherent sheaf of *graded $\mathcal{O}_X$-algebras*. This means that for each open set $U \subseteq X$, the group $\mathcal{R}(U)$ is a graded ring with degree 0 isomorphic to $\mathcal{O}_X(U)$.

For an open affine $U \subset X$, set $Y(U) = \operatorname{Proj} \mathcal{R}(U)$, with projection $\pi : Y(U) \to U$ induced by the natural map $\operatorname{Proj} \mathcal{R}(U) \to \operatorname{Spec} \mathcal{O}_X(U) = U$. If $V \subset U$ is another affine, the map $\mathcal{R}(U) \to \mathcal{R}(V)$ is a map of graded rings, this induces a map $Y(V) \to Y(U)$. Checking that the conditions of Proposition 24.1 are satisfied is similar to the Relative Spec-case. We call the resulting scheme $\mathbf{Proj}(\mathcal{R}) \to X$ the 'relative Proj of $\mathcal{R}$'.

Example 24.7 (Projective bundles)**.**

Example 24.8 (Hirzebruch surfaces)**.**

Example 24.9 (Blow-ups)**.**

24.4 Pushouts of affine schemes

Gluing schemes along open subschemes have been a central theme in this book. In some cases, we can also glue two schemes along a common *closed* subsheme. In this section, we explain how this can be done for affine schemes.

Let A, B, C be rings and let $f : A \to C$, $g : B \to C$ be surjections. From this data, we can form the pullback ring $A \times_C B$ arising in the pullback diagram

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

Explicitly, the ring $A \times_C B$ is defined by

$$A \times_C B = \{ (a, b) \in A \times B \mid f(a) = g(b) \}.$$

The diagram above induces a *pushout diagram* of schemes

$$\begin{array}{ccc} \operatorname{Spec} C & \hookrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec}(A \times_C B) \end{array}$$

This means that $\operatorname{Spec} A \times_C B$ satisfies a universal property dual to that of the fiber product: it is universal among diagrams of the form (24.4) with $\operatorname{Spec}(A \times_C B)$ replaced by some other scheme.

Proposition 24.10. As a topological space, $\operatorname{Spec}(A \times_C B)$ is homeomorphic to

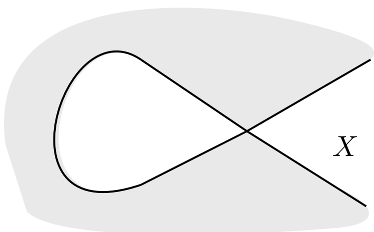
$$(\operatorname{Spec} A) \cup_{\operatorname{Spec} C} (\operatorname{Spec} B) \tag{24.2}$$

Example 24.11. The nodal cubic curve can be obtained from this construction; it is obtained by identifying two points of $\mathbb{A}^1_k$. [ADD MORE DETAILS.]

Example 24.12. Here is an example of a non-normal surface with an isolated singularity. We let X be the scheme obtained by identifying two points in $\mathbb{A}^2_k$; X is the affine variety given by the k-algebra

$$A = \{ f \in k[x, y] \mid f(0,0) = f(0,1) \}.$$

Then the normalization $\overline{X}$ is the affine plane.



The algebra A is generated by the 4 polynomials

$$x, xy, y^2 - y, y^3 - y \tag{24.3}$$

To see this, note that if $f(x, y)$ is any polynomial satisfying $f(0,0) = f(0,1)$, we may subtract products of the form $b(x)(y^2-y)^k$ or $b(x)(y^2-y)^k(y^3-y)$ until the y-degree of f is at most 1; the remaining polynomials can be written as polynomials in x and xy.

The polynomials (24.3) define a morphism $X \to \mathbb{A}^4$, onto the closed subscheme $V(I) \subset \mathbb{A}^4$, where I is the ideal of relations

$$\begin{aligned} I = (z_1z_3 + z_2z_3 - z_1z_4, z_3^3 - 2z_3^2 + 3z_3z_4 - z_4^2, \\ z_2z_3^2 + z_2z_3z_4 - z_1z_4^2 - 2z_2z_3 + z_2z_4, \\ z_2^2z_3 + z_1^2z_4 - z_1z_2z_4 + z_1z_2 - z_2^2) \end{aligned}$$

In some cases, it is possible to glue two schemes X and Y along a common *closed* subscheme Z. In this case, the gluing is represented by a pushout diagram

Example 24.13 ($\operatorname{Spec} \mathbb{Z}\sqrt{-3}$)**.** The spectrum of the ring $R = \mathbb{Z}[\sqrt{-3}]$ is rather interesting. It can be viewed as a sort of singular curve over $\operatorname{Spec} \mathbb{Z}$. As such it shares many properties with the nodal cubic curve of Example 13.16.

Note first that R is not a unique factorization domain. For example,

$$2 \cdot 2 = (1+\sqrt{-3})(1-\sqrt{-3}).$$

It is also not normal, because the element $\omega = (1+\sqrt{-3})/2 \in \mathbb{Q}(\sqrt{-3})$ satisfies the monic equation $x^2 + x + 1 = 0$, but $\omega \notin R$. In fact, the integral closure of R is given by the ring the ring of Eisenstein integers $\mathbb{Z}[\omega]$, and $\mathbb{Z}[\omega]$ *is* a unique factorization domain. In particular, $\mathbb{Z}[\omega]$ is normal, and equals the integral closure of R inside the fraction field $\mathbb{Q}(\sqrt{-3})$. It follows that the morphism

$$\operatorname{Spec} \mathbb{Z}[\omega] \to \operatorname{Spec} \mathbb{Z}[\sqrt{-3}],$$

induced by the inclusion $\mathbb{Z}[2\omega] \to \mathbb{Z}[\omega]$, is the normalization map.

Note that R is not a Dedekind domain: an integral domain is Dedekind if and only if each of its localizations is a discrete valuation ring. However, the localization $R_{\mathfrak{p}}$ at the prime ideal $\mathfrak{p} = (2, 1+\sqrt{-3})$ is not a discrete valuation ring; the maximal ideal requires two generators. However the square of $\mathfrak{p}$ is principal; it satisfies

$$\mathfrak{p}^2 = (2) \subset \mathbb{Z}[\sqrt{-3}].$$

There are two ring maps $\phi, \psi \colon \mathbb{Z}[\omega] \to \mathbb{F}_2[x]/(x^2+x+1) = \mathbb{F}_4$, one sending ω to x, and the other sending ω to $x+1$. The subring of $\mathbb{Z}[\omega]$ where these coincide is exactly $\mathbb{Z}[2\omega] = R$. We get a pushout diagram

$$\begin{array}{ccc} \mathbb{F}_4 & \xleftarrow[\psi]{} & \mathbb{Z}[\omega] \\ {\scriptstyle\phi}\uparrow & & \uparrow \\ \mathbb{Z}[\omega] & \longleftarrow & \mathbb{Z}[\sqrt{-3}] \end{array}$$

This induces a homeomorphism between $\operatorname{Spec} \mathbb{Z}[\sqrt{-3}]$ and $\operatorname{Spec} \mathbb{Z}[\omega]$ with two points identified. Thus $\operatorname{Spec} R$ is obtained by identifying two points in the spectrum of the Eisenstein integers.

24.5 Multigraded rings

The Proj-construction has the following multigraded analogue.

Let $R = k[x_1, \dots, x_n]$ be a ring graded by the group $\mathbb{Z}^n$. This means that each variable x_i is assigned a degree $e_i \in \mathbb{Z}^n$. Let $f \in R$ denote a homogeneous element with respect to the $\mathbb{Z}^n$-grading. Let $(R_f)_0$ denote all the elements in the localization of degree $\mathbf{0} \in \mathbb{Z}^n$.

Definition 24.14. For $\mathbf{w} \in \mathbb{Z}^n$, we define the subring

$$R(\mathbf{w}) = \bigoplus_{n \geqslant 0} R_{n\mathbf{w}} \subset R$$

The $\mathbf{w}$-irrelevant ideal as the graded ideal

$$I_{\mathbf{w}} = \bigoplus_{n \geqslant 0} R_{n\mathbf{w}} \subset R$$

Note that $R(\mathbf{w})$ and $I_{\mathbf{w}}$ are graded $R_{\mathbf{0}}$-modules. In fact, $R(\mathbf{w})$ is a graded algebra over $R_{\mathbf{0}}$.

Definition 24.15. For a given $\mathbf{w} \in \mathbb{Z}^n$, we define the multigraded projective spectrum $\mathbf{w}$-Proj(R) as the set of homogeneous prime ideals $\mathfrak{p}$ that do not contain the irrelevant ideal $I_{\mathbf{w}}$.

As in the usual Proj-construction, the set $\mathbf{w}$-Proj(R) inherits a Zariski-topology, by declaring that the closed sets are exactly the sets $V(\mathfrak{a})$ of homogeneous prime ideals $\mathfrak{p} \supset \mathfrak{a}$ (not containing the irrelevant ideal $I_{\mathbf{w}}$). There is also the set of distinguished opens $D_+(f) = \mathbf{w}$-Proj$(R) - V(f)$, defined for $\mathbb{Z}^n$-homogeneous f. As before, these give a basis for the topology on $\mathbf{w}$-Proj(R).

Next, we define the structure sheaf on $X = \mathbf{w}$-Proj(R). We define it on the basis consisting on distinguished opens by

$$\mathcal{O}_X(D_+(f)) = (R_f)_{\mathbf{0}}$$

and the restriction maps are as usual given by the localization maps $R_f \to R_g$ for $D_+(f) \supset D_+(g)$. This defines a sheaf of rings $\mathcal{O}_X$ on X whose stalks at are the local rings

Proposition 24.16. The locally ringed space

24.6 Examples

Consider the polynomial ring $R = k[x_0, x_1, x_2, x_3]$ with the grading given by the columns of the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Let us choose the vector $w = (1, 1)$ as the degree vector. Then the irrelevant ideal is generated by all monomials $x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3}$ of degree

$$e = a_0 \begin{pmatrix} 1 & 0 \end{pmatrix} + a_1 \begin{pmatrix} 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 1 \end{pmatrix}$$

such that $e \cdot w > 0$, i.e., $a_0 + a_1 + a_2 + a_3 > 0$.

The multigraded spectrum is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

In this case the irrelevant ideal is given by

$$(x_0, x_1) \cap (y_0, y_1)$$

The localizations are given by

$$k[\frac{x_1}{x_0}, \frac{y_1}{y_0}]$$

24.7 Toric ideals

Let $\mathcal{A}$ be an $m \times n$ with non-negative integer entries, and let the column vectors be $\mathbf{a}_1, \ldots, \mathbf{a}_n$. We use the multinomial notation, i.e., for $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ we write $z^{\mathbf{a}}$ for the monomial $z_1^{a_1} \cdot z_m^{a_m}$.

The matrix $\mathcal{A}$ allows us to define a ring map

$$\begin{aligned} \phi : k[x_1, \ldots, x_n] &\longrightarrow k[z_1^{\pm 1}, \ldots, z_1^{\pm 1}] \\ x_i &\mapsto z^{\mathbf{a}_i} \end{aligned} \tag{24.4}$$

and a corresponding morphism of schemes

$$f : \operatorname{Spec} k[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \longrightarrow k[x_1, \ldots, x_n]$$

We are interested in the (closure of the) image of f, i.e., the kernel of ϕ. This is described by the following intersection

$$I_{\mathcal{A}} = (z_1 - x^{\mathbf{a}_1}, \ldots, z_m - x^{\mathbf{a}_m}) \cap k[x_1, \ldots, x_n]$$

inside $k[x_1, \ldots, x_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$. More concretely, we have the following description:

Proposition 24.17. The kernel of ϕ is given by the ideal

$$I_{\mathcal{A}} = (x^u - x^v | u - v \in \operatorname{Ker} \mathcal{A}) \tag{24.5}$$

Proof It is clear that the ideal $I_{\mathcal{A}}$ is contained in the kernel, so we prove the opposite inclusion using a monomoial ordering argument. More precisely, we will consider the lexicographic ordering $\prec$ on monomials in $k[x_1, \ldots, x_n]$, so that

$$x_1^2 > x_1 x_2 > x_2^2 > x_1 x_3 > x_2 x_3 > x_3^2$$

and so on.

If $g \operatorname{Ker} \phi$ is any element, we can write it as

$$g = c_u x^u + \sum_{v \prec u} c_v x^v.$$

where $c_u x^u \neq 0$ is the leading term with respect to $\prec$. Applying ϕ, we get

$$0 = \phi(g) = c_u z^{\mathcal{A}u} + \sum_{v \prec u} c_v z^{\mathcal{A}v}.$$

This is an identity of polynomials in $k[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$, so there must be some cancellations between the monomials happening. In other words, there must be some v with $v \prec u$ such that $\mathcal{A}u = \mathcal{A}v$. But then replacing g with $g - c_u(x^u - x^v)$, we obtain a polynomial which has a leading term which is strictly smaller than g with respect to $\prec$. Note that $x^u - x^v$ belongs to the ideal $I_{\mathcal{A}}$. Continuing in this manner, we eventually obtain the zero polynomial, which means that g is an element of $I_{\mathcal{A}}$. □

Thus $I_{\mathcal{A}}$ is a prime ideal defined by binomials. To find a finite generating set, a few more computations are usually needed.

There is a smaller generating set of the ideal $I_{\mathcal{A}}$ given as follows. Let $S = \{u_1, \ldots, u_r\}$ be a $\mathbb{Z}$-basis for $\operatorname{Ker} \mathcal{A}$ and let

$$I'_{\mathcal{A}} = (x^{u_+} - x^{u_-} \mid u \in S)$$

where we decompose $u = u_+ - u_- \in \mathbb{Z}^n$ in terms of its non-negative and non-positive entries. Then $I_{\mathcal{A}}$ is the saturation of $I'_{\mathcal{A}}$ with respect to the maximal ideal at the origin, i.e., $I_{\mathcal{A}} = I'_{\mathcal{A}} : (x_1, \ldots, x_n)^\infty$.

Example 24.18. For $\mathcal{A} = (2\,3)$, we obtain the ideal $I = (x_2^2 - x_1^3)$, which is the ideal of the cuspidal cubic, parameterized by $t \mapsto (t^2, t^3)$.

Example 24.19. For $\mathcal{A} = (1\,2\,3)$, we obtain the ideal $I = (x_1^2 - x_2, x_1^3 - x_2)$, which is the ideal of the twisted cubic, parameterized by $t \mapsto (t, t^2, t^3)$.

Example 24.20. For

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

the corresponding morphism is the map $\mathbb{A}^2 \to \mathbb{A}^4$ given by $(x, y) \mapsto (1, x, y, xy)$. The image is the quadric $x_1 x_4 - x_2 x_3$

Example 24.21. For

$$\mathcal{A} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$

the corresponding toric ideal is given by

$$I_{\mathcal{A}} = (x_3x_5 - x_4^2, x_1x_5 - x_2x_4, x_1x_4 - x_2x_3, x_0x_5 - x_2^2, x_0x_4 - x_1x_2, x_0x_3 - x_1^2)$$

This is the affine cone over the Veronese surface.

Example 24.22. For

$$\mathcal{A} = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix}$$

the corresponding toric ideal is given by

$$I_{\mathcal{A}} = (x_1^2 x_3 - x_2^3, x_2 x_4^2 - x_3^3, x_1 x_4 - x_2 x_3)$$

This is the cone over the rational quartic curve in $\mathbb{P}^3_k$.

Appendix A

Some results from Commutative Algebra

A.1 Direct and inverse limits

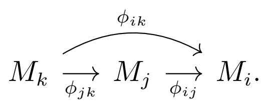
A.1.1 Direct limits

Recall that a *preordered* set is a set endowed with a relation $i \leqslant j$ which is symmetric; that is, $i \leqslant i$ for all i, and transitive; that is, if $i \leqslant j$ and $j \leqslant k$, then $i \leqslant k$. A preordered set resembles a partially ordered set, but lacks the anti-symmetry property: it might be that $i \leqslant j$ and $j \leqslant i$ without i and j being equal. We say that a preordered set I is *directed set* if the following condition holds: for any two elements i and j there is a $k \in I$ such that $k \geqslant i$ and $k \geqslant j$.

Definition A.1. A *directed system of A-modules* (M_i, ϕ_{ij}) is a collection $\{M_i\}_{i \in I}$ of A-modules, indexed by a directed set I, and a collection of A-linear maps $\phi_{ij} \colon M_j \to M_i$, one for each pair (i, j) with $j \leqslant i$, satisfying the two conditions

(i) $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ whenever $k \leqslant j \leqslant i$;
(ii) $\phi_{ii} = \mathrm{id}_{M_i}$.

The first condition may be illustrated by the commutative diagram:



Definition A.2. The *direct limit* of the directed system (M_i, ϕ_{ij}) of A-modules is an A-module $\varinjlim M_i$ together with a collection of A-linear maps

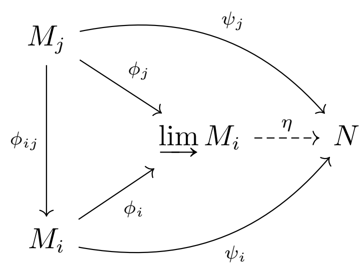
$$\phi_i \colon M_i \to \varinjlim M_i$$

which satisfy $\phi_i \circ \phi_{ij} = \phi_j$, and which are universal with respect to this property.

The limit having the universal property, means that for any A-module N and any given system of A-linear maps

$$\psi_i \colon M_i \to N$$

such that $\psi_i \circ \phi_{ij} = \psi_j$, there is a unique map $\eta\colon \varinjlim M_i \to N$ satisfying $\psi_i = \eta \circ \phi_i$.

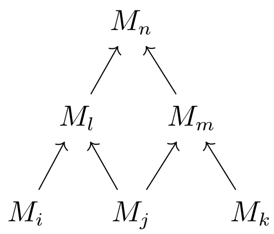


The definition of the direct limit may be formulated in any category: just replace the words 'A-module' with 'object' and 'A-linear map' by 'arrow'. In general categories it may easily happen that direct limits do not exist. However, the category of modules over a ring is a well behaved category, and here all limits exist.

Proposition A.3. Let A be any ring. Every directed system (M_i, ϕ_{ij}) of modules over A has a direct limit, which is unique up to a unique isomorphism.

Proof We begin with introducing an equivalence relation on the disjoint union $\coprod_i M_i$. Essentially, two elements are to be equivalent if they become equal somewhere far out in the hierarchy of the M_i's. In precise terms, $x \in M_i$ and $y \in M_j$ are to be equivalent when there is an index k larger than both i and j such that x and y map to the same element in M_k; that is, $\phi_{ki}(x) = \phi_{kj}(y)$, and we write $x \sim y$ to indicate that x and y are equivalent. The first point to verify is that this is an equivalence relation.

Obviously the relation is symmetric, since $\phi_{ii} = \mathrm{id}_{M_i}$ it is reflexive, and it being transitive follows from the system being directed: assume that $x \sim y$ and $y \sim z$, with x, y and z sitting in respectively M_i, M_j and M_k. This means that there are indices l dominating i and j, and m dominating j and k so that the two equalities $\phi_{li}(x) = \phi_{lj}(y)$ and $\phi_{mj}(y) = \phi_{mk}(z)$ hold true. Because the system is directed, there is an index n larger than both l and m.



By the first requirement in Definition A.1 above, we find

$$\phi_{ni}(x) = \phi_{nl}(\phi_{li}(x)) = \phi_{nl}(\phi_{lj}(y)) = \phi_{nm}(\phi_{mj}(y)) = \phi_{nm}(\phi_{mk}(z)) = \phi_{nk}(z),$$

and so we conclude that $x \sim z$. The underlying set of the A-module $\varinjlim M_i$ is the quotient $\coprod_i M_i/\sim$, and the maps ϕ_i are the ones induced by the inclusions of the M_i's in the disjoint union.

The rest of the proof consists of putting an A-module structure on $\varinjlim M_i$ and checking the universal property. To this end, the salient observation is that any two equivalence classes $[x]$ and $[y]$ in the limit may be represented by elements x and y from the same M_k; indeed, if $x \in M_i$ and $y \in M_j$, choose a k that dominates both i and j and replace x and y by their images in M_k. Forming linear combinations is then possible by the formula $a[x] + b[y] = [ax + by]$ where the last combination is formed in any M_k where both x and y live; this is independent of the particular k used (the system is directed, and the ϕ_{ij}'s are A-linear). The module axioms follow since any equality involving a finite number of elements from the limit may be checked in an M_k where all involved elements have representatives.

Finally, checking the universal property is straightforward. The obvious map from the disjoint union $\coprod_i M_i$ into N induced by the ψ_i's is compatible with the equivalence relation and hence passes to the quotient; that is, it gives the desired map $\eta\colon \varinjlim M_i \to N$. And as always with universal properties, it ensures that the limit will be unique up to a unique isomorphism. □

Apart from the universal property, there are two 'working principles', reflecting the working principles for stalks, one should bear in mind when computing with direct limits:

- Every element in $\varinjlim M_i$ is of the form $\phi_j(x)$ for some j and some $x \in M_j$.
- An element $x \in M_j$ maps to zero in $\varinjlim M_i$ if and only if $\phi_{ij}(x) = 0$ for some $i \geqslant j$.

Examples

Example A.4 (Union as a direct limit)**.** If each M_i are submodules of some A-module M, and the maps $M_j \to M_i$ are given by inclusions $M_j \subset M_i$, then the direct limit is simply the union:

$$\varinjlim_i M_i = \bigcup_i M_i.$$

Example A.5 (Stalks as a direct limit)**.** Let X be a topological space, and consider the directed set I of open neighbourhoods U of a point $x \in X$ ordered by inclusion. If $\mathcal{F}$ is a presheaf on X, then setting $M_U = \mathcal{F}(U)$, the above construction of the direct limit $\varinjlim_U M_U$ is exactly the same as the previous definition of the stalk $\mathcal{F}_x$.

Example A.6 (Localization as a direct limit)**.** Let A be a ring and S a multiplicative subset. We put a preorder on S by declaring $s \leqslant t$ when $t = us$ for some $u \in S$, and this makes S a directed set. Next, for $s \leqslant t$ with $t = us$, there exists a ring homomorphism $f_{ts} \colon A_s \to A_t$, which is defined by $f_{ts}(as^{-n}) = au^n t^{-n}$. In this way the family of rings $\{A_s\}_{s\in S}$ forms a directed system of rings, and one easily checks that the properties required of a directed family hold.

For each $s \in S$, there is a localization map $A_s \to S^{-1}A$, so from the universal property of the direct limit, we obtain a canonical A-linear map

$$\phi : \varinjlim_{s\in S} A_f \to S^{-1}A.$$

We contend this is an isomorphism. The map ϕ is surjective: any element in $S^{-1}A$ is of the form as^{-1} with $s \in S$; this element lies in A_s and hence in the image of ϕ. The map ϕ is injective: if $as^{-n} \in A_f$ is mapped to 0 in $S^{-1}A$, then for some $t \in S$ it holds that $ta = 0$, hence $as^{-n} = 0 \in A_{st}$, and ϕ is injective.

Functoriality

The the direct limit enjoys functoriality in two ways. A subset $J \subset I$ inherits a preorder from I, and if additionally it is directed, there is an induced A-linear map

$$\varinjlim_J M_j \to \varinjlim_I M_i. \tag{A.1}$$

This is clear since the inclusion of disjoint unions $\coprod_{i\in J} M_i \subset \coprod_{i\in I} M_i$ respects the equivalence relation, and therefore passes to the quotients and induces a map as in (A.1), which one without much effort checks is A-linear.

One says that a directed subset J is *cofinal* or *filtering* in I if for each element $i \in I$ there is a j in J with $j \geqslant i$.

Lemma A.7. If J is cofinal in I, the map in (A.1) is an isomorphism

$$\varinjlim_{j \in J} M_j \simeq \varinjlim_{i \in I} M_i.$$

Note that, in particular, if the index set I has a largest element i_0, then $\varinjlim_{i \in I} M_i = M_{i_0}$.

Proof The map in (A.1) is surjective since any $x \in M_i$ is equivalent to an element $\phi_{ji}(x)$ lying in an M_j with $j \in J$ and $j \geqslant i$. That some $x \in M_j$ is mapped to zero in $\varinjlim_{i \in I} M_i$, means that it maps to zero in some $M_{i'}$ with $i' \geqslant i$, but then it maps to zero in some M_j with $j \in J$ and $j \geqslant i'$ as well, and so is zero in $\varinjlim_{j \in J} M_j$. □

Assume then that (N_i, ψ_{ij}) and (M_i, ϕ_{ij}) are two directed systems, both indexed by the same directed set I. A map between them is a family $\{\rho_i\}$ of A-linear maps $\rho_i \colon N_i \to M_i$ such that $\rho_i \circ \psi_{ij} = \phi_{ij} \circ \rho_j$. In a straight forward manner, these data give rise to an A-linear map

$$\rho \colon \varinjlim_i N_i \to \varinjlim_i M_i,$$

which is compatible with he natural maps of the two limits; indeed, the family $\{\rho_i\}$ yields a map already between the disjoint unions which respects the equivalence relations. The direct systems of A-modules with maps as above form a category, and the induced map ρ above depends functorialy on the family ρ_i.

Inverse limits

The dual concept of a direct limit is the *inverse limit* (also called the *projective limit* or just the *limit*) of an *inverse system* $\{M_i\}_{i \in I}$. These systems and their limits are defined similarly to the direct systems, just with the arrows reversed. In fact, an inverse system indexed by I is nothing but a direct system indexed by the opposite ordered set I^{op}, though the limits will have rather different properties.

To be precise, the staging is as follows:

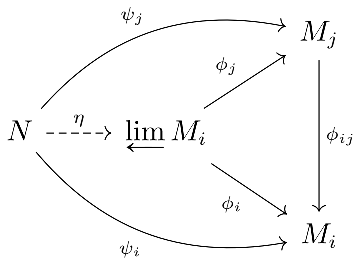
Definition A.8. An inverse system (M_i, ϕ_{ij}) of A-modules indexed by a directed set I is a family M_i of A-modules indexed by I, and for each pair i, j from I with $i \leqslant j$ an A-linear map $\phi_{ij} \colon M_j \to M_i$ subjected to the conditions

(i) $\phi_{ii} = \mathrm{id}_{M_i}$;
(ii) $\phi_{ik} \circ \phi_{jk} = \phi_{ji}$ whenever $i \leqslant k \leqslant j$.

Note that the maps ϕ_{ij} go 'backwards' relatively to the order in I; or in a functorial language, the dependence of M_i on i is contravariant. The definition of the inverse limit is word for word the same as the definition of the direct limit except that all arrows are reversed:

Definition A.9. The inverse limit of an inverse system (M_i, ϕ_{ij}) is an A-module $\varprojlim M_i$ together with a collection of A-linear maps $\phi_i \colon \varprojlim M_i \to M_i$ so that $\phi_i = \phi_{ji} \circ \phi_j$, which are universal in this respect.

The universal property is illustrated with the diagram



Proposition A.10. Every directed inverse system of A-modules has a limit.

Proof Define a submodule L of the product $\prod_i M_i$ by

$$L = \{\, (x_i) \mid x_i = \phi_{ij}(x_j) \text{ for all pairs } i, j \text{ with } i \leqslant j \,\} \tag{A.2}$$

The projections induce maps $\phi_i \colon L \to M_i$, and we claim that L together with these maps constitutes the inverse limit of the system. A family of maps $\psi_i \colon N \to M_i$ gives rise to a map $\eta \colon N \to \prod_i M_i$ by the assignment $x \mapsto (\psi_i(x))$, and it takes values in L when the ψ_i's satisfy the compatibility constraints $\psi_i = \phi_{ij} \circ \psi_j$. This map is clearly unique, and hence we get the desired universal property. □

Just as for injective limits, one has functoriality both on the level of the indexing set and on the level of modules. In deed, if $J \subset I$ is a directed subset, there is a canonical projection

$$\prod_{i \in I} M_i \to \prod_{i \in J} M_i$$

that just remembers the coordinates indexed by j. The submodules L are respected, and we get a map

$$\varprojlim_{i \in I} M_i \to \varprojlim_{i \in J} M_i \tag{A.3}$$

One easily proves the following along the same lines as the proof of Lemma A.7:

Lemma A.11. If J is cofinal in I, the map in (A.3) is an isomorphism.

Examples

Example A.12 (Inverse limits and intersections)**.** If all the M_i's are submodules of some fixed module M, and the maps $M_j \to M_i$ are the inclusions, the inverse limit will simply be the intersection of the M_i's:

$$\varprojlim_{i \in I} M_i = \bigcap_{i \in I} M_i \subset M.$$

Example A.13 (The p-adic integers)**.** An important application of inverse limits is to form so-called 'completions of rings'. The primary example is the p-adic numbers. Let p be a prime number and consider the modules $\mathbb{Z}/p^i\mathbb{Z}$. They form an inverse system indexed by $\mathbb{N}$ with ϕ_{ij} being just the canonical reduction map $\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$, that for $j \geqslant i$ sends a class $[n]_{p^j}$ mod p^j to the class $[n]_{p^i}$ mod p^i. The system may be visualized by the sequence

$$\ldots \longrightarrow \mathbb{Z}/p^{i+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^i\mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

The inverse limit is denoted by $\mathbb{Z}_p$ and is called the *ring of p-adic integers*.

Example A.14 (Inverse limits and sections)**.** Whereas direct limits gives us stalks, inverse limits give a way to compute sections. In the context of sheaves, the slogan is: 'Direct limits have a localizing effect, while inverse limits effectuate globalizations.'

Consider an open set U of the topological space X and a sheaf $\mathcal{F}$ on X. Assume given an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U which is directed under inclusion; i.e. the intersection of two members from $\mathcal{U}$ contains a third, then the restriction maps induce an isomorphism $\mathcal{F}(U) \simeq \varprojlim_{i \in I} \mathcal{F}(U_i)$. Indeed, the restriction maps $\rho_{UU_i} \colon \mathcal{F}(U) \to \mathcal{F}(U_i)$ comply with the compatibility request $\rho_{UU_i} = \rho_{U_jU_i} \circ \rho_{UU_j}$ for $U_i \subset U_j$, and they thus give a canonical map $\mathcal{F}(U) \to \varprojlim_{i \in I} \mathcal{F}(U_i)$.

In view of the description in (A.2) this is an isomorphism: that s maps to zero, means that $\rho_{UU_i}(s) = s|_{U_i} = 0$ for each i, which by the Locality axiom entails that $s = 0$. Furthermore, sections $s_i \in \mathcal{F}(U_i)$ so that $s_j|_{U_i} = s_i$ for each inclusion $U_i \subset U_j$ may, by the Gluing axiom, be glued together to give a section of $\mathcal{F}$ over U, and the map is surjective.

In fact, with slightly more care one can establish that if $\mathcal{F}$ is a presheaf, the sections of the sheafification $\mathcal{F}^+$ is given as the inverse limit

$$\mathcal{F}^+(U) \simeq \varprojlim_{i \in I} \mathcal{F}(U_i). \tag{A.4}$$

Exercise A.1.1. Convince yourself that (A.4) holds true.

Exercises

Exercise A.1.2. Show that arbitrary direct and inverse limits exist in the category Sets and Rings of sets, respectively of rings. HINT: Adapt the proofs above.

Exercise A.1.3. Show that the inverse limits exist unconditionally in the category of topological spaces. Show that the inverse limit of compact Hausdorff spaces is compact and Hausdorff. HINT: Express the limit as the intersection of inverse images of graphs, and use Tychonoff's theorem.

Exercise A.1.4. Exhibit a directed system in the category $\mathsf{FiniteSets}$ of finite sets that does not have a direct limit in $\mathsf{FiniteSets}$.

Exercise A.1.5. Exhibit a inverse system of finite sets indexed by $\mathbb{N}$ whose inverse limit is empty. Show that the inverse limit of a system of compact spaces with surjective maps is non-empty.

Exercise A.1.6. Let $(M_i, \phi_{ij})_{i\in I}$ be a directed (respectively inverse) system of A-modules. Assume that I is *discrete*; that is, that no two elements are comparable (in other words, $i \leqslant j$ only when $i = j$). Show that $\varinjlim_{i\in I} M_i = \bigoplus_i M_i$, respectively $\varprojlim_{i\in I} M_i = \prod_i M_i$.

Exercise A.1.7. Assume that I is a directed set in which every element is dominated by a maximal element. Let $(M_i, \phi_{ij})_{i\in I}$ be a direct (respectively inverse) system of A-modules indexed by I. Show that $\varinjlim_{i\in I} M_i$ is isomorphic to the direct sum $\bigoplus M_j$, respectively $\varprojlim_{i\in I} M_i$ is isomorphic to the product $\prod M_j$, where the sum (respectively the product) extends over all maximal elements in I.

Exercise A.1.8 (Direct limits are exact)**.** Let (M_i, ϕ_{ij}), (M_i', ϕ_{ij}') and (M_i'', ϕ_{ij}'') be three directed systems of A-modules. Suppose given exact sequences

$$0 \longrightarrow M_i' \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} M_i'' \longrightarrow 0$$

with the collections $\{\alpha_i\}$ and $\{\beta_i\}$ being maps of direct systems (i.e. they are compatible with the transition maps). Show that the induces sequence of limits

$$0 \longrightarrow \varinjlim M_i' \xrightarrow{\alpha} \varinjlim M_i \xrightarrow{\beta} \varinjlim M_i'' \longrightarrow 0$$

is exact; in short, the inductive limit is an exact functor.

Exercise A.1.9 (Inverse limits are left exact)**.** With setting as in the previous exercise except that the systems are inverse systems, show that the sequence of inverse limits

$$0 \longrightarrow \varprojlim M_i' \xrightarrow{\alpha} \varprojlim M_i \xrightarrow{\beta} \varprojlim M_i''$$

is exact. Show by finding an example that β is not always surjective; hence the inverse limit is merely left exact.

Exercise A.1.10. Let A be a ring and $a \in A$ an element. Let a direct system indexed by $\mathbb{N}$ be given by $M_i = A$ for all i and $\phi_{ij}(x) = a^{j-i}x$ for $i \leqslant j$. Determine the direct limit $\varinjlim_{i\in I} M_i$.

Exercise A.1.11. Let A be a ring. Show that the inverse limit of the inverse system

$$\ldots \to A[x]/\mathfrak{m}^{i+1} \to A[x]/\mathfrak{m}^i \to \ldots \to A[x]/\mathfrak{m}^2 \to A[x]/\mathfrak{m}$$

where $\mathfrak{m} = (x)$, and the maps are the canonical reduction maps, is isomorphic to the ring of formal power series $A[[x]]$.

Exercise A.1.12. Let p be a prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers (as in Example A.13). Show that $\mathbb{Z}_p$ is a Noetherian local domain with maximal ideal generated by p. Show that $\mathbb{Z}_p$ is compact when endowed with the limit topology. Show that the map $\mathbb{Z} \to \mathbb{Z}_p$ sending n to $([n]_{p^i})_i$ is an injective ring homomorphism. Show that the assignment $x \mapsto p$ defines an isomorphism $\mathbb{Z}[[x]]/(x-p) \to \mathbb{Z}_p$.

Exercise A.1.13 (Alternative description of the direct limit)**.** Let $(M_i, \phi_{ij})_{i \in I}$ be a directed system of modules over a ring A. Define an A-module homomorphism

$$\psi\colon \bigoplus_{j \in I} M_i \to \bigoplus_{i \in I} M_j$$

by the assignment

$$\psi((m_i j))_i = \begin{cases} m_i - \phi_{ij} m_j & \text{when } i \geqslant j \\ 0 & \text{else} \end{cases}$$

Show that the cokernel $\operatorname{Coker} \psi$ is isomorphic to the direct limit $\varinjlim M_i$. HINT: Verify the universal property.

A.2 Localization

A nonempty subset S of a commutative ring A is called a *multiplicative set* if it is closed under multiplication and contains the identity element of A.

The localization of A with respect to a multiplicative set S, denoted $S^{-1}A$, is the set of fractions a/s with $a \in A$ and $s \in S$. There is a well-defined addition and multiplication making $S^{-1}A$ into a ring. Formally, $S^{-1}A$ is constructed by definiing an equivalence relation on $A \times S$ by $(a, s) \sim (a', s')$ if there exists an element $t \in S$ such that $t(as' - a's) = 0$ in A. The elements of $S^{-1}A$ are denoted by a/s or $\frac{a}{s}$.

There is a canonical localization map

$$\rho : A \to S^{-1}A, x \mapsto x/1$$

which makes $S^{-1}A$ into an A-module. The map ρ is injective if A contains no zerodivisors: $a/1 = 0$ means that $t \cdot a = 0$ for some $t \in S$.

The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$ (if $0 \in S$, then $a/s = 0/1$ by definition).

If M is an A-module, one also defines a localization $S^{-1}M$ as the set of fractions m/s, for $m \in M$, $s \in S$, using the equivalence relation $(m, s) \sim (m', s')$ if $t(ms' - m's) = 0$ in M. As above, there is a canonical localization map $M \to S^{-1}M$. Also, $S^{-1}M$ is naturally an $S^{-1}A$-module.

Example A.15. The first prototype example is when $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$. In this case the localization $S^{-1}A$ is denoted $A_{\mathfrak{p}}$. The ring $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. The elements not in $\mathfrak{p}$ become units in $A_{\mathfrak{p}}$, and every non-unit in $A_{\mathfrak{p}}$ is in the maximal ideal.

Example A.16. The second prototype example is when $S = \{1, f, f^2, \dots\}$ for some $f \in A$. In this case the localization $S^{-1}A$ is denoted A_f. Elements of A_f are of the form a/f^n where $a \in A$ and $n > 0$.

Prime ideals in localizations. Note that if S is a multiplicative set and $\mathfrak{p}$ is a prime ideal, so that $S \cap \mathfrak{p} \neq \varnothing$, then $S^{-1}\mathfrak{p} = (1)$ in $S^{-1}A$. Thus the prime ideals that intersect S map to non-proper ideals in $S^{-1}A$.

For the other prime ideals, the map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p} = \mathfrak{p}S^{-1}A$. gives a one-to-one correspondence between the prime ideals of A that do not intersect S and the prime ideals of $S^{-1}A$; the inverse is given by $\rho^{-1}(\mathfrak{q})$.

A.3 Tensor products

Let M and N be two A-modles. We define the tensor product $M \otimes_A N$ as the quotient of the free module $A^{M \times N}$ with basis $(e_{m,n})$ modulo the relations

$$e_{m_1+m_2,n} - e_{m_1,n} - e_{m_2,n} = 0 \tag{A.5}$$

$$e_{m,n_1+n_2} - e_{m,n_1} - e_{m,n_2} = 0 \tag{A.6}$$

$$e_{am,n} - e_{m,an} = ae_{m,n} - e_{am,n} = 0 \tag{A.7}$$

for all $m \in M$, $n \in N$, $a \in A$. We write $m \otimes n$ for the class of $e_{m,n}$ in $M \otimes_A N$.

The assignment $(m,n) \mapsto m \otimes_A n$ defines map $\gamma : M \times N \to M \otimes_A N$ which is bilinar as a map of A-modules. It satisfies the following universal property: for any bilinear map $\phi : M \times N \to P$, there is a unique map $\overline{\phi} : M \otimes_A N \to P$ so that $\phi = \overline{\phi} \circ \gamma$.

A.4 Basic formulas

In the formulas below, M, N, L denote A-modules; $S \subset A$ is a multiplicative set; $\mathfrak{p}$ is a prime ideal of A. Each equality '=' between two modules means that there is a canonical isomorphism between them.

Localization identities:

(i) $S^{-1}(M/N) \simeq S^{-1}M/S^{-1}N$

(ii) If M is finitely presented: $S^{-1}\operatorname{Hom}_A(M,N) = \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$.

Tensor product identities:

(i) $A \otimes_A M = M$

(ii) $M \otimes_A N = N \otimes_A M$

(iii) $M \otimes_A (N \otimes_A P) = (M \otimes_A N) \otimes_A P$

(iv) $(\bigoplus_{i\in I} M_i) \otimes_A N = (\bigoplus_{i \in I} M_i \otimes_A N)$

(v) If $A \to B$ is a ring map; M, N are A-modules and P is a B-module, then there is a canonical isomorphism of B-modules

$$M \otimes_A (N \otimes_B P) = (M \otimes_A N) \otimes_B P.$$

(vi) $\operatorname{Hom}_A(M \otimes_A N, P) = \operatorname{Hom}_A(M, \operatorname{Hom}(N,P))$.

(vii) $S^{-1}M = M \otimes_A S^{-1}A$.

(viii) $S^{-1}(M \otimes_A N) = S^{-1}M \otimes_A S^{-1}N$.

Exactness properties. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then

$$0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$$

is exact for every multiplicative set S;

$$\begin{aligned} &0 \to \operatorname{Hom}(L, M') \to \operatorname{Hom}(L,M) \to \operatorname{Hom}(L, M'') \\ &0 \to \operatorname{Hom}(M'', L) \to \operatorname{Hom}(M,L) \to \operatorname{Hom}(M', L) \end{aligned} \tag{A.8}$$

are exact for every A-module L; and

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is exact for any A-module N.

A.5 Noetherian rings

A.6 Dimension theory

A.7 Exactness properties

A.8 Integral Extensions

There is a collection of results, the Cohen–Seidenberg Theorems, about prime ideals in integral extension with important applications to finite morphisms. We summarize them here without proofs. They are formulated with the more general hypothesis that the extension is integral, but finite ring extensions are integral.

Theorem A.17. Let $A \subset B$ be an integral extension of rings.
(i) (Lying–Over) If $\mathfrak{p}$ prime ideal in A, there is prime ideal $\mathfrak{q}$ in B so that $\mathfrak{q} \cap A = B$;
(ii) If $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals in B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then $\mathfrak{q} = \mathfrak{q}'$;
(iii) (Going–Up) If $\mathfrak{p} \subset \mathfrak{p}'$ are two prime ideals in A and $\mathfrak{q} \in \operatorname{Spec} B$ with $\mathfrak{q} \cap A = \mathfrak{p}$, there is a $\mathfrak{q}' \in \operatorname{Spec} B$ with $\mathfrak{q}' \cap A = \mathfrak{q}$;
(iv) (Going–Down) Assume that A is integrally closed and that $\mathfrak{p}' \subset \mathfrak{p}$ are two prime ideals. If $\mathfrak{q} \in \operatorname{Spec} B$ is such that $\mathfrak{q} \cap A = \mathfrak{p}$, then there is a $\mathfrak{q}' \in \operatorname{Spec} B$ such that $\mathfrak{q}' \cap A = \mathfrak{p}'$.

A.9 Normal rings

An integral domain A is said to be *normal* if it is integrally closed in its fraction field $K = k(A)$. In other words, any element $z \in K$ which satisfies a monic equation with coefficients in A, is already contained in A.

The following is a non-trivial result from commutative algebra about the integral closure:

Theorem A.18 (Finite generation of integral closure). Let A be an integral domain, $K = K(A)$ its fraction field, and let $K \supset L$ be a finite separable field extension. Let B be the integral closure of A in L (that is, the elements of L which are integral over A). Then
(i) If A is integrally closed, then B is a finitely generated A-module
(ii) If A is finitely generated as a k-algebra, then B is a finitely generated A-module.

The second part does not hold in general: there are non-noetherian rings where the integral closure is not finitely generated.

A.10 Regular local rings

A Noetherian local ring A with $\dim A = n$ and with maximal ideal $\mathfrak{m}$ is said to be *regular* if the maximal ideal can be generated by n elements. Nakayama's lemma implies that the minimal number of generators of $\mathfrak{m}$ equals the dimension of the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $\dim_{A/\mathfrak{m}}$. A general ring A is *regular* if all the local rings $A_\mathfrak{p}$ are regular.

Discrete valuation rings

When it comes to one-dimensional rings, A is regular if and only if $\mathfrak{m}$ is principal. This has many equivalent formulations, and we list the few we shall need.

Proposition A.19. Let A be a Noetherian local domain with maximal ideal $\mathfrak{m}$ of dimension one. Then the following are equivalent

(i) The maximal ideal $\mathfrak{m}$ is principal;
(ii) A is a PID and all ideals are powers of $\mathfrak{m}$;
(iii) A is integrally closed.
(iv) A is regular, i.e., $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.

Proof i)⇒ ii). Let x a generator for the maximal ideal $\mathfrak{m}$ and let $\mathfrak{a} \subset A$ be a non-zero ideal. Let n be the largest integer such that $\mathfrak{a} \subset \mathfrak{m}^n$. Krull's intersection theorem asserts that $\bigcap_i \mathfrak{m}^i = 0$, and the ideal $\mathfrak{a}$ is therefore not contained in all powers of $\mathfrak{m}$ and such an n exists. Since $\mathfrak{a} \nsubseteq \mathfrak{m}^{n+1}$, there is an $a \in \mathfrak{a}$b such that $a = bx^n$ with $b \notin \mathfrak{m}$; that is, b is a unit since the ring is local. It follows that $(x^n) \subset \mathfrak{a}$, and we are done.

ii)⇒ iii). Every PID is a UFD and all UFD's are integrally closed.

iii)⇒ i). Finally, assume that A is integrally closed and let $x \in \mathfrak{m}$ be any non-zero element. Since A is Noetherian and of dimension one, the maximal ideal $\mathfrak{m}$ is associated to (x) (indeed, $\mathfrak{m}$ is the only non-zero prime ideal in A), and we conclude by Lemma A.20 below.

$(i) \Leftrightarrow (iv)$. If $\mathfrak{m} = (x)$, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by the class of x modulo $\mathfrak{m}^2$. We also have $\mathfrak{m} \neq \mathfrak{m}^2$ (since A has dimension 1), so $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$. The converse implication follows by Nakayama's lemma. □

Lemma A.20. Let A be a Noetherian local normal domain and assume that the maximal ideal $\mathfrak{m}$ is associated to a principal ideal. Then $\mathfrak{m}$ is principal.

Proof Let $x \in A$ be such that $\mathfrak{m}$ is associated to (x). This means that there exists an $z \in (x)$ so that $(0 : z) = \mathfrak{m}$.

some $y \in A$ with $y \notin (x)$ it holds that $y\mathfrak{m} \subset (x)$. Then $\mathfrak{m}yx^{-1} \subset A$, but $yx^{-1} \notin A$. If $\mathfrak{m}yx^{-1} \subset \mathfrak{m}$ the element yx^{-1} would be integral over A by the third criterion of Proposition **??** on page ?? (because A is Noetherian, $\mathfrak{m}$ is finitely generated, and it is faithful as all ideals are). But this is impossible because A is normal and $yx^{-1} \notin A$. We deduce that $\mathfrak{m}yx^{-1} = A$, and consequently there is a relation $zyx^{-1} = 1$ with $z \in \mathfrak{m}$. Then $w = (wyx^{-1})z$ for all $w \in \mathfrak{m}$ (note that wyx^{-1} lies in A), and hence $\mathfrak{m} = (z)$. □

A ring as in the proposition is also a *discrete valuation ring*. If t is a generator for the maximal ideal $\mathfrak{m}$, one calls t a *uniformizing parameter* of A. In fact, the above proof shows that any element of $\mathfrak{m} - \mathfrak{m}^2$ is a uniformizing parameter.

In a discrete valuation ring A, all non-zero ideals are of the form (t^ν) with $\nu \in N_0$, and therefore any non-zero element in the fraction field $K = K(A)$ may be written as αt^ν with α a unit in A and ν an integer. Indeed, if $f \in A$ and $f \neq 0$, we let $v(f)$ be the unique non-negative integer such that $(f) = \mathfrak{m}^{\nu(f)}$, then $f = \alpha t^{v(f)}$ with α being a unit, and for a general non-zero element fg^{-1} of the fraction field, one finds $fg^{-1} = \alpha t^{\nu(f)-\nu(g)}$ with α a unit.

The function $v\colon A - \{0\} \to \mathbb{Z}$ sending f to the unique integer such that $f = \alpha t^{v(f)}$ with α a unit, is called the *valuation* associated to A. It resembles the well-known order function from complex analysis (recall that every meromorphic function has an order at a point, positive if its a zero and negative in case of a pole), and it share several of its properties. For instance, the two following identities hold:

- $\nu(fg) = \nu(f) + \nu(g)$;
- $\nu(f+g) \geqslant \min\{\nu(f), \nu(g)\}$,

with equality in the latter when $\nu(f) \neq \nu(g)$. Any function $A - \{0\} \to \mathbb{Z}$ satisfying these two properties is called a *discrete valuation* on A. We sometimes extend this definition to include 0, by assigning $v(0) = \infty$; in that case v is a map from $v : A \to \mathbb{Z} \cup \infty$. We will also sometimes extend the valuation to the whole fraction field $K = K(A)$ by defining $v(a/b) = v(a) - v(b)$.

Given the valuation $v : K \to \mathbb{Z} \cup \infty$, we can recover the valuation ring as the subring of K given by

$$A = \{x \in K^\times | v(x) \geqslant 0\} \cup \{0\}$$

and the maximal ideal is given by

$$\mathfrak{m} = \{x \in K^\times | v(x) \geqslant 1\} \cup \{0\}$$

The group of units in A is given by the subgroup

$$A^\times = \{x \in K | v(x) = 0\}.$$

Note also that for any $x \in K$, either $x \in A$ or $x^{-1} \in A$.

Example A.21. Let $K = k(x)$ be the field of rational functions in one variable. Let $f \in k[x]$ be an irreducible polynomial. Then any element $y \in K$ can be written as $y = f^d g/h$ where $d \in \mathbb{Z}$; and g, h are coprime to f. We can define a valuation $v_f : K^\times \to \mathbb{Z}$ by setting $v(y) = d$. In this case, the valuation ring is the localization of $k[x]$ at f:

$$A = k[x]_{(f)}$$

Example A.22. Let $K = k(x)$ be the field of rational functions in one variable. Define the valuation $v_\infty : K^\times \mathbb{Z}$ by setting

$$v_\infty(\frac{f}{g}) = \deg g - \deg f$$

One can check that this defines a valuation on $k(x)$. The valuation v_∞ is supposed to measure the order of a pole 'at infinity'. The corresponding valuation ring is

$$R = \{f/g \in k(x) | \deg f \leqslant \deg g\}.$$

with maximal ideal $\mathfrak{m} = \{f/g \in k(x) | \deg f < \deg g\}$.

Example A.23. Let $K = \mathbb{Q}$ be the field of rational numbers, and let p be a prime number. Any $y \in \mathbb{Q}$ can be expressed as $y = p^d a/b$ where $d \in \mathbb{Z}$ and a, b are coprime to p. We can define the *p-adic* valuation $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ by setting $v(y) = d$. In this case, the valuation ring is the localization of $\mathbb{Z}$ at (p):

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q} \,|\, \gcd(p, n) = 1 \right\}$$

Exercise A.10.1. Assume that ν is a discrete valuation on a field K. Show that the set $A = \{ x \in K \mid \nu(x) \geqslant 0 \}$ is discrete valuation ring by showing that $\{ x \in K \mid \nu(x) > 0 \}$ is a maximal ideal generated by one element.

A.11 Unique factorization domains

Lemma A.24. Let A be a noetherian domain. Then A is a UFD if and only if every height 1 prime ideal is principal

Proof Suppose that A is a UFD. Let $\mathfrak{p}$ be a height 1 prime ideal. Take $x \in \mathfrak{p}$ non-zero and let $x = x_1 \cdots x_n$ be a factorization into irreducible elements. Since $\mathfrak{p}$ is prime, we must have, say, $x_1 \in \mathfrak{p}$. However, also (x_1) is prime (since A is UFD), so since $\mathfrak{p}$ has height 1, we must have $\mathfrak{p} = (x_1)$.

Conversely, suppose that every height 1 prime is principal. Since A is noetherian, every non-zero non-unit x has a factorization into irreducible elements. It suffices to prove that an irreducible element is prime. Let $(x) \subset \mathfrak{p}$ be a minimal prime over (x). Then $\mathfrak{p}$ has height 1 (localize at $\mathfrak{p}$ and use minimality to see why).

□

A.12 Normal domains

Seidenberg's criterion

The criterion we are about to give seems first to have been published by Seidenberg, so we name it after him. It is closely related to the more famous Serre's R_1-S_2-criterion, but there is a more geometric flavour to it.

The proof does not require much preparation; it relies only on the simple lemma below. Before stating the lemma, let us recall that a ideal quotient $(b : a) = \{ x \mid xa \in (b) \}$ equals the annihilator of the class of a in $A/(b)A$, and a basic result from the theory about primary decomposition in Noetherian rings asserts that each proper annihilator is contained in a maximal annihilator, and in the present case, these are precisely the prime ideals associated to (b). Here comes the lemma:

Lemma A.25. A Noetherian domain A equals the intersection $\bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through the prime ideals associated to principal ideals.

Proof Seeking a contradiction, we assume there is an element ab^{-1} in the fraction field of A that lies in $\bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$, but not in A. The transporter ideal $(b : a) = \{ x \in A \mid xa \in (b) \}$ is a proper ideal since $ab^{-1} \notin A$, and so $(b : a)$ is contained in a maximal transporter; that is, a prime $\mathfrak{p}$ associated to (b). Then $ab^{-1} \in A_{\mathfrak{p}}$ by assumption, and we may write $ab^{-1} = cd^{-1}$ with $c, d \in A$ but with $d \notin \mathfrak{p}$. Hence $ad = bc$, and so $d \in (b : a) \subset \mathfrak{p}$, which is absurd. □

Recall that an ideal is said to *unmixed* if all its associated primes are of the same height. Krull's Hauptidealsatz states that primes minimal over a principal ideal (f) are of height one, so in that case the common height must be one, and moreover, if the ideal is not unmixed, any associated prime of larger height must be embedded.

Theorem A.26 (Seidenberg). Let A be a Noetherian domain. Then A is normal if and only if the two following conditions are are fulfilled:

(i) The local ring $A_{\mathfrak{p}}$ at each height one prime ideal $\mathfrak{p}$ is a DVR;

(ii) Each principal ideal is 'unmixed'; i.e. it has no embedded components.

Proof We begin with observing that A is normal when the two conditions are fulfilled: DVR's are normal and intersections of normal rings are normal, and the second condition combined with Krull's Principal Ideal Theorem ensures that all primes associated to a principal ideal are of height one, and we conclude by the lemma.

As to the other implication: The local rings at height one primes are one dimensional and normal since being integrally closed is a local property, hence they are DVR's. Let then $\mathfrak{p}$ be a prime in A associated to a principal ideal (b). Consider the local ring $A_{\mathfrak{p}}$. Its maximal ideal $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$ persists being associated to $(b)A_{\mathfrak{p}}$, and citing Lemma A.20 above, we conclude that the maximal ideal $\mathfrak{m}$ is principal. Then $A_{\mathfrak{p}}$ is a discrete valuations ring; consequently $\mathfrak{p}$ is of height one, and therefore it can not be embedded. □

The first condition of the criterion, has when fulfilled, the consequence that the $\mathfrak{p}$-primary ideals of height one are rather well understood (at least in principle). The only ones are the symbolic powers $\mathfrak{p}^{(\nu)} = A \cap \mathfrak{p}^{v} A_{\mathfrak{p}}$. Indeed, if $\mathfrak{q}$ is $\mathfrak{p}$-primary, it holds that $\mathfrak{q} = A \cap \mathfrak{q}A_{\mathfrak{p}}$, and $A_{\mathfrak{p}}$ being a DVR, all ideals in $A_{\mathfrak{p}}$ are powers of the maximal ideal.

Secondly, when a principal ideal is unmixed, all the primary components are of height one, and hence in a normal ring the primary decomposition takes the form

$$(f) = \mathfrak{p}_1^{(\nu_1)} \cap \cdots \cap \mathfrak{p}_r^{(\nu_r)}.$$

The exponents ν_i completely determine (f); that is, they determine f up to an invertible factor. In a domain that is not normal, two principal ideal (f) and (g) whose primary decompositions have the same height one part, might have different embedded components, and so f and g would not be related by a unit. If v_i denotes the valuation on fraction field of A attributed to the valuation ring $A_{\mathfrak{p}_i}$, it holds that $\nu_i = v_i(f)$.

Exercises

Exercise A.12.1 (Primary decomposition and quartic space curves)**.** We keep the notation from Example **??**. Let $\mathfrak{p} \subset A$ be the ideal $\mathfrak{p} = (t_0, t_1, t_3)$.

a) Show that $\mathfrak{p}$ is a prime ideal and that $Z(\mathfrak{p}) \subset \mathbb{A}_k^4$ is the line connecting the point $(0, 0, 0, 1)$ to the origin.

b) Show that the symbolic power $\mathfrak{p}^{(4)}$ is given as

$$\mathfrak{p}^{(4)} = (t_0, t_1 t_2^3) = (u^4, u^6 u^{10}).$$

c) Show that (t_0, t_1^3, t_4^3) is $\mathfrak{m}$-primary and that a primary decomposition of t_0 is given as

$$(t_0) = \mathfrak{p}^{(4)} \cap (t_0, t_1^3, t_4^3).$$

Exercise A.12.2 (Serre's R_1 and S_2 conditions)**.** Let A be a Noetherian domain. Show that condition (ii) in Seidenberg's criterion is fulfilled if and only if for each non-unit $f \in A$ the quotient $A/(f)A$ has no associate primes of height two or more. Readers aquatinted with the concept of depth, should recognize this as equivalent to saying that $\operatorname{depth} A_\mathfrak{p} \geqslant \min(2, \dim A_p)$ for all $\mathfrak{p} \in \operatorname{Spec} A$, which is just the S_2 criterion of Serre. The R_1 condition is identical to condition (i) in A.26.

Exercise A.12.3 (Eben Matlis)**.** Let A be a Noetherian domain. Show that every prime ideal in A associated to a principal ideal is of the form $(a : b)$. HINT: Start with a primary decomposition of (f).

Hartog's theorem

The Seidenberg criterion has a corollary important in geometry, which in a geometric parlance loosely says that rational functions on normal varieties can be extended over codimension two subsets; or equivalently, that the loci where they are not defined, are of codimension one. It is commonly referred to as *Hartog's Extension Theorem*, even though it merely is an algebraic reflection of a much deeper result from complex function theory proved by FriedrichHartogs.

Theorem A.27 (Hartogs' extension theorem). A normal Noetherian domain A satisfies $A = \bigcap_\mathfrak{p} A_\mathfrak{p}$ where the intersection extends over all prime ideals $\mathfrak{p}$ of height one.

Proof According to Lemma A.25 tells us that $A = \bigcap_\mathfrak{p} A_\mathfrak{p}$ with $\mathfrak{p}$ running through the primes associated to principal ideal, but according to Seidenberg's criterion, those are precisely the height one primes. □

A.13 Projective modules

Lemma A.28. Let A be a local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated projective A-module. Then M is free.

Proof This is a standard application of Nakayama's lemma. Let $k = A/\mathfrak{m}$ denote the residue field, and consider the module $M \otimes_A k = M/\mathfrak{m}M$. Since M is finitely generated, this is a finite dimensional vector space over k. Let $m_1, \ldots, m_r \in M$ denote a collection of elements in M that map to a basis for $M \otimes_A k$. We obtain a map $\phi : A^r \to M$ sending the standard basis vector e_i to m_i for each $i = 1, \ldots, r$. Note that $\phi \otimes id_k$ is an isomorphism, so by Nakayama's lemma ϕ is surjective. We thus get a short exact sequence

$$0 \to K \to A^r \xrightarrow{\phi} M \to 0,$$

where $K = \operatorname{Ker} \phi$. When M is a projective module, this sequence splits **?**. Hence it stays exact when tensorized by k. Again, since $\phi \otimes id_k$ is an isomorphism, we get that $K \otimes_A k = 0$, and hence $K = 0$, once more by Nakayama's lemma (note that K is finitely generated, being a direct summand of a finitely generated module). It follows that $M \simeq A^r$ is free. □

A.14 Dimension theory

A.14.1 The length of a module

A.14.2 Krull's Principal Ideal Theorem

Theorem A.29 (Krull's Principal Ideal Theorem). Let A be a Noetherian ring and $I = (f_1, \ldots, f_r)$ a proper ideal of A. Then each minimal prime ideal over I has height at most r.

In the special case when A is a finitely generated k-algebra, ..

Appendix B

More on sheaf cohomology

In Chapter **??**, we introduced the Cech cohomology of sheaves, which is well suited for computations, and in an fact is most efficient road (if not the only) to find the explicit necessary results on cohomology. There is however another standard way of introducing cohomology which works in greater generality. It goes by the so-called *derived functors*, in our case the *right derived functors* (there is also the notion of left derived functors).

The idea is to approximate an object A (in any abelian category) by 'cohomologically trivial objects'. Such an approximation, or a *acyclic resolution* as it is called, is an exact complex $(\mathscr{C}^\bullet, d^\bullet)$ with an isomorphism $A \to \operatorname{Ker} d^0$; it displays as

$$0 \to A \to \mathscr{C}^0 \to \mathscr{C}^1 \to \mathscr{C}^2 \to \dots \tag{B.1}$$

and, the key point, the $\mathscr{C}^i$ are 'cohomologically trivial' (we'll come back with what that means in our concrete situation, typically the $\mathscr{C}^i$ will be so-called 'injective' objects). [1][2cm] Then one applies the functor F to $\mathscr{C}^\bullet$ and thus obtains the complex $F(\mathscr{C}^\bullet)$, which displayed appears as

$$F(\mathscr{C}^0) \to F(\mathscr{C}^1) \to F(\mathscr{C}^2) \to \dots$$

The value of the (right) derived functor (or the i-cohomology group) of F at A will be the homology of that complex; that is, for each $i \in \mathbb{N}_0$ one has $R^iF(A) = H^i(F(\mathscr{C}^\bullet))$.

There is a serious issue brought on by the choices involved in this process. The homology $H^i(F(\mathscr{C}^\bullet))$ must be well-defined so it must, in some sense, be independent of the choice of the complex (B.1), and the precise condition is it be unique up to a unique isomorphism. Uniqueness of the isomorphism is required to have the necessary functorial properties, one wants *equalities* between induced maps.

We shall not dive into the deep see of abelian categories and homological algebra, but merely concentrate on our present interest, the global section functor. And we shall circumvent the unicity issues by using so-called *flabby sheaves* as resolving objects; with those there is a completely canonical resolution of any abelian sheaf, which also depends functorially on $\mathcal{F}$.

Part of the story is also to show that the two definitions of cohomology coincide in most situations. In the case of general (separated) schemes this hinges on a theorem of Henri Cartan with a longish proof, which we refrain from giving. We contend ourselves with a proof for Noetherian seperated schemes; then things are considerably much easier.

B.1 Flabby sheaves

Let X be a topological space and $\mathcal{F}$ a sheaf X. One calls $\mathcal{F}$ as *flabby* if the restriction map

$$\mathcal{F}(X) \to \mathcal{F}(U)$$

is surjective for every open subset $U \subset X$. Flabby sheaves are quite different from the coherent sheaves one usually encounters in algebraic geometry, and they tend to be rather large and 'formless'. Here are two prototypical examples:

Example B.1 (Godement sheaves)**.** Back in Chapter **??**, we constructed the Godement

[1] Recall that a complex is exact if the kernel of each map is equals the image of the preceding one; that is $\operatorname{Im} d^i = \operatorname{Ker} d^{i+1}$.

sheaves $\mathcal{A}$. They were constructed by choosing an arbitrary family of abelian groups A_x, one for each point $x \in X$, whose group of sections over an open U is

$$\mathcal{A}(U) = \prod_{x \in U} A_x$$

and whose restriction maps are induced appropriate projections. These sheaves are obviously flabby. Indeed, the restriction map $\mathcal{A}(X) = \prod_{x \in X} A_x \to \prod_{x \in U} A_x = \mathcal{A}(U)$ is just the projection that keeps the components of (a_x) with indices $x \in U$ and throws the others away.

In particular the Godement sheaf $\Pi(\mathcal{F})$ associated to an abelian sheaf F belongs to the class of flabby sheaves; just let the family of abelian groups be the family of stalks $\mathcal{F}_x$.

Example B.2. If $X = \operatorname{Spec} A$ is affine and M is a divisible A-module (that is, all multiplication maps $x \mapsto fx$ with $f \neq 0$ are surjective), then $\widetilde{M}$ is flabby. Indeed, the localization maps $M \to M_f$ are surjective. In particular, this applies to injective modules over an integral domain.

So to the words ‘cohomologically trivial’. Heuristically, the origin of cohomology of sheaves is that taking global section does not preserve surjections, and the next lemma may be view as an indication that flabby sheaves are ‘cohomologically trivial’:

Lemma B.3. Given an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of sheaves $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ on the topological space X. If $\mathcal{F}$ is flabby, the corresponding sequence of global sections

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$$

is exact for every open set $U \subseteq X$.

Proof By restricting $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ to U, it suffices to prove the statement for $U = X$. The global section functor is left exact, so we only need to check that the sequence is exact on the right. Let $\sigma \in \mathcal{H}(X)$. Consider the family Σ of pairs (U, s) of open subsets U of X and sections $s \in G(U)$ that maps to $\sigma|_U$. The set Σ has a partial order for which $(U, s) \leqslant (U', s')$ if $U \subset U'$ and $s = s'|_U$, and it is quite clear that under this ordering every ascending chain in Σ is bounded. So Zorn’s lemma ensures there is a maximal pair (U_0, s_0).

Aiming for a contradiction, assume that U_0 is not the entire space X and pick a point $x \in X - U_0$. Let U_1 be an open neighbourhood of x small enough that $\sigma|_{U_1}$ lifts to a section s_1 in $\mathcal{G}(U_1)$. On the intersection $V = U_0 \cap U_1$ both sections $s_0|_V$ and $s_1|_V$ maps to $\sigma|_V$, and hence their difference $s_0|_V - s_1|_V$ belongs to $\mathcal{F}(V)$. Now $\mathcal{F}$ is flabby, so the difference is the restriction of a section $t \in \mathcal{F}(X)$. Then $s_1 + t|_{U_1}$ maps to $\sigma|_{U_1}$ and coincides with s_0 on V. Hence the two can be glued together to a section of G over $U_0 \cup U_1$ that maps to $\sigma|_{U_0 \cup U_1}$, contradicting the maximality of (U_0, s_0). □

Lemma B.4. Suppose we are given an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

If $\mathcal{F}$ and $\mathcal{G}$ are flabby, then so is $\mathcal{H}$.

Proof Let $U \subset X$ be a subset of X. Then each section $h \in \mathcal{H}(U)$ is represented by a section $g \in \mathcal{G}(U)$ by the previous lemma. Since $\mathcal{G}$ is flabby, g can be extended to a section g' of $\mathcal{G}(X)$. Then g' maps to an element $h' \in \mathcal{H}(X)$ extending h; that is, $h'|_U = h$. □

Lemma B.5. Suppose we are given an exact complex of flabby sheaves

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \cdots \longrightarrow \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \longrightarrow \ldots \tag{B.2}$$

Then for each open set $U \subset X$, the complex

$$0 \longrightarrow \mathcal{F}^0(U) \xrightarrow{d^0(U)} \mathcal{F}_1(U) \xrightarrow{d^1(U)} \cdots \longrightarrow \mathcal{F}^i(U) \xrightarrow{d^i(U)} \mathcal{F}^{i+1}(U) \longrightarrow \ldots \tag{B.3}$$

is exact.

Proof One chops the complex (B.3) up into short exact sequences

$$0 \longrightarrow \operatorname{Im} d^i \longrightarrow \mathcal{F}^{i+1} \longrightarrow \operatorname{Im} d^{i+1} \longrightarrow 0.$$

Bearing the two preceding lemmas in mind, we see by induction that each $\operatorname{Im} d_i$ is flabby (the base of the induction follows as $\operatorname{Im} d^0 = \mathcal{F}_0$ which is flabby by assumption) and that each sequence

$$0 \longrightarrow \operatorname{Im} d^i(U) \longrightarrow \mathcal{F}^{i+1}(U) \longrightarrow \operatorname{Im} d^{i+1}(U) \longrightarrow 0$$

is exact. □

B.2 The Godement resolution

Given a sheaf $\mathcal{F}$ on a topological space X, we are about to construct a resolution of $\mathcal{F}$ in terms of flabby sheaves which we shall use to define the cohomology of $\mathcal{F}$. There are no choices involved, so the construction is canonical, and moreover it has the virtue of being functorial (in every conceivable way) so we get unambiguously defined cohomology groups, and all their functorial properties come almost for free.

To explain how this works, recall the Godement sheaf $\Pi(\mathcal{F})$ with sections over U being

$$\Pi(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x$$

and restriction maps the appropriate projection, and the canonical inclusion $\kappa_{\mathcal{F}} : \mathcal{F} \to \Pi(F)$ which over an open set U sends a section $s \in \mathcal{F}(U)$ to the element $(s_x)_{x \in U}$. Defining $\mathscr{C}^0\mathcal{F} = \Pi(()\mathcal{F})$ and $\mathscr{Z}^1\mathcal{F}$ as the cokernel of κ, we get a canonical exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathscr{C}^0\mathcal{F} \longrightarrow \mathscr{Z}^1\mathcal{F} \longrightarrow 0.$$

Remember that Π is a functor $\mathsf{AbSh}_X \to \mathsf{AbSh}_X$ which is compatible with κ; that is, it holds true that $\Pi(\alpha) \circ \kappa_{\mathcal{F}} = \kappa_{\mathcal{G}} \circ \Pi(\alpha)$ for each map $\alpha\colon \mathcal{F} \to \mathcal{G}$. Thus $\Pi(\alpha)$ passes to the quotient, and we have commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathscr{C}^0(\mathcal{F}) & \longrightarrow & \mathscr{Z}^1\mathcal{F} & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathscr{C}^0\alpha} & & \downarrow{\scriptstyle\mathscr{Z}^1\alpha} & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathscr{C}^0(\mathcal{G}) & \longrightarrow & \mathscr{Z}^1\mathcal{G} & \longrightarrow & 0. \end{array}$$

This makes $\mathscr{Z}^1$ a functor.

The Godement functor $\Pi(\mathcal{F})$ is even an exact functor. This hinges on the fundamental quality that being exact is a local property of sequences of sheaves; so if the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is exact, the sequence of sections over an open U

$$0 \longrightarrow \prod_{x\in U} \mathcal{F}_x \longrightarrow \prod_{x\in U} \mathcal{G} \longrightarrow \prod_{x\in U} \mathcal{H} \longrightarrow 0$$

is exact for all U; indeed, it is obtain by taking the product (which preserves exactness) of the stalk-wise sequences (which are exact). The snake lemma then shows that also $\mathcal{Z}^1$ is an exact functor.

We now iterate this construction and recursively put $\mathscr{C}^{i+1}\mathcal{F} = \mathscr{C}^0\mathscr{Z}^i\mathcal{F}$ and $\mathscr{Z}^{i+1}\mathcal{F} = \mathscr{Z}^1\mathscr{Z}^i\mathcal{F}$. These sheaves all fit into short exact sequences, one for each i, shaped like

$$0 \longrightarrow \mathscr{Z}^i\mathcal{F} \longrightarrow \mathscr{C}^i\mathcal{F} \longrightarrow \mathscr{Z}^{i+1} \longrightarrow 0.$$

Proceeding to assemble the Godement resolution we splice these sequences together to a complex $\mathscr{C}^\bullet\mathcal{F}$. The sheaves in this complex will of course be the $\mathscr{C}^i\mathcal{F}$'s, and the differentials $d^i\colon \mathscr{C}^i \to \mathscr{C}^{i+1}$ will be the compositions $\mathscr{C}^i \to \mathscr{Z}^i \to \mathscr{C}^{i+1}$:

Proposition B.6. Given a topological space X and an abelian sheaf $\mathcal{F}$ on X.

(i) The Godement complex $\mathscr{C}^\bullet\mathcal{F}$ is a flabby resolution of $\mathcal{F}$.

(ii) The complex $\mathscr{C}^\bullet$ depends functorially on $\mathcal{F}$, and the functor $\mathscr{C}^\bullet\colon \mathsf{ShAb}_X \to \mathsf{CpxShAb}_X$ is an exact functor.

Proof Most has already been done. By construction the sheaves $\mathscr{C}^i\mathcal{F}$ are flabby and $\mathscr{C}^\bullet$ is exact in positive degrees. For $i = 0$ it holds, also by construction, that $\operatorname{Ker} d^0 \simeq \mathcal{F}$. This takes care of (i).

Claim (ii) is an immediate consequence of $\mathscr{C}^i$ and $\mathscr{Z}^i$ being exact functors. □

B.3 Sheaf cohomology

We are now ready for defining the cohomology of an abelian sheaf $\mathcal{F}$. The procedure is: first form the Godement resolution

$$\mathscr{C}^\bullet\mathcal{F}: \qquad \mathscr{C}^0\mathcal{F} \to \mathscr{C}^1\mathcal{F} \to \mathscr{C}^2\mathcal{F} \to \ldots \tag{B.4}$$

then take global section of $\mathscr{C}^\bullet\mathcal{F}$, which yields a complex of abelian groups

$$\mathscr{C}^\bullet\mathcal{F}(X): \qquad \mathscr{C}^0\mathcal{F}(X) \to \mathscr{C}^1\mathcal{F}(X) \to \mathscr{C}^2\mathcal{F}(X) \to \dots \tag{B.5}$$

and finally, take the homology of that complex:

Definition B.7. Let $\mathcal{F}$ be an abelian sheaf on the topological space X. We define the i-th cohomology group $H^i(X,\mathcal{F})$ by the formula

$$H^i(X,\mathcal{F}) = H^i(\mathscr{C}^\bullet\mathcal{F}(X)).$$

For each map $\alpha\colon \mathcal{F} \to \mathcal{G}$ we define $H^i(X,\alpha)\colon H^i(X,\mathcal{F}) \to H^i(X,\mathcal{G})$ by the formula

$$H^i(X,\alpha) = H^i(\mathscr{C}^\bullet\alpha(X)).$$

The notation $H^i(X,\alpha)$ is exceptionally cumbersome and one usually abbreviats it to α^i_* or sometimes even to α_* with the index i being tacitly understood. The cohomology is a functor in that $(\alpha\circ\beta)_* = \alpha_*\circ\beta_*$ whenever α and β are composable maps between abelian sheaf and of course $\mathrm{id}_* = \mathrm{id}$.

Recall that any short exact sequence of complexes of groups induces a long exact sequence in homology. And for any functor to have the right to bear the title 'a cohomology theory' an absolute requirement is similarly to induce long exact sequences from short ones:

Proposition B.8 (Long exact sequence). With each short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

of abelian sheaf on the topological space X and each non-negative integer i is associated a connecting map $\delta\colon H^i(X,\mathcal{F}) \to H^{i+1}(X,\mathcal{H})$ so that the long sequence

$$\dots \longrightarrow H^i(X,\mathcal{G}) \xrightarrow{\beta_*} H^i(X,\mathcal{H}) \xrightarrow{\delta} H^{i+1}(X,\mathcal{F}) \xrightarrow{\alpha_*} H^{i+1}(X,\mathcal{G}) \longrightarrow \dots$$

is exact. Moreover, the connecting map δ depends functorially on the sequence.

Again, including the dependence on the sequence and on i in in the notation δ would make it unnecessarily cluttered; but of course, when needed any appropriate decoration is possible. That δ depends functorially on the sequence means that for any map between two exact sequence, that is a set up like

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{H}' & \longrightarrow & 0 \end{array}$$

with squares commuting, it holds true that $\alpha^{i+1}_*\circ\delta = \delta\circ\gamma^i_*$; or for lovers of diagrams, that for all i the middle red square in the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & H^i(X,\mathcal{G}) & \longrightarrow & H^i(X;\mathcal{H}) & \xrightarrow{\delta} & H^{i+1}(X,\mathcal{F}) & \longrightarrow & H^{i+1}(X,\mathcal{G}) & \longrightarrow \ldots \\ & \beta^i_*\downarrow & & \gamma^i_*\downarrow & & \downarrow\alpha^{i+1}_* & & \downarrow\beta^{i+1}_* & \\ \ldots \longrightarrow & H^i(X,\mathcal{G}') & \longrightarrow & H^i(X,\mathcal{H}') & \xrightarrow[\delta]{} & H^{i+1}(X,\mathcal{F}') & \longrightarrow & H^{i+1}(X,\mathcal{G}') & \longrightarrow \ldots \end{array}$$

The other squares commute as well simply because the cohomology $H^\bullet(X,\mathcal{F})$ is functorial in $\mathcal{F}$.

Proof Proposition B.6 tells us that the sequence

$$0 \longrightarrow \mathscr{C}^\bullet\mathcal{F} \longrightarrow \mathscr{C}^\bullet\mathcal{G} \longrightarrow \mathscr{C}^\bullet\mathcal{H} \longrightarrow 0$$

formed from (B.8) is an exact sequence of complexes. In each degree there is an exact sequence of sheaves which is exact and consists of flabby sheaves, and by Lemma B.3 it follows that it persists being exact after global sections are taken. But that means precisely that the complex

$$0 \longrightarrow \mathscr{C}^\bullet\mathcal{F}(X) \longrightarrow \mathscr{C}^\bullet\mathcal{G}(X) \longrightarrow \mathscr{C}^\bullet\mathcal{H}(X) \longrightarrow 0$$

of abelian groups is exact, and taking homology yields a long exact sequence of homology groups (see Chapter **??**).

The second statement about functoriality follows from the corresponding property of complexes of abelian groups since both $\mathscr{C}^\bullet$ and $\Gamma(X,-)$ are functors. □

Proposition B.9. If $\mathcal{F}$ is flabby all higher cohomology of $\mathcal{F}$ vanish; i.e. $H^i(X,\mathcal{F}) = 0$ for $i \geqslant 1$.

Example B.10. Flabby resolutions furnish good tools for establishing general formal statements, but in concrete situations they are usually rather difficult to study in an explicit manner. There are however a few exceptions, and here comes one:

Let $X = \operatorname{Spec} A$ be a reduced and irreducible affine scheme; that is, the ring A is an integral domain. The field of fractions K of A induces the sheaf $\widetilde{K}$ on X, and since K is divisible, this sheaf is flabby. One effortlessly checks that also the quotient K/A is divisible, hence $\widetilde{K/A}$ is flabby, and we have the flabby resolution

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \widetilde{K} \longrightarrow \widetilde{K/A} \longrightarrow 0.$$

It follows using acyclicity of flabby sheaves and the long exact sequence that $H^i(X,\mathcal{O}_x) = 0$ for $i > 1$, and $H^1(X,\mathcal{O}_X) = 0$ since the global section of the map $\widetilde{K} \to \widetilde{K/A}$ is just the surjection $K \to K/A$.

Example B.11. Let X be an integral scheme, In Chapter **??** we defined the group of $CaDiv$ as $\Gamma(X,\mathscr{K}^*/\mathcal{O}_X^*)$ where $\mathscr{K}_X$ is the constant sheaf with value the function field $K(X)$ of X, and Cartier divisor class group as the cokernel of the map $K \to \Gamma(X,\mathscr{K}^*/\mathcal{O}_X^*)$ induced from the exact sequence

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathscr{K}_X^* \longrightarrow \mathscr{K}_X^*/\mathcal{O}_X^* \longrightarrow 1$$

We saw that $\mathscr{K}_X^*$ is a flabby and hence it follows that $CaCl(X) \simeq H^1(X, \mathcal{O}_X^*) = \operatorname{Pic}(X)$.

Exercise B.3.1. Let X be a topological space and let $\iota\colon Z \to X$ be the inclusion of a subset. Show that for a sheaf $\mathcal{F}$ on Z,

$$H^i(X, i_*\mathcal{F}) = H^i(Z, \mathcal{F}) \tag{B.6}$$

for all i.

B.3.1 Acyclic sheaves

Since the Godement resolution often is difficult to handle and the involved sheaves are both rather enormous and structureless, one looks for other and more workable resolutions. The following proposition, where resolutions by so-called acyclic sheaves are used, gives this flexibility.

Definition B.12. A sheaf $\mathcal{F}$ on the topological space X is called *acyclic* if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Consider a resolution $\mathscr{C}^\bullet$ of $\mathcal{F}$,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathscr{C}^0 \longrightarrow \mathscr{C}^1 \longrightarrow \mathscr{C}^2 \longrightarrow \dots ,$$

(which by definition means that the sequence is exact), and the resulting complex $\mathscr{C}^\bullet(X)$ of abelian groups

$$\mathscr{C}^0(X) \longrightarrow \mathscr{C}^1(X) \longrightarrow \mathscr{C}^2(X) \longrightarrow \dots$$

Of course, this may in general fail to be exact. Our main goal now is to show that if the $\mathcal{C}_i$'s are acyclic, we get back the cohomology of $\mathcal{F}$:

Lemma B.13. If the sheaves $\mathscr{C}^i$ in (B.3.1) are acyclic, then there is a natural isomorphism

$$H^i(X, \mathcal{F}) \simeq H^i(\mathscr{C}^\bullet(X))$$

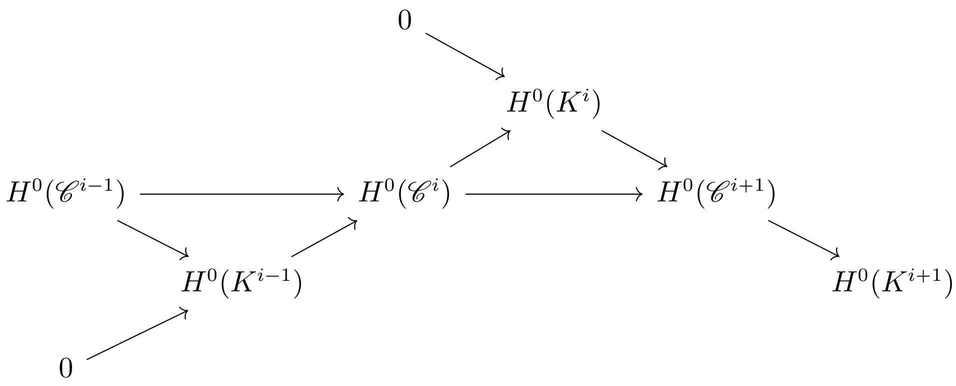
Proof Define $K^{-1} = \mathcal{F}$, and $K^i = \operatorname{Ker}(\mathscr{C}^{i+1} \to \mathscr{C}^{i+2})$ for $i \geqslant 0$. By exactness of the complex $\mathscr{C}^\bullet$, we have for each $i \geqslant 0$ an exact sequence

$$0 \to K^{i-1} \to \mathscr{C}^i \to K^i \to 0.$$

Taking the long exact sequence, we get

$$0 \to H^0(K^{i-1}) \to H^0(\mathscr{C}^i) \to H^0(K^i) \to H^1(K^{i-1}) \to H^1(\mathscr{C}^i) = 0 \tag{B.7}$$

where the right-most group it zero because $\mathscr{C}^i$ is acyclic. Also, the same sequence shows that $H^p(K^i) = H^{p+1}(K^{i-1})$ for every $p \geqslant 1$. The maps in these sequences fit into the diagram



From this, we see that

$$\operatorname{Im}\left(H^0(\mathscr{C}^i) \to H^0(K^i)\right) = \operatorname{Im}\left(H^0(\mathscr{C}^i) \to H^0(\mathscr{C}^{i+1})\right)$$

and that

$$H^0(K^{i-1}) = \operatorname{Ker}\left(H^0(\mathscr{C}^i) \to H^0(\mathscr{C}^{i+1})\right).$$

In particular,

$$H^0(\mathcal{F}) = \operatorname{Ker}\left(H^0(\mathscr{C}^0) \to H^0(\mathscr{C}^1)\right) = H^0(\mathscr{C}^\bullet(X)),$$

and the theorem holds in degree $i = 0$. By the same token, we have

$$H^0(K^i) = \operatorname{Ker}\left(H^0(\mathscr{C}^{i+1}) \to H^0(\mathscr{C}^{i+2})\right).$$

From (B.7), and the isomorphisms $H^p(K^i) \simeq H^{p+1}(K^{i-1})$ we therefore get

$$\begin{aligned}
H^{i+1}(\mathscr{C}^\bullet(X)) &= \operatorname{Ker}\left(H^0(\mathscr{C}^{i+1}) \to H^0(\mathscr{C}^{i+2})\right)/\operatorname{Im}\left(H^0(\mathscr{C}^i) \to H^0(\mathscr{C}^{i+1})\right) && \text{(B.8)}\\
&= H^0(K^i)/\operatorname{Im}\left(H^0(\mathscr{C}^i) \to H^0(K^i)\right) && \text{(B.9)}\\
&= H^1(K^{i-1}) && \text{(B.10)}\\
&= H^2(K^{i-2}) && \text{(B.11)}\\
&= \cdots && \text{(B.12)}\\
&= H^{i+1}(K^{-1}) && \text{(B.13)}\\
&= H^{i+1}(\mathcal{F}).
\end{aligned}$$

□

B.4 Cech vs sheaf cohomology

We have introduced two definitions of sheaf cohomology, one given by the Godement resolution and then Cech cohomology. In this section we shall show that they coincide for quasi-coherent sheaves on Noetherian separated schemes. The Noetherian hypothesis is in fact not necessary, but disposing of it requires a rather long proved result of Henri Cartan, which we will merely state for reference.

Note the important point that in Leray's theorem, only uses a fixed cover - this is indispensable when it comes to concrete calculations.

The tactic in the proof of Leray's theorem is to first exhibit a resolution of the sheaf in question by making a complex of sheaves out of the Cech resolution associated with an affine cover. Then we verify the salient point that these Cech sheaves will be acyclic, once we know that affine schemes are cohomologically trivial.

B.4.1 The Cech resolution

We start by introducing the sheafy version of the Cech complex. The setting is a scheme X with a quasi-coherent sheaf $\mathcal{F}$. We are further given an finite open affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, and if $\alpha = (i_0, \ldots, i_q)$ is a sequence of indices from I we use the notation $U_\alpha = U_{i_0\ldots i_q} = U_{i_0} \cap \cdots \cap U_{i_q}$. These are all affine because X is separated. Moreover, we let $\iota_\alpha \colon U_\alpha \to X$ be the open inclusion of U_α into X.

The covering $\mathcal{U} = \{U_i\}_{i\in I}$ induces a covering $U_V = \{U_i \cap V\}_{i\in I}$ of each open subset V in X, and with it is associated a Cech complex $\mathscr{C}^\bullet(U_V, \mathcal{F}|_V)$ as in Section 17.2 on page 283. Furthermore there are for each open subset $V' \subset V$ obvious restriction maps $\mathscr{C}^p(U_V, \mathcal{F}|_V) \to \mathscr{C}^p(U_{V'}, \mathcal{F}|_V)$ (they are simple cases of the refinment maps described in (**??**)), and these make each $\mathscr{C}^p(U_V, \mathcal{F}|_V)$ a sheaf; which we shall denote by $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$. The sections over an open V are given as

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})(V) = \prod_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(V \cap U_{i_0} \cap \cdots \cap U_{i_p})$$

and with a few moments of reflection, one convinces oneself that this means that

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod_{\alpha=(i_0,\ldots,i_p)\in I^{p+1}} \iota_{\alpha *}\mathcal{F}|_{U_{i_0}\cap\cdots\cap U_{i_p}}.$$

The restrictions of the sheaves $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ are compatible with the coboundary maps of the Cech complexes, and hence we obtain a complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of sheaves. The sheaf version of the formula given in Chapter **??** for the coboundary map reads

$$(d\sigma)_{i_0\ldots i_p} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0\ldots\hat{i_j}\ldots i_p}|_{V\cap U_{i_0}\cap\ldots U_{i_p}}$$

where σ is a section in $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})(V)$.

Lemma B.14. This gives a resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^2} .. \qquad \text{(B.14)}$$

Moreover, $\Gamma(X, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})) = \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$.

Proof The second statement and that B.14 is a complex, follow from the the definition of $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$.

So the main content is that — contrary to the ordinary Cech complex — the sheafy version of the Cech complex is exact. Since this is a sequence of sheaves, we may check exactness on on stalks.

The proof consists of writing down a *homotopy operator* on the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)_x$ of stalks at a point $x \in X$. For a general complex $\mathscr{C}^\bullet$ with differential d, a homotopy operator is a map $k\colon \mathscr{C}^\bullet \to \mathscr{C}^\bullet$ of degree -1 (that is, a bunch of maps $k^p\colon \mathscr{C}^p \to \mathscr{C}^{p-1}$, one for each $p > 0$ so that $kd + dk = \mathrm{id}_{\mathscr{C}^\bullet}$). Having such a homotopy operator forces the complex to be exact in positive degrees; indeed, if $d\sigma = 0$, one has $\sigma = dh\sigma + kd\sigma = dk\sigma$, and so σ is a coboundary.

We are about to define a map $k^p\colon \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})_x \to \check{\mathcal{C}}^{p-1}(\mathcal{U}, \mathcal{F})_x$: An element in the stalk $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})_x$ is induced from a section (σ_α) over an open neighbourhood V of x, and we can assume that some $r \in I$ it holds that $V \subset U_r$ (just shrink V if needed). Then $V \cap U_{i_0 \ldots i_{p-1}} \subset V \cap U_{ri_0 \ldots i_{p-1}}$, and there is a restriction map

$$\rho\colon \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})(V \cap U_{ri_0 \ldots i_{p-1}}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})(V \cap U_{i_0 \ldots i_{p-1}})$$

which allows us to define

$$(k^p\sigma)_{i_0 \ldots i_{p-1}} = \rho(\sigma_{ri_0 \ldots i_{p-1}}).$$

Now the crux is that k is a homotopy operator on the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})_x$ of stalks; that is,

$$dk + kd = \mathrm{id}.$$

Establishing this is just a matter of writing down the definitions: on the one hand we have

$$\begin{aligned}(dk\sigma)_{i_0 \ldots i_p} &= \sum_{j=0}^{p} (-1)^j (k\sigma)_{i_0 \ldots \hat{i_j} \ldots i_p} = \\ &= \sum_{j=0}^{p} (-1)^j \sigma_{ri_0 \ldots \hat{i_j} \ldots i_p},\end{aligned}$$

and on the other hand

$$\begin{aligned}(kd\sigma)_{i_0 \ldots i_p} &= d\sigma_{ri_0 \ldots i_p} = \\ &= \sigma_{i_0 \ldots i_p} + \sum_{j=0}^{p} (-1)^{j+1} \sigma_{ri_0 \ldots \hat{i_j} \ldots i_p}),\end{aligned}$$

and adding the two yields the formula. □

Theorem B.15 (Leray). Assume that X is a topological space with a sheaf $\mathcal{F}$ on X and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. If all sheaf cohomology groups $H^p(U_{i_0} \cap \cdots \cap U_{i_q}, \mathcal{F}) = 0$ for all $p > 0$ and all sequences (i_j) of indices, then the Cech cohomology and the sheaf cohomology of $\mathcal{F}$ coincide; more precisely, $\check{H}(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$.

There two comments to make: firstly, the conclusion is that actually Cech cohomology of just one covering gives the sheaf cohomology, a property important for the computations. Secondly, we underline that it is the *sheaf cohomology* $H^p(U_{i_0} \cap \ldots U_{i_q}, \mathcal{F})$ that is supposed to vanish as opposed to the Cech cohomology. As mentioned in the introduction, there is a related result:

Theorem B.16 (Cartan). Let X be a topological space and $\mathcal{F}$ an abelian sheaf on X. If there is a set of open subsets $\mathcal{B}$, forming a basis for the topology and being closed under finite intersections, and is such that $\check{H}^p(U, F) = 0$ for all $U \in \mathcal{B}$ and all $p > 0$, then Cech and sheaf cohomology of F coincide.

Proof The sheaves $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ will be acyclic and we can activate Lemma B.13 on page 436. Indeed, in view of Exercise B.3.1, this ensues from the expression in (B.4.1) for the Cech complex. □

The affine case

Theorem B.17. Assume that $X = \operatorname{Spec} A$ is Noetherian affine scheme and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

The condition that $\mathcal{F}$ be quasi-coherent is essential. For instance, as we observed in Example B.11 in good cases one has $\operatorname{Pic} X \simeq H^1(X, \mathcal{O}_X^*)$, and rather many affine schemes have a non trivial divisor class group. Examples can be $\operatorname{Spec} A$ for any Dedekind ring that is not a UFD (e.g. any affine elliptic curve).

As mentioned above, the result holds true without the Noetherian hypothesis (see EGA III 1.3.1??):

Corollary B.18. Let X be a Noetherian scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module.

(i) If X is separated, sheaf- and Cech cohomology on X agree: it holds that $\check{H}^i(X, \mathcal{F}) \simeq H(X, \mathcal{F})$ for all $i > 0$;

(ii) If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open affine covering so that any finite intersection $U_{i_1} \cap \cdots \cap U_{i_r}$ of members of $\mathcal{U}$ is affine, then $H^i(X, \mathcal{F}) = \check{H}^i(X, \mathcal{F}) = H^1(U, \mathcal{F})$.

If there is a covering U_i of affines closed under finite intersections, the result still is true (and the proof still holds water.).

Example B.19. Glue the spectrum $X = \operatorname{Spec} A$ of a DVR A to it self at the generic point η. Then X is covered by two open affine subsets $U_i = \operatorname{Spec} A$ whose intersection is the open affine $\{\eta\} = \operatorname{Spec} K$. Sheaf- and Cech cohomology coincide, and to compute $H^i(X, \mathcal{O}_X)$ we have the sequence Cech complex.

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow A \times A \xrightarrow{\alpha} K \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

where $\alpha(a, b) = a - b$. Thus $H^0(X, \mathcal{O}_X) = A$, and $H^1(X, \mathcal{O}_X) = K/A$. This is a rather large module. For instance, in case $A = \mathbb{Z}_p$ for a prime p, it equals the group $\mathbb{Z}_{p^\infty}$ of roots of unity a power of p.

There are three parts, in the first we establish the theorem for the special case of the structure sheaf $\mathcal{F} = \mathcal{O}_X$ of an integral scheme, subsequently for coherent sheaves and finally reduce it to that case.

To begin with we assume X integral and $\mathcal{F} = \mathcal{O}_X = \tilde{A}$. If K is the fraction field of A, the sheaf $\mathcal{K} = \tilde{K}$ is constant and therefore flabby. One easily see that the quotient K/A is divisible that sheaf is flabby as well, and we have the flabby resolution

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O}_X \longrightarrow 0$$

of $\mathcal{O}_X$. The globale sections of $\mathcal{K} \to \mathcal{K}/\mathcal{O}_X$ is surjective so $H^1(X, \mathcal{O}_X) = 0$, and the long exact sequence yields $H^i(X, \mathcal{O}_X) = H^{i-1}(X, \mathcal{K}/\mathcal{O}_X) = 0$.

Secondly we consider any coherent sheaf $\mathcal{F}$ and write $\mathcal{F} = \widetilde{M}$ with M a finitely generated A-module (A is Noetherian). A result from commutative algebra (xxxx) tells us there is a descending sequence $\{M_j\}$ of submodules of M such that each subquotient $M_{j-1}/M_j = A/\mathfrak{p}_j$ with $\mathfrak{p}_j$'s being pime ideals. Hence

$$0 \longrightarrow \mathcal{F}_j \longrightarrow \mathcal{F}_{j-1} \longrightarrow \mathcal{O}_{\mathcal{X}_|} \longrightarrow 0$$

where $\mathcal{F}_i = \tilde{M}_i$ and $X_i = V(\mathfrak{p}_i)$. Induction on i and the first point above yields that $H^i(X, \mathcal{F}_i) = 0$ for all i and j, and this establishes the theorem for coherent modules.

Finally we treat the case that $\mathcal{F}$ is quasi-coherent, and to reduce the proof to the previous case, we write M as the union $\bigcup_j M_j = M$ of its finitely generated submodules M_j.

Quite generally, if $\mathcal{F}$ is the union of a bunch of subsheaves $\{\mathcal{F}_i\}$, one readily verifies that the Godement resolution $\Pi(\mathcal{F})$ is the union of the $\Pi(F_i)$'s $\Pi(F_i)/F_i$ (the sections of $\Pi^\bullet(F)$ over opens are just products of stalks, and forming stalks is an exact operation). Hence the Godement resolution $\Pi^\bullet(F)$ has sub complexes $\Pi(F_i)$ such that each $\Pi^j(F) = \bigcup_i \Pi^j(F_j)$.

In our case, the subsheaves $F_i = \tilde{M}_j$ are coherent and each a $\Pi^\bullet(F_i)$ is exact by the second point above, and so we may finish the proof by the following little observation:

Lemma B.20. If $(C\bullet, d)$ is a complex with subcomplexes $(C_j^\bullet, d_j)$ and each $\mathscr{C}^i = \bigcup_j \mathscr{C}_j^i$ is exact for $i > 0$, then $\mathscr{C}^\bullet$ is exact for $i > 0$.

Proof Indeed, if x is a cocycle in $\mathscr{C}^\bullet$, that is $dx = 0$. For some index j the element x belongs to $C_j^\bullet$ and persists being cocycle, so because $\mathscr{C}_j^\bullet$ is exact, it is coboubdary $dy = d_j y = x$. □

Proposition B.21. If $\mathcal{F}$ is flasque, then so are the sheaves $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ for $p > 0$. Hence (**??**) is an acyclic resolution for $\mathcal{F}$ and

$$H^p(X, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$$

Proof If $\mathcal{F}$ is flasque, then so is each restriction to each $U_{i_0 \cap \cdots \cap U_p}$, and products of flasque sheaves are flasque, so $\prod_{i_0 < \cdots < i_p} i_* \mathcal{F}|_{U_{i_0 \cap \cdots \cap U_p}}$ is flasque. □

B.5 Godement vs. Cech

It remains to see why these two definitions are equivalent. So let $\mathcal{U} = \{U_i\}$ be a covering for $\mathcal{F}$. We will assume that this is *Leray* in the sense that $H^i(U_I, \mathcal{F}) = 0$ for all multi-indices I

and $i > 0$. We claim that there is a natural isomorphism

$$H^i(X, \mathcal{F}) \simeq \check{H}^i(\mathcal{U}, \mathcal{F}),$$

where we, in order to avoid confusion, let $\check{H}^i(\mathcal{U}, \mathcal{F})$ denote the Cech cohomology group. The statement is clearly true for $i = 0$, since both coincide with $\Gamma(X, \mathcal{F})$.

Lemma B.22. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence. If $\mathcal{U}$ is Leray, there is a long exact sequence

$$0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{G}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{H}) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{H}) \longrightarrow \cdots$$

Proof Since $\mathcal{U}$ is Leray, we have $H^1(U_I, \mathcal{F}) = 0$ for all multi-indexes I (in fact, this is the only property we need from the covering $\mathcal{U}$). Hence the following sequences are exact:

$$0 \to \mathcal{F}(U_I) \to \mathcal{G}(U_I) \to \mathcal{H}(U_I) \to 0$$

Then applying the Cech complex, we get an exact sequence of complexes

$$0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{G}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

Now the claim follows from Proposition 17.1. □

Hence we get our desired theorem:

Theorem B.23 (Leray). Suppose $\mathcal{U}$ is a cover of X and $H^q(U_1 \cap \cdots \cap U_p, \mathscr{F}) = 0$ for all $p, q > 0$ and all $U_1, \ldots, U_p \in \mathcal{U}$. Then there is a natural isomorphism between cohomology and Čech cohomology:

$$H^p(X, \mathcal{F}) \simeq \check{H}^p(\mathcal{U}, \mathcal{F})$$

Proof We use induction on p. For $p = 0$, the claim is clear. Note that we have the exact sequence

$$0 \to \mathcal{F} \to \Pi(()\mathcal{F}) \to \mathscr{Z}^1 \to 0$$

and $H^1(X, \mathcal{F}) = \operatorname{Coker}(\Gamma(X, \Pi(()\mathcal{F})) \to \Gamma(X, \mathscr{Z}^1\mathcal{F}))$, and

$$H^p(X, \mathcal{F}) = H^{p-1}(X, \mathscr{Z}^1\mathcal{F})$$

for $p \geqslant 2$. On the other hand, we also have the corresponding result for Cech cohomology:

$$0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, \Pi(()\mathcal{F})) \longrightarrow \check{H}^0(\mathcal{U}, \mathscr{Z}^1) \longrightarrow \check{H}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^1(\mathcal{U}, \Pi(()\mathcal{F})) = 0$$

where $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ by Lemma **??**, since $\Pi(()\mathcal{F})$ is acyclic. Hence also

$$\check{H}^1(X, \mathcal{F}) = \operatorname{Coker}(\Gamma(X, \Pi(()\mathcal{F})) \to \Gamma(X, \mathscr{Z}^1\mathcal{F})) = H^1(X, \mathcal{F})$$

Hence the theorem also holds for $p = 1$.

We continue by induction on p. Since also $\check{H}^i(\mathcal{U}, \Pi(()F)) = 0$ for all $i > 0$, same long exact sequence of Cech cohomology also shows that $\check{H}^p(\mathcal{U}, \mathcal{F}) = \check{H}^{p-1}(\mathcal{U}, \mathscr{Z}^1)$. Moreover, the cover $\mathcal{U}$ is also Leray with respect to $\mathscr{Z}^1$: $H^i(\mathcal{U}_I, \mathscr{Z}^1) = H^{i+1}(\mathcal{U}_I, \mathcal{F}) = 0$). Hence replacing $\mathcal{F}$ with $\mathscr{Z}^1$, we get the desired conclusion. □