Introduction to Schemes

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Introduction

0.1 Introduction to the schemes formalism

If $X$ is an affine variety over an algebraically closed field $k$, then it has an affine coordinate ring $A(X)$ of regular functions $X \rightarrow k$, and $A(X)$ completely characterises $X$ up to isomorphism. This sets up a bijective correspondence between algebra and geometry: affine varieties correspond to finitely generated $k$-algebras without zero-divisors, and morphism between varieties corresponds to $k$-algebra homomorphisms. The modern formulation of algebraic geometry, due to Grothendieck, vastly generalizes this picture, replacing the $k$-algebras without zero-divisors with the more general category of commutative rings.

This has the tremendous advantage that many of the arguments and intuition carry over from the classical setting, and can be used to study more general problems in other branches of mathematics. For instance, studying diophantine equations over non-closed fields (such as $\mathbb{Q}$) is clearly of interest in number theory.

We should emphasise that the main goal of algebraic geometry is still to understand algebraic varieties over a field. However, enlarging our category to schemes offers a tremendous flexibility. Let us give a few examples of this.

0.1.1 Nilpotents and zero-divisors

Nilpotents and zero-divisors appear already in Bezout’s theorem: The conic $C = Z(y - x^2)$ intersects a general line in two points, but the line $y = 0$ only at the origin (with ‘multiplicity’ 2). Looking at the ideals, it is intuitive that the intersection should correspond to the ideal $(y - x^2, y) = (x^2, y)$ which gives us the intersection number (‘multiplicity 2’), but that ideal does not correspond to an algebraic variety, as it is not radical.

Similar phenomena appear naturally when you take ‘limits’ (or ‘deformations’)


of algebraic varieties. As a basic example, consider the family of curves in $\mathbb{A}^2$ given by $V_t = Z(y^2 - tx)$. Then for $t \neq 0$, $V_t$ is a smooth conic, whereas for $t = 0$ it is a ‘double line’. The limit as $t \to 0$ is problematic; if you insist that it should correspond to the affine variety $Z(y)$, which has degree 1, then the degree of $V_t$ is not a continuous function in $t$.

These problems vanish when passing to schemes: it is completely unproblematic to work with geometric objects (affine schemes) associated to the coordinate rings $k[x, y]/(x^2, y)$ or $k[x, y]/(y^2)$ respectively. Note however that we have to adjust our viewpoint about regular functions: For instance, in the second example the element ‘$y$’ is non-zero in the ring $k[x, y]/(y^2)$ - but it takes the value 0 at each of the closed points of the underlying topological space (the $x$-axis).

0.1.2 Gluing

Just like in differential geometry, we do not like to consider manifolds as embedded in some $\mathbb{R}^n$ (although Whitney’s theorem says that such an embedding exists). It is much more fruitful to study manifolds $M$ locally, using how it is glued together by a bunch of open subsets in $\mathbb{R}^n$. For instance, the concept of a tangent vector of $M$ at a point $p$ can be defined using equivalence classes of locally parameterized curves passing through $p$; gluing these sets together for each $p$, we obtain the tangent bundle of $M$.

In a similar fashion, we will see that we get a lot of flexibility by talking about varieties or schemes as glued objects rather than embedded objects in some $\mathbb{A}^n$ or $\mathbb{P}^n$. We can define the normalization of a variety by normalizing each chart; we can talk about tangent and cotangent bundles by gluing together local trivializations.

Moreover, by gluing together affine schemes we obtain much more interesting spaces which may indeed fail to be embeddable in any $\mathbb{P}^n$. And a posteri non-quasi-projective varieties are not so obscure as one would think - they really do occur in many contexts in algebraic geometry.

0.1.3 General base rings

Working with a non-algebraically closed base field is obviously important in number theory: If we want to talk about rational points on the curve given by the equation $x^3 + y^3 = z^3$ we need to be able to think of elliptic curves as schemes over the field $\mathbb{Q}$. There is a naive fix, to just go to the algebraic closure, but this misses the main points: The curves $x^3 + y^3 = z^3$ and $x^3 + y^3 = 9z^3$ are isomorphic over $\overline{\mathbb{Q}}$, but the first has only a few trivial points over $\mathbb{Q}$ and the second has infinitely many.

Going from fields to more general rings offers even greater flexibility. For instance, the variety $Z(y^2 - x(x - 1)(x - t))$ can be viewed simultaneously a
surface $S$ in $\mathbb{A}_k^3$ and a curve $C$ of genus 1 curve over the ring $k[t]$. The $k[t]$-points here correspond to rational curves on $S$. This sort of interplay is very useful in many situations.

### 0.1.4 Prime ideals, rather than maximal ideals.

By the Nullstellensatz, the points of an affine variety are in bijection with the maximal ideals of its coordinate ring. We will see that it is in fact much more natural to include all the prime ideals of this ring. This is something that is almost forced on us by functoriality: Think about the inclusion $k[x] \hookrightarrow k(x)$. The maximal ideal $(0)$ of $k(x)$ pulls back to a prime ideal of $k[x]$ which is not maximal. Including the set of all prime ideals, we obtain a very nice category which behaves essentially just like the category of commutative rings.

Beside of these categorical advantages, there are also a number of places in more classical settings where thinking of points defined by prime ideals provides an extra conceptual clarity. We will try to highlight a few examples where this happens in Chapter 2.
Chapter 1
Sheaves

The concept of a sheaf was conceived in the German camp for prisoners of war called Oflag XVII where French officers taken captive during the fighting in France in the spring 1940 were imprisoned. Among them was the mathematician and lieutenant Jean Leray. In the camp he gave a course in algebraic topology(!!) during which he introduced some version of the theory of sheaves. Leray was aiming to compute the cohomology of a total space of a fibration in terms of invariants of the base and the fibres (and naturally the fibration). To achieve this, in addition to the concept of sheaves, he also invented spectral sequences.

After the war, Henri Cartan and Jean Pierre Serre developed the theory further, and finally the theory was brought to the state as we know it today by Alexandre Grothendieck.

1.1 Sheaves and presheaves

A common theme in mathematics is to study spaces by describing them through their local properties. A manifold is a space which looks locally like Euclidean space; a complex manifold is a space which is locally biholomorphic to \( \mathbb{C}^n \); an algebraic variety is a space that looks locally like the zero set of a set of polynomials. Here it is clear that point set topology alone is not enough to fully capture these three notions: In each case, the space comes equipped with a natural set of functions: \( C^\infty \)-functions, holomorphic functions, and the polynomials respectively, and these are preserved under the local identifications.

Sheaves are introduced to make these ideas precise. To explain this better, let us consider the primary example of a sheaf: The sheaf of continuous maps on
1.1. Sheaves and presheaves

a topological space $X$. By definition, $X$ comes with a collection of ‘open sets’, and these encode what it means for a map $f : X \to Y$ to another topological space $Y$ to be continuous: For every open $U \subseteq Y$, the set $f^{-1}(U)$ should be open in $X$. For two topological spaces $X$ and $Y$, we can define for each open $U \subseteq X$, a set of continuous maps

$$C(U, Y) = \{ f : U \to Y \mid f \text{ is continuous} \}.$$ 

Note that if $V \subseteq U$ is open, then the restriction $f|_V$ of a continuous function $f$ to $V$ is again continuous, so we obtain a map

$$\rho_{UV} : C(U, Y) \to C(V, Y) \quad f \mapsto f|_V.$$ 

Moreover, note that if $W \subseteq V \subseteq U$, we can restrict to $W$ by first restricting to $V$, and so $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. The collection of the sets $C(U, Y)$ together with their restriction maps $\rho_{UV}$ constitutes the sheaf of continuous maps from $X$ to $Y$.

An essential feature of continuity is that it is a local property; $f$ is continuous if and only it is continuous in a neighbourhood of every point, and of course two continuous maps that are equal in a neighbourhood of every point, have to be equal everywhere. Moreover, continuous functions can be glued: Given an open covering $\{U_i\}_{i \in I}$ of an open set $U$, and continuous functions $f_i \in C(U_i, Y)$, so that for each $x \in X$ the value $f_i(x)$ does not depend on $i$, we can patch the maps $f_i$ together to form a continuous map $f : U \to Y$, so that $f|_{U_i} = f_i$ for each $i$: Just define $f(x) = f_i(x)$ for any $i$ such that $x \in U_i$.

Essentially, a sheaf on a topological space $X$ is a structure that encodes these properties. On a topological space, a sheaf can be thought of as a distinguished set of functions, but they can also be more general objects that behave as sets of functions. The main aspect is that we want the distinguished properties to be preserved under restrictions to open sets, and that they are determined from their local properties.

1.1.1 Presheaves

We begin with the definition of a presheaf.

**Definition 1.1.** Let $X$ be a topological space. A *presheaf of abelian groups* $\mathcal{F}$ on $X$ consists of the following two sets of data:

(i) for each open $U \subseteq X$, an abelian group $\mathcal{F}(U)$;

(ii) for each pair of nested opens $V \subseteq U$ a group homomorphism (restriction
Chapter 1. Sheaves

maps)

\[ \rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V). \]

The restriction maps must furthermore satisfy the following two conditions:

(a) for any open \( U \subseteq X \), one has \( \rho_{UU} = \text{id}_{\mathcal{F}(U)} \);

(b) for any nested opens \( W \subseteq V \subseteq U \), one has \( \rho_{UW} = \rho_{VW} \circ \rho_{UV} \).

We will usually write \( s|_V \) for \( \rho_{UV}(s) \) when \( s \in \mathcal{F}(U) \). The elements of \( \mathcal{F}(U) \) are usually called ‘sections’ (or ‘sections over \( U \)’). You may think of them as ‘abelian group-valued functions on \( U \).

We will often also write \( \Gamma(U, \mathcal{F}) \) for the group \( \mathcal{F}(U) \); here \( \Gamma \) is the ‘global sections’-functor (it is functorial in both \( U \) and \( \mathcal{F} \)).

Of course the notion of a presheaf is not confined to presheaves of abelian groups. One may speak about presheaves of sets, rings, vector spaces or whatever you want: Indeed, for any category \( \mathcal{C} \) one may define presheaves with values in \( \mathcal{C} \). The definition goes just like for abelian groups, the only difference being that one requires the ‘spaces’ of sections \( \mathcal{F}(U) \) over open sets \( U \) to be objects in the category \( \mathcal{C} \) and no longer abelian groups, and of course the restriction maps are all required to be morphisms in \( \mathcal{C} \). One may phrase this definition purely in categorical terms by introducing the small category \( \text{open}_X \) of open sets in \( X \) whose objects are the open sets, and the morphisms are the inclusion maps between open sets. With that definition up our sleeve, a presheaf with values in the category \( \mathcal{C} \) is just a contravariant functor

\[ F : \text{open}_X \to \mathcal{C}. \]

We are certainly going to meet sheaves with a lot more structure than the mere structure of abelian groups e.g., like sheaves of rings, but they will all have an underlying structure of abelian group, so we start with those. That being said, sheaves of sets play a great role in mathematics, and in algebraic geometry, so we should not completely wipe them under the rug. Most results we establish for sheaves of abelian groups can be proved mutatis mutandis for sheaves of sets as well, as long as they can be formulated in terms of sets.

1.1.2 Sheaves

We are now ready to give the main definition of this chapter:

**Definition 1.2.** A presheaf \( \mathcal{F} \) is a sheaf if it satisfies the two conditions:

1. (Locality axiom) Given an open subset \( U \subseteq X \) with an open covering \( \mathcal{U} = \{U_i\}_{i \in I} \), and a section \( s \in \mathcal{F}(U) \), then if \( s|_{U_i} = 0 \) for all \( i \), then \( s = 0 \in \mathcal{F}(U) \).
(ii) (Gluing axiom) If $U$ and $U'$ are as in (i), and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections matching on the overlaps, i.e.,

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j \in I,$$

then there exists a section $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$ for all $i$.

Note that the condition (i) says that sections are uniquely determined from their restrictions to smaller open sets. The condition (ii) says that you can patch together local sections to a global one, provided they agree on overlaps, just like in the example of continuous functions.

**Remark 1.3.** There is a nice concise way of formulating the two sheaf axioms at once. For each open over $U = \{U_i\}$ of an open set $U \subseteq X$ there is a sequence

$$0 \to \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

where the maps $\alpha, \beta$ are defined by $\alpha(s) = (s|_{U_i})_i$, and $\beta(s_i) = (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i,j}$. Then $\mathcal{F}$ is a sheaf if and only if these sequences are exact. Indeed, you should check that the Locality axiom is equivalent to $\alpha$ being injective, and the Gluing axiom to $\ker \beta = \mathrm{im} \alpha$. This reformulation is sometimes handy when proving that a given presheaf is a sheaf.

**Example 1.4** (The empty set). There is a subtle point about taking $U$ to be the empty set in the definition of a sheaf. Indeed, if $\mathcal{F}$ is a sheaf we are forced to define $\mathcal{F}(\emptyset) = 0$. Indeed, note that the empty set is covered by the empty open covering, and the empty product is 0, so the sheaf sequence looks like $0 \to \mathcal{F}(\emptyset) \to 0$.

### 1.1.3 Subsheaves and saturation of subpresheaves

If $\mathcal{F}$ is a presheaf on $X$, a **subpresheaf** $\mathcal{G}$ is a presheaf such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for every open $U$, and such that the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, of course $\mathcal{G}$ is called a **subsheaf**.

Let $\mathcal{F}$ be a sheaf on $X$ and $\mathcal{G} \subseteq \mathcal{F}$ a subpresheaf. We say that a section $s \in \mathcal{F}(U)$ **locally lies** in $\mathcal{G}$ if for some open covering $\{U_i\}_{i \in I}$ one has $s|_{U_i} \in \mathcal{G}(U_i)$ for each $i$.

**Definition 1.5.** We define the **sheaf saturation** $\overline{\mathcal{G}}$ of $\mathcal{G}$ in $\mathcal{F}$ by letting the sections of $\overline{\mathcal{G}}$ over $U$ be the sections of $\mathcal{F}$ over $U$ that locally lie in $\mathcal{G}$.

The sheaf saturation $\overline{\mathcal{G}}$ is again a subpresheaf of $\mathcal{F}$ (with restriction maps being the ones induced from $\mathcal{F}$). In fact, $\overline{\mathcal{G}}$ is, almost by definition, a sheaf. The
locality axiom is satisfied for $\mathcal{G}$, since it holds for $\mathcal{F}$. Furthermore, given a set of patching data for $\mathcal{G}$; that is, an open covering $\{U_i\}_{i \in I}$ of an open set $U$ and sections $s_i$ over $U_i$ matching on intersections, the $s_i$'s can be glued together in $\mathcal{F}$ since $\mathcal{F}$ is a sheaf, and since they are born to locally lie in $\mathcal{G}$, the patch lies locally in $\mathcal{G}$ as well and is a section of $\mathcal{G}$ over $U$. So also the Gluing axiom holds for $\mathcal{G}$.

The saturation $\mathcal{G}$ is basically the smallest subsheaf of $\mathcal{F}$ which contains $\mathcal{G}$. If $\mathcal{G}$ already is a sheaf, we don’t get anything new, so that $\mathcal{G} = \mathcal{G}$.

### 1.2 A bunch of examples

**Example 1.6.** Take $X = \mathbb{R}^n$ and let $C(X, \mathbb{R})$ be the sheaf whose sections over an open set $U$ is the ring of continuous real valued functions on $U$, and the restriction maps $\rho_{UV}$ are just the good old restriction of functions. Then $C(X, \mathbb{R})$ is a sheaf of rings (functions can be added and multiplied), and both the sheaf axioms are satisfied. You should convince yourself that this is true.

**Example 1.7.** For a second familiar example, let $X \subseteq \mathbb{C}$ be any open set. On $X$ one has the sheaf $\mathcal{O}_X$ of holomorphic functions. That is, for any open $U \subseteq X$ the sections $\mathcal{O}_X(U)$ is the ring of holomorphic (i.e., complex analytic) functions on $U$. One can relax the condition of holomorphy to get the larger sheaf $\mathcal{K}_X$ of meromorphic functions on $X$. This sheaf contains $\mathcal{O}_X$, and the sections over an open $U$ are the meromorphic functions on $U$. In a similar way, one can get smaller sheaves contained in $\mathcal{O}_X$ by imposing vanishing conditions on the functions. For example if $p \in X$ is any point, one has the sheaf denoted $\mathcal{m}_p$ of holomorphic functions vanishing at $p$. As the name indicates the sections of $\mathcal{m}_p$ over $U$ are holomorphic functions in $U$, and if $p \in U$, one requires additionally that they should vanish at $p$. Convince yourself that this indeed is a subsheaf of $\mathcal{O}_X$.

**Exercise 1.** Let $X \subseteq \mathbb{C}$ be an open set, and assume $a_1, \ldots, a_n$ are distinct points in $X$ and $n_1, \ldots, n_r$ be natural numbers. Define $\mathcal{F}(U)$ to be the set of those meromorphic functions $f \in \Gamma(U, \mathcal{K}_X)$ holomorphic away from the $a_i$'s and having a pole order bounded by $n_i$ at $a_i$. Show that $\mathcal{F}$ is a sheaf of abelian groups. Is it a sheaf of rings?

**Example 1.8 (A presheaf that is not a sheaf).** The sheaf $C(X, \mathbb{R})$ from Example 1.6 has a subpresheaf $\mathcal{F}$ which is not a sheaf: For each open $U$ let $\mathcal{F}(U) = C_b(U, \mathbb{R})$, the group of bounded continuous functions. $\mathcal{F}$ is not a sheaf, because you can’t glue; e.g., the function $f(x) = x$ is bounded on any open ball $B_r(0) = \{x \in \mathbb{R}^n ||x|| < r\}$, but the restrictions $f|_{B_n}$ do not give you a bounded continuous function on all of $\bigcup_{n \geq 0} B_n = \mathbb{R}^n$.
1.2. A bunch of examples

In this example, any continuous function is locally bounded, so the saturation of \( C_b(X, \mathbb{R})^+ \) in \( C(X, \mathbb{R}) \) is simply all of \( C(X, \mathbb{R}) \).

**Example 1.9.** Let us continue the set-up in Example 1.7 to make another example of a presheaf which is not a sheaf. Let \( X = \mathbb{C} \), and let \( \mathcal{O}_X \) denote the sheaf of holomorphic functions. \( \mathcal{O}_X \) contains the subpresheaf given by

\[
\mathcal{F}(U) = \{ f \in \mathcal{O}_X(U) | f = g^2 \text{ for some } g \in \mathcal{O}_X(U) \}.
\]

This is not a sheaf: The function \( f(z) = z \) has a holomorphic square root near any point \( x \in X \), but there is no global \( \sqrt{z} \) on \( X \).

**Example 1.10.** (Constant presheaf) For any space \( X \) and any abelian group \( A \) one has the **constant presheaf** whose group of sections over any nonempty open set \( U \) equals \( A \) and equals 0 if \( U = \emptyset \). This is not a sheaf, since if \( U \cup U' \) is a disjoint union, any choice of elements \( a, a' \in A \) will give sections over \( U \) and \( U' \) respectively, and they match on the empty intersection! But if \( a \neq a' \), they cannot be glued. In fact, this is a sheaf if and only if any two non-empty open subsets of \( X \) have non-empty intersection. Algebraic varieties with the Zariski topology are examples of such spaces.

There is a quick fix for this. We can define the following sheaf \( A_X \) by defining \( A_X(U) \) to be the group of continuous maps \( f : U \to A \) (with the discrete topology on \( A \)). For a connected open set \( U \) we have \( A_X(U) = A \). More generally, since \( f \) must be constant on each connected component of \( U \), we have

\[
A_X(U) = \prod_{\pi_0(U)} A,
\]

(1.2.1)

where \( \pi_0(U) \) denotes the set of connected components of \( U \). As before, we also define \( A_X(\emptyset) = 0 \).

The new presheaf \( A_X \) is now a sheaf, the **constant sheaf** on \( X \) with values in \( A \). That being said, the sheaf \( A_X \) is not quite worthy of the name as it is not quite constant.

**Example 1.11** (Skyscraper sheaf). Let \( A \) be a group. For \( x \in X \) define a presheaf \( A_x \) by \( A_x(U) = A \) if \( x \in U \) and \( A_x(U) = 0 \) otherwise. This is a sheaf, usually called a ‘skyscraper’ sheaf.

The two next examples suggest another way of thinking about sheaves: They are ‘sections’ of maps \( Y \to X \) into \( X \):

**Example 1.12** (Tautological sheaf on projective space). Let \( P(V) \simeq \mathbb{P}^n \) be the projective space of lines in an \((n + 1)\)-dimensional vector space \( V \). Let

\[
\mathbb{L} \subseteq P(V) \times V
\]
be the set of pairs ([l], v) where [l] denotes the point corresponding to the line \( l \subseteq V \) and \( v \in l \). Let \( \pi \) be the projection to the first factor. Note that the fiber over any point \( p \in P(V) \) is a 1-dimensional vector space \( (\pi^{-1}(p)) \) is the line in \( V \) corresponding to \( p \). This is a basic example of a non-trivial vector bundle. We will discuss these later in the course.

For now, we can use it to define the so-called tautological sheaf on \( P(V) \): For any \( U \subseteq P(V) \), let

\[
\mathcal{F}(U) = \{ \sigma : U \to \mathbb{L} | \sigma \text{ continuous and } \pi \circ \sigma = \text{id}_U \}
\]

This is indeed a sheaf, usually denoted by \( \mathcal{O}_{P(V)}(-1) \).

**Example 1.13.** Here is a related example from topology. Consider the Mobius strip \( M \) with its projection \( \pi : M \to S^1 \) to \( S^1 \). This is an example of a fiber bundle; any fiber of \( \pi \) is homeomorphic to the unit interval \([0, 1]\). The sheaf of sections of \( \pi \) is given over an open set \( U \subseteq S^1 \) by

\[
\mathcal{F}(U) = \{ s : U \to M | \sigma \text{ continuous and } \pi \circ s = \text{id}_U \}
\]

Again, this is a sheaf.

There is also the trivial fiber bundle \( \pi : S^1 \times [0, 1] \to S^1 \) given by the projection to the first factor. This is not homeomorphic to the bundle above. [Brain teaser: Prove this.]

Our main interest in this course will still be the following:

**Example 1.14.** Let \( V \) be an algebraic variety (e.g., quasi-projective, as in Hartshorne Ch. I) with the Zariski topology. For each open \( U \subseteq X \), define \( \mathcal{O}_V(U) \) to be the ring of regular functions \( U \to k \). This is certainly a presheaf, and in fact, a sheaf.
1.3 Stalks

Given a presheaf $F$ of abelian groups on $X$. With every point $x \in X$ there is an associated abelian group $F_x$ called the stalk\(^1\) of $F$ at $x$. The elements of $F_x$ are called germs of sections\(^2\) near $x$.

The definition goes as follows: We begin with the disjoint union $\bigsqcup_{x \in U} F(U)$ whose elements we think of as pairs $(s, U)$ where $U$ is any open neighbourhood of $x$ and $s$ is a section of $F$ over $U$. We want to identify sections that coincide near $x$; that is, we declare $(s, U)$ and $(s', U')$ to be equivalent, and write $(s, U) \sim (s', U')$, if there is an open $V \subseteq U \cap U'$ with $x \in V$ such that $s$ and $s'$ coincide on $V$; that is one has

$$s|_V = s'|_V.$$

This is clearly a reflexive and symmetric relation. And it is transitive as well: if $(s, U) \sim (s', U')$ and $(s', U') \sim (s'', U'')$, one may find open neighbourhoods $V \subseteq U \cap U'$ and $V' \subseteq U' \cap U''$ of $x$ over which $s$ and $s'$, respectively $s'$ and $s''$, coincide. Clearly $s$ and $s''$ then coincide over the intersection $V' \cap V$, and the relation $\sim$ is an equivalence relation.

**Definition 1.15.** The stalk $F_x$ at $x \in X$ is by definition the set of equivalence classes

$$F_x = \bigsqcup_{x \in U} F(U) / \sim.$$

In case $F$ is a sheaf of abelian groups, the stalks $F_x$ are all abelian groups. This is not a priori obvious, since sections over different open sets can not be added. However if $(s, U)$ and $(s', U')$ are given, the restrictions $s|_V$ and $s'|_V$ to any open $V \subseteq U \cap U'$ can be added, and this suffices to define an abelian group structure on the stalks.

1.3.1 The germ of a section

For any neighbourhood $U$ of $x \in X$, there is a map $F(U) \to F_x$ sending a section $s$ to the equivalence class where the pair $(s, U)$ belongs. This class is called the germ of $s$ at $x$, and a common notation for it is $s_x$. The map is a homomorphism of abelian groups (rings, modules...) as one easily verifies. Clearly one has $s_x = (s|_V)_x$ for any other open neighbourhood $V$ of $x$ contained

\(^1\)Norsk: 'stilk'

\(^2\)Norsk: 'seksjonskimer' eller bare 'kimer'
in $U$, or expressed in the lingo of diagrams, the following diagram commutes:

$$\begin{align*}
\mathcal{F}(U) & \longrightarrow \mathcal{F}_x \\
\rho_{UV} & \\
\mathcal{F}(V) & \swarrow
\end{align*}$$

When working with sheaves and stalks, it is important to remember the three following properties; the third one is easily deduced from the two first.

- The germ $s_x$ of $s$ vanishes if and only if $s$ vanishes on a neighbourhood of $x$, i.e., there is an open neighbourhood $U$ of $x$ with $s|_U = 0$.

- All elements of the stalk $\mathcal{F}_x$ are germs, i.e., of the shape $s_x$ for some section $s$ over an open neighbourhood of $x$.

- The abelian sheaf $\mathcal{F}$ is the zero sheaf if and only if all stalks are zero, i.e., $\mathcal{F}_x = 0$ for all $x \in X$.

**Example 1.16.** Let $X = \mathbb{C}$, and let $\mathcal{O}_X(U)$ be the ring of holomorphic functions on $U$. If $f$ and $g$ are two sections of $\mathcal{O}_X$ over a neighbourhood of $p$ having the same germ at $p$, i.e., $f = g \in \mathcal{O}_{X,p}$, then $f = g$ where they are both defined. If you replace ‘holomorphic’ by $C^\infty$, then $f$ and $g$ having the same germ at $p$ only implies that the derivatives of $f$ and $g$ at $p$ of all orders of coincide, i.e., $f^{(n)}(p) = g^{(n)}(p)$ for all $n \geq 0$, but not much else.

### 1.3.2 A primer on limits

Another notation for the stalk of $\mathcal{F}$ at $x$ is

$$\mathcal{F}_x = \lim_{\mathcal{U} \ni x} \mathcal{F}(U).$$

This is the **direct limit** (or ‘colimit’) of all $\mathcal{F}(U)$ when $U$ runs over the partially ordered set of open sets containing $x$. Taking the direct limit is a general construction in algebra, so let us give a few more details here for future reference.

A **directed set** $I$ is a partially ordered set such that for each pair of elements $i, j \in I$ there is a third element $k$ such that $i < k$ and $j < k$. If $I$ is a directed set and $\mathcal{C}$ is a category, a **directed system of objects** in $\mathcal{C}$ is a collection $\{G_i\}_{i \in I}$ of objects in $\mathcal{C}$, such that for all $i < j$ there is a morphism $f_{ij} : G_i \to G_j$, with $f_{ii} = id$ and $f_{jk} \circ f_{ij} = f_{ik}$.
1.4. Morphisms between (pre)sheaves

Direct limits

The direct limit of \( G_i \), denoted by \( G = \varinjlim_{i \in I} G_i \) is an object in \( C \), equipped with morphisms \( g_i : G_i \to G \) which satisfy the following universal property: For any object \( H \in C \) with maps \( h_i : G_i \to H \) such that \( h_i = h_j \circ f_{ij} \) for each \( i \leq j \), there is a unique map \( h : G \to H \) making the following diagram commute:

\[
\begin{array}{ccc}
G_i & \xrightarrow{h_i} & H \\
\downarrow{f_i} & & \downarrow{h} \\
G & \xrightarrow{h} & H
\end{array}
\]

Heuristically, two elements in the direct limit represent the same element if they are ‘eventually equal.’

If the \( G_i \) are sets (or groups, rings, ..), an explicit construction for this is the quotient \( \bigsqcup_{i \in I} G_i / \sim \), where \( g \sim h \), with \( g \in G_i \) and \( h \in G_j \), if there exists a \( k \in I \) such that \( f_{ik}(g) = f_{jk}(h) \). In the case \( I \) is the set of opens, and \( G_U = F(U) \), we recover the previous definition of the stalk \( F_x \).

Inverse limits

By reversing all the arrows, we can define the ‘inverse limit’ or ‘projective limit’ of a directed system \( G_i \) is defined as above. That is, the maps \( G_i \to G_j \) are defined for \( j < i \), and \( \varprojlim_{i \in I} G_i \) is an element of \( C \) equipped with universal maps to each of the \( G_i \).

1.4 Morphisms between (pre)sheaves

A morphism \( \phi : \mathcal{F} \to \mathcal{G} \) of (pre)sheaves on a space \( X \) is collection of maps \( \phi_U : \mathcal{F}(U) \to \mathcal{G}(U) \) compatible with the restriction maps. In other words, the following diagram commutes for each inclusion \( V \subseteq U \):

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\
\downarrow{\rho_{UV}} & & \downarrow{\rho_{UV}} \\
\mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V)
\end{array}
\]

In this way the abelian sheaves on \( X \) form a category \( \text{AbSh}_X \) whose objects are the abelian sheaves on \( X \) and the morphisms being the maps between them. The composition of two maps of sheaves is defined in the obvious way as the composition of the maps on sections.

Remark 1.17. If one considers the sheaves \( \mathcal{F} \) and \( \mathcal{G} \) as contravariant func-
tors on the category open\textsubscript{X}, a map between them is what is called a natural transformation between the two functors.

As usual, a map \( \phi \) between two (pre)sheaves \( F \) and \( G \) is an isomorphism if it has a two-sided inverse, i.e., a map \( \psi : G \to F \) such that \( \phi \circ \psi = \text{id}_G \) and \( \psi \circ \phi = \text{id}_F \).

A map \( \phi : F \to G \) between two presheaves \( F \) and \( G \) induces for every point \( x \in X \) a map \( \phi_x : F_x \to G_x \) between the stalks. Indeed, one may send a pair \( (s, U) \) to the pair \( (\phi_U(s), U) \), and since \( \phi \) behaves well with respect to restrictions, this assignment is compatible with the equivalence relations; that is, if \( (s, U) \) and \( (s', U') \) are equivalent and \( s \) and \( s' \) coincide on an open set \( V \subseteq U \cap U' \), one has
\[
\phi_{U'}(s)|_V = \phi_{V'}(s)|_V = \phi_{V'}(s'|_V) = \phi_{U'}(s'|_V).
\]

Obviously \( (\phi \circ \psi)_x = \phi_x \circ \psi_x \) and \( (\text{id}_F)_x = \text{id}_{F_x} \), so the assignment \( \phi \mapsto \phi_x \) is a functor from the category of abelian sheaves to the category of abelian groups.

**Example 1.18.** In the case \( X = \mathbb{R} \) let \( C^r(X) \) be the subsheaf of \( C(X, \mathbb{R}) \) consisting of functions being \( r \) times continuously differentiable (check that this is indeed a subsheaf!). The differential operator \( D = d/dx \) defines a map \( D : C^r \to C^{r-1} \).

**Exercise 2.** In the same vein, the differential operator gives a map \( D : \mathcal{O}_X \to \mathcal{O}_X \), where as previously \( X \subseteq \mathbb{C} \) is an open set. Show that the assignment
\[
\mathcal{A}(U) = \{ f \in \mathcal{O}_X(U) \mid Df = 0 \}
\]
defines a subsheaf \( \mathcal{A} \) of \( \mathcal{O}_X \). Show that if \( U \) is a connected open subset of \( X \), one has \( \mathcal{A}(U) = \mathbb{C} \). In general for a not necessarily connected set \( U \), show that \( \mathcal{A}(U) = \prod_{\pi_0 U} \mathbb{C} \) where the product is taken over the set \( \pi_0 U \) of connected components of \( U \).

**1.5 Kernels and images**

Let \( \phi : F \to G \) be a map between two abelian sheaves on \( X \).

**Definition 1.19.** The kernel ker \( \phi \) of \( \phi \) is a subsheaf of \( F \) whose sections over \( U \) are just ker \( \phi_U \), i.e., the sections in \( F(U) \) mapping to zero under \( \phi_U : F(U) \to G(U) \).

The requirement in the definition is compatible with the restriction maps since \( \phi_U(s|_V) = \phi_U(s)|_V \), for any section \( s \) over the open set \( U \) and any open \( V \subseteq U \). Thus we have defined a subpresheaf of \( F \). This is indeed a subsheaf:
the Locality axiom holds because $\mathcal{F}$ is a sheaf. Moreover, if $\{ s_i \}$ are gluing data for the kernel, one may glue the $s_i$’s to a section $s$ of $\mathcal{F}$ over $U$. One has $\phi(s)|_{U_i} = \phi(s)|_{U_i} = \phi(s_i) = 0$, and from the locality axiom for $\mathcal{G}$ it follows that $\phi(s) = 0$. We leave it to the reader to verify that the stalk $(\ker \phi)_x$ of $\ker \phi$ at $x$ equals $\ker \phi_x$. We have proven the following

**Lemma 1.20.** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves. The kernel $\ker \phi$ is a subsheaf of $\mathcal{F}$ having the two properties

- Taking the kernel commutes with taking sections: $\Gamma(U, \ker \phi) = \ker \phi_U$,
- Forming the kernel commutes with forming stalks: $(\ker \phi)_x = \ker \phi_x$.

One says that the map $\phi$ is injective if $\ker \phi = 0$. This is, in view of the previous lemma, equivalent to the condition $\ker \phi_x = 0$ for all $x$, i.e., that all $\phi_x$ are injective. One often expresses this in a slightly imprecise manner by saying that $\phi$ is injective on all stalks.

When it comes to images the situation is not as nice as for kernels. One defines the image presheaf contained in $\mathcal{G}$ by letting the sections over $U$ be equal to $\text{im} \phi_U$. However this is not necessarily a sheaf. If $s_i = \phi_U(t_i)$ are gluing data for the image presheaf there is no reason for the $t_i$’s to match on the intersections $U_{ij} = U_j \cap U_j$ even if the $s_i$’s do; the differences $t_i|_{U_{ij}} - t_j|_{U_{ij}}$, may very well be non-zero sections of the kernel of $\phi_{U_{ij}}$! In fact, we will see several explicit examples of this later.

To remedy this situation, we simply make the following definition:

**Definition 1.21.** For a morphism $\phi : \mathcal{F} \to \mathcal{G}$ we define the sheaf $\text{im} \phi$ to be the saturation of the image presheaf $U \mapsto \text{im} \phi_U$, i.e., the smallest subsheaf containing the images.

Forming the image of a map of sheaves does not always commute with taking sections, but as we shall verify in the upcoming lemma, forming images commutes with forming stalks:

**Lemma 1.22.** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves. The image $\text{im} \phi$ is a subsheaf of $\mathcal{G}$.

- For all open subsets $U$ of $X$ one has $\text{im} \phi_U \subseteq \Gamma(U, \text{im} \phi)$.
- For all $x \in X$ one has $(\text{im} \phi)_x = \text{im} \phi_x$.

**Proof.** We only have to verify the last statement, so let $t_x \in \text{im} \phi_x$ and pick an $s_x \in \mathcal{F}_x$ with $\phi_x(s_x) = t_x$. We may extend these elements to sections $s, t$ over some open neighbourhood $V$, so that $\phi_V(s) = t$, and $t$ is a section of $\text{im} \phi$ over $V$. This shows that $\text{im} \phi_x \subseteq (\text{im} \phi)_x$. Conversely, if $t$ is a section of $\mathcal{G}$ over an open $U$ containing $x$ locally lying in image presheaf, the restriction $t \subseteq V$ lies in $\text{im} \phi_V$ for some smaller neighbourhood $V$ of $x$, hence the germ $t_x$ lies in $\text{im} \phi_x$. 

The map $\phi: \mathcal{F} \to \mathcal{G}$ is said to be surjective if the image sheaf $\text{im} \phi = \mathcal{G}$. This is equivalent to all the stalk-maps $\phi_x$ being surjective (one says $\phi$ is surjective on stalks). However, this condition does not imply that all maps $\phi_U$ are surjective for any $U$. Here is a counterexample:

**Example 1.23.** Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X = \mathbb{CP}^1$ and let $p, q \in X$ be two distinct points. Let $\mathcal{O}_X(-p)$ and $\mathcal{O}_X(-q)$ be the two subsheaves of $\mathcal{O}_X$ of holomorphic functions vanishing at $p$ and $q$ respectively and consider the map

$$
\phi: \mathcal{O}_X(-p) \oplus \mathcal{O}_X(-q) \to \mathcal{O}_X
$$

defined by $(f, g) \mapsto f + g$. Then $\phi_x$ is surjective for every $x \in X$ [Check this!], but $\phi_X$ cannot be surjective, since by Liouville’s theorem, the only globally holomorphic functions are the constants, and so $\mathcal{O}_X(-p)(X) = \mathcal{O}_X(-q)(X) = 0$ and $\mathcal{O}_X(X) = \mathbb{C}$.

In particular, since $\text{im} \phi(X) = 0$, this shows that the naive image presheaf $\mathcal{F}(U) = \text{im} \phi_U$ is not a sheaf.

However, for the map $\phi$ to be an *isomorphism*, one has the following

**Proposition 1.24.** Let $\phi: \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves. Then the following four conditions are equivalent

- The map $\phi$ is an isomorphism.
- For every $x \in X$ the map on stalks $\phi_x: \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism,
- One has $\ker \phi = 0$ and $\text{im} \phi = \mathcal{G}$,
- For all open subsets $U \subseteq X$ the map on sections $\phi_U: \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

**Proof.** Most of the implications are straightforward from what we have done so far and are left to the reader. We comment just on the two main salient points.

Firstly, assume that all the stalk maps $\phi_x: \mathcal{F}_x \to \mathcal{G}_x$ are isomorphism, and let us deduce from this that all the maps $\phi_U: \mathcal{F}(U) \to \mathcal{G}(U)$ on sections are isomorphisms. It is clear that $\phi_U$ is injective since forming kernels commute with taking sections as in lemma 1.20 on page 26. So take an element $t \in \mathcal{G}(U)$. For each $x \in U$ there is a germ $s_x$ induced by a section $s(x)$ of $\mathcal{F}$ over some open neighbourhood $U_x$ of $x$ satisfying $\phi_x(s_x) = t_x$. Let $t_{(x)} = t|_{U_x}$.

After shrinking the neighbourhood $U_x$, we may assume that $\phi_{U_x}(s_{(x)}) = t_{(x)}$. Note that the $t_{(x)}$'s match on the intersections $U_x \cap U_y$ – they are all restrictions of the section $t$ – and therefore the $s_{(x)}$'s match as well because $\phi_{U_x}$ is injective (as we just observed above). Hence, the sections $s_{(x)}$'s patch together to a section $s$ of $\mathcal{F}$ that must map to $t$ since it does so locally.
Secondly, if all the $\phi_U$’s are isomorphisms, we have all the inverse maps $\phi_U^{-1}$ 

at our disposal. They commute with restrictions since the maps $\phi_U$ do. Indeed, from $\phi_V \circ \rho_{UV} = \rho_{UV} \circ \phi_U$ one obtains $\rho_{UV} \circ \phi_U^{-1} = \phi_V^{-1} \circ \rho_{UV}$ and the $\phi_U^{-1}$’s thus define a map $\phi^{-1} : G \to F$ of sheaves, which of course, is inverse to $\phi$. We conclude that $\phi$ is an isomorphism. \hfill $\square$

1.6 Exact sequences of sheaves.

Given a sequence

$$
\ldots \xrightarrow{\phi_{i-2}} F_{i-1} \xrightarrow{\phi_{i-1}} F_i \xrightarrow{\phi_i} F_{i+1} \xrightarrow{\phi_{i+1}} \ldots
$$

of maps of abelian sheaves, we say that the sequence is exact at $F_i$ if $\ker(\phi_i) = \text{im}(\phi_{i-1})$. The short exact sequences are the ones one most frequently encounters. They are sequences of the form

$$
0 \to F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \to 0 \tag{1.6.1}
$$

that are exact at each stage. This is just another and very convenient way of simultaneously saying that $\phi$ is injective, that $\psi$ is surjective and that $\text{im}(\phi) = \ker(\psi)$.

**Proposition 1.25.** For a short exact sequence $0 \to F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \to 0$, we have the following induced exact sequence

$$
0 \to F'(U) \xrightarrow{\phi_U} F(U) \xrightarrow{\psi_U} F''(U) \tag{1.6.2}
$$

**Proof.** The map $\phi$ is injective as a map of sheaves, hence injective on all open sets $U$, so the sequence above is exact at $\Gamma(U, F')$. To see that it is also exact at the bottom, we show that $\ker(\psi_U) = \text{im}(\phi_U)$. (The image presheaf is then given by the kernel of a morphism of sheaves, which is indeed a sheaf.)

It might be helpful to look at the following diagram, for $x \in U$:

$$
\begin{array}{ccc}
0 & \to & F'(U) \\
\downarrow & & \downarrow \\
0 & \to & F_x
\end{array}
\quad
\begin{array}{ccc}
F'(U) & \xrightarrow{\phi_U} & F(U) \\
\downarrow & & \downarrow \\
F_x & \xrightarrow{\phi_x} & F_x
\end{array}
\quad
\begin{array}{ccc}
F(U) & \xrightarrow{\psi_U} & F''(U) \\
\downarrow & & \downarrow \\
F_x & \xrightarrow{\psi_x} & F_x
\end{array}
\quad
\begin{array}{ccc}
F''(U) & \xrightarrow{\psi_U} & 0 \\
\downarrow & & \downarrow \\
F_x & \xrightarrow{\psi_x} & F''
\end{array}
$$

Note that the bottom row is exact, since the sheaf sequence is exact.

$\ker(\psi_U) \supseteq \text{im}(\phi_U)$: Let $s \in \Gamma(U, F')$ and consider for each $x \in U$ the germ
of $\psi_U(\phi_U(s))$ in the stalk $\mathcal{F}_x^n$:

$$(\psi_U(\phi_U(s)))_x = \psi_x(\phi_x(s_x)).$$

But by exactness, $\psi_x(\phi_x(s_x)) = 0$ for all $x \in U$. Hence $\psi_U(\phi_U(s)) = 0$, so $\text{im}(\phi_U) \subseteq \ker(\psi_U)$.

$\ker(\psi_U) \subseteq \text{im}(\phi_U)$: Let $t \in \ker(\psi_U)$, so $\psi_U(t) = 0$. Then for all $x \in U$ we have that $\psi_x(t_x) = (\psi_U(t))_x = 0$, so the germ $t_x$ is an element in $\ker(\psi_x) = \text{im}(\phi_x)$ (where we use exactness again). That means that for every $x \in U$ there is an element $s'_x \in \mathcal{F}_x'$, say represented by $(s'_x(V(x)))$ for some open neighborhood $V(x) \subseteq U$ of $x$ and $s'(x) \in \mathcal{F}(V(x))$, such that $\phi_x(s'_x) = t_x$. Then we have that for $x, y \in U$

$$\phi_{V(x) \cap V(y)}(s'_x|_{V(x) \cap V(y)}) = t|_{V(x) \cap V(y)} = \phi_{V(x) \cap V(y)}(s'_y|_{V(x) \cap V(y)}),$$

so that by the injectivity of $\phi_{V(x) \cap V(y)}$ (which we have already proved), we get the required condition

$$s'_x|_{V(x) \cap V(y)} = s'_y|_{V(x) \cap V(y)}$$

for the gluing of the $s'_x$ for $x \in U$. Therefore we have a section $s \in \Gamma(U, \mathcal{F})$ with the property that for all $x \in U$

$$s|_{V(x)} = s'_x.$$ 

Now we can conclude that for every $x \in U$

$$(\phi_U(s))_x = \phi_x(s_x) = \phi_x(s'_x) = t_x,$$

since $s_x = s'_x$, which gives $\phi_U(s) = t$ as desired. 

One way of phrasing this is to say that taking sections over an open set $U$, that is, $\Gamma(U, -)$, is a left exact functor. However it is not right exact in general. The defect of this lacking surjectivity is a fundamental problem in every part of mathematics where sheaf theory is used, and to cope with it one has developed cohomology.

Let us give a few examples where the surjectivity on the right fails:

**Example 1.26** (The exponential sequence). Let $X = \mathbb{C} - \{0\}$. The non-vanishing holomorphic functions in an open set $U \subseteq X$ form a multiplicative group, and there is a sheaf $\mathcal{O}^*_X$ with these groups as sections. For any $f$ holomorphic in $U$ the exponential $\exp f(z)$ is a section of $\mathcal{O}^*_X$. Hence there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}^*_X \longrightarrow 0,$$

where $\mathbb{Z}_X$ is the sheaf associated to the sheaf of locally constant functions with values in $\mathbb{Z}$.
where the first map sends 1 to $2\pi i$. It is surjective on the right since non-vanishing functions locally have logarithms. However, the map $\exp(U)$ is not surjective over $U = X$, since the non-vanishing function $f(z) = z$ is not the exponential of a global holomorphic function.

**Example 1.27** (Differential operators). Let $X = \mathbb{C}$ and recall the sheaf $\mathcal{O}_X$ of holomorphic functions and the map $D: \mathcal{O}_X \to \mathcal{O}_X$ sending $f(z)$ to the derivative $f'(z)$. There is an exact sequence

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \xrightarrow{D} \mathcal{O}_X \longrightarrow 0.$$ 

This hinges on the two following facts. Firstly, a function whose derivative vanishes identically is locally constant; hence the kernel $\ker D$ equals the constant sheaf $\mathbb{C}_X$. Secondly, in small open disks any holomorphic function has a primitive function, i.e., is a derivative, e.g., if $f(z) = \sum_{n \geq 0} a_n (z - a)^n$ in a small disk around $a$, the function $g(z) = \sum_{n \geq 0} a_n (n + 1)^{-1} (z - a)^{n+1}$ has $f(z)$ as derivative. However, taking sections over open sets $U$ we merely obtain the sequence

$$0 \longrightarrow \Gamma(U, \mathbb{C}_X) \longrightarrow \Gamma(U, \mathcal{O}_X) \xrightarrow{D_U} \Gamma(U, \mathcal{O}_X).$$

Whether $D_U$ is surjective or not, depends on the topology of $U$. If $U$ is simply connected, one deduces from Cauchy’s integral theorem that every holomorphic function in $U$ is a derivative, so in that case $D_U$ is surjective. On the other hand, if $U$ not simply connected, $D_U$ is not surjective; e.g., if $U = \mathbb{C} - \{0\}$, the function $z^{-1}$ is not a derivative in $U$.

**Example 1.28.** Consider an algebraic variety $X$ with the sheaf $\mathcal{O}_X$ of regular functions on $X$. For any point $x \in X$ let $k(x)$ denote the skyscraper sheaf (see Example 1.11) whose only non-zero stalk is the field $k$ located at $x$. There is a map of sheaves $\text{ev}_x: \mathcal{O}_X \to k(x)$ sending a function that is regular in a neighbourhood of $x$ to the value it takes at $x$. This maps sits in the exact sequence of sheaves

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathcal{O}_X \xrightarrow{\text{ev}_x} k(x) \longrightarrow 0,$$

where $\mathfrak{m}_x$ by definition is the kernel of $\text{ev}_x$ (the sections of $\mathfrak{m}_x$ are the functions vanishing at $x$). Taking two distinct points $x$ and $y$ in $X$ we find a similar exact sequence

$$0 \longrightarrow \mathcal{I}_{x,y} \longrightarrow \mathcal{O}_X \xrightarrow{\text{ev}_{x,y}} k(x) \oplus k(y) \longrightarrow 0,$$

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where \( \mathcal{I}_{x,y} \) is the sheaf of functions vanishing on \( x \) and \( y \), and

\[
ev_{x,y}(f) = \begin{cases} 
(0,0) & \text{if } x, y \notin U, \\
(f(x),0) & \text{if } x \in U \text{ but } y \notin U, \\
(0,f(y)) & \text{if } y \in U \text{ but } x \notin U, \\
(f(x),f(y)) & \text{if } x, y \in U.
\end{cases}
\]

If for example \( X = \mathbb{P}^1 \) (or any other complete variety), there are no global regular functions on \( X \) other than the constants, and hence \( \Gamma(X, \mathcal{O}_X) = k \). But of course, \( \Gamma(\mathbb{P}^1, k(x) \oplus k(y)) = k \oplus k \), so the map \( \ev_{x,y} \) can not be surjective on global sections.

### 1.7 \( \mathcal{B} \)-sheaves

Recall that a basis for a topology on \( X \) is a collection of open subsets \( \mathcal{B} \) such that any open set of \( X \) can be written as a union of elements of \( \mathcal{B} \). In many situations it turns out to be convenient to define a sheaf by saying what it should be on a specific basis for the topology on \( X \). In this section we will see that there exists a unique way to produce a sheaf, given sections that glue over open subsets in \( \mathcal{B} \).

Let us first make the following definition:

**Definition 1.29.** A \( \mathcal{B} \)-presheaf \( \mathcal{F} \) consists of the following data: (i) For each \( U \in \mathcal{B} \), an abelian group \( \mathcal{F}(U) \); and (ii) for all \( U \subseteq V \), with \( U, V \in \mathcal{B} \), a restriction map \( \rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V) \). As usual, these are required to satisfy the relations \( \rho_{UU} = \text{id}_{\mathcal{F}(U)} \) and \( \rho_{WV} = \rho_{UV} \circ \rho_{WU} \). A \( \mathcal{B} \)-sheaf is a \( \mathcal{B} \)-presheaf satisfying the Locality and Gluing axioms for open sets in \( \mathcal{B} \).

**Proposition 1.30.** Let \( X \) be a topological space and let \( \mathcal{B} \) be a basis for the topology on \( X \). Then

(i) Every \( \mathcal{B} \)-sheaf \( \mathcal{F} \) extends uniquely to a sheaf on \( X \).

(ii) If \( \phi : \mathcal{F} \to \mathcal{G} \) is a morphism of \( \mathcal{B} \)-sheaves, then \( \phi \) extends uniquely to a morphism between the corresponding sheaves.

Indeed, for any open set \( U \subseteq X \), we can write \( U \) as a union of open sets \( U_i \in \mathcal{B} \), and then we can define \( \mathcal{F}(U) \) to be the set of elements \( s_i \in \prod_i \mathcal{F}(U_i) \) such that \( s_i|_{U \cap U_j} = s_j|_{U \cap U_j} \). This is indeed gives a well-defined sheaf, and it must be unique, since we can see stalks using open sets in \( \mathcal{B} \). We leave the verification of the point (\( ii \)) as an exercise.
1.8 A family of examples – Godement sheaves

We will consider a class of rather peculiar sheaves, called Godement sheaves, to demonstrate the versatility and the generality of the notion of sheaves. They will also be important later, when we define the sheafification of a presheaf.

Let $X$ be any topological space. Assume that we for each point $x \in X$ are given an abelian group $A_x$. The groups $A_x$ can be chosen in a completely arbitrary way, at random if you will. The choice of these groups gives rise to a sheaf $A$ on $X$ whose sections over an open set $U \subseteq X$ are given as

$$
\Gamma(U, A) = \prod_{x \in U} A_x,
$$

and whose restriction maps are defined as the natural projections

$$
\rho_{UV}: \prod_{x \in U} A_x \to \prod_{x \in V} A_x,
$$

where $V \subseteq U$ is any pair of open subsets of $X$. The restriction map just “throws away” the components at points in $U$ not lying in $V$.

**Proposition 1.31.** $A$ is a sheaf.

**Proof.** The Locality condition holds since if the family $\{U_i\}_i$ of open sets covers $U$, any point $x_0 \in U$ lies in some $U_{i_0}$, so if $s = (a_x)_{x \in U} \in \Gamma(U, A)$ is a section, the component $a_{x_0}$ survives in the projection onto $\Gamma(U_{i_0}, A) = \prod_{x \in U_{i_0}} A_x$. Hence if $s|_{U_i} = 0$ for all $i$, it follows that $s = 0$.

The Gluing condition holds: Assume given an open cover $\{U_i\}_i$ of $U$ and sections $s_i \in U_i$ matching on the intersections $U_i \cap U_j$. The matching conditions imply that the component of $s_i$ at a point $x$ is the same whatever $i$ is as long as $x \in U_i$. Hence we get a well-defined section $s$ of $A$ over $U$ by using this common component as the component of $s$ at $x$. It is clear that $s|_{U_i} = s_i$. \qed

**Definition 1.32.** The sheaf $A$ is sometimes called the Godement sheaf of the collection $\{A_x\}$.

The construction is not confined to abelian groups, but works for any category where general products exist; like sets, rings, etc.

So what is the stalk of $A$ at a point $x$? Don’t ask that question! The group can be fairly complicated. Of course there is a map $A_x \to A_x$, but that is in general the best you can say. For example, suppose $A_y = \mathbb{Z}/2$ for all $y \in X$, and that there is a sequence $x_n$ of points in $X$ converging to $x \in X$, with $x_n \neq x_m$ for all $n \neq m$. Then $A_x$ maps onto the (infinite) set of tails of sequences of 0s and 1s. Namely, every open neighbourhood of $x$ contains almost all of the $x_n$. 

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On the other hand, if every neighbourhood of $x$ contains a point $y$ such that $A_y = 0$, then $A_x = 0$.

### 1.8.1 Skyscraper sheaves.

This is a very special instance of a Godement sheaf where all the abelian groups $A_x$ are zero except for one. So we fix just one abelian group $A$, and we choose a closed point $x \in X$. The corresponding Godement sheaf is denoted $A(x)$ and is called a \textit{skyscraper sheaf}. The sections are described by

$$\Gamma(U, A(x)) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}$$

Contrary to the general case, in this case the stalks are easy to describe: they are zero everywhere except at $x$, where the stalk equals $A$. Indeed, if $y \neq x$, then $y$ lies in the open set $X - \{x\}$ over which all sections of $A(x)$ vanish.

In most spaces concerning us there are plenty of non-closed points. For such points one still has the Godement sheaf $A(x)$, but the argument above does not work, and the description of the stalks are somehow more complicated.

**Exercise 3.** Assume that $x$ is not closed and let $Z = \overline{\{x\}}$ be the closure of the singleton $\{x\}$. Show that the stalks of $A(x)$ are $(A(x))_y = 0$ if $y \notin Z$ and $(A(x))_y = A$ for points $y$ belonging to $Z$.

Slightly generalising the construction of a skyscraper sheaf, one may form the Godement sheaf $\mathcal{A}$ defined by a finite set of distinct closed points $x_1, \ldots, x_r$ and corresponding abelian groups $A_1, \ldots, A_r$. Then one sees, having in mind that an empty direct sum is zero, that the sections of $\mathcal{A}$ over an open set $U$ is given as $\Gamma(U, \mathcal{A}) = \bigoplus_{x_i \in U} A_i$. The stalks of $\mathcal{A}$ are

$$\mathcal{A}_x = \begin{cases} 0 & \text{when } x \neq x_i \text{ for all } i \\ A_i & \text{when } x = x_i. \end{cases}$$

Perhaps a suitable name for such a sheaf would be a \textit{barcode sheaf}?

**Example 1.33** (A set of peculiar examples). Of course if $Z$ is a discrete subset of $X$, not necessarily finite, one may associate a Godement sheaf $\mathcal{A}$ with any family $\{A_z\}_{z \in Z}$ of abelian groups indexed by $Z$ (letting $A_x = 0$ if $x \notin Z$). The sections over an open set $U$ is by definition

$$\Gamma(U, \mathcal{A}) = \prod_{z \in Z \cap U} A_z$$
1.8. A family of examples – Godement sheaves

Just as in the finite case the stalks will be

\[ \mathcal{A}_x = A_x \]

However if \( Z \) has accumulation points this is no longer true. Assume for simplicity that \( \{ z_i \} \) is a sequence converging to \( z \). Two sequences of elements \( a_i \) and \( a'_i \) are said to define the same tail if there is some \( N \) such that \( a_i = a'_i \) for \( i \geq N \). Then the stalk at \( z \) will be the space of tails from the \( A_i \), that is, \( A_z = \prod_i A_i / \sim \).

1.8.2 The Godement sheaf associated with a presheaf

Assume \( \mathcal{F} \) is a given abelian presheaf on \( X \). The stalks \( \mathcal{F}_x \) of \( \mathcal{F} \) of course give a collection of abelian groups indexed by points in \( X \), good as any other, and we may form the corresponding Godement sheaf which we denote by \( \Pi(\mathcal{F}) \). The sections of \( \Pi(\mathcal{F}) \) are given by

\[ \Pi(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x, \quad (1.8.1) \]

and the restriction maps are the projections like for any Godement sheaf. This sheaf is sometimes called the sheaf of discontinuous sections of \( \mathcal{F} \).

There is an obvious and canonical map

\[ \kappa_\mathcal{F}: \mathcal{F} \to \Pi(\mathcal{F}) \]

sending a section \( s \in \mathcal{F}(U) \) to the element \( (s_x)_{x \in U} \) of the product in (1.8.1). This map is functorial in \( \mathcal{F} \), for if \( \phi: \mathcal{F} \to \mathcal{G} \) is a map, one has the stalkwise maps \( \phi_x: \mathcal{F}_x \to \mathcal{G}_x \), and by taking appropriate products of these, we obtain a map \( \Pi(\phi): \Pi(\mathcal{F}) \to \Pi(\mathcal{G}) \). Over an open set \( U \), we have

\[ \Pi(\phi)((s_x)_{x \in U}) = (\phi_x(s_x))_{x \in U} \]

and there is a commutative diagram

\[ \begin{array}{ccc}
\mathcal{F} & \xrightarrow{\kappa_\mathcal{F}} & \Pi(\mathcal{F}) \\
\phi \downarrow & & \Pi(\phi) \downarrow \\
\mathcal{G} & \xrightarrow{\kappa_\mathcal{G}} & \Pi(\mathcal{G}).
\end{array} \quad (1.8.2) \]

It is not hard to check that \( \Pi(\text{id}_\mathcal{F}) = \text{id}_{\Pi(\mathcal{F})} \), and \( \Pi(\psi \circ \phi) = \Pi(\psi) \circ \Pi\phi \) for morphisms \( \phi: \mathcal{F} \to \mathcal{G}, \psi: \mathcal{G} \to \mathcal{H} \), so that \( \Pi \) defines a functor from the category of presheaves on \( X \) to sheaves on \( X \).
1.9 Sheafification

Given any presheaf $\mathcal{F}$ on $X$, there is a canonical way of defining a sheaf $\mathcal{F}^+$ that in some sense is the sheaf that best approximates it. What prevents the presheaf $\mathcal{F}$ from being a sheaf is of course the failure of one or both of the sheaves axioms. To remedy this one must factor out all sections of $\mathcal{F}$ whose germs are everywhere zero, and one has to enrich $\mathcal{F}$ by adding enough new sections to be able to glue from local sections.

A nice and canonical way of doing this is by using the Godement sheaf $\Pi(\mathcal{F})$ associated with $\mathcal{F}$. Recall the canonical map $\kappa: \mathcal{F} \to \Pi(\mathcal{F})$ that sends a section $s$ of $\mathcal{F}$ over an open $U$ to the sequence of germs $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x = \Gamma(U, \Pi(\mathcal{F}))$. This map clearly kills the doomed sections, i.e., those whose germs all vanish. Now we can get an actual sheaf, by taking the image of $\kappa$ in $\Pi(\mathcal{F})$:

**Definition 1.34.** For a presheaf $\mathcal{F}$ on $X$, we define its sheafification $\mathcal{F}^+$ as the image sheaf $\text{im} \kappa$ in $\Pi(\mathcal{F})$. (In other words, $\mathcal{F}^+$ is the saturation of the subpresheaf $U \mapsto \text{im} \kappa_U$ in $\Pi(\mathcal{F})$).

Abusing notation slightly, we also write $\kappa_{\mathcal{F}}$, or simply $\kappa$, for the canonical map $\mathcal{F} \to \mathcal{F}^+$.

So what are the sections of $\mathcal{F}^+$? Over an open set $U \subseteq X$ they are characterized by those locally coming from $\mathcal{F}$. To be completely explicit about it, $\mathcal{F}^+$ is the subsheaf of $\Pi(\mathcal{F})$ given by

$$\mathcal{F}^+(U) = \{ (s_x) \in \prod_{x \in U} \mathcal{F}_x | (s_x) \text{ locally lies in } \mathcal{F} \}$$

where, as before, the sentence in the bracket means the following: for each $p \in U$, there exists an open neighbourhood $V \subseteq U$ containing $p$, and a section $t \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = t_v$ in $\mathcal{F}_v$.

**Lemma 1.35.** The sheafification $\mathcal{F}^+$ depends functorially on $\mathcal{F}$. Moreover, if $\mathcal{F}$ is a sheaf, $\kappa: \mathcal{F} \to \mathcal{F}^+$ is an isomorphism, so that $\mathcal{F}$ and $\mathcal{F}^+$ are canonically isomorphic.

**Proof.** Assume that $\phi: \mathcal{F} \to \mathcal{G}$ is a map between two presheaves. Let $s$ be section of $\Pi(\mathcal{F})$ over some open set $U$, so that $s$ locally lies in $\mathcal{F}$. In other words there is a covering $\{U_i\}$ of $U$ and sections $s_i$ of $\mathcal{F}$ over $U_i$ with $s|_{U_i} = \kappa_{\mathcal{F}}(s_i)$. Hence by (1.8.2) one has

$$\Pi(\phi)(s|_{U_i}) = \Pi(\phi)(\kappa_{\mathcal{F}}(s_i)) = \kappa_{\mathcal{G}}(\phi(s_i)).$$

This means that $\Pi(\phi)(s)$ lies locally in $\mathcal{G}$, and $\Pi(\phi)$ takes $\mathcal{F}^+$ into $\mathcal{G}^+$. Moreover,
there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\kappa_\mathcal{F}} & \mathcal{F}^+ \\
\phi & \Downarrow & \phi^+ \\
\mathcal{G} & \xrightarrow{\kappa_\mathcal{G}} & \mathcal{G}^+ \\
\end{array} \rightarrow \Pi(\mathcal{F}) \rightarrow \Pi(\mathcal{G})
\]

In case \( \mathcal{F} \) is a sheaf, the map \( \kappa_\mathcal{F} \) maps \( \mathcal{F} \) injectively into \( \Pi(\mathcal{F}) \) and \( \mathcal{F} = \text{im} \kappa_\mathcal{F} \) is its own saturation, hence \( \kappa_\mathcal{F} \) is an isomorphism.

**Proposition 1.36.** Given a presheaf \( \mathcal{F} \) on \( X \). Then the sheaf \( \mathcal{F}^+ \) and the natural map \( \kappa : \mathcal{F} \rightarrow \mathcal{F}^+ \) enjoys the universal property that any map of presheaves \( \mathcal{F} \rightarrow \mathcal{G} \) where \( \mathcal{G} \) is sheaf, factors through \( \mathcal{F}^+ \) in a unique way. This property characterises \( \mathcal{F}^+ \) up to unique isomorphism.

**Proof.** If \( \mathcal{G} \) in the diagram above is a sheaf, the map \( \kappa_\mathcal{G} : \mathcal{G} \rightarrow \mathcal{G}^+ \) is an isomorphism and \( \phi^+ \) takes values in \( \mathcal{G} \) and provides the wanted factorization. The uniqueness statement follows formally: Given two sheaves \( \mathcal{F}^+ \), \( \mathcal{F}'^+ \) satisfying the above, we get by the universal properties two maps \( \mathcal{F}^+ \rightarrow \mathcal{F}' \) and \( \mathcal{F}' \rightarrow \mathcal{F}^+ \), whose compositions are the identity.

From what we have proved above, we have that for a presheaf \( \mathcal{F} \) and a sheaf \( \mathcal{G} \) a bijection

\[ \text{Hom}_{\text{PrSh}}(\mathcal{F}, i(\mathcal{G})) = \text{Hom}_{\text{Sh}}(\mathcal{F}^+, \mathcal{G}) \]

where on the right hand side, \( i(\mathcal{G}) \) denotes \( \mathcal{G} \) but considered as a presheaf. A fancy way of restating this is that the sheafification functor \( \mathcal{F} \mapsto \mathcal{F}^+ \) is an adjoint to the forgetful functor \( i : \text{Sh} \rightarrow \text{PrSh} \) from sheaves to presheaves.

**Lemma 1.37.** Sheafification preserves stalks: \( \mathcal{F}_x = (\mathcal{F}^+_x) \) via \( \kappa_x \).

**Proof.** The map \( \kappa_x : \mathcal{F}_x \rightarrow (\mathcal{F}^+_x) \) is injective, since \( \mathcal{F}_x \rightarrow (\Pi(\mathcal{F}))_x \) is injective. To show that it is surjective, suppose that \( \overline{s} \in (\mathcal{F}^+_x) \). We can find an open neighbourhood \( U \) of \( x \) such that \( \overline{s} \) is the equivalence class of \( (s,U) \) with \( s \in \mathcal{F}^+ (U) \). By definition, this means there exists an open neighbourhood \( V \subseteq U \) of \( x \) and a section \( t \in \mathcal{F}(V) \) such that \( s|_V \) is the image of \( t \) in \( \Pi(\mathcal{F})(V) \). Clearly the class of \( (t,V) \) defines an element of \( \mathcal{F}_x \) mapping to \( \overline{s} \).

**Example 1.38 (Constant sheaves).** Recall Example 1.10 in which we showed that the constant presheaf given by \( A_X(U) = A \) is usually not a sheaf (where \( A \) is an abelian group). In this case, the sheafification is exactly the sheaf \( A_X \) defined by

\[ \Gamma(U, A_X) = \prod_{\pi_0(U)} A, \quad (1.9.1) \]
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where $\pi_0(U)$ denotes the set of connected components of the open set $U$. Indeed, a section $s$ of $\Pi(A')$ over $U$ is simply a collection of elements $(a_x)_{x \in U}$ where $a_x \in A$. Saying that this locally lies in $A$ means that for any $x \in U$ we have $a_y = a_x$ for all $y$ in some neighbourhood $V \subseteq U$. This implies that the section $s$ is constant on the connected component containing $U$, and hence (1.9.1) holds.

Example 1.39. By similar reasoning, the sheafification of the presheaf $C_b(X, \mathbb{R})$ of bounded continuous functions on $X = \mathbb{R}^n$ from Example 1.8 is just the sheaf of continuous functions $C(X, \mathbb{R})$.

1.10 Quotient sheaves and cokernels

The main application of the sheafification process we just described is to show that one may form quotient sheaves. So assume that $G \subseteq F$ is an inclusion of sheaves, and define a presheaf whose sections over $U$ is the quotient $F(U)/G(U)$. The restriction maps of $F$ and $G$ respect the inclusions $G(U) \subseteq F(U)$ and hence pass to the quotients $F(U)/G(U)$. We use these maps as the restriction maps for the quotient presheaf. The quotient sheaf $F/G$ is by definition the sheafification of this quotient presheaf. The quotient sits in the exact sequence

$$0 \rightarrow G \rightarrow F \rightarrow F/G \rightarrow 0.$$  

The cokernel $\text{coker} \phi$ of the map $\phi: F \rightarrow G$ of abelian sheaves is then just the quotient sheaf $G/\text{im} \phi$; it fits in the exact sequence

$$0 \rightarrow \text{ker} \phi \rightarrow F \xrightarrow{\phi} G \rightarrow \text{coker} \phi \rightarrow 0.$$  

Example 1.40. To illustrate why we have to sheafify in these constructions, consider again the exponential map in Example 1.26. The naive presheaf $U \mapsto \text{coker} \exp(U)$ is not a sheaf: the class of the function $f(z) = z$ restricts to 0 in $\text{coker} \exp$ on sufficiently small open sets, but it is itself not zero (since otherwise we would be able to define a global logarithm on $\mathbb{C} - 0$).

1.11 The pushforward of a sheaf

So far we have been interested in various constructions of sheaves on a fixed space $X$. These constructions become more interesting if we involve continuous maps between spaces, and try to transfer sheaves between them.

Let $X$ and $Y$ be two topological spaces with a continuous map $f: X \rightarrow Y$ between them. Assume that $F$ is an abelian sheaf on $X$. This allows us to define an abelian sheaf $f_*F$ on $Y$ by specifying the sections of $f_*F$ over the open set
1.11. The pushforward of a sheaf

$U \subseteq Y$ to be:

$$(f_*F)(U) = F(f^{-1}U),$$

and the restriction maps are those of $F$.

**Definition 1.41.** The sheaf $f_*F$ is called the **pushforward sheaf** or the **direct image** of $F$.

It is straightforward to see that $f_*F$ is a sheaf and not just a presheaf. Indeed, if $\{U_i\}$ is an open covering of $U$ a set of gluing data consists of sections $s_i \in \Gamma(U_i, f_*F) = \Gamma(f^{-1}U, F)$. That they match on the intersections means that they coincide in $\Gamma(U_i \cap U_j, f_*F) = \Gamma(f^{-1}U_i \cap f^{-1}U_j, F)$, and therefore can be glued to a section in $\Gamma(f^{-1}U, F) = \Gamma(U_i, f_*F)$ since $F$ is a sheaf. The Locality axiom also follows for $f_*F$ because it holds on $F$.

If you want, the pushforward sheaf $f_*F$ is just the restriction of $F$ to the subcategory of the open sets in $X$ which are inverse images of open sets in $Y$.

**Example 1.42.** Let $i : \{x\} \to X$ be the inclusion of a closed point in $X$. If $G$ is the constant sheaf of a group $G$ on $\{x\}$, then $i_*G$ is the skyscraper sheaf $G(x)$ from Example 1.11.

**Example 1.43.** Consider an affine variety $X \subseteq \mathbb{A}^n$ and let $i : X \to \mathbb{A}^n$ be the inclusion. For each open $U \subseteq \mathbb{A}^n$ define

$$\mathcal{I}_X(U) = \{ f \in \mathcal{O}_{\mathbb{A}^n}(U) | f(x) = 0 \ \forall \ x \in X \}.$$

Then $\mathcal{I}_X$ is a sheaf (of ideals) and we have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{A}^n} \to i_*\mathcal{O}_X \to 0$$

The pushforward is a functorial construction:

**Lemma 1.44.** If $g : X \to Y$ and $f : Y \to Z$ are continuous maps between topological spaces, and $F$ is a sheaf on $X$, one has

$$(f \circ g)_*F = f_*(g_*F).$$

(This is indeed an equality, not merely an isomorphism.)

**Proof.** This is a good exercise in manipulating sections. \qed

In particular, this means that $f_*$ defines a functor $\text{Sh}_X \to \text{Sh}_Y$.

**Lemma 1.45.** The functor $f_*$ is left exact. That is: Given an exact sequence of sheaves on $X$

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$
then the following sequence is exact

\[
0 \longrightarrow f_* F' \longrightarrow f_* F \longrightarrow f_* F''.
\]

**Proof.** As \( f_* F \) are sections of \( F \) over inverse images of opens in \( Y \), the lemma follows readily from the exactness of the sequence (1.6.2) on page 28 above. \( \square \)

**Exercise 4.** Denote by \( \{ * \} \) a one point set. Let \( f: X \to \{ * \} \) be the one and only map. Show that \( f_* F = \Gamma(X, F) \) (where strictly speaking \( \Gamma(X, F) \) stands for the constant sheaf on \( \{ * \} \)).

### 1.12 Inverse image sheaf

If \( F \) is a sheaf on \( X \) and \( U \subseteq X \) is an open subset, then \( F \) defines a sheaf on \( U \) by restriction of sections. For an arbitrary subset \( Z \subseteq X \), this naive restriction does not give a sheaf on \( Z \) so easily, because an open subset \( V \subseteq Z \) is usually not open in \( X \). However, as in the definition of stalks, we can take limits of \( F(U) \) as \( U \) runs over smaller and smaller open subsets containing \( V \). This gives the following definition:

**Definition 1.46.** If \( i: Z \to X \) is the inclusion of a closed subset \( Z \subseteq X \), we define the restriction \( F|_Z \) as the sheafification of the following presheaf:

\[
V \mapsto \lim_{U \supseteq V} F(U)
\]

We can extend this idea to any continuous map \( f: X \to Y \) and a sheaf \( G \) on \( Y \). This gives the *inverse image of \( G \)*, denoted \( f^{-1} G \), which we define as follows. As above, we know the values of \( G \) on open subsets \( V \) on \( Y \), but we want to define a sheaf on \( X \). Here the image of an open set \( U \), \( f(U) \) need not be open in \( Y \), but we can look at equivalence classes of \( G(V) \) as \( V \) gets closer and closer to \( f(U) \). This leads us to the following:

**Definition 1.47.** Let \( f: X \to Y \) be a continuous map; \( G \) a presheaf on \( Y \). We define the inverse image \( f^{-1} G \) as the sheafification of the following presheaf:

\[
f_{p}^{-1} G : U \mapsto \lim_{V \supseteq f(U)} G(V).
\]

(1.12.1)

Note in particular that the stalk of \( f^{-1} G \) at a point \( x \in X \) is isomorphic to
\[ \mathcal{G}_{f(x)} \]. Indeed, it suffices to verify this on the level of presheaves:

\[
(f_p^{-1}\mathcal{G})_x = \lim_{\rightarrow U \ni x} f_p^{-1}\mathcal{G}(U) = \lim_{\rightarrow U \ni x} \lim_{\rightarrow V \ni f(U)} \mathcal{G}(V)
\]

\[
= \lim_{\rightarrow V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}
\]

### 1.12.1 Adjoint property

The definition of \( f^{-1}\mathcal{G} \) is natural, but a little bit hard to work with for actual computations, since it involves both taking a direct limit over open sets and a sheafification. What’s more important is what this sheaf does: It is the adjoint of the functor \( f_* \) as a functor from \( \text{Sh}_X \) to \( \text{Sh}_Y \). The precise meaning behind that statement is the following:

**Theorem 1.48.** Let \( f : X \to Y \) be a morphism, and let \( \mathcal{F} \) be a sheaf on \( X \) and let \( \mathcal{G} \) be a presheaf on \( Y \). Then we have a natural bijection

\[
\text{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \simeq \text{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})
\]

which is functorial in \( \mathcal{F} \) and \( \mathcal{G} \).

This shows that the sheaf \( f^{-1}\mathcal{G} \) constructed before satisfies a very natural universal property. Sheaf morphisms \( \phi : \mathcal{G} \to f_*\mathcal{F} \) correspond bijectively to maps \( f^{-1}\mathcal{G} \to \mathcal{F} \). In particular, applying this to \( \mathcal{F} = f^{-1}\mathcal{G} \), and \( \mathcal{G} = f_*\mathcal{F} \) with the identity maps, we see that there are canonical maps

\[
\eta : \mathcal{G} \to f_*f^{-1}\mathcal{G},
\]

which is functorial in \( \mathcal{G} \), and

\[
\nu : f^{-1}f_*\mathcal{F} \to \mathcal{F}
\]

which is functorial in \( \mathcal{F} \).

To prove this theorem, it will be convenient to introduce some notation. Let us define an \( f\)-map \( \Lambda : \mathcal{G} \to \mathcal{F} \) to be a collection of maps \( \Lambda_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V)) \) indexed by open subsets \( V \subseteq Y \) such that

\[
\mathcal{G}(V) \xrightarrow{\Lambda_V} \mathcal{F}(f^{-1}V)
\]

commutes for all \( V' \subseteq V \subseteq Y \) open. With this definition in mind, we can now prove the following lemma, which implies the above theorem:
Lemma 1.49. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf on $X$ and let $\mathcal{G}$ be a sheaf on $Y$. There are canonical bijections between the following four sets:

1. The set of $f$-maps $\Lambda : \mathcal{G} \to \mathcal{F}$.
2. The set of sheaf maps $\mathcal{G} \to f_* \mathcal{F}$.
3. The set of sheaf maps $f^{-1} \mathcal{G} \to \mathcal{F}$.
4. The set of presheaf maps $f_p^{-1} \mathcal{G} \to \mathcal{F}$.

Proof. The bijection between (3) and (4) follows by the adjoint property of sheafification (because $\mathcal{F}$ is a sheaf), so it suffices to consider the sets in (1), (2) and (4).

Let us first consider (1) and (2). If we are given a map of sheaves $\phi : \mathcal{G} \to f_* \mathcal{F}$, we have a map $\phi_V : \mathcal{G}(V) \to f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}V)$ for each open set $V \subseteq Y$. By the definition of a sheaf map, this commutes with the various restriction mappings to smaller opens $V' \subseteq V$, so we get a well-defined $f$-map $\Lambda : \mathcal{G} \to \mathcal{F}$. Conversely, it is clear that any $f$-map $\Lambda$ appears from a map of sheaves in this manner, so we have established the desired bijection.

For the bijection between (1) and (4), suppose we are given a map of presheaves $f_p^{-1} \mathcal{G} \to \mathcal{F}$, so that we have a map

$$\lim_{W \supseteq f(U)} \mathcal{G}(W) \to \mathcal{F}(U).$$

Applying this to $U = f^{-1}(V)$, and composing with the map $\mathcal{G}(V)$ into the direct limit, we get a map $\mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. Again, this is compatible with the restriction maps to smaller open sets $V' \subseteq V$, so we get a well-defined $f$-map $\Lambda : \mathcal{G} \to \mathcal{F}$. Conversely, it is clear that any $f$-map $\Lambda$ arises in this manner, i.e., each $\Lambda_V$ factors as

$$\mathcal{G}(V) \xrightarrow{\Lambda_V} \mathcal{F}(f^{-1}V) \xrightarrow{\phi_{f^{-1}V}} \lim_{W \supseteq f(U)} \mathcal{G}(W)$$

for some map of presheaves $\phi : f_p^{-1} \mathcal{G} \to \mathcal{F}$: Just define $\phi_U$ for $U \subseteq X$ by composing $\Lambda_W$ with the restriction map to get a map $\mathcal{G}(W) \to \mathcal{F}(f^{-1}W) \to \mathcal{F}(U)$ for $W \supseteq f(U)$ - the fact that $\Lambda$ is an $f$-map means that we get an induced map in the direct limit. Over $U = f^{-1}V$, this $\phi$ makes the above diagram commute. This completes the proof of the lemma.

\[\square\]
Chapter 2

Schemes

2.1 The spectrum of a ring

Affine schemes are the building blocks in the theory of schemes. Every scheme is locally an affine scheme, i.e., every point has a neighbourhood which is isomorphic to an affine scheme. This is similar to the notion of a manifold, where a topological space is locally diffeomorphic to an open set in $\mathbb{R}^n$. However, affine schemes are infinitely more complicated than open sets in $\mathbb{R}^n$ and can have extremely bad singularities.

The affine schemes are basically just a geometric embodiment of rings. There is a one-to-one correspondence between rings and affine schemes, and furthermore, given two rings, the ring homomorphisms between them correspond are in bijective correspondence with the scheme maps between the two corresponding schemes (but the maps change direction). The situation is in perfect analogy with what happens for affine varieties, which are described by their coordinate rings. However, in that case there are heavy constraints on the rings involved; they are required to be integral domains of finite type over an algebraically closed field.

Following the Grothendieck maxim of always working in the maximal generality the theory allows, we start out working with a commutative\footnote{Many attempts have been made on defining non-commutative geometry, and there are some versions around. None however are so far completely satisfying.} ring $A$ with a unit element. We are going to define an affine scheme, written $\text{Spec} \, A$, called the prime spectrum of $A$ or just the spectrum of $A$ for short. As for any scheme the structure of $\text{Spec} \, A$ will have two layers: There is an underlying topological
2.1. The spectrum of a ring

space on which there lives a sheaf of rings.

Recall the situation for affine varieties and assume for a moment that $A$ is an algebra of finite type over the algebraically closed field $k$ that is an integral domain. In other words, $A$ is the coordinate ring of a variety, say $X$, and the elements of $A$ are the regular functions on $X$.

We know by Hilbert’s Nullstellensatz that points of $X$ correspond to the maximal ideals in $A$, a point $x$ corresponding to the ideal $m_x$ of regular functions vanishing at $x$, and conversely every maximal ideal is the vanishing ideal of a point.

Closed algebraic sets contained in $X$ are zero sets of ideals $a$ in $A$; i.e., they are of the shape $V(a) = \{ x \in X \mid f(x) = 0 \text{ all } f \in a \}$, or with the correspondence of points and maximal ideals in mind this may be phrased as $V(a) = \{ m \mid m \subseteq A \text{ maximal and } a \subseteq m \}$. The variety $X$ comes with the Zariski topology whose open sets are the just the complements of the closed algebraic sets $V(a)$.

There is a natural way of generalizing this to more general rings, and this involves replacing maximal ideals with prime ideals.

**Definition 2.1.** For a ring $^2A$ we define its *spectrum* as

$$\text{Spec } A = \{ p \mid p \subseteq A \text{ is a prime ideal } \}.$$  

The set Spec $A$ has a topology, which generalizes the Zariski topology on a variety, and the definitions are very similar: The closed sets in Spec $A$ are defined to be those of the form

$$V(a) = \{ p \in \text{Spec } A \mid p \supseteq a \}$$

where $a$ is any ideal in $A$. Of course one has verify that the axioms for a topology are satisfied. For closed sets the wording of the axioms is that the union of two closed and the intersection of any number (finite or infinite) must be closed. And of course, both the whole space and the empty set must be closed. The family of subsets of the form $V(a)$ indeed satisfies these axioms, as we prove in the following lemma:

**Lemma 2.2.** Let $A$ be a ring and assume that $\{ a_i \}_{i \in I}$ is a family of ideals in $A$. Let $a$ and $b$ be two ideals in $A$. Then the following three statements hold true:

- $V(a \cap b) = V(a) \cup V(b) = V(ab)$,
- $V(\sum_i a_i) = \bigcap_i V(a_i)$,
- $V(A) = \emptyset$ and $V(0) = \text{Spec } A$.

---

$^2$From now on the term ‘ring’ stands for a commutative ring having a unit element.
Chapter 2. Schemes

Proof. Prime ideals are by definition proper ideals, so \( V(A) = \emptyset \). Also, the zero-ideal is contained in every ideal, so \( V(0) = \text{Spec } A \). This proves the last statement. The second follows as easily, since the sum of a family of ideals is contained in an ideal if and only each of the ideals is.

The first of the three statements needs an argument. The inclusion \( V(a) \cup V(b) \subseteq V(a \cap b) \) is clear, so we need to show that \( V(a \cap b) \subseteq V(a) \cup V(b) \). Let \( p \) be a prime ideal such that \( a \cap b \subseteq p \). If \( b \notin p \), there is an element \( b \in b \) with \( b \notin p \). But since \( ab \in a \cap b \) for all \( a \in a \), and \( p \) contains \( a \cap b \), one has \( ab \in p \). The ideal \( p \) is prime, so we deduce that \( a \in p \), and consequently one also has the inclusion \( a \subseteq p \).

Corollary 2.3. The set \( \{ V(a) \} \) where \( a \) runs through all the ideals in \( A \) is the family of closed sets for a topology on \( \text{Spec } A \).

The next lemma is about what inclusions between the closed sets of \( \text{Spec } A \) hold, and we recognise them all from the theory of varieties.

Lemma 2.4. One has

\[ V(a) \subseteq V(b) \text{ if and only if } \sqrt{a} \supseteq \sqrt{b}. \text{ In particular, one has } V(a) = V(\sqrt{a}). \]

\[ V(a) = \emptyset \text{ if and only if } a = A. \]

\[ V(a) = \text{Spec } A \text{ if and only if } a \subseteq \sqrt{A}. \]

Proof. The main point is that the radical of an ideal equals the intersection all the prime ideals containing it, or expressed with a formula:

\[ \sqrt{a} = \bigcap_{p \supseteq a} p. \tag{2.1.1} \]

In particular, we see that \( a \) and \( \sqrt{a} \) are contained in the same prime ideals, so that \( V(a) = V(\sqrt{a}) \). To show the first claim in the lemma we begin with assuming that \( V(a) \subseteq V(b) \). From (2.1.1) we then obtain

\[ \sqrt{a} = \bigcap_{p \in V(a)} p \supseteq \bigcap_{p \in V(b)} p = \sqrt{b}. \]

Now, if \( p \) lies in \( V(a) \) and \( \sqrt{a} \supseteq \sqrt{b} \), the chain of inclusions \( p \supseteq \sqrt{a} \supseteq \sqrt{b} \supseteq b \) imply the inclusion \( p \in V(b) \) as well. This proves the first claim.

The second claim follows from the fact that \( \sqrt{a} = A \) if and only if \( a = A \), and the third holds as the radical of \( A \) by definition satisfies \( \sqrt{A} = \sqrt{(0)} \). \( \square \)
Remark 2.5 (A few words about notation). Many authors, unlike Hartshorne, prefer to denote points in \( \text{Spec} \, A \) in a way that distinguish points in \( \text{Spec} \, A \) from the corresponding ideals in \( A \), for example by denoting points by latin letters like \( x \) and \( y \), and the corresponding prime ideals by \( p_x \) or \( p_y \). This might seem superfluous, but there is a reason: When we start working with general schemes, points do no more correspond to prime ideals in a canonical way, so in that case its not only natural but required.

2.1.1 Generic points

The Zariski topology on \( \text{Spec} \, A \) differs greatly from the standard topology in the sense that points can fail to be closed. In fact, the next proposition implies that a point \( x \in \text{Spec} \, A \) is closed if and only if the corresponding ideal is maximal.

Proposition 2.6. If \( p \) is a prime ideal of \( A \), the closure \( \{ p \} \) of the one-point set \( \{ p \} \) in \( \text{Spec} \, A \) equals the closed set \( V(p) \).

Proof. If the point \( p \) is contained in a smaller closed set than \( V(p) \), there is an ideal \( a \) with \( p \in V(a) \subseteq V(p) \). By lemma 2.4 above this implies that \( \sqrt{p} \subseteq \sqrt{a} \subseteq p \), from which it follows that \( p = \sqrt{a} \), and hence we conclude that \( V(a) = V(p) \).

In general there are of course typically a lot of prime ideals which are not maximal. In fact, the rings having the property that every prime ideal is maximal are quite special: in the noetherian case they corresponds exactly to the artinian rings (in which case the spectrum consists of a finite set of points).

Definition 2.7. We say that a point \( x \) is a generic point of the closure \( V(x) = \{ x \} \) of the singleton \( \{ x \} \).

So for example, for an integral domain \( A \), the zero ideal \((0)\) is prime, and corresponds to the generic point of all of \( \text{Spec} \, A \).

2.1.2 Irreducible closed subsets

Recall that a topological space \( X \) is said to be irreducible if it can not be written as the union two proper closed subsets; that is, if \( X = Z \cup Z' \) with \( Z \) and \( Z' \) both being closed subsets, then either \( Z = X \) or \( Z' = X \). This notion is well known from the theory of varieties, where we recall that a closed algebraic subset \( V(a) \) of the affine space \( \mathbb{A}^n(k) \) is irreducible if and only if the radical \( \sqrt{a} \) is a prime. The corresponding statement is true in \( \text{Spec} \, A \):

Lemma 2.8. A closed subset \( Z \subseteq \text{Spec} \, A \) is irreducible if and only if \( Z \) is of the form \( Z = V(p) \) for some prime ideal \( p \).
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Proof. We have seen that the closure \( \{p\} \) is irreducible. For the reverse direction, we let \( V(a) \subseteq \text{Spec } A \) be an a closed subset. Recall that one has \( \sqrt{a} = \bigcap_{p \subseteq a} p \), and if \( \sqrt{a} \) is not prime, there are more than one prime involved in the intersection. We may divide them into two different groups thus representing \( \sqrt{a} \) as the intersection \( \sqrt{a} = b \cap b' \) where \( b \) and \( b' \) are ideals both different from \( a \). One concludes that \( V(a) = V(b) \cup V(b') \), so it is not irreducible.

2.2 Functoriality

Let \( A \) and \( B \) be two rings and let \( \phi: A \to B \) be a ring homomorphism. Let us check that the inverse image of a prime ideal \( p \subseteq B \) really is a prime ideal: That \( ab \in \phi^{-1}(p) \) means that \( \phi(ab) = \phi(a)\phi(b) \in p \), so at least one of \( \phi(a) \) or \( \phi(b) \) lies in \( p \). Hence sending \( p \) to \( \phi^{-1}(p) \) gives us a well defined map \( \text{Spec } B \to \text{Spec } A \), and we shall denote this map by \( \phi^* \).

Lemma 2.9. Assume that \( \phi: A \to B \) is a map of rings. Then the induced map between the ring spectra \( \phi^*: \text{Spec } B \to \text{Spec } A \) is continuous.

Proof. We need to show that inverse images of closed sets are closed, so let \( a \subseteq A \) be an ideal. Then the following equalities hold

\[
(\phi^*)^{-1}(V(a)) = \{ p \subseteq B \mid \phi^{-1}(p) \supseteq a \} = \{ p \subseteq B \mid p \supseteq \phi(a) \} = V(\phi(a)B),
\]

since \( p \supseteq \phi(a) \) if and only if \( \phi^{-1}(p) \supseteq a \) as \( \phi^{-1}(\phi(a)) \supseteq a \). Hence \( (\phi^*)^{-1}(V(a)) \) is closed.

Note that is also a functorial construction since clearly \( \phi^{-1}(\psi^{-1}(p)) = (\psi\phi)^{-1}(p) \) where \( \phi \) and \( \psi \) are two composable ring homomorphisms, and of course one has \( \text{id}_A^* = \text{id}_{\text{Spec } A} \). This means that we have defined a contravariant functor from the category rings of rings to the category top of topological spaces. It sends a ring \( A \) to the prime spectrum \( \text{Spec } A \) and a map \( \phi: A \to B \) of rings to the continuous map \( \phi^*: \text{Spec } B \to \text{Spec } A \).

We include two prototype examples:

Example 2.10 (Spec\( (A/a) \)). If \( a \subseteq A \) is an ideal, the ring homomorphism \( A \to A/a \) induces a map \( \text{Spec } (A/a) \to \text{Spec } A \). Note that prime ideals in \( A/a \) correspond exactly to prime ideals \( p \) in \( A \) containing \( a \): This is exactly our closed set \( V(a) \). Thus the map on spectra corresponds to the inclusion of \( V(a) \) in \( \text{Spec } A \). This is the prototype of a closed immersion. We will discuss these in detail later.

Example 2.11 (Spec \( A_f \)). For an element \( f \in A \) we can consider the localization \( A_f \) of \( A \) in which \( f \) is inverted, and the corresponding ring homomorphism
2.3. Examples

A \rightarrow A_f$. Recall that the prime ideals in the localized ring $A_f$ are in a natural one-to-one correspondence with the prime ideals $p$ of $A$ not containing $f$, that is, the complement $D(f) = \text{Spec } A - V(f)$. Thus the induced map $\text{Spec } A_f \rightarrow \text{Spec } A$ corresponds to the inclusion of the open set $D(f)$ of $\text{Spec } A$. This is an example of an open immersion.

2.3 Examples

Example 2.12 (Fields). If $K$ is a field, the prime spectrum $\text{Spec } K$ has only one element the zero-ideal being the only ideal $i K$. This also holds true for local rings with the property that all elements in the maximal ideals are nilpotent, i.e., the radical $\sqrt{(0)}$ of the ring is a maximal ideal. For noetherian rings this is equivalent to the ring being an artinian local ring.

Exercise 5. Show that the converse of the last statement in the example above is true. That is if $\text{Spec } A$ has just one element, $A$ is a local ring all whose non-units are nilpotent.

Example 2.13. The ring $R = \mathbb{C}[x]/(x^2)$ is not a field, but it has only one prime ideal (namely the ideal $(x)$). Note that $(0)$ is not prime, since $x^2 = 0$, but $x \not\in (0)$.

Example 2.14 (Discrete valuation rings). A discrete valuation ring $A$ has only two prime ideals, the maximal ideal $m$ and the zero ideal $(0)$. So its prime spectrum $\text{Spec } A$ has just two points, and $\text{Spec } A = \{ \eta, x \}$ with $x$ corresponding to the maximal ideal $m$ and $\eta$ corresponding to $(0)$. The point $x$ is closed in $\text{Spec } A$, and therefore $\{ \eta \} = X - x$ is open. So $\eta$ is an open point! The point $\eta$ is the generic point of $\text{Spec } A$; its closure is the whole $\text{Spec } A$.

The open sets of $X$ are $\emptyset, X, \{ \eta \}$. In particular $\text{Spec } A$ is not Hausdorff, as $\eta$ is contained in the only open set containing $x$, the whole space.

Example 2.15 (Spec $\mathbb{Z}$). There are two types of prime ideals in $\mathbb{Z}$. There is the zero-ideal and there are the maximal ideals $(p)\mathbb{Z}$, one for each prime $p$. These are closed points in $\text{Spec } \mathbb{Z}$, however one has $V(0) = \text{Spec } \mathbb{Z}$, so the point corresponding to the zero-ideal is a generic point, the closure is the whole of $\text{Spec } \mathbb{Z}$.
The reduction mod $p$-map $\mathbb{Z} \to \mathbb{F}_p$ induces a map $\text{Spec } \mathbb{F}_p \to \text{Spec } \mathbb{Z}$. The one and only point in $\text{Spec } \mathbb{F}_p$ is sent to the point in $\text{Spec } \mathbb{Z}$ corresponding to the maximal ideal $(p)$ generated by $p$. On the level of structure sheaves, the map is just the restriction map.

The inclusion $\mathbb{Z} \subseteq \mathbb{Q}$ of the integers in the field of rational numbers induces likewise a map $\text{Spec } \mathbb{Q} \to \text{Spec } \mathbb{Z}$, that sends the unique point in $\text{Spec } \mathbb{Q}$ to the generic point $\eta$ of $\text{Spec } \mathbb{Z}$.

Every ring $A$ has (since by our convention it contains a unit element) a prime ring, i.e., the subring generated by 1. Hence there is a canonical (and in fact, unique) map $\text{Spec } A \to \text{Spec } \mathbb{Z}$.

This canonical map factors through the map $\text{Spec } \mathbb{F}_p \to \text{Spec } \mathbb{Z}$ described above if and only if $A$ is of characteristic $p$. This is clear if one considers the diagram on the ring level:

$$\begin{array}{ccc}
\mathbb{Z} & \rightarrow & \mathbb{F}_p \\
\downarrow & & \downarrow \\
A & \rightarrow & A
\end{array}$$

the canonical map $\mathbb{Z} \to A$ goes via $\mathbb{F}_p$ if and only if $A$ is of characteristic $p$.

**Exercise 6.** In the same vain, show that a ring $A$ is $\mathbb{Q}$-algebra (that is, contains a copy of $\mathbb{Q}$) is and only if the canonical map $\text{Spec } A \to \text{Spec } \mathbb{Z}$ factors through the generic point $\text{Spec } \mathbb{Q} \to \text{Spec } \mathbb{Z}$.

**Example 2.16.** If $R$ is an integral domain, if an ideal of the form $(r)$ for $r \in R$ is prime, then $r$ is irreducible (meaning, if $r = ab$, then either $a$ or $b$ is a unit). Indeed, if $r = ab$, then since $(r)$ is prime, then without loss of generality $a \in (r)$, so $a = xr$ for some $x \in R$, and hence $r = ab = xrb$, so $bx = 1$ (since $R$ is an integral domain).

The converse however is not true. Let $R = \mathbb{Z}[\sqrt{-5}]$. Then $2 \in R$ is irreducible, and $2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, so the ideal (2) is not prime (the factors involving $\sqrt{-5}$ are not multiples of 2).

**Example 2.17** (PIDs). If $R$ is a principal ideal domain (PID), any ideal is of the form $(r)$ for some $r \in R$. Also, every PID is a unique factorization domain (same argument as over $\mathbb{Z}$), so the points of $\text{Spec } R$ corresponds to $(0)$ and the set of irreducible elements (modulo units).
2.3. Examples

Example 2.18. For $R = \mathbb{C}[[x]]$, the non-zero ideals are $(x^n)$. $R$ is a PID, so the prime ideals are $(0)$ and $(x)$.

Example 2.19. The ring $R = \mathbb{C}[[x^2, x^3]]$ is not a PID. The non-zero ideals are $(x^n + cx^{n+1})$ and $(x^n, x^{n+1})$. Of these, only $(x^2, x^3)$ is prime (Prove this!).

Example 2.20 (Spec $\mathbb{Z}[i]$). The inclusion $\mathbb{Z} \subseteq \mathbb{Z}[i]$ induces a continuous map

$$\phi : \text{Spec } \mathbb{Z}[i] \rightarrow \text{Spec } \mathbb{Z}.$$ 

We will study Spec $\mathbb{Z}[i]$ by studying the fibers of this map. If $p \in \mathbb{Z}$ is a prime, the fiber over $(p)\mathbb{Z}$ consists of those primes that contain $(p)\mathbb{Z}[i]$. These come in three flavours:

- $p$ stays prime in $\mathbb{Z}[i]$ and the fiber over $(p)\mathbb{Z}$ has one element, namely $(p)\mathbb{Z}[i]$. This happens if and only if $p \equiv 3 \mod 4$.

- $p$ splits into a product of two different prime ideals, and the fiber consists of these two points. This happens if and only if $p \equiv 1 \mod 4$.

- $p$ factors into a product of repeated primes (such a prime is said to ‘ramify’ and only (2) does this here).

Let us explain the last point. The ideal $(2)\mathbb{Z}[i]$ is not radical: Note that

$$(2)\mathbb{Z}[i] = (2i)\mathbb{Z}[i] = (1 + i)^2\mathbb{Z}[i].$$

So the fiber consists of the single prime $(1 + i)\mathbb{Z}[i]$.

The following picture shows Spec($\mathbb{Z}[i]$):

![Diagram of Spec($\mathbb{Z}[i]$)](image)

Now here’s where the Galois group comes into play. The group $G = \text{Gal}(\mathbb{Q}[i]/\mathbb{Q})$ permutes the primes in Spec($\mathbb{Z}[i]$) sitting over any $p$ in Spec($\mathbb{Z}$). In this case

$$G = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$$
where \( g \) is the order two element given by complex conjugation. So for instance if you look at the primes sitting over say \((5)\), namely \((2 + i)\) and \((2 - i)\), you see that complex conjugation maps one into the other.

So we think of \( \text{Spec}(\mathbb{Z}[i]) \) as sitting over \( \text{Spec}(\mathbb{Z}) \) and \( G \) permuting the fibers over in the base space \( \text{Spec}(\mathbb{Z}) \).

**Exercise 7.** (i) Show that the fiber of \( \phi \) over a prime \( p \) is homeomorphic to

\[
\text{Spec} \mathbb{F}_p[x]/(x^2 + 1)
\]

and that \( \dim_{\mathbb{F}_p} \mathbb{F}_p[x]/(x^2 + 1) = 2 \). Hint: Use that \( \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2 + 1) \).

(ii) Show that \( \mathbb{F}_p[x]/(x^2 + 1) \) is a field if and only if \( x^2 + 1 \) does not have a root in \( \mathbb{F}_p \).

(iii) Show that \( \mathbb{F}_p[x]/(x^2 + 1) \) is a field if and only if \((p)\mathbb{Z}[i] \) is a prime ideal.

**Example 2.21 (\( \text{Spec} \mathbb{R}[t] \)).** It is instructive to take a closer look at the prime spectrum \( \text{Spec} \mathbb{R}[t] \). The ring \( \mathbb{R}[t] \) being a principal ideal domain, any prime ideal has the shape \((f(t))\) where \( f(t) \) is an irreducible polynomial, and we may well assume \( f(t) \) to be monic. From the the fundamental theorem of algebra it follows that either \( f \) is linear, that is, \( f(t) = t - a \) for \( a \in \mathbb{R} \), or \( f \) is quadratic with to conjugate complex and non-real roots; That is, \( f(t) = (t - a)(t - \overline{a}) \) with \( a \in \mathbb{C} \) but \( a \notin \mathbb{R} \). This shows that the closed points in \( \text{Spec} \mathbb{R}[t] \) may be identified the set of pairs \( \{a, \overline{a}\} \) for \( a \in \mathbb{C} \). And of course, there is the generic point \( \eta \) corresponding to the zero ideal.

The Galois group \( G = \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{C}[t] \) via the conjugation map \( \sigma \); that is, the map that sends a polynomial \( f(t) = \sum a_i t^i \) into \( \sum \sigma_i a_i t^i \). The corresponding scheme map from \( \text{Spec} \mathbb{C}[t] \) to \( \text{Spec} \mathbb{C}[t] \) defines an action of \( \mathbb{Z}/2\mathbb{Z} \) on \( \text{Spec} \mathbb{C}[t] \), and the points of \( \text{Spec} \mathbb{R}[t] \) are the quotient of points of \( \text{Spec} \mathbb{C}[t] \) by this action. We just saw this holds for closed points, and clearly the generic point of \( \text{Spec} \mathbb{C}[t] \) is invariant and corresponds to the generic point of \( \text{Spec} \mathbb{R}[t] \). The quotient map is the map \( \text{Spec} \mathbb{C}[t] \to \text{Spec} \mathbb{R}[t] \) induced by the inclusion \( \mathbb{R}[t] \subseteq \mathbb{C}[t] \).

**Example 2.22.** Generalizing the previous example, let \( K \subseteq L \) be a Galois extension of fields with Galois group \( G \); i.e., \( G \) acts on \( L \) and \( K = L^G \), the subfield of invariant elements. This action induces and action on \( L[t] \) by letting the action of a group element \( g \) on the polynomial \( f(t) = \sum a_i t^i \) be \( g(f) = \sum g(a_i) t^i \).

The map \( K[t] \subseteq L[t] \) induces a map \( \pi : \text{Spec} L[t] \to \text{Spec} K[t] \). Show that \( \pi \) has the following universal property: For any affine scheme \( \text{Spec} A \) and any invariant morphism \( \psi : \text{Spec} L[t] \to \text{Spec} A \), i.e., a morphism such that \( \psi \circ g = \psi \) for all \( g \), the map \( \psi \) factors through \( \pi \); that is, there is a \( \psi' : \text{Spec} K[t] \to \text{Spec} A \)
2.3. Examples

with \( \psi = \psi' \circ \pi \). The diagram looks like

\[
\begin{array}{ccc}
\text{Spec } L[t] & \xrightarrow{g} & \text{Spec } L[t] \\
\downarrow & & \downarrow \\
\text{Spec } K[t] & & \text{Spec } A
\end{array}
\]

**Exercise 8.** Let \( A' \to A \) be a surjection of commutative rings whose kernel is nilpotent. Show that the map \( \text{Spec } A \to \text{Spec } A' \) is a homeomorphism.

**Exercise 9 (Direct products of rings).** Assume that \( e_1, \ldots, e_r \) is a complete set of orthogonal idempotents in the ring \( A \), meaning that one has \( 1 = e_1 + e_2 + \cdots + e_r \), that \( e_i e_j = 0 \) when \( i \neq j \), and that \( e_i^2 = e_i \). Such a set of idempotents corresponds to a decomposition of \( A \) as the direct product \( A = A_1 \times A_2 \times \cdots A_r \) where \( A_i = e_i A \).

(i) Let \( \mathfrak{a} \) be an ideal in \( A \) and let \( \mathfrak{a}_i = \mathfrak{a} e_i \). Show that \( \mathfrak{a}_i \) is an ideal in \( A_i \) and that \( \mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2 + \cdots + \mathfrak{a}_r \).

(ii) Show that \( \mathfrak{a} \) is a prime ideal if and only \( \mathfrak{a}_i = A_i \) for all but one index \( i_0 \) and \( \mathfrak{a}_{i_0} \) is a prime ideal in \( A_{i_0} \).

(iii) Show that \( \text{Spec } A \) is not connected: \( \text{Spec } A = \text{Spec } A_1 \cup \text{Spec } A_2 \cup \cdots \cup \text{Spec } A_r \), the union being disjoint and each \( \text{Spec } A_i \) being open in \( \text{Spec } A \).

**Exercise 10.** Show that \( \text{Spec } A \) is connected if and only if the only two idempotents in \( A \) are 1 and 0.

**Exercise 11.** Let \( \mathfrak{a} \subseteq A \) be an ideal. Show that \( \sqrt{\mathfrak{a}} = \bigcap_{p \supseteq \mathfrak{a}} p \). Hint: If \( f \notin \sqrt{\mathfrak{a}} \) the ideal \( \mathfrak{a} A_f \) is an ideal in the localization \( A_f \), hence contained in a maximal ideal.

**Exercise 12.** Let \( k \) be an algebraically closed field. Let \( X \subseteq \mathbb{A}^n_k \) be the set of closed points with the induced topology. Show that \( X \) is homeomorphic to the variety \( \mathbb{A}^n(k) \), that is \( k^n \) with the Zariski topology.

2.3.1 The world of schemes vs the world of varieties

Comparing the world of varieties with the world of schemes, one notices many similarities, but there are also quite a few dramatic differences. In some sense the category of varieties over an algebraically closed field \( k \) is a full subcategory of the category of schemes, as we are going to elucidate in full later on. Now we just give a glimpse into the situation taking a look at the simplest case, namely the affine varieties \( \mathbb{A}^n(k) \).

If one lets \( A = k[x_1, \ldots, x_n] \) denote the ring of polynomials over \( k \) in the variables \( x_1, \ldots, x_n \), one knows thanks to Hilbert’s Nullstellensatz that then
maximal ideals in \( A \) stand in a one-to-one correspondence with the points of the affine space \( \mathbb{A}^n(k) \); they are all of the form \( (x_1 - a_1, \ldots, x_n - a_n) \) with the \( a_i \)'s being elements in \( k \).

The affine variety \( \mathbb{A}^n(k) \) is the subset of the scheme \( \mathbb{A}^n_k = \text{Spec} \ A \) consisting of the closed points, or the points in \( \text{Spec} \ A \) corresponding to maximal ideals. The good old Zariski topology on the variety \( \mathbb{A}^n_k \) is the induced topology. Indeed, the closed sets of the induced topology are by definition all of the form \( V(a) \cap \mathbb{A}^n(k) \), which clearly equals the “old” \( V(a) \) from the world of varieties. The scheme \( \text{Spec} \ A \) has however many more points that are not closed. There are a lot of prime ideals in \( A \) that are not maximal; at least if \( n \geq 2 \).

Example 2.23 (The affine line \( \mathbb{A}^1_k = \text{Spec} \ k[x] \)). In the polynomial ring \( k[x] \) all ideals are principal, and all non-zero prime ideals are maximal. They are of the form \((f(x))\) where \( f(x) \) is an irreducible polynomial, hence of the form \((x - a)\) as we have assumed that \( k \) is algebraically closed. There is only one non-closed point in \( \text{Spec} \ k[x] \), the generic point \( \eta \) corresponding to the zero-ideal. The closure \( \{ \eta \} \) is the whole line \( \mathbb{A}^1_k \).

Example 2.24 (The affine plane \( \mathbb{A}^2_k = \text{Spec} \ k[x_1, x_2] \)). In this case the maximal ideals are of the shape \((x_1 - a_1, x_2 - a_2)\) and constitute all the closed points of \( \mathbb{A}_k^2 \). Every irreducible polynomial \( f(x_1, x_2) \) generates a prime ideal \( p_f = (f) \) which together with the zero ideal are all non-maximal prime ideals. In addition to the point \( p_f \), the points of of the closed set \( V(f(x_1, x_2)) \) are the closed points corresponding to ideals \((x_1 - a_1, x_2 - a_2)\) containing \( f(x_1, x_2) \). This condition being equivalent to \( f(a_1, a_2) = 0 \), so the closed points of \( V(f(x_1, x_2)) \) correspond to what one in the world of varieties would call the curve given by the equation \( f(x_1, x_2) = 0 \).

2.4 Distinguished open sets

There is no way to describe the open sets in \( \text{Spec} \ A \) as simply and elegantly as the closed sets can be described. However there is a natural basis for the topology on \( \text{Spec} \ A \) whose sets are easily defined, and it turns out to be very useful. For an element \( f \in A \), we let \( D(f) \) be the complement of the closed set \( V(f) \), that is

\[
D(f) = \{ p \mid f \notin p \} = X - V(f).
\]

These are clearly open sets and are called distinguished open sets.

Lemma 2.25. The open sets \( D(f) \) form a basis for the topology of \( \text{Spec} \ A \) when \( f \) runs through \( A \).

Proof. We need to show that any open subset \( U \) of \( \text{Spec} \ A \) can be written as the union of distinguished open sets, that is, of sets of the form \( D(f) \). Let the
complement $U^c$ of $U$ be given as $U^c = V(a)$, and choose a set $\{ f_i \}$ of generators for the ideal $a$ (not necessarily a finite set!). Then we have

\[ U = V(a)^c = V(\sum_i (f_i)^c) = (\bigcap_i V(f_i))^c = \bigcup_i D(f_i). \]

\[ \square \]

**Exercise 13.** Show that $D(f) = \emptyset$ if and only if $f$ is nilpotent. Hint: Use that $\sqrt{(0)} = \bigcap_{p \in \text{Spec } A} p$.

### 2.4.1 Basic properties of distinguished sets

We now give some of the basic properties of the distinguished open sets that will be needed later.

**Lemma 2.26.** A family $\{ D(f_i) \}$ forms an open covering of $\text{Spec } A$ if and only if one may write $1 = \sum_i a_i f_i$ with the $a_i$'s being elements from $A$ only a finite number of which are non-zero.

**Proof.** One has $V(\sum_i (f_i)^c) = (\bigcap_i V(f_i))^c = \bigcup_i D(f_i)$, so the open sets $D(f_i)$ constitute a covering if and only if the ideal generated by the $f_i$'s is the whole ring $A$, that is 1 belongs there. \[ \square \]

This shows in particular that any covering by distinguished open sets may be reduced to a finite one. As the distinguished sets form a basis for the topology, any open cover may be refined to one whose sets all are distinguished, and hence it can be reduced to a finite covering. A topological space with this property is quasi-compact.\(^3\)

**Lemma 2.27.** One has $D(f) \cap D(g) = D(fg)$.

**Proof.** If $p$ is a prime ideal, $f \notin p$ and $g \notin p$ both hold true if and only if $fg \notin p$. \[ \square \]

**Lemma 2.28.** One has $D(g) \subseteq D(f)$ if and only if $g^n \in (f)$ for a suitable natural number $n$. In particular one has $D(f) = D(f^n)$ for all natural numbers $n$.

**Proof.** The inclusion $D(g) \subseteq D(f)$ holds if and only if $V(f) \subseteq V(g)$, and by Lemma 2.4 om page 45 this is true if and only if $(g) \subseteq \sqrt{(f)}$, i.e., if and only if $g^n \in (f)$ for a suitable $n$. \[ \square \]

\(^3\)The terminology is a little bit unfortunate: spaces in which every open cover has a finite subcover are usually called ‘compact’. However, some authors reserve the term ‘compact’ for quasi-compact and Hausdorff, and this jargon has caught on in algebraic geometry literature.
Chapter 2. Schemes

In fact, the inclusion $D(g) \subseteq D(f)$ is equivalent to the localization map $\tau: A \to A_g$ extending to a map $A_f \to A_g$. Indeed, $\tau$ extends $\Leftrightarrow \tau(f)$, i.e., $f$ regarded as an element in $A_g$ is invertible $\Leftrightarrow$ there is an $c \in A$ and an $m \in N$ with $g^m(fc - 1) \Leftrightarrow g^m = cf$ for some $c$ and some $m$.

2.5 The structure sheaf on $\text{Spec } A$

We have now come to point where we define a structure sheaf on the topological space $\text{Spec } A$. This is a sheaf of rings $\mathcal{O}_{\text{Spec } A}$ whose stalks all are local rings, so that the pair $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a ringed space.

The two paramount properties of the structure sheaf $\mathcal{O}_{\text{Spec } A}$ are the following:

- Sections over distinguished opens: $\Gamma(D(f), \mathcal{O}_{\text{Spec } A}) = A_f$.
- Stalks: $\mathcal{O}_{\text{Spec } A, x} = A_{p_x}$.

These are two properties that one uses all the time when working with the structure sheaf on an affine scheme. Moreover, as we will see, they even characterize the structure sheaf uniquely.

2.5.1 Motivation

The model for the structure sheaf $\mathcal{O}_{\text{Spec } A}$ on the prime spectrum $\text{Spec } A$ of a ring $A$ is the sheaf of regular functions on an affine variety $X$, and when we subsequently define it we mimic what happens on varieties, so it is worthwhile recalling that situation.

Let $X$ be a variety with coordinate $A$; that is, $A$ is the ring of globally defined regular functions on $X$. The fraction field $K$ of $A$ is the field of rational functions on $X$, and the regular functions on any open set are elements in $K$.

For different open sets $U$ the set of functions regular in $U$ form different subrings of $K$, and if $V \subseteq U$ is an open contained in $U$, the ring of regular functions $\Gamma(V, \mathcal{O}_X)$ on $V$ of course contains the ring $\Gamma(U, \mathcal{O}_X)$ of those regular on the bigger set $U$. The restriction map is nothing but the inclusion $\Gamma(U, \mathcal{O}_X) \subseteq \Gamma(V, \mathcal{O}_X)$; it does nothing to the functions in $K$; just considers the functions in $\Gamma(U, \mathcal{O}_X)$ to lie in $\Gamma(V, \mathcal{O}_X)$.

Those function regular on the distinguished open set $D(f)$ are allowed to have powers of $f$ in the denominator, and they constitute the subring $A_f \subseteq K$ of elements of the form $af^{-n}$. If $D(g)$ is another distinguished open set that is contained in $D(f)$, i.e., $D(g) \subseteq D(f)$, one has $f = cg^n$, and hence $A_g \subseteq A_f$ (since $f^{-1} = cg^{-n}$).

The general situation differs only significantly from the situation of varieties in that the localization maps $A_f \to A_g$ are no more necessarily injective, but the formalism is the same.
2.5. The structure sheaf on Spec $A$

The standard example of restriction maps not being injective is the union of two varieties; to make it simple say the union of two line in the plane given by the equation $xy = 0$; i.e., $X = \text{Spec } k[x, y]/(xy)$. The regular function $x$ vanishes identically on the open $D(y)$ where $y \neq 0$, and the regular function $y$ vanishes on $D(x)$. So this is by no means a big mystery, it naturally appears once we allow reducible spaces into the mix.

2.5.2 Definition of the structure sheaf $\mathcal{O}_{\text{Spec } A}$

One may straight away write down a definition of the structure sheaf, but many students experience this as coming out of the blue. So we find it instructive to make a detour and start with the definition of a $B$-presheaf that has a virtue of being intuitive.

Definition 2.29. Let $\mathcal{B}$ be the collection of distinguished open subsets $D(f)$. We define the $\mathcal{B}$-presheaf $\mathcal{O}$ by

$$\mathcal{O}(D(f)) = A_f$$

and for $D(g) \subseteq D(f)$ we let the restriction map be localization map $A_f \to A_g$.

Let $S_{D(f)}$ be the multiplicative system $\{s \in A \mid s \not\in p \forall p \in D(f)\}$. There is a localization map $\tau: A_f \to S^{-1}_{D(f)}A$ since $f \in S_{D(f)}$. The following lemma says that the ring $\mathcal{O}(D(f))$ is independent of the representative $f$ for $D(f)$:

Lemma 2.30. The map $\tau$ is an isomorphism, permitting us to identify $A_f = S^{-1}_{D(f)}A$.

Proof. The point is that any element $s \in S_{D(f)}$ does not lie in $p$ for any $p \in D(f)$, in other words, one has $D(s) \subseteq D(f)$. This is equivalent to $\sqrt{(f)} \subseteq \sqrt{(s)}$, and one may write $f^n = cs$ for an appropriate $c \in A$ and $n \in \mathbb{N}$. Assume that $af^{-m} \in A_f$ maps to zero in $S^{-1}_{D(f)}A$. This means that $sa = 0$ for some $s \in S_{D(f)}$. But then $f^n a = csa = 0$, and therefore $a = 0$ in $A_f$. This shows that the map $\tau$ is injective. To see that is surjective, take any $as^{-1}$ in $S^{-1}_{D(f)}A$ and write is as $as^{-1} = ca(cf^n)^{-1} = caf^{-n}$. $\square$

Proposition 2.31. $\mathcal{O}$ is a $\mathcal{B}$-sheaf of rings.

The situation is as follows. We are given a distinguished set $D(f)$ and an open covering $D(f) = \bigcup_{i \in I} D(f_i)$. Of course then $D(f_i) \subseteq D(f)$, and we have localization maps $\tau_i: A_f \to A_{f_i}$ and $\tau_{ij}: A_{f_i} \to A_{f_if_j}$. The statement in the Proposition is equivalent to the exactness of the following sequence

$$0 \longrightarrow A_f \overset{\alpha}{\longrightarrow} \prod_i A_{f_i} \overset{\rho}{\longrightarrow} \prod_{i,j} A_{f_if_j}$$ (2.5.1)
where \( \alpha(a)_i = \tau_i(a) \) and \( \rho((a_i))_{i,j} = (\tau_{ij}(a_i) - \tau_{ji}(a_j)). \) It is clear that \( \alpha \circ \rho = 0 \) since \( \tau_{ij} \circ \tau_i = \tau_{ji} \circ \tau_j. \)

**Lemma 2.32.** The sequence (2.5.1) is exact.

**Proof.** We start by observing that we assume \( A = A_f \)—that is \( f = 1 \)— indeed, one has \((A_f)_{f_i} = A_{f_i}\) and \((A_f)_{f_i f_j} = A_{f_i f_j} \) since \( f_i^{n_i} = h_i f \) for suitable natural numbers \( n_i. \)

Then to the proof: That \( \alpha(a) = 0 \) means that \( a \) is mapped to zero in each of the localizations \( A_{f_i} \), just means that a power of \( f_i \) kills \( a \). So for each \( i \) one has \( f_i^{n_i} a = 0 \) for natural numbers \( n_i. \) The open sets \( D(f_i) \) covers \( D(f) \) which then is covered by the \( D(f_i^{n_i})'s \) as well. Thus one we may write \( 1 = \sum_i b_i f_i^{n_i} \), which upon multiplication by \( a \) gives

\[
a = \sum_i b_i f_i^{n_i} a = 0.
\]

In down-to-earth terms, the equality \( \ker \rho = \text{im} \alpha \) means the following: Assume given a sequence of elements \( a_i \in A_{f_i} \) such that \( a_i \) and \( a_j \) are mapped to the same element in \( A_{f_i f_j} \) for every pair \( i, j \) of indices. Then there ought to be an \( a \in A \), such that every \( a_i \) is the image of \( a \) in \( A_{f_i} \), i.e., \( \tau_i(a) = a_i. \)

Each \( a_i \) can be written as \( a_i = b_i f_i^{n_i} \) where \( b_i \in A \), and since the indices are finite in number one may replace \( n_i \) with \( n = \max_i n. \) That \( a_i \) and \( a_j \) induce the same element in the localization \( A_{f_i f_j} \) means that we have the equations

\[
f_i^N f_j^N (b_i f_j^m - b_j f_i^m) = 0, \tag{2.5.2}
\]

where \( N \) a priori depends on \( i \) and \( j \), but again due to there being only finitely many indices, it can be chosen independent of the indices. Equation (2.5.2) gives

\[
b_i f_i^N f_j^m - b_j f_j^N f_i^m = 0 \tag{2.5.3}
\]

where \( m = N + n. \) Putting \( b_i' = b_i f_i^N \) we see that \( a_i \) equals \( b_i' f_i^m \) in \( A_{f_i} \), and equation (2.5.3) takes the form

\[
b_i' f_j^m - b_j' f_i^m = 0. \tag{2.5.4}
\]

Now \( D(f_i^m) = D(f_i) \), and the distinguished open sets \( D(f_i^m) \) form an open covering of \( \text{Spec } A. \) Therefore we may write \( 1 = \sum_i c_i f_i^m. \) Letting \( a = \sum_i c_i b_i' \), we find

\[
a f_i^m = \sum_i c_i b_i' f_i^m = \sum_i c_i b_j' f_i^m = b'_j \sum_i c_i f_i^m = b'_j
\]

and hence \( a = b'_j / f_i^m. \)
2.5. The structure sheaf on Spec $A$

Using Proposition 1.30 of Chapter 1, we may now make the following definition:

**Definition 2.33.** We let $\mathcal{O}_{\text{Spec } A}$ be the unique sheaf extending the $\mathcal{B}$-sheaf $\mathcal{O}$.

To end this section, we prove that our sheaf indeed satisfies the two properties we want:

**Proposition 2.34.** The sheaf $\mathcal{O}_{\text{Spec } A}$ on Spec $A$ as defined above is a sheaf of rings satisfying the two paramount properties above, namely

- **Sections over distinguished opens:** $\Gamma(D(f), \mathcal{O}_{\text{Spec } A}) = A_f$.
- **Stalks:** $\mathcal{O}_{\text{Spec } A, x} = A_{p_x}$.

In particular, $\Gamma(X, \mathcal{O}_X) = A$.

**Proof.** We defined $\mathcal{O}$ so that the first property would hold, so only the second point needs proof. Using the first part, and the fact that you can compute stalks by running through open sets of the form $D(f)$, it suffices to show that the canonical homomorphism

$$\phi : \lim_{\substack{\longrightarrow \\ f \notin p}} A_f \rightarrow A_p$$

isomorphism. $\phi$ is surjective: Any element in $A_p$ is of the form $a/f$ with $f \notin p$. This element lies in $A_f$ and so it lies in the image of $\phi$. $\phi$ is injective: if $af^{-n} \in A_f$ is mapped to 0 in $A_p$, then $\exists g \notin p$ such that $ga = 0$, hence $af^{-n} = 0 \in A_{gf}$ and so $\phi$ is injective.

The last statement follows by taking $f = 1$. \qed

2.5.3 Maps between the structure sheaves of two spectra

Previously, we assigned to any map $\phi : A \rightarrow B$ of rings a continuous map $\phi^* : \text{Spec } B \rightarrow \text{Spec } A$.

We climb one step in the hierarchy of structures and shall associate to $\phi$ a map of sheaves of rings

$$\phi^\# : \mathcal{O}_{\text{Spec } A} \rightarrow \phi_* \mathcal{O}_{\text{Spec } B}.$$

By a simple patching argument, it suffices to tell what $\phi^\#$ should do to the sections over open sets belonging to a basis for the topology as long as this definition is compatible with the restriction maps; and of course, the basis we shall use is the basis of the distinguished open sets. Everything follows from the following lemma:
Lemma 2.35. Let \( \phi : A \to B \) be a map of rings and let \( f \in A \) be an element. Then \((\phi^*)^{-1}(D(f)) = D(\phi(f))\)

Proof. We have

\[
(\phi^*)^{-1}(D(f)) = \{ p \subseteq B \mid f \notin \phi^{-1}(p) \} = \{ p \subseteq B \mid \phi(f) \notin p \} = D(\phi(f)).
\]

This means that \( \Gamma(D(f), \phi_* \mathcal{O}_{\text{Spec} B}) = B(\phi(f)) \), and we know that \( \Gamma(D(f), \mathcal{O}_{\text{Spec} A}) = A_f \). The original map of rings \( \phi : A \to B \) now localizes to a map \( A_f \to B_{\phi(f)} \), sending \( af^{-n} \) to \( \phi(a)\phi(f)^{-n} \), and that shall be the map \( \phi^\# \) on sections over the distinguished open set \( D(f) \).

It is obvious, but anyhow the matter of some easy work, to check that this definition is compatible with the restriction maps among distinguished open sets: Indeed, if \( D(g) \subseteq D(f) \), we can as usual write \( g^n = cf \), and the localization map \( A_f \to A_g \) sends \( af^{-m} \) to \( ac^m g^{-nm} \). One has \( \phi(g)^n = \phi(c)\phi(f) \), and the diagram below is commutative:

\[
\begin{array}{ccc}
A_f & \longrightarrow & A_g \\
\downarrow & & \downarrow \\
B_{\phi(f)} & \longrightarrow & B_{\phi(g)}
\end{array}
\]

which is the required compatibility.

2.6 Schemes

We would like to define a ‘scheme’ to be something that locally looks like a \( \text{Spec} A \) for some ring \( A \). To be able to make sense of such a definition, we need a suitable category of spaces to work with. We will use the two pieces of data we have; the topological space \( \text{Spec} A \) together its sheaf of rings \( \mathcal{O}_X \). This motivates the following definition:

Definition 2.36. A ringed space is a pair \((X, \mathcal{O}_X)\) where \( X \) is a topological space and \( \mathcal{O}_X \) is a sheaf of rings on \( X \).

A morphism of ringed spaces is a pair \((f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) where \( f : X \to Y \) is continuous, and

\[
f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X
\]

is a map of sheaves of rings on \( Y \) (so that \( f^\#(U) \) is a ring homomorphism for each open \( U \subseteq X \)).
2.6. Schemes

Note that we specify very little data here: The sheaf $\mathcal{O}_X$ can really be any sheaf of rings and the maps above $f^#$ can also be completely arbitrary. Of course, we want to eventually think about sections of $\mathcal{O}_X$ as ‘regular functions’ on $X$, so there are some additional requirements we would like to impose to get a decent category. We take the following as a guiding example:

**Example 2.37.** Consider a manifold $X$ with the sheaf $C^\infty(X)$ of smooth functions. The pair $(X, C^\infty(X))$ forms a ringed space. Indeed, a continuous $f : X \to Y$ of manifolds is smooth if and only if for every local section $h$ of $C^\infty(Y)$ the composition $f^#(h) = h \circ f$ is a local section of $C^\infty(X)$. This gives a map of sheaves

$$f^# : C^\infty(Y) \to f_* C^\infty(X)$$

Thus a map $f$ gives rise in a natural way to a morphism of ringed spaces

$$f : (X, C^\infty(X)) \longrightarrow (Y, C^\infty(Y))$$

Let us consider what happens to functions locally: For a point $x \in X$, the ring of germs of functions $C^\infty(X)_x$ is a local ring with maximal ideal $m_x$ the functions which vanish at $x$ (similarly for $C^\infty(Y)_y$ where $y = f(x) \in Y$). Note that the induced map on stalks

$$f^#_y : C^\infty(Y)_y \to C^\infty(X)_x$$

maps the maximal ideal into the maximal ideal: this is simply because $f(x) = y$. In other words, $f_y$ is a local homomorphism.

Based on this, we make the following definition:

**Definition 2.38.** A locally ringed space is a pair $(X, \mathcal{O}_X)$ as above, but with the additional requirement that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

A morphism of locally ringed spaces is a pair $(f, f^#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as above, with the additional requirement that for every $x \in X$, $f^#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism; that is, the map

$$f^#_{Y,f(x)} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

satisfies

$$(f^#_{Y,f(x)})^{-1}(m_x) = m_{f(x)}$$

where $m_x \subseteq \mathcal{O}_{X,x}$ and $m_{f(x)} \subseteq \mathcal{O}_{Y,f(x)}$ are the maximal ideals.

Once again, the intuition behind the local homomorphism condition comes from thinking about sections of the structure sheaf as regular functions: we want $g \in \mathcal{O}_Y$ to vanish at a point $y = f(x)$ if and only if $f^#(g)$ vanishes at $x$.

Finally, we can give the formal definition of a scheme.
Definition 2.39. An affine scheme is a locally ringed space \((X, \mathcal{O}_X)\) which is isomorphic to \((\text{Spec } A, \mathcal{O}_{\text{Spec } A})\) for some ring \(A\). A scheme is a locally ringed spaced \((X, \mathcal{O}_X)\) that is locally isomorphic to an affine scheme.

So a scheme is a topological space \(X\) and \(\mathcal{O}_X\) is a sheaf of rings such that all stalks are local rings. A locally ringed space being locally affine means that it possesses an open cover \(\{U_i\}\) of \(X\) with each open \((U_i, \mathcal{O}_X|_{U_i})\) being isomorphic (as locally ringed spaces) to an affine scheme.

A morphism or map for short between two schemes \(X\) and \(Y\) is simply a map \(\phi\) between \(X\) and \(Y\) regarded as locally ringed spaces. It has two components, a continuous map, that we denote \(\phi\) as well slightly abusing the language, and a map of sheaves of rings \(\phi^\#: \mathcal{O}_Y \to \phi_* \mathcal{O}_X\) with the additional requirement that \(\phi^\#\) induce local homomorphisms on the stalks. This is to say, for all \(x \in Y\) the map \(\phi_x^\#: \mathcal{O}_{Y, \phi(x)} \to \phi_* \mathcal{O}_{X, x}\) maps the maximal ideal in \(\mathcal{O}_{Y, x}\) into the one in \(\phi_x^\# \mathcal{O}_{X, x}\).

In this way the schemes form a category, which we denote by \(\text{Sch}\). We denote by \(\text{AffSch}\) the subcategory of affine schemes.

2.6.1 Relative schemes

There is also the notion of relative schemes where a base scheme \(S\) is chosen. A scheme over \(S\) is scheme \(X\) with a morphism \(\pi: X \to S\) called the structure map. If two such schemes over \(S\) are given, say \(X \to S\) and \(Y \to S\), then a map between them is a map \(X \to Y\) compatible with the two structure maps; that is, such that the diagram below is commutative

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
S & \to & S
\end{array}
\]

The schemes over \(S\) form a category \(\text{Sch}/S\), and the set morphisms as defined above is denoted by \(\text{Hom}_S(X, Y)\). Composition of maps makes it is a functor in both \(X\) and \(Y\). It is a functor from \(\text{Sch}/S\) to \(\text{Sets}\), contravariant in \(X\) and covariant in \(Y\). If the base scheme \(S\) is affine, say \(S = \text{Spec } A\), a short hand notation for \(\text{Sch}/\text{Spec } A\) is \(\text{Sch}/A\).

As there is a canonical map from any scheme \(X\) to \(\text{Spec } \mathbb{Z}\), every scheme is a \(\mathbb{Z}\)-scheme. On the level of categories one may express this a \(\text{Sch} = \text{Sch}/\mathbb{Z}\).
2.7 Open immersions and open subschemes

If $X$ is a scheme and $U \subseteq X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on $U$, making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is even a scheme, since if $X$ is covered by affines $V_i = \text{Spec} \, A_i$, then each $U \cap V_i$ is open in $V_i$, hence can be covered by affine schemes. It follows that there is a canonical scheme structure on $U$, and we call $(U, \mathcal{O}_X|_U)$ an open subscheme of $X$ and say that $U$ has the induced scheme structure.

As a special case, consider $U = \text{Spec} \, A_f \subseteq \text{Spec} \, A = X$. It follows from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_U$ coincides with the structure sheaf on $\text{Spec} \, A_f$.

Remark 2.40. A word of warning: An open subscheme of an affine scheme might not not itself be affine (we will see an example of this in Chapter 3).

2.8 Residue fields

For varieties, we construct the sheaf $\mathcal{O}_X$ from the regular functions on which we think of as continuous maps $X \rightarrow k$. However, in the world of schemes, we do not have the luxury of having a field $k$ to map into – all we know is that locally $\mathcal{O}_X$ is built from elements of a ring.

We can still define an analogy between the elements $f$ of $A$ and some sort of functions on $\text{Spec} \, A$. If $x$ is a point in $\text{Spec} \, A$ corresponding to the prime ideal $\mathfrak{p}$, the localization $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and one obtains the field $k(x) = A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})$. The element $f$ reduced modulo $\mathfrak{p}$ gives an element $f(x) \in k(x)$, which may considered as the “value” of $a$ at $x$; clearly $f(x) = 0$ if and only if $f \in \mathfrak{p}$.

Definition 2.41. The field $k(x)$ is called the residue field of $X$ at $x$.

Example 2.42. Consider $X = \mathbb{A}^1_k = \text{Spec} \, k[t]$. When $k$ is algebraically closed, there are two types of points (maximal ideals $(t-a)$ and $(0)$). The residue fields are of the form $k(a) = k[t]_{(t-a)}/(t-a) \simeq k$ and $k(0) = k[t]_{(0)} = k(t)$.

When $k$ is not algebraically closed, we have more interesting residue fields: $\mathfrak{p} = (x^2+1)$ defines a point in $\mathbb{A}^1_k$ with residue field $\mathbb{C}$.

Be aware that the “values” of the element $f$ are in lying in different fields that might vary with the point. For instance, the element $f = 17 \in \mathbb{Z}$ defines a function on $X = \text{Spec} \, \mathbb{Z}$. Some of its “values” are given by

$$f((2)) = \overline{1}, f((3)) = \overline{1}, f((7)) = \overline{3}, f((11)) = \overline{6}, f((17)) = \overline{0}, f(((19)) = \overline{17}$$
and the values have to be interpreted in the appropriate residue field $\mathbb{Z}/p$. Thus we tweak our notion of a 'regular function' on $X$; they are not maps into some fixed field, but rather maps into the disjoint union $\bigsqcup_{x \in X} k(x)$.

**Definition 2.43.** For a scheme $X$, we can define the residue field $k(x)$ at a point $x \in X$ by choosing some affine neighborhood $U = \text{Spec } A$ of $x$ and using the definition above. Equivalently, the residue field is given by $k(x) = \mathcal{O}_{X,x}/m_x$.

If $U \subseteq X$ is an open set containing $x$, and $s \in \mathcal{O}_X(U)$ (or if $s$ is an element of $\mathcal{O}_{X,x}$), we let $s(x)$ denote the class of $s$ modulo $m_x$ in $k(x)$. 
Chapter 3

Gluing and first results on schemes

3.1 Gluing sheaves

The setting is a scheme $X$ with an open covering $\{U_i\}_{i \in I}$ with a sheaf $\mathcal{F}_i$ on each open subset $U_i$. As usual the sheaves can take values in any category, but the principal situation we have in mind is when the sheaves are sheaves of abelian groups. The intersections $U_i \cap U_j$ are denoted by $U_{ij}$, and triple intersections $U_i \cap U_j \cap U_k$ are written as $U_{ijk}$.

The gluing data in this case consists of isomorphisms $\tau_{ji} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$. The idea is to identify sections of $\mathcal{F}_i|_{U_{ijk}}$ with $\mathcal{F}_j|_{U_{ijk}}$ using the isomorphisms $\tau_{ij}$. For the gluing process to be possible, the $\tau_{ij}$’s must satisfy the three conditions

- $\tau_{ii} = \text{id}_{\mathcal{F}_i}$,
- $\tau_{ji} = \tau_{ij}^{-1}$,
- $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$,

where the last identity takes place where it makes sense; on the triple intersection $U_{ijk}$. Observe that the three conditions parallel the three requirements for a relation being an equivalence relation; the first reflects reflectivity, the second symmetry and the third transitivity.

The third requirement is obviously necessary in order to glue sections: A section $s_i$ of $\mathcal{F}_i|_{U_{ijk}}$ will be identified with its image $s_j = \tau_{ji}(s_i)$ in $\mathcal{F}_j|_{U_{ijk}}$, and in its turn, $s_j$ is going to be equal to $s_k = \tau_{kj}(s_j)$. Then, of course, $s_i$ and $s_k$ are forced to be equal, which means that $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$.
3.1. Gluing sheaves

**Proposition 3.1.** In the setting as above, there exists a unique sheaf $\mathcal{F}$ on $X$ such that there are isomorphisms $\nu_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ satisfying $\nu_j = \tau_{ji} \circ \nu_i$ over the intersections $U_{ij}$.

**Proof.** Let $V \subseteq X$ be an open set and let $V_i = U_i \cap V$ and $V_{ij} = U_{ij} \cap V$. We are going to define the sections of $\mathcal{F}$ over $V$, and they are of course obtained by gluing sections of the $\mathcal{F}_i$'s along $V_i$ using the isomorphisms $\tau_{ij}$. We define

$$\mathcal{F}(V) = \{ (s_i)_{i \in I} \mid \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \} \subseteq \prod_{i \in I} \mathcal{F}_i(V_i). \quad (3.1.1)$$

The $\tau_{ij}$'s are maps of sheaves and therefore are compatible with all restriction maps, so if $W \subseteq V$ is another open set, we have $\tau_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}$ if $\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}$. By this, the defining condition (3.1.1) is compatible with componentwise restrictions, and they can therefore be used as the restriction maps in $\mathcal{F}$. We have thus defined a presheaf.

The first step in what remains of the proof, is to establish the isomorphisms $\nu_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$. To avoid getting confused by the names of the indices, we work with a fixed index $\alpha \in I$. Suppose $V \subseteq U_\alpha$ is an open set. Then naturally one has $V = V_\alpha$, and projecting from the product $\prod_i \mathcal{F}_i(V_i)$ onto the component $\mathcal{F}(V_\alpha) = \mathcal{F}_\alpha(V_\alpha)$ gives us a map of presheaves\(^1\) $\nu_\alpha : \mathcal{F}|_{V_\alpha} \to \mathcal{F}_\alpha$. This map sends $(s_i)_{i \in I}$ to $s_\alpha$. This is summarized in the following diagram

$$\begin{array}{ccc}
\mathcal{F}(V) & \longrightarrow & \prod_{i \in I} \mathcal{F}_i(V_i) \\
\downarrow_{\nu_\alpha} & & \downarrow_{p_\alpha} \\
\mathcal{F}_\alpha(V). & & &
\end{array}$$

We will now show that the $\nu_\alpha$'s give the desired isomorphism.

To begin with, on the intersections $V_{\alpha\beta}$ the requirement in the proposition that $\nu_\beta = \tau_{\beta\alpha} \circ \nu_\alpha$ is fulfilled. This follows immediately since by the second property of the glueing, one has $s_\beta = \tau_{\beta\alpha}(s_\alpha)$

$\nu_\alpha$ is surjective: Take a section $\sigma$ of $\mathcal{F}_\alpha$ over some $V \subseteq U_\alpha$ and define $s = (\tau_{i\alpha}(\sigma|_{V_{i\alpha}}))_{i \in I}$. Then $s$ satisfies the condition in (3.1.1), and is an honest element of $\Gamma(V, F)$; indeed, by the third gluing condition we obtain

$$\tau_{ji}(\tau_{i\alpha}(\sigma|_{V_{j\alpha}})) = \tau_{j\alpha}(\sigma|_{V_{j\alpha}})$$

for every $i, j \in I$, and that is just the condition in (3.1.1). As $\tau_{\alpha\alpha}(\sigma|_{V_{\alpha\alpha}}) = \sigma$ by the first property of the glue, the element $s$ projects to the section $\sigma$ of $\mathcal{F}_\alpha$.

\(^1\)Restrictions operating componentwise, it is straightforward to verify this map being compatible with restrictions.
$v_\alpha$ is injective: This is clear, since if $s_\alpha = 0$ if follows that $s_i|_{V_\alpha} = \tau_\alpha(s_\alpha) = 0$ for all $i \in I$. Now $F_\alpha$ is a sheaf and the $V_\alpha$ constitute an open covering of $V_\alpha$. We conclude that $s = 0$ by the locality axiom for sheaves.

The next step is to show that $F$ is a sheaf, and we start with the patching axiom: So assume that $\{ V_\alpha \}$ is an open covering of $V \subseteq X$ and that $s_\alpha \in \Gamma(V_\alpha, F)$ is a bunch of sections matching on the intersections $V_\alpha \cap V$. Since $F|_{U_i \cap V}$ is a sheaf—we just checked that $F|_{U_i}$ is isomorphic to $F_i$—the sections $s_\alpha|_{V_\alpha \cap U_i}$ patch together to give sections $s_i$ in $\Gamma(U_i \cap V, F)$ matching on $U_i \cap V$. This last condition means that $\tau_{ij}(s_i) = s_j$. By definition $(s_i)_{i \in I}$ then this is a section in $\Gamma(V, F)$ restricting to $s_i$, and we are done. The locality axiom is easy to verify and is left to reader to verify (do it!).

Exercise 14. Show the uniqueness statement in the proposition.

### 3.2 Gluing maps of sheaves

This is the easiest gluing situation we encounter in this course. The setting is as follows. We are given two sheaves $F$ and $G$ on the scheme $X$ and an open covering $\{ U_i \}_i$ of $X$. On each open set $U_i$ we are given a map $\phi_i : F|_{U_i} \to G|_{U_i}$ of sheaves, and the gluing conditions

- $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$

are assumed to be satisfied for all $i, j \in I$. In this context we have

**Proposition 3.2.** There exists a unique map of sheaves $\phi : F \to G$ such that $\phi|_{U_i} = \phi_i$.

**Proof.** The salient point is this: Take any $s \in \Gamma(V, F)$ where $V \subseteq X$ is open, and let $V_i = U_i \cap V$. Then $\phi_i(s|_{V_i})$ is a well defined element in $\Gamma(V_i, G)$, and it holds true that $\phi_i(s|_{V_i}) = \phi_j(s|_{V_{ij}})$ by the gluing condition. Hence the sections $\phi_i(s|_{V_i})$’s of $G|_{V_{ij}}$ patch together to a section of $G$ over $V$ which we define to be $\phi(s)$. The checking of all remaining details is hassle-free and left to the zealous student.

### 3.3 The gluing of schemes

The possibility of gluing different schemes together along open subschemes, gives rise to many new schemes. The most prominent one being the projective spaces. The gluing process is also an important part in many general existence proofs, like in the construction of the fiber-product, which as we are going to show, exists without restrictions in the category of schemes.
3.3. The gluing of schemes

In the present context of scheme gluing we are given a family \( \{ X_i \}_{i \in I} \) of schemes indexed by the set \( I \). In each of the schemes \( X_i \) we are given a collection of open subschemes \( X_{ij} \), where the second index \( j \) runs through \( I \). They form the glue lines in the process; i.e., the contacting surfaces that are to be glued together. In the glued scheme they will be identified and will be equal to the intersections of \( X_i \) and \( X_j \). The identifications of the different pairs of the \( X_{ij} \)'s are encoded by a family of scheme-isomorphisms \( \tau_{ji} : X_{ij} \to X_{ji} \). Furthermore, we let \( X_{ijk} = X_{ik} \cap X_{ij} \) — this are the different parts of the triple intersections before the gluing has been done—and we have to assume that \( \tau_{ij}(X_{ijk}) = X_{jik} \).

Notice that \( X_{ijk} \) is an open subscheme of \( X_i \).

The three following gluing condition, very much alike the ones we saw for sheaves, must be satisfied for the gluing to be doable:

- \( \tau_{ii} = \text{id}_{X_i} \).
- \( \tau_{ij} = \tau_{ji}^{-1} \).
- The isomorphism \( \tau_{ij} \) takes \( X_{ijk} \) into \( X_{jik} \) and one has \( \tau_{ki} = \tau_{kj} \circ \tau_{ji} \) over \( X_{ijk} \).

**Proposition 3.3.** Given gluing data as above. Then there exists a scheme \( X \) with open immersions \( \psi_i : X_i \to X \) such that \( \psi_i|_{X_{ij}} = \psi_j|_{X_{ji}} \circ \tau_{ji} \), and such that the images \( \psi_i(X_i) \) form an open covering of \( X \). Furthermore one has \( \psi_i(X_{ij}) = \psi_i(X_i) \cap \psi_j(X_j) \). The scheme \( X \) is uniquely characterized by these properties.

**Proof.** To build the scheme \( X \) we first build the underlying topological space \( X \) and subsequently equip it with a sheaf of rings. For the latter, we rely on the patching technic for sheaves presented in proposition 3.1. And finally, we need to show that \( X \) is locally affine, this follows however immediately once the immersions \( \psi_i \) are in place—the \( X_i \)'s are schemes and therefore locally affine.

On the level of topological spaces, we start out with the disjoint union \( \coprod_i X_i \) and proceed by introducing an equivalence relation on it. We require that two points \( x \in X_{ij} \) and \( x' \in X_{ji} \) be equivalent if \( x' = \tau_{ji}(x) \)—observe that if the point \( x \) does not lie in any \( X_{ij} \) with \( i \neq j \), we leave it alone, and it will not be equivalent to any other point.

The three gluing conditions imply readily that we obtain an equivalence relation. The first requirement entails that the relation is reflexive, the second that it is symmetric, and the third ensures it is transitive. The topological space \( X \) is then defined to be the quotient of \( \coprod_i X_i \) by this relation, and we declare the topology on \( X \) to be the quotient topology. If \( \pi \) denotes the quotient map, a subset \( U \) of \( X \) is open if and only if \( \pi^{-1}(U) \) is open.

Topologically the maps \( \psi_i : X_i \to X \) are just the maps induced by the open inclusions of the \( X_i \)'s in the disjoint union \( \coprod_i X_i \). They are clearly injective.
since a point \( x \in X_i \) never is equivalent to another point in \( X_i \). Now, \( X \) has the quotient topology so a subset \( U \) of \( X \) is open if and only if \( \pi^{-1}(U) \) is open, and this holds if and only if \( \psi_i^{-1}(U) = X_i \cap \pi^{-1}(U) \) is open for all \( i \). In view of the formula

\[
\pi^{-1}(\psi_i(U)) = \bigcup_j \tau_{ji}(U \cap X_{ij})
\]

we conclude that each \( \psi_i \) is an open immersion.

To simplify notation we now identify \( X_i \) and \( \psi_i(X_i) \), which is in concordance with our intuitive picture of \( X \) as being the union of the \( X_i \)'s with points in the \( X_{ij} \)'s identified according to the \( \tau_{ij} \)'s. Then \( X_{ij} \) becomes \( X_i \cap X_j \) and \( X_{ijk} \) becomes the triple intersection \( X_i \cap X_j \cap X_k \).

On \( X_{ij} \) we have the isomorphisms \( \tau_{ji}^\# : \mathcal{O}_{X_j}|_{X_{ij}} \to \mathcal{O}_{X_i}|_{X_{ij}} \); the sheaf-parts of the scheme-isomorphisms \( \tau_{ji} : X_{ij} \to X_{ji} \). In view of the third gluing condition \( \tau_{ki} = \tau_{kj} \circ \tau_{ji} \), valid on \( X_{ijk} \), we obviously have \( \tau_{ki}^\# = \tau_{ji}^\# \circ \tau_{kj}^\# \). The two first gluing conditions translate into \( \tau_{ii}^\# = \text{id} \) and \( \tau_{ji}^\# = (\tau_{ji}^\#)^{-1} \). The end of the story is that the gluing properties needed to apply proposition 3.1 are satisfied, and we are enabled to glue the different \( \mathcal{O}_{X_i} \)'s together and thus to equip \( X \) with a sheaf of rings. This sheaf of rings restricts to \( \mathcal{O}_{X_i} \) on each of the open subsets \( X_i \), and therefore it is a sheaf local rings. So \((X, \mathcal{O}_X)\) is a locally ringed space that is locally affine, hence a scheme.

The uniqueness property is, as usual, left to the industrious student. \( \square \)

### 3.4 Global sections of glued schemes

The standard exact sequence for computing global sections from an open covering is a valuable tool in the setting of glued schemes. If \( X \) is obtained by gluing the open subschemes \( X_i \) along \( \tau_{ji} : X_{ij} \to X_{ji} \) it reads:

\[
0 \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\alpha} \bigoplus_i \Gamma(X_i, \mathcal{O}_{X_i}) \xrightarrow{\rho} \bigoplus_{i,j} \Gamma(X_{ij}, \mathcal{O}_{X_{ij}})
\]

where \( \rho(s_i)_{i \in I} = (s_i|_{X_{ij}} - \tau_{ij}^\#(s_j|_{X_{ji}})) \) and \( \alpha(s) = (\psi_i^\#(s))_{i \in I} \).
### 3.5 The gluing of morphisms

Assume given schemes $X$ and $Y$ and an open covering $\{U_i\}_{i \in I}$ of $X$. Assume further that there is given a family of morphisms $\phi: U_i \to Y$ which match on the intersections $U_{ij} = U_i \cap U_j$. The aim of this paragraph is to show that they can be glued together to give a morphism $X \to Y$:

**Proposition 3.4.** In the situation just described, there exists a unique map of schemes $\phi: X \to Y$ such that $\phi|_{U_i} = \phi_i$.

**Proof.** Clearly the underlying topological map is well defined, so if $U \subseteq Y$ is an open set, we have to define $\phi^\# : \Gamma(U, \mathcal{O}_Y) \to \Gamma(U, \phi_* \mathcal{O}_X) = \Gamma(\phi^{-1}U, \mathcal{O}_X)$. So take any section $s$ of $\mathcal{O}_Y$ over $U$. This gives sections $t_i = \phi_i^\#(s)$ of $\mathcal{O}_X|_{U_i}$, but since $\phi_i^\#$ and $\phi_j^\#$ restrict to the same map on $U_{ij}$, one has $t_i|_{U_{ij}} = t_j|_{U_{ij}}$. The $t_i$ therefore patch together to a section of $\mathcal{O}_X$ over $U$, which is the section we are aiming at: We define $\phi^\#(s)$ to be $t$. The checking of the remaining details is left to student (as usual). \hfill \Box

### 3.6 The category of affine schemes

The assignment $A \to \text{Spec} A$ and $\phi \to \text{Spec}(\phi)$ gives a contravariant functor from the category $\text{Rings}$ of rings to the category $\text{AffSch}$ of affine schemes. There is also a contravariant functor the other way around. Taking the global sections of the structure sheaf $\mathcal{O}_{\text{Spec} A}$ gives us the ring $A$ back. Furthermore, a map of affine schemes $f: \text{Spec} B \to \text{Spec} A$, comes with a map of sheaves $f^\# : \mathcal{O}_{\text{Spec} A} \to f_* \mathcal{O}_{\text{Spec} B}$. Taking global sections gives a ring homomorphism

$$A = \Gamma(\text{Spec} A, \mathcal{O}_{\text{Spec} A}) \to \Gamma(\text{Spec} A, \phi_* \mathcal{O}_{\text{Spec} B}) = B.$$ 

which coincides with $\phi$ in the case $f = \text{Spec}(\phi)$.

Let $X, T$ be two schemes and let $\text{Hom}_{\text{AffSch}}(X, Y)$ denote the set of morphisms $f: X \to Y$ between them. We can define a canonical map

$$\rho : \text{Hom}_{\text{AffSch}}(X, Y) \to \text{Hom}_{\text{Rings}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

which sends $(f, f^\#)$ to the map $f^\#(Y) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

**Lemma 3.5.** If $X$ and $Y$ are affine, the map $\rho$ is bijective.

**Proof.** Write $X = \text{Spec} B$ and $Y = \text{Spec} A$. By definition, we have $A = \mathcal{O}_Y(Y)$ and $B = \mathcal{O}_X(X)$. Given a morphism $f: X \to Y$, let $\phi = \rho(f) = f^\#(Y) : B \to A$ be the corresponding ring homomorphism. As we noted above, any ring homomorphism induces a morphism of the corresponding ring spectra, so we
obtain a morphism of affine scheme \( f_\phi : X \to Y \). To prove the lemma, we need only prove that \( f_\phi = f \).

Let \( x \in X \) be a point, corresponding to the prime ideal \( q \subseteq B \), and let \( p \subseteq A \) be the prime ideal corresponding to \( f(x) \in Y \). We have a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow & & \downarrow \\
A_p & \rightarrow & B_q
\end{array}
\]

where the vertical maps are the localization maps. We have \( \phi(A - p) \subseteq B - q \). Hence \( \phi^{-1}(q) \subseteq p \). Now \( f^\#_x \) is a local homomorphism, and so in fact \( \phi^{-1}(q) = p \). Hence \( f \) and \( f_\phi \) induce the same map on the underlying topological spaces. Moreover, we have \( f^\#_x = f^\#_{\phi,x} \) for every \( x \in X \), and so also \( f^\# = f^\#_\phi \). \( \square \)

We have established the all important theorem:

**Theorem 3.6.** The two functors \( \text{Spec} \) and \( \Gamma \) are mutually inverse and define an equivalence between the categories \( \text{Rings} \) and \( \text{AffSch} \).

Thus affine schemes \( X \) are characterized by their rings of global sections \( \Gamma(X, \mathcal{O}_X) \), and morphisms between affine schemes \( X \to Y \) are in bijective correspondence with ring homomorphisms \( \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X) \).

### 3.7 Universal properties of maps into affine schemes

For a general scheme \( X \) one can consider the associated affine scheme \( \text{Spec} \Gamma(X, \mathcal{O}_X) \), but this is in general very different from \( X \) (for the projective line introduced in the next section, \( \text{Spec} \Gamma(X, \mathcal{O}_X) \) is just a point). There is however still a canonical morphism \( X \to \text{Spec} \Gamma(X, \mathcal{O}_X) \) enjoying the following universal property:

**Proposition 3.7.** Let \( X \) be any scheme. Then there is a canonical (unique) map of schemes \( X \to \text{Spec} \Gamma(X, \mathcal{O}_X) \) inducing the identity on global sections of the structure sheaves.

This will follow by applying the following theorem to \( A = \Gamma(X, \mathcal{O}_X) \):

**Theorem 3.8.** For any scheme \( X \), the canonical map

\[ \Phi_X : \text{Hom}_{\text{Sch}}(X, \text{Spec} A) \to \text{Hom}_{\text{Rings}}(A, \Gamma(X, \mathcal{O}_X)) \]

given by \( f \to f^\# \), is bijective.
3.7. Universal properties of maps into affine schemes

Proof. Let \( \{U_i\} \) be an affine covering of \( X \), and consider the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(X, \text{Spec } A) & \xrightarrow{\Phi_X} & \text{Hom}(A, \Gamma(X, \mathcal{O}_X)) \\
\downarrow & & \downarrow \\
\prod_i \text{Hom}(U_i, \text{Spec } A) & \xrightarrow{\prod \Phi_{U_i}} & \prod_i \text{Hom}(A, \Gamma(U_i, \mathcal{O}_{U_i}))
\end{array}
\]

where the vertical maps are induced by the inclusion maps \( U_i \to X \). By the affine schemes case (Theorem 3.6), we know that \( \prod \Phi_{U_i} \) is bijective. In particular, this shows that \( \Phi_X \) is injective.

To show that \( \Phi_X \) is surjective, let \( \beta : A \to \Gamma(X, \mathcal{O}_X) \) be a ring homomorphism. This induces maps \( \beta_i : A \to \Gamma(U_i, \mathcal{O}_{U_i}) \), and hence, morphisms \( f_i : U_i \to \text{Spec } A \). We check that the \( f_i \) glue to a map \( f : X \to \text{Spec } A \). This is a consequence of the fact that the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma(U_i, \mathcal{O}_{U_i}) & \xrightarrow{\beta_i} & \Gamma(U_i) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\beta} & \Gamma(X, \mathcal{O}_X) \\
\downarrow & & \downarrow \\
\Gamma(U_i) & \xrightarrow{\beta_j} & \Gamma(U_i) \\
\end{array}
\]

Indeed, note that for \( V \subseteq U_i \cap U_j \) affine, the diagram implies that the restrictions \( f_i|_V \) and \( f_j|_V \) induce the same element in \( \text{Hom}(A, \Gamma(V, \mathcal{O}_X)) \), and so they are equal on \( V \) (using Theorem 3.6). Since this is true for any \( V \), they are equal on all of \( U_i \cap U_j \). So by gluing the \( f_i \), we obtain a morphism \( f : X \to \text{Spec } A \). We must have \( \Phi_X(f) = \beta \), by injectivity of \( \Phi_X \) and since \( f \) maps to \( \prod \beta_i \) via \( \prod \Phi_{U_i} \). This completes the proof.

Corollary 3.9. The canonical map \( \psi : X \to \text{Spec } \Gamma(X, \mathcal{O}_X) \) is universal among the maps from \( X \) to affine schemes; i.e., any map \( \alpha : X \to \text{Spec } A \) factors as \( \alpha = \alpha' \circ \psi \) for a unique map \( \alpha' : \text{Spec } \Gamma(X, \mathcal{O}_X) \to \text{Spec } A \).

Proof. In the theorem above, \( \psi \) corresponds to the identity map \( \text{id}_{\Gamma(X, \mathcal{O}_X)} \) on the right hand side. The morphism \( \alpha' \) is the map of Spec’s induced by the ring map \( \alpha^# : A \to \Gamma(X, \mathcal{O}_X) \). We check that it factors \( \alpha \): The morphism \( (\alpha' \circ \psi) : X \to \text{Spec } A \) satisfies \( (\alpha' \circ \psi)^# = \psi^# \circ \alpha^# = \alpha^# \) and hence it coincides with \( \alpha \) by the above theorem.

Remark 3.10. Let \( X \) be any scheme. By the above we have a bijection

\[
\text{Hom}_{\text{Sch}}(X, \text{Spec } \mathbb{Z}) = \text{Hom}_{\text{Rings}}(\mathbb{Z}, \Gamma(X, \mathcal{O}_X)).
\]
The set on the right is clearly a one point set and thus we find that \( \text{Spec} \mathbb{Z} \) is final object in the category \( \text{Sch} \).

\( \text{Sch} \) also has a initial object: The spectrum of the zero ring, \( \text{Spec} 0 \), which has the empty set as underlying topological space. Given any scheme \( X \) there is clearly a unique morphism \( \text{Spec} 0 \to X \), which on the level of sheaves sends every section to zero.

### 3.8 \( R \)-valued points

For a scheme \( X \), it also makes sense to study morphisms \( \text{Spec} R \to X \) from affine schemes into it. We call such morphisms \( R \)-valued points and the set of all such will be denoted by \( X(\mathbb{R}) \). The jargon here is justified from the following:

**Example 3.11.** Let \( \mathbb{A}^n = \text{Spec} \mathbb{Z}[x_1, \ldots, x_n] \) and let us study the \( R \)-valued points of \( \mathbb{A}^n \). A morphism \( g : \text{Spec} R \to \text{Spec} \mathbb{Z}[x_1, \ldots, x_n] \) is determined by the \( n \)-tuple \( a_i = g^*(x_1) \) for \( i = 1, \ldots, n \). Hence,

\[
\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n.
\]

Now, let \( X = \text{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I \) where \( I = (f_1, \ldots, f_r) \) is an ideal. The set of \( R \)-points can be found similarly: Indeed, any morphism

\[
g : \text{Spec} R \to \text{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I
\]

is determined by the \( n \)-tuple \( a_i = g^*(x_1) \) for \( i = 1, \ldots, n \), and those \( n \)-tuples that occur are exactly those such that \( f \mapsto f(a_1, \ldots, a_n) \) defines a homomorphism

\[
\mathbb{Z}[x_1, \ldots, x_n]/I \to \mathbb{R}.
\]

In other words, the \( a_i \) are elements in \( R \) which are solutions of \( f_1 = \cdots = f_r = 0 \).

The sets \( X(R) \) of points over \( R \) are obviously important in number theory, as they naturally generalizes the solution set of the polynomials \( f_1 = \cdots = f_r = 0 \). Even when \( R \) is a field, it can be quite difficult to describe the set \( X(K) \) of \( K \)-valued points \( \text{Spec} K \to X \), or even determining whether \( X(K) = \emptyset \) (consider for instance the case \( K = \mathbb{Q} \)).

These questions are interesting even for the simplest examples of affine schemes, namely the spectra of fields. If \( K \) is a field, then of course the underlying topological \( \text{Spec} K \) is very simple – it is a singleton set corresponding to the zero ideal. However, the structure sheaf \( \mathcal{O}_{\text{Spec} K} \) on \( \text{Spec} K \) carries more information: Morphisms \( \text{Spec} L \to \text{Spec} K \) (i.e., the elements of \( X(K) \)) correspond exactly to field extensions \( L \supseteq K \). In particular, \( \text{Spec} K \) and \( \text{Spec} L \) are isomorphic if and only if \( K \simeq L \).
The residue fields play an important role here. The following result says that they satisfy the following ‘universal’ property with respect to morphisms $\text{Spec } K \to X$:

**Proposition 3.12.** Let $X$ be a scheme and let $K$ be a field. Then to give a morphism of schemes $\text{Spec } K \to X$ is equivalent to giving a point $x \in X$ plus a map $k(x) \to K$.

**Proof.** Let $(f, f^\#) : \text{Spec } K \to X$ be the morphism and let $x$ be the image of $f$ (Spec $K$ consists of a single point, so this is a well-defined point of $X$.) The map on stalks is just $f_x^\# : \mathcal{O}_{X,x} \to \mathcal{O}_{\text{Spec } K,0} = K$, which gives a map on residue fields $k(x) = \mathcal{O}_{X,x}/m_x \to \mathcal{O}_{\text{Spec } K,0} = K$. As always with non-zero maps of fields, this has to be an injection.

Conversely, suppose we are given $x \in X$ and $k(x) \to K$. We construct a map of topological spaces $f : \text{Spec } K \to X$, which takes Spec $K$ to $x \in X$. We also construct a map of structure sheaves $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_{\text{Spec } K}$: for opens $U \subseteq X$ not containing $x$, $f^\#(U)$ is the zero map as $f^{-1}(U) = \emptyset$, while for opens $U \subseteq X$ containing $x$, we need maps $\mathcal{O}_X(U) \to K$. These maps are constructed using the given map $k(x) \to K$ via the compositions

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow k(x) \longrightarrow K.$$ 

This yields a morphism of schemes $(f, f^\#) : \text{Spec } K \to X$. \hfill $\square$

In particular, we have for $X$ a scheme over an algebraically closed field $k$

$$X(k) = \{\text{Spec } k \to X\} = \{\text{closed points of } X\}$$

More generally, we can for a fixed scheme $S$ define $X(S)$ to be the set of all morphisms $S \to X$ (the ‘$S$-valued points’ of $X$). In the example above, we have for any scheme $S$,

$$\mathbb{A}^n(S) = \text{Hom}_{\text{Sch}}(S, \mathbb{A}^n) = \Gamma(S, \mathcal{O}_S)^n.$$ 

In fancy terms, this says that $\mathbb{A}^n$ represents the functor taking a scheme to $n$-tuples of elements of $\Gamma(S, \mathcal{O}_S)$. We will see a similar functorial characterization of projective space $\mathbb{P}^n$ later in the book.

**Example 3.13** (The morphism $\text{Spec } \mathcal{O}_{X,x} \to X$). Let $X$ be a scheme and let $x \in X$ be a point. There is a canonical morphism

$$f : \text{Spec } (\mathcal{O}_{X,x}) \to X$$

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which is induced from all the morphisms $\text{Spec } A_p \to \text{Spec } A$. The image $S_x = \text{im } f \subseteq X$ of $f$ is exactly the intersection of all neighbourhoods of $x$ in $X$: it is a kind of “microgerm” of $X$ at $x$.

More geometrically, consider the irreducible closed subset $V = \{x\} \subseteq X$ whose generic point is $x$. Let $Y \subseteq X$ be a closed irreducible subset containing $V$ and let $\eta_Y$ be the generic point of $Y$. Then our set $S_x$ is exactly the set of all those generic points $\eta_Y$. We say that $S_x$ is the set of specializations of $x$.

For example, if $X = \mathbb{A}^2_k = \text{Spec}(k[x, y])$ and $x$ is a closed point corresponding to the maximal ideal $m = (x - a, x - b)$, then $S_x$ is the set consisting of $x$, the generic point of $\mathbb{A}^2_k$ and the generic points of all irreducible curves passing through $x$, like for example the curve $(y - b)^2 - (x - a)^3 = 0$.

### 3.9 The functor from varieties to schemes

We have already stated that schemes form a generalization of algebraic varieties. On the other hand, we have also seen that even the simplest schemes (e.g., $\mathbb{A}^2_k = \text{Spec}(k[x, y])$) behave differently than varieties in the sense that they usually have many non-closed points. Thus for this generalization to make sense, we should expect that there to be a canonical way to ‘add non closed points’ to an algebraic variety $V$, so that the resulting topological space has the structure of a scheme. Let us explain what this means more precisely.

Let $k$ be an algebraically closed field, and let $V$ be an algebraic variety over $k$. Let us first consider the case where $V$ is an affine variety, i.e., $V \subseteq \mathbb{A}^n_k$ is the zero-set of some ideal $I \subseteq k[x_1, \ldots, x_n]$. The coordinate ring $A = A(V) = k[x_1, \ldots, x_n]/I$ is the ring of regular functions $f : V \to k$. The scheme $V^{sh} = \text{Spec } A$ is an affine scheme whose closed points is in bijection with the points of $V$. So any affine variety determines a canonical affine scheme.

In the general case, a variety has an open cover by affine varieties, and gluing can be performed in both the category of varieties, as well as the categories of schemes, and it is a matter of checking that this gives a well-defined scheme.

This assignment works also for morphisms; a morphism of varieties $\phi : V \to W$ is determined by a ring homomorphism $A(W) \to A(V)$, and so taking Spec we obtain a morphism $\phi^{sh} : V^{sh} \to W^{sh}$, which extends $\phi$. It follows that the assignment $V \mapsto V^{sh}$ in fact determines a functor $i : \text{Var}/k \to \text{Sch}/k$. It is a theorem (Hartshorne Chapter II), that this functor is fully faithful, in the sense that

$$\text{Hom}_{\text{Var}/k}(V, W) = \text{Hom}_{\text{Sch}/k}(V^{sh}, W^{sh})$$

is bijective. So two varieties give rise to the same scheme if and only if they are isomorphic by a unique isomorphism. In particular, this tells us that the
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category of varieties $\text{Var}/k$ is equivalent to a full subcategory of $\text{Sch}/k$. 
Chapter 4

More examples

4.1 A scheme that is not affine

Consider $A = k[u,v]$ and let $X = \text{Spec } A$. Let $U = X - V(u,v)$; this is the affine plane $\mathbb{A}^2_k$ minus the origin. We claim that $U$ is not an affine scheme. The point of this example is that the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$ is an isomorphism. This shows that $U$ can not be an affine scheme, since in that case, by Theorem 3.8 above, the inclusion map would be an isomorphism which obviously is not true, since it is not surjective.

Let us check that that restriction map really is an isomorphism. The two distinguished open sets $D(u)$ and $D(v)$ form an open covering of $U$. The proof is based on the sequence from before associated to that covering:

$$0 \longrightarrow \Gamma(U, \mathcal{O}_U) \longrightarrow A_v \times A_u \overset{\rho}{\longrightarrow} A_{uv} \overset{i^\#}{\longrightarrow} A$$

where $\rho$ is the difference between the two localization maps; that is; it maps a pair $(av^{-m}, bu^{-n})$ to $av^{-m} - bu^{-n}$. We have included the restriction map $i^\#$ in the diagram, which is the just the action of the inclusion map $i: U \to X$ on global sections of the structure sheaves. It sends an element $a \in A$ to the pair $(a, a)$.

If the pair $(av^{-m}, bu^{-n})$ lies in kernel of $\rho$, we have the relation

$$au^n = bv^m,$$
4.2. The projective line

which (since \( A \) is a UFD) implies that there is an element \( c \in A \) with \( a = cv^m \) and \( b = cu^n \); that is, \( av^{-m} = bu^{-n} \). This shows that \( i^\# \) is surjective, and hence it is an isomorphism since it obviously is injective.

Note that the same example works for any is an integral domain \( A \) with an ideal \( a \) generated by two elements \( u \) and \( v \) having the following property: If there is a relation \( au^n = bv^m \) with \( a, b \in A \) and \( n, m \in \mathbb{N} \), one may find a \( c \in A \) such that \( a = cv^m \) and \( b = cu^n \).

4.2 The projective line

In elementary courses on complex function theory one learns about the Riemann sphere. That is the complex plane with one point added, the point at infinity. If \( z \) is the complex coordinate centered at the origin, the inverse \( 1/z \) is the coordinate centered at infinity. Another name for the Riemann sphere is the complex projective line, denoted \( \mathbb{P}^1_{\mathbb{C}} \).

The construction of \( \mathbb{P}^1_{\mathbb{C}} \) can be vastly generalized, and works in fact over any ring \( A \). Let \( u \) be a variable (“the coordinate function at the origin”), and let \( U_1 = \text{Spec} \ A[u] \). The inverse \( u^{-1} \) is a variable as good as \( u \) (“the coordinate at infinity”), and we let \( U_2 = \text{Spec} \ A[u^{-1}] \). Both are copies of the affine line \( \mathbb{A}^1_A \) over \( A \).

Inside \( U_1 \) we have the open set \( U_{12} = D(u) \) which is canonically equal to the prime spectrum \( \text{Spec} \ A[u, u^{-1}] \), the isomorphism coming from the inclusion \( A[u] \subseteq A[u, u^{-1}] \). In the same way, inside \( U_2 \) there is the open set \( U_{21} = D(u^{-1}) \). This is also canonical isomorphic to the spectrum \( \text{Spec} \ A[u^{-1}, u] \), the isomorphism being induced by the inclusion \( A[u^{-1}] \subseteq A[u^{-1}, u] \). Hence \( U_{12} \) and \( U_{21} \) are isomorphic schemes (even canonically), and we may glue \( U_1 \) to \( U_2 \) along \( U_{12} \). The result is called the projective line over \( A \) and is denoted by \( \mathbb{P}^1_A \).

\begin{center}
\begin{tikzpicture}[scale=0.5]
    \node (u) at (0,0) {$u$};
    \node (u2) at (0,-2) {$u_2$};
    \draw (u) circle (2);
    \draw (u2) circle (2);
    \draw (u) to[out=90,in=180] (u2);
\end{tikzpicture}
\end{center}

Gluing two affine lines to get \( \mathbb{P}^1 \)

In this example, we will carry out our first computations with sheaf cohomology using the Čech complex:

**Proposition 4.1.** We have \( \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_A}) = A \).
Proof. The sheaf axiom exact sequence gives us

\[
\begin{array}{c}
\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1}) \oplus \Gamma(U_2, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow \Gamma(U_{12}, \mathcal{O}_{\mathbb{P}^1}) , \\
\downarrow \sim \quad \downarrow \sim \\
A[u] \oplus A[u^{-1}] \longrightarrow \rho \longrightarrow A[u, u^{-1}] 
\end{array}
\]

where the map \( \rho \) sends a pair \((f(u), g(u^{-1}))\) of polynomials with coefficients in \( A \), one in the variable \( u \) and one in \( u^{-1} \), to their difference. We claim that the kernel of \( \rho \) equals \( A \); i.e., the polynomials \( f \) and \( g \) must both be constants.

So assume that \( f(u) = g(u^{-1}) \), and let \( f(u) = au^n + \) lower terms in \( u \), and in a similar way, let \( g(u^{-1}) = bu^{-m} + \) lower terms in \( u^{-1} \), where both \( a \neq 0 \) and \( b \neq 0 \), and without loss of generality we may assume that \( m \geq n \). Now, assume that \( m \geq 1 \). Upon multiplication by \( u^m \) we obtain \( b + uh(u) = umf(u) \) for some polynomial \( h(u) \), and putting \( u = 0 \) we get \( b = 0 \), which is a contradiction. Hence \( m = n = 0 \) and we are done.

In particular, the global sections of \( \mathcal{O}_X \) of \( X = \mathbb{P}^1 \) is just \( \mathbb{C} \). In particular, knowing \( \Gamma(X, \mathcal{O}_X) \) does not recover \( X \), in constrast to the case when \( X = \text{Spec} \ A \) is affine.

### 4.2.1 Affine line a doubled origin

Keeping the notation from the previous example, if we now glue \( U_1 = U_2 = \text{Spec} \ A[v] \) along \( U_{12} = \text{Spec} \ A[u, u^{-1}] \) using the identity map on the overlap, we obtain the ‘affine line with two origins’.

This scheme is not affine: Using the sheaf axiom sequence, we find that \( \Gamma(X, \mathcal{O}_X) = A[u] \). Note that both of the points above the origin would correspond to the same ideal, which is not possible on an affine scheme.

More strikingly, the cokernel of the map \( A[u] \otimes A[u^{-1}] \rightarrow A[u, u^{-1}] \) is \( A[u, u^{-1}]/A[u] \), and so \( H^1(X, \mathcal{O}_X) \) is not finitely generated as an \( A \)-module. We will see that this cannot happen on affine schemes. In fact, for \( X \) affine, all higher cohomology groups vanish: \( H^i(X, \mathcal{O}_X) = 0 \ i > 0 \).

### 4.3 The blow-up of the affine plane

#### 4.3.1 Blow-up as a variety

Recall the classical construction of a blow-up. There is a rational map \( f : A^2_k \rightarrow \mathbb{P}^1_k \) sending a point \((x, y)\) to \((x : y)\) (in homogeneous coordinates on \( \mathbb{P}^1_k \)).
This map is not defined at the origin, but we can still associate to it its graph
$X \subseteq \mathbb{A}^2_k \times \mathbb{P}^1_k$, consisting of all pairs $(x, y) \times (s : t)$ so that $f(x, y) = (s : t)$. From
the definition, we see that $X$ is defined by a single equation in $\mathbb{A}^2_k \times \mathbb{P}^1_k$, namely

$$X = Z(xt - ys) \subseteq \mathbb{A}^2_k \times \mathbb{P}^1_k$$

We also have two projection maps $p : X \to \mathbb{A}^2$ and $q : X \to \mathbb{P}^1$. The situation is
depicted in Figure 4.1

![Figure 4.1: The blow-up of the plane at a point](image)

Let us analyze the fibers of these maps.

The fibers of $p$ are easy to describe. If $(x, y) \subseteq \mathbb{A}^2_k$ is not the origin, then
$p^{-1}(x, y)$ consists of a single point; the equation $xt = ys$ allows us to determine
the point $(s : t)$ uniquely. However, when $(x, y) = (0, 0)$, any $s$ and $t$ satisfy
the equation, so $p^{-1}(0, 0) = (0, 0) \times \mathbb{P}^1$. In particular, that this inverse image is
1-dimensional - it is called the exceptional divisor of $X$, and is usually denoted
by $E$.

Similarly, if $(s : t) \in \mathbb{P}^1_k$ is a point, the the fiber $q^{-1}(s : t) = \{(x, y) \times (s : t) | xt = ys\} \subseteq \mathbb{A}^2_k \times (s : t)$ is a line in $\mathbb{A}^2$. The map $q$ is an example of a line
bundle; all of its fibers are $\mathbb{A}^1_k$’s. We will see this later on in the book.

The standard covering of $\mathbb{P}^1_k$ as a union of two $\mathbb{A}^1_k$’s gives an affine cover of $X$:
If $U \subseteq \mathbb{P}^1_k$ is the open set where $s \neq 0$, we can normalize so that $s = 1$, and solve
the equation $xt = sy$ for $y$. Hence $x$ and $t$ give coordinates on $q^{-1}(U) \simeq \mathbb{A}^2_{x,t}$.
In these coordinates, the morphism $p : X \to \mathbb{A}^2_k$, restricts to a map $\mathbb{A}^2_{x,t} \to \mathbb{A}^2_{x,y}$
given by $(x, t) \mapsto (x, xt)$. Similarly, $q^{-1}(V) = \mathbb{A}^2_{y,s}$ and the map $p$ is given here
as $(y, s) \mapsto (sy, y)$.

### 4.3.2 The blow-up as a scheme

From the above discussion, we can define the scheme-analogue of the blow-up
of $\mathbb{A}^2_k$ at a point. We will define this as a scheme over $\mathbb{Z}$, rather than over a
field $k$. Also, in addition to the scheme $X$, we also want a morphism of schemes $p : X \to \mathbb{A}^2_\mathbb{Z}$ having similar properties to that above.

Consider the affine plane $\mathbb{A}^2 = \text{Spec } \mathbb{Z}[x, y]$. The prime ideal $p = (x, y) \subseteq \mathbb{Z}[x, y]$ corresponds to the origin in $\mathbb{A}^2$. Consider the diagram

$$
\begin{array}{ccc}
\mathbb{Z}[x, y] & & \mathbb{Z}[y, s] \\
\downarrow & & \uparrow \\
\mathbb{Z}[x, t] & & R = \mathbb{Z}[x, y, s, t]/(xt - y, st - 1)
\end{array}
$$

Here the diagonal maps on the top are given by $(x, y) \mapsto (x, xt)$ and $(x, y) \mapsto (ys, y)$ respectively.

Note that the ring $R$ is isomorphic to $\mathbb{Z}[x, s, t]/(st - 1) = \mathbb{Z}[x, s, s^{-1}]$, as well as $\mathbb{Z}[y, s, t]/(st - 1) = \mathbb{Z}[y, s, s^{-1}]$. Since this is a localization of both $\mathbb{Z}[x, t]$ and $\mathbb{Z}[y, s]$ we can identity its spectrum as an open set of $\text{Spec } \mathbb{Z}[x, t]$ and $\text{Spec } \mathbb{Z}[y, s]$ respectively. This gives a digram

$$
\begin{array}{ccc}
\text{Spec } \mathbb{Z}[x, y] & & \\
\downarrow & & \uparrow \\
U = \text{Spec } \mathbb{Z}[x, t] & & V = \text{Spec } \mathbb{Z}[y, s] \\
\downarrow & & \uparrow \\
\text{Spec } R & & 
\end{array}
$$

Hence we can glue these two affine spaces along $\text{Spec } R$ to a new scheme $X$. By construction, the maps $\text{Spec } \mathbb{Z}[x, t] \to \text{Spec } \mathbb{Z}[x, y]$ and $\text{Spec } \mathbb{Z}[y, s] \to \text{Spec } \mathbb{Z}[x, y]$ coincide with the map $\text{Spec } R \to \text{Spec } \mathbb{Z}[x, y]$ which is induced by $\mathbb{Z}[x, y] \to R$. Hence they glue to a morphism $p : X \to \mathbb{A}^2_\mathbb{Z}$.

To complete the discussion, we should define the corresponding morphism $q : X \to \mathbb{P}^1$. Again we work locally. On the affine open $U = \text{Spec } \mathbb{Z}[x, t]$, we have a map $U \hookrightarrow \mathbb{A}^1 = \text{Spec } \mathbb{Z}[t]$ induced by the inclusion $\mathbb{Z}[t] \subseteq \mathbb{Z}[x, t]$. Similarly, on $V = \text{Spec } \mathbb{Z}[y, s]$, we have a map $V \hookrightarrow \mathbb{A}^1 = \text{Spec } \mathbb{Z}[s]$. To see if we can glue, we have to see what happens on the overlap $U \cap V = \text{Spec } R$. However, on $\text{Spec } R$, $t = s^{-1}$, so using the standard gluing on $\mathbb{P}^1$, we see that the maps $\mathbb{Z}[t] \to R$ and $\mathbb{Z}[s] \to R$ induce the desired morphism $q : X \to \mathbb{P}^1$. 

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4.4 Hyperelliptic curves

Let $k$ be a field and consider the scheme $X$ glued together by the affine schemes $U = \text{Spec } A$ and $V = \text{Spec } B$, where

$$A = \frac{k[x, y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1x)}$$

and

$$B = \frac{k[u, v]}{(-v^2 + a_{2g+1}u + \cdots + a_1u^{2g+1})}$$

Here we glue $D(x) = \text{Spec } A_{(x)}$ to $D(u) = \text{Spec } B_{(u)}$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$. These curves are so called hyperelliptic curves or double covers of $\mathbb{P}^1$. In the case $g = 1$ they are called elliptic curves. Here is an illustration of one of the affine charts:

![Illustration of an affine chart](https://via.placeholder.com/150)

The term ‘double cover’ comes from the fact that it admits a map $X \to \mathbb{P}^1$ coming from the inclusions $k[x] \subseteq A$ and $k[u] \subseteq B$; note that $u = x^{-1}$ gives the standard gluing of $\mathbb{P}^1$ by the two affine schemes $\text{Spec } k[x]$ and $\text{Spec } k[u]$. The corresponding morphism $X \to \mathbb{P}^1$ is finite of degree 2, as we are adjoining a square root of an element of $k[x]$. 

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5.1 Some topological properties

Many basic geometric concepts of schemes are formulated just in terms of their underlying topological space. For instance, a scheme $(X, \mathcal{O}_X)$ is said to be connected if the topological space of $X$ connected (i.e., it cannot be written as a union of two proper open sets.)

Example 5.1. If $R = \mathbb{Z} \times \mathbb{Z}$. Then Spec $R$ is not connected. Indeed, the proper prime ideals of $R$ are of the form $p \times \mathbb{Z}$ or $\mathbb{Z} \times p$, for $p \subseteq \mathbb{Z}$ a prime ideal. And so Spec $R = \text{Spec } \mathbb{Z} \cup \text{Spec } (\mathbb{Z})$.

More generally, Spec $A$ is disconnected if and only if $A \simeq B \times C$ for some non-zero rings $B, C$.

Likewise, we say that $X$ is irreducible (irreducible) if its space is not the union of two proper, closed subsets. That is, if $X = Z \cup Z'$ with $Z$ and $Z'$ both closed one has either $Z = X$ or $Z' = X$. An equivalent formulation is that any two nonempty open subsets of $X$ has a non empty intersection; indeed if $U$ and $U'$ are open, it holds that $U \cap U' = \emptyset$ is equivalent to $U^c \cup (U')^c = X$.

From the theory of varieties we know that the coordinate ring of a variety (which is assumed to be irreducible), is an integral domain, and very simple examples illustrate that reducibility is strongly connected to zero divisors in ring of functions:

Example 5.2. The scheme $X = \text{Spec } k[x, y] / (xy)$ is the prime example of something that is not irreducible. The coordinate functions $x$ and $y$ are zero-divisors in the ring $k[x, y] / (xy)$, and their zero-sets $V(x)$ and $V(y)$ show that $X$ has two components.
We have seen that the irreducible subsets of an affine scheme $\text{Spec} A$ correspond exactly to the prime ideals $p$ in $A$. Specializing to the irreducible components of $X$, we find:

**Proposition 5.3.** Let $X = \text{Spec} A$ be an affine scheme.

1. The irreducible components of $X$ are precisely the closed subsets of the form $V(p)$ where $p \subseteq A$ is a minimal prime.

2. $X$ is irreducible if and only if $A$ has a unique minimal prime, if and only if $\sqrt{0}$ is prime.

Note that the ring $R = k[t]/(t^2)$ has only one prime ideal $(t)$, so $X = \text{Spec} R$ is just point, hence irreducible. However $k[t]/(t^2)$ is not an integral domain.

### 5.2 Properties of the scheme structure

A ring $A$ is said to be *reduced* if it has no nilpotent elements. A scheme $(X, \mathcal{O}_X)$ is said to be *reduced* if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced. This condition holds if and only if for every open $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotents. Indeed, suppose $(X, \mathcal{O}_X)$ is reduced. Clearly, then, for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ has no nilpotents via the description of its elements as germs of $\mathcal{O}_X$ around $x$. The converse follows similarly: a nilpotent section of $\mathcal{O}_X(U)$ would pass to a nilpotent element in the local ring at any point $x \in U$.

Given $(X, \mathcal{O}_X)$, we can define a new ringed space $(X, (\mathcal{O}_X)_{\text{red}})$ whose underlying topological space is the same as that of $X$, but the structure sheaf of $\mathcal{O}_{X,\text{red}}$ is defined as the sheafification of the presheaf

$$\mathcal{F}(U) = \mathcal{O}_X(U)_{\text{red}}$$

where $\mathcal{O}_X(U)_{\text{red}} = \mathcal{O}_X(U)/\mathcal{N}(U)$ is the quotient of $\mathcal{O}_X(U)$ by its nilradical $\mathcal{N}(U)$ of nilpotent elements. The collection $\mathcal{N}(U)$ forms the so-called *nilradical sheaf* $\mathcal{N}$. You should convince yourself that $\mathcal{N}$ is indeed a sheaf.

**Lemma 5.4.** $(X, \mathcal{O}_{X,\text{red}})$ is a scheme.

*Proof.* We find that $\mathcal{O}_{X,\text{red},x} = (\mathcal{O}_{X,x})_{\text{red}} = \mathcal{O}_{X,x}/\sqrt{0}$ is the quotient of a local ring and hence local. It follows that $(X, (\mathcal{O}_X)_{\text{red}})$ is a locally ringed space. Furthermore, if $(X, \mathcal{O}_X)$ is locally isomorphic to affine schemes $(\text{Spec} A_i, \mathcal{O}_{\text{Spec} A_i})$, and then $(X, (\mathcal{O}_X)_{\text{red}})$ is locally isomorphic to affine schemes $(\text{Spec} A_i/N_i, \mathcal{O}_{\text{Spec} A_i/N_i})$, where $N_i$ is the nilradical of $A_i$. Hence $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. \hfill $\square$

We refer to $(X, \mathcal{O}_{X,\text{red}})$ as simply $X_{\text{red}}$, the *reduced scheme associated to $X$*. $X_{\text{red}}$ comes with a canonical morphism of schemes $r : X_{\text{red}} \rightarrow X$ defined as
follows. The underlying topological space of $X_{\text{red}}$ is that of $X$, and hence there is a homeomorphism $r : X_{\text{red}} \to X$. There is also a map of sheaves $r^\# : \mathcal{O}_X \to r_*\mathcal{O}_{X_{\text{red}}}$ that on each open $U \subseteq X$ is the quotient map $\mathcal{O}_X(U) \to (\mathcal{O}_X)_{\text{red}}(U)$. As the induced map on stalks is a quotient map as well, it is a local homomorphism, and we obtain a morphism $(r, r^\#) : X_{\text{red}} \to X$ of schemes.

The scheme $X_{\text{red}}$ and the morphism $r$ satisfy the following universal property:

**Proposition 5.5.** Let $f : Y \to X$ be a morphism of schemes, with $Y$ reduced. Then $f$ factors uniquely through the natural map $r : X_{\text{red}} \to X$, i.e. there exists a unique $g : Y \to X_{\text{red}}$ such that $f = r \circ g$.

**Proof.** The only difference between $X$ and $X_{\text{red}}$ is the structure sheaf, so define $g$ on the level of topological spaces by $f$. On the level of sheaves we find that (over any open $U \subseteq X$) the map $f^\# : \mathcal{O}_X(U) \to f_*\mathcal{O}_Y(U)$ takes all nilpotents to zero as $Y$ is reduced. By the universal property of quotients there must exist a unique morphism of rings $g^\#(U) : \mathcal{O}_{X_{\text{red}}}(U) \to f_*\mathcal{O}_X(U)$ such that $f^\# = g^\# \circ r^\#$. This gives the required morphism $(g, g^\#)$ of schemes.

### 5.2.1 Integral schemes

**Definition 5.6.** A scheme is integral if it is both irreducible and reduced.

In particular, an affine scheme $X = \text{Spec} A$ is integral if and only if $A$ is an integral domain. Indeed, $X$ is reduced if and only if $A$ has no nilpotents. $X$ is irreducible if and only if the nilradical of $A$ $N = \cap_p \mathfrak{p}$ is prime. These two statements imply that the zero-ideal ($= \cap \mathfrak{p}$) is prime, and so $A$ is an integral domain.

Moreover, it is not so hard to prove the following:

**Proposition 5.7.** A scheme $X$ is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open $U \subseteq X$.

If $X$ is an integral scheme, there is a unique generic point $\eta$ of $X$ which is dense in $X$. Indeed, take any $U = \text{Spec} A \subseteq X$ and let $\eta$ be the point corresponding to the zero ideal (which is prime). The local ring $\mathcal{O}_{X, \eta}$ is equal to the field of fractions of $A$. We define the function field $K(X)$ of $X$ as the field of fractions of $\mathcal{O}_{X, \eta}$ at the generic point.

**Example 5.8.** The function field of $\mathbb{A}^n_k = \text{Spec} k[x_1, \ldots, x_n]$ is $k(x_1, \ldots, x_n)$.

**Example 5.9.** The function field of $\text{Spec} \mathbb{Z}$ is $\mathbb{Q}$. 

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5.2. Properties of the scheme structure

5.2.2 Noetherian spaces and quasi-compactness

**Definition 5.10.** A topological space is called *noetherian* (noetersk) if every descending chain of closed subsets is eventually constant; that is, if \{ Z_i \}_{i \in \mathbb{N}} is a family of closed subsets \( Z_i \) with \( Z_{i+1} \subseteq Z_i \), there is an \( i_0 \) such that \( Z_i = Z_{i+1} \) for \( i \geq i_0 \).

An equivalent formulation is any any non-empty family of closed subsets has a minimal element. This happens if and only if every non-empty family of open subsets has a maximal element.

Clearly if \( A \) is a noetherian ring, the prime spectrum \( \text{Spec} \, A \) is a noetherian topological space. The converse however, is not true. A simple example is the following: Take the polynomial ring \( k[t_1, t_2, \ldots] \) in countably many variables \( t_i \) and mod out by the square \( m^2 \) of the maximal ideal generated by the variables, i.e., with \( m = (t_1, t_2, \ldots) \). The resulting ring \( A \) has just one prime ideal, the one generated by the \( t_i \)'s, so \( \text{Spec} \, A \) has just one point. The ring \( A \), however, is clearly not noetherian; the sole prime ideal requiring infinitely many generators, namely all the \( t_i \)'s.

Given this example, we take a different route to define noetherianness for schemes:

**Definition 5.11.**

(i) A scheme is *quasi-compact* if every open cover of \( X \) has a finite subcover.

(ii) A scheme is *locally noetherian* if it can be covered by open affine subsets \( \text{Spec} \, A_i \) where each \( A_i \) is a noetherian ring

(iii) A scheme is *noetherian* if it is both locally noetherian and quasi-compact.

An affine scheme \( X = \text{Spec} \, A \) is always quasi-compact: If \( X = \bigcup \text{Spec} \, A_i \), each \( \text{Spec} \, A_i \) can be covered by the distinguished opens \( D(f_{ij}) \), and \( X \) is covered by such subsets if and only if 1 belongs to the ideal generated by the \( f_{ij} \); and in this case finitely many of them will do. Hence a general scheme is noetherian if and only if it can be covered by finitely many \( \text{Spec} \, A_i \) where each \( A_i \) is noetherian.

In fact, now we have

**Proposition 5.12.** \( \text{Spec} \, A \) is noetherian (as a scheme) if and only if \( A \) is noetherian.

This is really a purely algebraic fact: Refining the cover, we may assume that each \( A_i = A_{f_i} \). By a theorem in commutative algebra, \( A \) is noetherian provided that each \( A_{f_i} \) is noetherian and \( 1 \in (f_1, \ldots, f_r) \).

**Proposition 5.13.** If \( X \) is a noetherian scheme, then its underlying topological space is noetherian.
By covering \( X \) with open affine subsets, it suffices to show the proposition for \( X = \text{Spec} \ A \), for \( A \) a noetherian ring. In this case a descending chain of closed subsets is of the form \( V(I_1) \supseteq V(I_2) \supseteq \cdots \), where we may assume that the ideals \( I_n \) are radical. Then the condition that \( V(I_n) \) is decreasing, corresponds the sequence \((I_n)\) being increasing, and so it has to be stationary, because \( A \) is noetherian.

**Proposition 5.14.** Let \( X \) be a (locally) noetherian scheme. Then any open subscheme of \( X \) is also (locally) noetherian.

This is clear for the locally noetherian case. For the noetherianness, we need to show that \( U \) is also quasi-compact. Since \( X \) is noetherian, \( U \) is noetherian as a topological space. Now if \((V_i)_{i \in I} \) is a cover of \( U \), then the collection of unions of the form \( \bigcup_{i \in F} V_i \) for \( F \) finite, has a maximal element, which gives a finite subcover of \( U \).

We also see that if \( X \) is locally noetherian, then all the local rings \( \mathcal{O}_{X,x} \) are noetherian. The converse however, does not hold:

**Example 5.15.** Let \( A = \mathbb{C}[y, x_1, x_2, \ldots] / I \) where \( I \) is the ideal generated by \( x_i - (y-i)x_{i+1}, x_i^2 \) for \( i = 1, 2, \ldots \). Then \( A \) is not noetherian, but \( A_p \) is noetherian for every prime ideal \( p \). The nilradical \( \mathcal{N} \) is \( (x_1, x_2, \ldots) \) – the \( x_i \) are nilpotent, and \( A/(x_1,\ldots) \) is a domain. In particular, \( A \) is not noetherian. We have \( A/\mathcal{N} \simeq \mathbb{C}[y] \), so the maximal ideals of \( A \) are of the form \( m = (y-a, x_1, x_2, \ldots) \) for some \( a \in \mathbb{C} \). If \( a \) is not a positive integer, then \( A_m \) is the localization of a ring generated by \( y \) and \( x_1 \), and hence is noetherian. If \( a = n \) is a positive integer, we have \( x_{i+1} = x_i/(y-i) \) for \( i \neq n \) in \( m \). Hence \( A_m \) is the localization of a ring generated by \( y, x_n \), and hence is Noetherian.

Here is a similar example of a non-noetherian scheme which is locally noetherian and all local rings noetherian.

**Example 5.16.** Let \( k \) be an infinite field, let \( R = k[x] \), and for each \( \alpha \in k \) let \( R_\alpha = R[y_\alpha]/((x-\alpha)y_\alpha, y_\alpha^2) \). Then \( R_\alpha \) is an affine line with an embedded prime \( p_\alpha = (x-\alpha, y_\alpha) \) at \( x = \alpha \), sticking out in the \( y_\alpha \)-direction. Finally, let

\[
R_\infty = \bigotimes_{\alpha \in k} R_\alpha = \lim_{\text{finite} \ I \subseteq k} \bigotimes_{\alpha \in I} R_\alpha
\]

be their tensor product over \( R \) (*not* over \( k \)); that is

\[
R_\infty = k[x] \{y_\alpha \}_{\alpha \in k} / \sum_{\alpha \in k} ((x-\alpha)y_\alpha, y_\alpha^2).
\]

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This is not a Noetherian ring, because the radical \( r = \{ y_\alpha \}_{\alpha \in k} \) is not finitely generated. But \( \text{Spec } R_\infty \) agrees as a topological space with \( \text{Spec } R_\infty^{\text{red}} = A_k^1 \), hence \( |\text{Spec } R_\infty| \) is a Noetherian topological space.

On the other hand, the map \( R \to R_\alpha \) is an isomorphism away from \( \alpha \), and similarly \( R_\alpha \to R_\infty \) induces isomorphisms on the stalks at \( \alpha \). Thus, the stalk \( (R_\infty)_{q_\alpha} = (R_\alpha)_{p_\alpha} \) at \( q_\alpha = p_\alpha R_\infty + r \) is Noetherian. Similarly, the stalk at the generic point \( r \) is just \( R_{(0)} = k(x) \). Thus, we conclude that all the stalks of \( R_\infty \) are Noetherian.

\[ \square \]

**Proposition 5.17.** Let \( X \) be a noetherian scheme. Then any closed subset \( Y \subseteq X \) can be written as a finite union \( Y = \bigcup Y_i \) where each \( Y_i \) is irreducible.

**Proof.** If \( W \) is the set of closed subsets that cannot be written as a finite union of closed irreducible subsets, and \( W \) is non-empty, it has a minimal element \( Z \in W \). \( Z \) is not irreducible, so it can be written as a union of two proper closed subsets which do not lie in \( W \), a contradiction. \[ \square \]

The \( Y_i \) are called the *irreducible components* of \( Y \). They are a priori defined only at the level of topological spaces, but we will see in Chapter 7 (Lemma 8.37) that one can define a scheme structure on them in a natural way, so that they become ‘closed subschemes’ of \( X \).

### 5.3 The dimension of a scheme

Recall that the *Krull dimension* of a ring \( A \) is defined as the supremum of the length of all chains of prime ideals in \( A \). For a scheme, we make the following similar definition:

**Definition 5.18.** Let \( X \) be a scheme. The dimension of \( X \) is the supremum of all integers \( n \) such that there exists a chain

\[ Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \]

of distinct closed irreducible closed subsets of \( X \).

Note that this supremum might not be a finite number, in which case we say that \( \dim X = \infty \). Note also that the dimension of \( X \) only depends on the underlying topological space.

In the case where \( X = \text{Spec } A \) is affine, we know that the closed irreducible subsets are of the form \( V(p) \) where \( p \) is a prime ideal. Using this observation we find

**Proposition 5.19.** The dimension of \( X = \text{Spec } A \) equals the Krull dimension of \( A \).
Example 5.20.  1. The dimension of $\mathbb{A}^n_k = \text{Spec } A[x_1, \ldots, x_n]$ is $n + \dim A$.
   In particular, when $A = k$ is a field, $\mathbb{A}^n_k$ has dimension $n$.
   
   2. $\dim \text{Spec } \mathbb{Z}$ is 1.
   
   3. $\dim \text{Spec } k[\epsilon]/\epsilon^2 = 0$.

Remark 5.21. Having finite dimension does not guarantee that the scheme is noetherian. For an explicit counterexample, take $\text{Spec } R$ where $R$ is the ring of integers in $\mathbb{Q}_p$.

More surprisingly, there might may be noetherian rings having infinite Krull dimension! First counterexamples were constructed by Nagata. In short the counterexample involves taking the polynomial ring in infinitely many variables $k[x_1, x_2, \ldots]$ and dividing out by a certain ideal, ensuring that all prime chains are finite, but there is a sequence of prime chains whose lengths tend to infinity!

Definition 5.22. Let $Y \subseteq X$ be a closed subset of $X$. We define the codimension of $Y$ as the suprenum of all integers $n$ such that there exists a chain

$$Y = Y_0 \subseteq Y_1 \subseteq \cdots Y_n$$

of distinct irreducible closed subsets of $X$.

The codimension of $V(p)$ in $\text{Spec } A$ is the height of the prime $p$ in $A$.

Note that the formula $\dim Y + \text{codim } Y = \dim X$ does not always hold, even if $X$ is the spectrum of a ring. Indeed, let $X = \text{Spec } R[t]$ where $R = k[[t]]$. The prime $p = (tu - 1)$ has height 1, but $A/p \simeq R[1/t]$ is a field, hence of dimension 0. However, $\dim A = \dim R + 1 = 2$.

On the other hand, for integral schemes, we can say something:

Theorem 5.23. Let $X$ be an integral scheme of finite type over a field $k$, with function field $K$. Then

(i) $\dim X$ equals the trancendence degree of $K$ over $k$ (in particular, $\dim X < \infty$).

(ii) For each $U \subseteq X$ open, $\dim U = \dim X$.

(iii) For $p \in X$ closed, $\dim X = \dim \mathcal{O}_{X,p}$.

Proof. Note that (ii) follows from (i) since $X$ and an open subset $U$ have the same function field. To prove (i) we may assume that $X = \text{Spec } A$ is affine. The hypothesis on $X$ gives that $A$ is a finitely generated $k$-algebra with quotient field $K$. In this case, (i) is a consequence of Noether normalization (see Chapter 12.
of Atiyah–MacDonald). To prove (iii), we may again assume that \( X = \text{Spec} \ A \), and use the formula
\[
\dim A/p + \text{ht } p = \dim A
\]
which folds for prime ideals in finitely generated \( k \)-algebras.

For schemes which aren’t integral but still of finite type, we still have a good control over its dimension. First of all, the dimension of \( X \) is the same as \( X_{\text{red}} \), so we may assume that \( X \) is reduced. Then if \( X = \bigcup X_i \) is the decomposition into irreducible components, we have \( X_i \) integral, and \( \dim X \) is the supremum of all \( \dim X_i \).

## 5.4 Properties of morphisms

In case of affine schemes say \( S = \text{Spec} \ A \) and \( X = \text{Spec} \ B \), giving a map \( X \to S \) is equivalent to giving a ring homomorphism \( A \to B \), or said differently giving \( B \) the structure of an \( A \)-algebra. So if \( S = \text{Spec} \ A \) the category \( \text{Aff}/S \) of affine schemes over \( S \) is equivalent to the category \( \text{Alg}/A \) of \( A \)-algebras.

**Definition 5.24.** (i) A morphism \( f : X \to Y \) is of **locally finite type** if \( Y \) has a cover consisting of affines \( V_i = \text{Spec} \ B_i \) such that \( f^{-1} V_i \) can be covered by affine subsets of the form \( \text{Spec} \ A_{ij} \), where each \( A_{ij} \) is finitely generated as a \( B_i \)-module.

(ii) \( f \) is of **finite type** if, in (i), one can do with a finite number of \( \text{Spec} A_{ij} \).

(iii) \( f \) is **finite** if \( f^{-1} V_i = \text{Spec} A_i \), where \( A_i \) is a finite \( B_i \)-module.

We say that a scheme over \( k \) is of **locally finite type** (resp. **of finite type**, resp. **finite**) over \( k \), if the morphism \( X \to \text{Spec} k \) has this property.

For an affine scheme these conditions translate into the following:

- \( \text{Spec} A \) is a scheme over \( k \) if and only if there is a homomorphism \( k \to A \), or equivalently, that \( A \) is a \( k \)-algebra.

- \( \text{Spec} B \to \text{Spec} A \) is a morphism of schemes over \( k \) if and only if the induced map \( A \to B \) is a homomorphism of \( k \)-algebras.

- \( \text{Spec} A \) is of finite type over \( k \) if and only if \( A \) is a finitely generated \( k \)-algebra.
5.5 More examples

The rings $\mathbb{Z}(2)$ and $\mathbb{Z}(3)$ are both discrete valuation rings with maximal ideal being (2) and (3) respectively. Their fraction field are both equal to $\mathbb{Q}$. Let $X_1 = \text{Spec} \mathbb{Z}(2)$ and $A_2 = \text{Spec} \mathbb{Z}(3)$. Both have a generic point that is open, so there is a canonical open immersion $\text{Spec} \mathbb{Q} \to X_i$ for $i = 1, 2$. Hence we can glue. We obtain a scheme $X$ with one open point $\eta$ and two closed points. Let us compute the global sections of $\mathcal{O}_X$ using the now classical restriction sequence for the open covering $\{ X_1, X_2 \}$:

\[
\begin{align*}
\Gamma(X, \mathcal{O}_X) &\longrightarrow \Gamma(X_1, \mathcal{O}_X) \oplus \Gamma(X_2, \mathcal{O}_X) \longrightarrow \Gamma(X_1 \cap X_2, \mathcal{O}_X) \rightarrow 0.
\end{align*}
\]

The map $\rho$ sends a pair $(an^{-1}, bm^{-1})$ to the difference $an^{-1} - bm^{-1}$, hence the kernel consists of the diagonal, so to speak, in $\mathbb{Z}(2) \oplus \mathbb{Z}(3)$, which is isomorphic to the intersection $\mathbb{Z}(2) \cap \mathbb{Z}(3)$. This is a semi local ring with the two maximal ideals (2) and (3). Hence there is a map $X \to \text{Spec} \mathbb{Z}(2) \cap \mathbb{Z}(3)$ and it is left as an exercise to show this is an isomorphism.

More generally, if $P = \{ p_1, \ldots, p_r \}$ is a finite set different prime numbers, one may let $X_p = \text{Spec} \mathbb{Z}(p)$ for $p \in P$. There is, as in the previous case, canonical open embedding $\text{Spec} \mathbb{Q} \to X_p$. Let the image be $\{ \eta_p \}$. Obviously the gluing conditions are all satisfied (the transition maps are all equal to id$_{\text{Spec} \mathbb{Q}}$ and $X_{pq} = \{ \eta_p \}$ for all $p$). We do the gluing and obtain a scheme $X$. Again, to compute the global sections, we use the sequence

\[
\begin{align*}
\Gamma(X, \mathcal{O}_X) &\longrightarrow \bigoplus_{p \in P} \Gamma(X_p, \mathcal{O}_X) \longrightarrow \bigoplus_{p,q \in P} \Gamma(X_1 \cap X_2, \mathcal{O}_X) \rightarrow 0.
\end{align*}
\]

The map $\rho$ sends the sequence $(a_p)_{p \in P}$ to the sequence $(a_p - a_q)_{p,q \in P}$ and the kernel of $\rho$ is the intersection $A_P = \bigcap_{p \in P} \mathbb{Z}(p)$, which is a semilocal ring whose maximal ideals are the $(p)A_p$'s for $p \in P$. There is a canonical morphism $X \to \text{Spec} A_P$, and again we leave it to the industrious student to verify that this is an isomorphism.

5.5.1 A more fancy example

In the previous example we worked with a finite set of primes, but the hypotheses of the gluing theorem impose no restrictions on the number of schemes to be
5.5. More examples

glued together, and we are free to take $\mathcal{P}$ infinite, for example we can use the set $\mathcal{P}$ of all primes! The glued scheme $X_{\mathcal{P}}$ is a peculiar animal: It is neither affine nor noetherian, but it is locally noetherian. There is a map $\phi : X_{\mathcal{P}} \rightarrow \text{Spec } \mathbb{Z}$ which is bijective and continuous, but not a homeomorphism, and it has the property that for all open subsets $U \subseteq \text{Spec } \mathbb{Z}$ the map induced on sections $\phi^\#: \Gamma(U, \mathcal{O}_{\text{Spec } \mathbb{Z}}) \rightarrow \Gamma(\phi^{-1}U, \mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism!

As before we construct the scheme $X_{\mathcal{P}}$ by gluing the different $\text{Spec } \mathbb{Z}(p)$'s together along the generic points. However, when computing the global sections, we see things change. The kernel of $\rho$ is still $\bigcap_{p \in \mathcal{P}} \mathbb{Z}(p)$, but now this intersection equals $\mathbb{Z}$! Indeed, a rational number $\alpha = a/b$ lies in $\mathbb{Z}(p)$ precisely when the denominator $b$ does not have $p$ as factor, so lying in all $\mathbb{Z}(p)$, means that $b$ has no non-trivial prime-factor. That is, $b = \pm 1$, and $\alpha \in \mathbb{Z}$.

There is a morphism $X_{\mathcal{P}} \rightarrow \text{Spec } \mathbb{Z}$ which one may think about as follows. Each of the schemes $\text{Spec } \mathbb{Z}(p)$ maps in a natural way into $\text{Spec } \mathbb{Z}$, the mapping being induced by the inclusions $\mathbb{Z} \subseteq \mathbb{Z}(p)$. The generic points of the $\text{Spec } \mathbb{Z}(p)$'s are all being mapped to the generic point of $\text{Spec } \mathbb{Z}$. Hence they patch together to give a map $X_{\mathcal{P}} \rightarrow \text{Spec } \mathbb{Z}$. This is a continuous bijection by construction, but it is not a homeomorphism! Indeed, the subsets $\text{Spec } \mathbb{Z}(p)$ are open in $X_{\mathcal{P}}$ by the gluing construction, but they are not open in $\text{Spec } \mathbb{Z}$, since their complements are infinite, and the closed sets in $\text{Spec } \mathbb{Z}$ are just the finite sets of maximal ideals.

The topology of the scheme $X_{\mathcal{P}}$ is not noetherian since the subschemes $\text{Spec } \mathbb{Z}(p)$ form an open cover that obviously can not be reduced to a finite cover. However, it is locally noetherian the open subschemes $\text{Spec } \mathbb{Z}(p)$ being noetherian. The sets $U_p = X_{\mathcal{P}} - \{ (p) \}$ map bijectively to $D(p) \subseteq \text{Spec } \mathbb{Z}$ and $\Gamma(U_p, \mathcal{O}_{X_{\mathcal{P}}}) = \mathbb{Z}_p$, but $U_p$ and $D(p)$ are not isomorphic.

**Exercise 15.** Glue $X_2$ to itself along the generic point to obtain a scheme $X$. Show that $X$ is not affine. Hint: Show that $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}_4$. 

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Chapter 6
Projective schemes

6.1 Motivation

Let us consider the usual construction of complex projective space: As a topological space, \( \mathbb{CP}^n \) is the quotient space

\[
\mathbb{CP}^n = \left( \mathbb{C}^{n+1} - 0 \right) / \mathbb{C}^*
\]

where \( \mathbb{C}^* \) acts on \( \mathbb{C}^{n+1} \) by scaling the coordinates. We can translate this into algebra as follows: If \( f \) is a function on \( \mathbb{C}^{n+1} \) and \( \lambda \in \mathbb{C}^* \) then we get a (right) action of \( \mathbb{C}^* \) on \( f \) by defining \( f.\lambda(x) = f(\lambda x) \). In other words, \( \mathbb{C}^* \) acts on the ring \( \mathbb{C}[x_0, \ldots, x_n] \). Now, the actions \( \mathbb{C}^* \) are well-understood: Any time \( \mathbb{C}^* \) acts on a complex vector space \( V \), we can decompose that vector space as a direct sum

\[
V = \sum_{d \in \mathbb{Z}} V_d
\]

where \( \lambda \in \mathbb{C}^* \) acts on \( v \in V_d \) by \( \lambda.v = \lambda^d v \). Thus we obtain a grading on \( \mathbb{C}[x_0, \ldots, x_n] \). More generally, affine schemes over \( \mathbb{C} \) with an action of \( \mathbb{C}^* \) correspond to graded \( \mathbb{C} \)-algebras.

We want to take the quotient of \( \mathbb{A}^{n+1}_\mathbb{C} - 0 = \text{Spec} \mathbb{C}[x_0, \ldots, x_n] - V(x_0, \ldots, x_n) \) by this action. We write \( \mathbb{P}^n_\mathbb{C} \) for the corresponding quotient space. The notation \( \mathbb{P}^n_\mathbb{C} \), rather than \( \mathbb{CP}^n \), is used to emphasise that the quotient is taken with the Zariski topology, and not the usual topology.

We can try to put a scheme structure on \( \mathbb{P}^n_\mathbb{C} \) by looking for reasonable open covers. Note that the open subsets of \( \mathbb{P}^n_\mathbb{C} \) correspond to \( \mathbb{C}^* \)-invariant open subsets of \( \mathbb{A}^{n+1}_\mathbb{C} - 0 \). If is not too hard to see that \( D(f) \subseteq \mathbb{A}^{n+1}_\mathbb{C} \) is \( \mathbb{C}^* \)-invariant if and
only if $f$ is a homogeneous polynomial. We write $D_+(f) \subseteq \mathbb{P}^n_C$ for the open subset corresponding to $D(f) \subseteq \mathbb{A}^{n+1}_C - 0$.

To define a structure sheaf we should figure out what $\mathcal{O}_{\mathbb{P}^n_C}(D_+(f))$ should be. While it is true that $D(f)$, being an affine scheme, has a structure sheaf, we have to take more care in deciding which sections to take, to make things compatible with the $\mathbb{C}^*$-action: A function on $D_+(f)$ should be a function on $D(f)$ that is invariant under the action of $\mathbb{C}^*$. That is, we should have $g.\lambda = g$, which means precisely that $g$ has graded degree zero. The functions of with this property are exactly the ones of graded degree zero, thus we define

\[ \mathcal{O}_P(D_+(f)) = \mathbb{C}[x_0, \ldots, x_n, f^{-1}]_0. \]

where the degree means that we take the degree 0 part.

We can generalize the above for any $\mathbb{C}$-scheme with an action of $\mathbb{C}^*$. Such a scheme corresponds to a graded $\mathbb{C}$-algebra $R$. To make a reasonable quotient, we must remove the locus in Spec $R$ that is fixed by $\mathbb{C}^*$. It is not too hard to prove the following:

**Lemma 6.1.** The fixed locus of $\mathbb{C}^*$ acting on Spec $R$ is $V(R_+)$, where $R_+$ denotes the ideal generated by element of positive degree.

So we then attempt to take a quotient of Spec $R - V(R_+)$ by $\mathbb{C}^*$. Again, the $\mathbb{C}^*$-invariant open subsets are of the form $D_+(f)$ where $f$ is homogeneous, and these correspond to open subsets $D_+(f) \subseteq P = (\text{Spec } R - V(R_+))/\mathbb{C}^*$. We can then define a $\mathcal{B}$-sheaf on $P$ by setting $\mathcal{O}_P(D_+(f)) = \mathcal{O}_{\text{Spec } S(D(f))_0}$, and check that we get a scheme $P$.

Beside of inducing a grading on $S$, the action of $\mathbb{C}^*$ plays very little role here. Realizing this, we can in fact build a scheme $P$ from any graded ring $R$: We construct the topological space of $P$ from the set of homogeneous prime ideals of $R$ (with the induced Zariski topology), and define a structure sheaf on it by the formula like to the one above. This is essentially the ‘Proj’-construction.

### 6.2 Basic remarks on graded rings

A *graded ring* $R$ is a ring with a decomposition

\[ R = R_0 \oplus R_1 \oplus \cdots \]

such that $R_m \cdot R_n \subseteq R_{m+n}$ for each $m, n \geq 0$, and such that $R_0$ is a subring of $R$. We say that an $R$-module $M$ is graded if it has a similar decomposition $M = \bigoplus_{n \geq 0} M_n$ such that $R_m \cdot M_n \subseteq M_{m+n}$ for each $m, n \geq 0$. As usual we say that an element $f \in R$ (resp. $m \in M$) is *homogeneous* of degree $d$ if it lies in $R_d$.
An ideal \( I \subseteq R \) is homogeneous if the homogeneous components of any element in \( I \) belongs to \( I \).

We will write \( R_+ \) for the sum \( \bigoplus_{n > 0} R_n \); this is naturally a homogeneous ideal of \( R \), which we call the irrelevant ideal.

We let \( R^{(d)} \) denote the subring of \( R \) given by \( \bigoplus_{n \geq 0} R_{nd} \).

### 6.2.1 Localization

Given a homogeneous prime ideal \( p \) let \( T(p) \) be the set of homogeneous elements \( f \in R - p \). We obtain a grading on the localization \( T(p)^{-1}R \) by setting \( \deg(f/g) = \deg f - \deg g \).

**Definition 6.2.** For a homogeneous prime ideal \( p \subseteq R \) and \( f \in R \) homogeneous, define for an \( R \)-module \( M \)

- \( M_{(p)} = (T(p)^{-1}M)_0 \)
- \( M_{(f)} = (\{1, f, f^2, \ldots\}^{-1}M)_0 \)

where the subscript means the degree 0 part.

**Example 6.3.** For \( R = A[x_0, \ldots, x_n] \), we have \( R(x_i) = A[x_i^{-1}x_j]_{0 \leq j \leq n} \).

### 6.3 The Proj construction

**Definition 6.4.** Let \( \text{Proj} \ R \) denote the set of homogeneous prime ideals of \( R \) that do not contain the irrelevant ideal \( R_+ \).

We can put a topology on \( \text{Proj} \ R \) by setting, for a homogeneous ideal \( b \),

\[
V(b) = \{ p \in \text{Proj} \ R : p \supseteq b \}.
\]

These sets satisfy the following identities

1. \( V(\sum b_i) = \bigcap V(b_i) \).
2. \( V(ab) = V(a) \cup V(b) \).
3. \( V(\sqrt{a}) = V(a) \).

In particular, the \( V \)'s do in fact yield a topology on \( \text{Proj} \ R \) (setting the open sets to be complements of the \( V \)'s). Notice that this is nothing by the induced topology from the inclusion \( \text{Proj} \ R \subseteq \text{Spec} \ R \).

As with the affine case, we can define the distinguished open sets. For \( f \) homogeneous of positive degree, define \( D_+(f) \) to be the collection of homogeneous
ideals (not containing $R_+$) that do not contain $f$. These are open sets with respect to the Zariski topology.

Let $a$ be a homogeneous ideal. Then we claim that:

**Lemma 6.5.** $V(a) = V(a \cap R_+)$.

**Proof.** Indeed, suppose $p$ is a homogeneous prime not containing $R_+$ such that all homogeneous elements of positive degree in $a$ (i.e., anything in $a \cap R_+$) belongs to $p$. We will show that $a \subseteq p$.

Choose $a \in a \cap R_0$. It is sufficient to show that any such $a$ belongs to $p$ since we are working with homogeneous ideals. Let $f$ be a homogeneous element of positive degree that is not in $p$. Then $af \in a \cap R_+$, so $af \in p$. But $f \notin p$, so $a \in p$.  

Thus, when constructing these closed sets $V(a)$, it suffices to work with ideals contained in the irrelevant ideal. In fact, we could take $a$ in any prescribed power of the irrelevant ideal, since taking radicals does not affect $V$. Note incidentally that we would not get any more closed sets if we allowed all ideals $b$, since to any $b$ we can consider its “homogenization.”

**Proposition 6.6.** We have $D_+(f) \cap D_+(g) = D_+(fg)$. Also, the $D_+(f)$ form a basis for the topology on $\text{Proj } R$.

**Proof.** The first part is evident, by the definition of a prime ideal. We prove the second. Note that $V(a)$ is the intersection of the $V((f))$ for the homogeneous $f \in a \cap R_+$. Thus $\text{Proj } R - V(a)$ is the union of these $D_+(f)$. So every open set is a union of sets of the form $D_+(f)$.  

**Proposition 6.7.** Let $f \in R$ be homogeneous of degree $d$. There is a canonical homeomorphism $\phi_f : D_+(f) \to \text{Spec } R(f)$ given by

$$\phi_f : p \to pR_f \cap R(f)$$

sending homogeneous prime ideals of $R$ not containing $f$ into primes of $R(f)$. Moreover,

(i) For $g \in R$ such that $D_+(g) \subseteq D_+(f)$. If $u = g^d f^{-\deg g} \in R(f)$, then

$$\phi(D_+(g)) = D(u).$$

(ii) For any graded $R$-module $M$, there is a canonical homomorphism $M(f) \to M(g)$, which induces an isomorphism $(M(f))_u \simeq M(g)$.

(iii) If $I \subseteq R$ is a homogeneous ideal, then $\phi(V(I) \cap D_+(f)) = V(IR_f \cap R(f))$.  

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Proof. Note that \( \phi \) is just the restriction of the canonical map \( \text{Spec} R_f \to \text{Spec} R(\mathfrak{f}) \) induced by the inclusion map \( R(\mathfrak{f}) \subseteq R_f \). Therefore it is continuous. We now check that it is a bijection.

We construct the inverse map \( \psi : \text{Spec}(R(\mathfrak{f})) \to D_+(f) \) as follows. Given a prime ideal \( \mathfrak{p} \in \text{Spec}(R(\mathfrak{f})) \), define \( \psi(\mathfrak{p}) = \bigoplus_{n \geq 0} \psi(\mathfrak{p})_n \) where

\[
\psi(\mathfrak{p})_n = \{ x \in R_n | x^d/f^n \in \mathfrak{p} \}
\]

(Recall that \( d = \deg f \)). We need to check that this is a homogeneous prime ideal. Given \( x, y \in \psi(\mathfrak{p})_n \), we have, by definition, that \( x^d/f^n \in \mathfrak{p} \) and \( y^d/f^n \in \mathfrak{p} \), and so by the Binomial theorem, \( (x + y)^{2d}/f^{2n} \in \mathfrak{p} \). Therefore, \( (x + y)^d/f^n \in \mathfrak{p} \), and so \( \psi(\mathfrak{p}) \) is closed under addition. The other conditions for being a homogeneous ideal are proved similarly. To show that \( \psi(\mathfrak{p}) \) is prime, let \( x \in R_m \) and \( y \in R_n \) so that \( (xy)^d/f^{m+n} \in \mathfrak{p} \). Then \( (xy)^d \in \mathfrak{p} \), and so either \( x \in \mathfrak{p} \) or \( y \in \mathfrak{p} \) since \( \mathfrak{p} \) is prime. Hence either \( x^d/f^m \in \psi(\mathfrak{p})_m \) or \( y^d/f^n \in \psi(\mathfrak{p})_n \), and so \( \psi(\mathfrak{p}) \) is prime.

The fact that \( \phi \) and \( \psi \) are mutually inverse follows almost by construction. Indeed, to verify \( \psi(\mathfrak{p} R_f \cap R(\mathfrak{f})) = \mathfrak{p} \), note that \( x \in \mathfrak{p}_n \subseteq R_n \) satisfies \( x^d/f^n \in \mathfrak{p} R_f \cap R(\mathfrak{f}) \). Conversely, if \( x \in \psi(\mathfrak{p} R_f \cap R(\mathfrak{f})) \), then \( x^d/f^n \in \mathfrak{p} R_f \cap R(\mathfrak{f}) \), and so there is an \( N > 0 \) so that \( f^N x^d \in \mathfrak{p} \). As \( \mathfrak{p} \) is prime, and \( f \not\in \mathfrak{p} \), we have \( x^d \in \mathfrak{p} \), and hence \( x \in \mathfrak{p} \). Proving \( \phi \circ \psi = \text{id} \) is similar. This shows that \( \phi \) is bijective. The fact that \( \phi \) is a homeomorphism follows from the fact that it is continuous, and the point (i) showing that it is open.

(i) Let \( g \in R \) be an element with \( D_+(g) \subseteq D(f) \). Then for \( \mathfrak{p} \in D_+(f) \), we have \( \mathfrak{p} \in D_+(g) \iff g^r/f^s \not\in \mathfrak{p} R_f \) for some \( r, s > 0 \iff g^r/f^s \not\in \mathfrak{p} R_f \) for all \( r, s > 0 \iff g^{\deg f}/f^{\deg g} \not\in \mathfrak{p} \cap R(\mathfrak{f}) = \phi(\mathfrak{p}) \). Hence \( \phi(D_+(g)) = D(u) \).

(ii) Write \( g^k = af \). The localization map \( M_f \to M_g \) is given by \( \frac{x}{f^m} \mapsto \frac{a^m x}{g^{mk}} \), where \( x \in M \). This induces a map \( M_f \to M_g \) (since \( \deg x + m(\deg a - k \deg g) = \deg x - m \deg f \)). The element \( u \) acts as an invertible element on \( R_g \), so the map \( M_f \to M_g \) factors via a map

\[
\rho : (M_f)_u \to M_g
\]

We claim that this is an isomorphism.

\( \rho \) surjective: Explicitly, we have

\[
\rho \left( \frac{x f^{-n}}{u^m} \right) = \frac{f^{tm-n} x}{g^{tm}}
\]

where \( t = \deg g \). Take an element \( y/g^l \in M_g \) where \( \deg y = tl \). Choose \( m \) large so that \( dm \geq l \). Define \( x = \frac{g^{dm-l} y}{f^{tm-n}} \in M(f) \). We have \( \deg x = nd \), and hence \( \frac{x f^{-n}}{u^m} \in (M_f)_u \) is an element that maps to the given \( \frac{y}{g^l} \).

\( \rho \) is injective: If \( x/f^n \in M(f) \) maps to 0 in \( M(g) \), then there is an \( l > 0 \).
6.3. The Proj construction

so that \( g^{l_d}a^n x = 0 \in M \). Multiplying up by powers of \( a \) and \( f \), we get a relation of the form \( g^{(l+n)d}x = 0 \in M \), and hence \( u^{(l+n)d}x = 0 \in M(f) \). But then \( x/f^n = 0 \in (M(f))_u \).

(iii) Follows by the definition of \( \phi \). Let \( p \in V(I) \cap D_+(f) \), so \( p \supseteq I \) and \( p \not\supseteq f \). Hence \( p R_f \cap R(f) \supseteq IR_f \cap R(f) \) which gives the ‘\( \subseteq \)’ containment. Conversely, given a prime ideal \( p \subseteq R(f) \) such that \( p \supseteq IR_f \cap R(f) \), then its preimage \( p' \) in \( R \) is a homogeneous prime ideal not containing \( f \), and \( \phi(p') = p' R_f \cap R(f) \supseteq IR_f \cap R(f) \).

\[ \square \]

6.3.1 Proj \( R \) as a scheme

We shall now make \( X = \text{Proj} R \) into a locally ringed space. Let \( \mathcal{B} \) be the base of \( \text{Proj} R \) made up by the distinguished open subsets. For each \( D_+(f) \) we let

\[ \mathcal{O}_X(D_+(f)) = R(f) \]

The previous proposition shows that this gives a well-defined \( \mathcal{B} \)-presheaf of rings, and using the homeomorphism \( \phi \) to \( \text{Spec} R(r) \), we see that it is a actually a \( \mathcal{B} \)-sheaf. (Alternatively, we could modify the proof for the case of \( \text{Spec} \) to see this directly). We will denote the unique sheaf extension by \( \mathcal{O}_X \).

It follows that \( X \) is has the structure of a ringed space. This is in fact a locally ringed space, because the stalk \( \mathcal{O}_p \) is just \( R(p) \), which is a local ring. Indeed, the unique maximal ideal is generated by \( p \). Moreover, the previous discussion has shown that the basic open sets \( D_+(f) \) are each isomorphic as locally ringed spaces to \( \text{Spec} R(f) \), which are affine schemes, and so \( \text{Proj} R \) is a scheme.

**Definition 6.8.** For a graded ring \( R \), we call the scheme \( (\text{Proj} R, \mathcal{O}) \) the projective spectrum of \( R \).

In fact, \( \text{Proj} R \) is naturally a scheme over \( R_0 \): The homomorphisms \( R_0 \to R(f) \) induce a morphism

\[ \text{Proj} R \to \text{Spec}(R_0) \]

Moreover, if \( R \) is a finitely generated over \( R_0 \), this is of finite type over \( \text{Spec} R_0 \). This follows by looking at the distinguished opens, \( D_+(f) \) - each \( R(f) \) is finitely generated as an \( R_0 \)-algebra if \( R \) is.

**Example 6.9.** Let \( A \) be a ring and let \( R = A[t] \) with the grading given by \( \deg t = 1 \) and \( \deg a = 0 \) for all \( a \in A \). Then \( \text{Proj} R \cong \text{Spec} A \).

**Example 6.10.** Let us study the case of a polynomial ring in two variables. \( R = \mathbb{Z}[s, t] \) where \( s \) and \( t \) have degree 1, and \( k \) is any ring. To see that \( X = \text{Proj} R \) coincides with \( \mathbb{P}^1 \) as defined in Chapter 3, we see how it is glued from two affine schemes. Note that \( X \) is covered by \( D_+(s) \) and \( D_+(t) \) (since
these generate the irrelevant ideal). Write for simplicitly $U = D_+(s) \simeq R(s)$
and $V = D_+(t) \simeq R(t)$. Let us check how $X$ is glued together from $U, V$ along
$U \cap V = D_+(st) \simeq \text{Spec } R_{(st)}$.

Note first that $R_s \simeq \mathbb{Z}[s, s^{-1}, t]$ has degree 0 part equal to $\mathbb{Z}[s^{-1}t] \subseteq R_s$. Similarly, $R_{(t)} = \mathbb{Z}[st^{-1}]$. The intersection $D_+(st)$ is the degree 0 part of $R_{(st)}$, i.e.,
$R_{(st)} = \mathbb{Z}[st^{-1}]$. In other words, if we write $u = s^{-1}t$, we have $R_{(s)} = \mathbb{Z}[u]$, $R_{(t)} = \mathbb{Z}[u^{-1}]$ and $R_{(st)} = \mathbb{Z}[u, u^{-1}] = \mathbb{Z}[u^2]$. It follows that $U \simeq \text{Spec } \mathbb{Z}[u] = \mathbb{A}_\mathbb{Z}^1$
and $V \simeq \text{Spec } \mathbb{Z}[u^{-1}] \simeq \mathbb{A}_\mathbb{Z}^1$ are glued along $\text{Spec } R_{(st)}$ which is an open set in both. From this we see that $\text{Proj } R$ coincides with our previous definition of $\mathbb{P}^1$.

**Example 6.11.** Let $R = k[x, y]/(xy)$. Spec $R$ is the union of the $x$- and $y$-axes. So Spec $R - V(x, y)$ is the axes excluding the origin. Proj $R$ consists of just two points: We obtain Proj $R$ by gluing Spec($R_{(x)}$) and Spec($R_{(y)}$). Now,
$$R_{(x)} = k[x, y]_{(x)}/xy = k[x, y]_{(x)}/y = k[x]_{(x)} = k$$
and the corresponding chart of Proj $R$ is Spec $k$. Similarly, the other chart is Spec $k$. We have $R_{(xy)} = 0$, so the overlap is empty and Proj $R$ is two points.

**Definition 6.12.** We define the **projective $n$-space** to be the scheme
$$\mathbb{P}^n = \text{Proj } \mathbb{Z}[x_0, \ldots, x_n].$$

More generally, for a ring $A$, the **projective space over $A$**
$$\mathbb{P}^n_A = \text{Proj } A[x_0, \ldots, x_n]$$

In this notation, $\mathbb{P}^n_A$ is a scheme whose closed $k$-points $\mathbb{P}^n(k)$ coincides with the variety of projective $n$-space.

**Proposition 6.13.** Let $R$ be a graded ring.

(i) If $R$ is noetherian, then Proj $R$ is noetherian.

(ii) If $R$ is finitely generated over $R_0$, then Proj $R$ is of finite type over Spec $R_0$.

(iii) If $R$ is an integral domain, then Proj $R$ is integral.

**Proof.** The properties listed are properties one can verify on an affine covering. In our case Proj $R$ is covered by the affines Spec $R_{(f)}$ which are noetherian (resp. of finite type, integral) provided $R$ is noetherian (resp. finitely generated, an integral domain). \qed
6.4 Functoriality

Unlike the case of affine schemes, a graded ring homomorphism \( \phi : R \to S \) does not induce a morphism between the projective spectra \( \text{Proj} S \) and \( \text{Proj} R \). The reason is that some primes in \( S \) may pullback to \( R \) to contain the irrelevant ideal \( R_+ \). However, as we will see shortly, this is the only obstruction to defining a morphism.

Given a homomorphism \( \phi : R \to S \), we define the set \( G(\phi) \subseteq \text{Proj} S \) to be the set of prime ideals \( p \) such that \( p \subseteq S \) that do not contain \( \phi(R_+) \), in particular, such that \( \phi^{-1}(p) \) is in \( \text{Proj} R \). This sets up a map

\[
\psi : G(\phi) \to \text{Proj} R
\]

\[
p \mapsto \phi^{-1}(p)
\]

Note that \( G(\phi) \) is an open set of \( \text{Proj} S \), so in particular it has an induced scheme structure. We claim that the map \( G(\phi) \to \text{Proj} R \) is a morphism of schemes.

First of all, the map \( \psi \) is continuous, as the Zariski topologies are induced from those of \( \text{Spec} S \) and \( \text{Spec} R \). More explicitly, the inverse image \( \phi^{-1}(D_+(f)) \) is equal to \( G(\phi) \cap D_+(\phi(f)) \), which is open.

Write \( X = G(\phi) \) and \( Y = \text{Proj} R \). We now define the map \( \psi \) on the level of sheaves, i.e., a map

\[
\psi^\#: \mathcal{O}_Y \to \psi_*\mathcal{O}_X
\]

On the distinguished open set \( D_+(f) \subseteq \text{Proj} R \), we can define this map using the isomorphism to \( \text{Spec} R(f) \). We have a map \( R(f) \to S_{\phi(f)} \) induced by \( \phi \). Moreover, since \( \psi^{-1}(D_+(f)) \) is open in \( \text{Spec} S_{\phi(f)} \), we get the map

\[
\mathcal{O}_Y(D_+(f)) = R(f) \to \mathcal{O}_X(\psi^{-1}(D_+(f)))
\]

by restriction. We have shown the following

**Proposition 6.14.** Let \( \phi : R \to S \) be a morphism of graded rings. The map \( \psi : G(\phi) \to \text{Proj} R \) is a morphism of schemes.

**Example 6.15.** To see why restriction to the open set \( G(\phi) \) is necessary, we consider the case where \( R = k[x_0, x_1] \), \( S = k[x_0, x_1, x_2] \) and \( \phi \) is the inclusion map. Note that the prime ideal \( a = (x_0, x_1) \) defines an element in \( \text{Proj} S \), but its restriction to \( R \) is the whole irrelevant ideal of \( R \). In fact, \( G(\phi) = \text{Proj} S - V(a) \), and the map

\[
\psi : \mathbb{P}^2_k - V(a) \to \mathbb{P}^1_k
\]

is nothing but the projection from the point \([0, 0, 1]\). It is a good exercise to prove that there can be no morphisms \( \mathbb{P}^m_k \to \mathbb{P}^n_k \) for \( m > n \) in general.
6.4.1 Closed immersions

The primary example of the above construction is considering the graded quotient homomorphism \( \phi : R \to R/I \), where \( I \subseteq R \) is a homogeneous ideal. Here we have \( G(\phi) = \text{Proj}(R/I) \), so we get a map

\[
\psi : \text{Proj}(R/I) \to \text{Proj} R.
\]

We claim that this is a closed immersion. As usual, homogeneous primes in \( R/I \) not containing \( R_+ \) correspond to homogeneous primes in \( R \) containing \( I \) but not \( R_+ \). It follows that \( \psi \) is injective, mapping onto the closed subset \( V(I) \) in \( \text{Proj} R \). Finally, \( \psi^\# \) is surjective on stalks (where it is just the map \( R_+(p) \to (R/I)_+(p) \)), and so \( \psi \) is a closed immersion. We will see later that there is a converse to this statement, under some mild assumptions on \( R \).

6.5 The Veronese embedding

Let \( R \) be a graded ring and let \( d > 1 \) be an integer. The inclusion \( \phi : R^{(d)} \to R \) induces a morphism

\[
\nu_d : \text{Proj} R \to \text{Proj} R^{(d)}
\]

Indeed, in this case \( G(\phi) = \text{Proj} R \), since any prime \( p \) such that \( p \supseteq R_+ \cap R^{(d)} \) must also contain all of \( R_+ \). If \( r \in R_+ \), note that \( r^d \in R_+ \cap R^{(d)} \) and so \( r \in p \) as well! This map is called the Veronese embedding, or \( d \)-uple embedding of \( X \).

**Proposition 6.16.** The Veronese embedding \( \nu_d \) is an isomorphism.

**Proof.** \( \nu_d \) is injective: If \( p, q \in \text{Proj} R \) such that \( p \cap R^{(d)} = q \cap R^{(d)} \). Then for a homogeneous element \( x \in R \) we have

\[
x \in p \iff x^d \in p \iff x^d \in q \iff x \in q
\]

And so \( p = q \). To show that \( \nu_d \) is surjective, let \( q \in \text{Proj} R^{(d)} \), and define the homogeneous ideal in \( R \) by

\[
p = \bigoplus_{n=0}^{\infty} \{ x \in R_n | x^d \in q \}.
\]

It is not too hard to check that \( p \) is prime, and that \( p \cap R^{(d)} = q \). Hence \( \nu_d \) is bijective, and even a homeomorphism. \( \nu_d \) induces an isomorphism when restricted to the open affines \( D_+(f) \) as well, and so we are done. \( \square \)
6.5. The Veronese embedding

6.5.1 Remark on rings generated in degree 1

We will frequently assume that the ring $R$ is generated in degree 1, that is, $R$ is generated as an $R_0$-algebra by $R_1$. The reason for this will become clear in the next section. Intuitively, it is because we want $\text{Proj } R$ to be covered by the 'affine coordinate charts' $D_+(x)$ where $x$ should have degree 1.

We remark that this assumption is in fact not too restrictive: Any projective spectrum of a finitely generated ring is isomorphic to the Proj of a ring generated in degree 1. This is because of the basic algebraic fact that if $R$ is finitely generated, then some subring $R^{(d)}$ will have all of its generators in one degree, and since $\text{Proj } R^{(d)} \simeq \text{Proj } R$, we don’t change the Proj by replacing $R$ with $R^{(d)}$.

6.5.2 The Blow-up as a Proj

Consider the ring $A = k[x,y]$ and the ideal $I = (x,y)$. We can form a new graded ring by introducing a formal variable $t$ and setting $R = \bigoplus_{m \geq 0} I^k t^k$ where $I^0 = A$. In $R$, $t$ has degree 1, and all other variables have degree 0. This gives a morphism $\pi : \text{Proj } R \rightarrow \text{Spec } A = A^2_k$.

This Proj $R$ is glued together by two affine schemes $\text{Spec } R_{(xt)}$ and $\text{Spec } R_{(yt)}$ which are both isomorphic to $A^2_k$. It is a nice exercise to check that this coincides with the gluing of the blow-up of $A^2_k$, and that $\pi$ is the blow-up morphism.

Claim: We have $R \simeq A[u,v]/(xu - yv)$.

Consider the map $\phi : A[u,v] \rightarrow R$ given by $\phi(u) = xt, \phi(v) = yt$. This is clearly surjective. It suffices to show that $\ker \phi = I$ where $I = (xu - yv)$. The $i \supseteq j'$-direction is clear. Conversely, we can write, modulo $I$, any element $p$ of $k[x,u,y,v]$ as

$$\sum a_{i,j,k} x^i u^j v^k + \sum b_{i',j',k'} x^{i'} y^{j'} v^{k'}.$$ 

If $p \in \ker \phi$, we have

$$0 = \phi(p) = \sum a_{i,j,k} x^{i+j} u^k t^{i+k} + \sum b_{i',j',k'} x^{i'} y^{j'} v^{k'}.$$ 

Hence the coefficients $a_{i,j,k}, b_{i',j',k'}$ vanish except possibly when we have the same monomials appearing in each sum i.e. when $i + j = i', k = j' + k', i = i' + j'$ and $j + k = k'$, in which case we have $a_{i,j,k} = -b_{i',j',k'}$. These conditions imply that $j = -j'$ which must then both be 0, and so $i = i', k = k'$. Hence $p = 0 \mod I$ and so $\ker \phi \subseteq I$. 

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Now, writing $R = A[u,v]/(xv - yu)$, we see that Proj $R$ is covered by the two open sets $D_+ (u) = \text{Spec } R(v)$ and $D_+ (v) = \text{Spec } R(u)$. Here

$$R(u) \simeq (A[u,v]_{u}/(xv - yu))_0 = k[x,v/u]$$

and

$$R(v) \simeq (A[u,v]_{v}/(xv - yu))_0 = k[y,u/v].$$

These are glued along

$$R(\ uv) \simeq (A[u,v]_{uv}/(xv - yu))_0 = k[x,y,u/v,v/u]/(x \cdot v/u - y) \simeq k[x,u/v,v/u]$$

so we see that Proj $R$ coincides with the previous blow-up description.
Chapter 7

Fiber products

7.1 Introduction

One of the most fundamental properties of the category of schemes is that all fiber products exist. The fiber product is extremely useful in many situations and takes on astonishingly versatile roles.

The aim of this chapter is to prove the existence theorem for fiber products and explain the various contexts where fiber products appear. We begin with recalling the definition of the fibre product of sets, then transition into a very general situation to discuss fibre product in general categories, for then to return to the present context of schemes. At the end of the chapter we will go through a series of examples.

7.1.1 Fiber products of sets.

As a warming up we use some lines on recalling the fiber product in the category Sets of sets. The points of departure is two sets $X_1$ and $X_2$ both equipped with a map to a third set $S$; i.e., we are given a diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\psi_1} & S \\
\downarrow & & \downarrow \\
X_2 & \xleftarrow{\psi_2} & S \\
\end{array}
\]
The fibre product $X_1 \times_S X_2$ is the subset of the cartesian product $X \times Y$ consisting of the pairs whose two components have the same image in $S$; that is, we have

$$X_1 \times_S X_2 = \{ (x_1, x_2) \mid \psi_1(x_1) = \psi_2(x_2) \}.$$ 

Clearly the diagram below where $\pi_1$ and $\pi_2$ denote the restrictions of the two projections to the fiber product—that is, $\pi_i(x_1, x_2) = x_i$—is commutative,

\[
\begin{array}{ccc}
X_1 \times_S X_2 & \xrightarrow{\pi_1} & X_1 \\
& & \downarrow \psi_1 \\
X_1 & \xrightarrow{\psi_1} & S \\
\end{array}
\begin{array}{ccc}
& & \downarrow \psi_2 \\
X_2 & \xleftarrow{\pi_2} & S \\
\end{array}
\]

(7.1.1)

And more is true; the fibre product enjoys a universal property: Given any two maps $\phi_1: Z \to X_1$ and $\phi_2: Z \to X_2$ such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ there is a unique map $\phi: Z \to X_1 \times_S X_2$ satisfying $\pi_1 \circ \phi = \phi_1$ and $\pi_2 \circ \phi = \phi_2$. To lay your hands on such a $\phi$, just use the map whose two components are $\phi_1$ and $\phi_2$ and observe that it takes values in $X_1 \times_S X_2$ since the relation $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ holds. Giving the two $\phi_i$’s is to give a commutative diagram like 7.1.1 above with $Z$ replacing the product $X \times_S Y$, and the universal property is to say that 7.1.1 is universal—a more precise usage would be to say it is final among such diagrams.

### 7.1.2 The fiber product in general categories

The notion of a fiber product—formulated as the solution to a universal problem as above—is mutatis mutandis meaningful in any category $C$. Given any two arrows $\psi_i: X_i \to S$ in the category $C$, an object—that we shall denote by $X_1 \times_S X_2$—is said to be the fiber product (fiberproduktet) of the objects $X_i$, or more precisely of the two arrows $\psi_i: X_i \to S$, if the following two conditions are fulfilled:

- There are two arrows $\pi_i: X_1 \times_S X_2 \to X_i$ in $C$ such that $\psi_1 \circ \pi_1 = \psi_2 \circ \pi_2$ (called the projections).

- For any two arrows $\phi_i: Z \to X_i$ in $C$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, there is a unique arrow $\phi: Z \to X_1 \times_S X_2$ satisfying $\pi_i \circ \phi = \phi_i$ for $i = 1, 2$.

The two arrows $\pi_1 \circ \phi$ and $\pi_2 \circ \phi$ that determine the arrow $\phi: Z \to X_1 \times_S X_2$, are called the components (komponentene) of $\phi$, and the notation $\phi = (\phi_1, \phi_2)$ is sometimes used. If $\phi_1: Y_1 \to X_1$ and $\phi_2: Y_2 \to X_2$ are two arrows over $S$, there is a unique arrow denoted $\phi_1 \times \phi_2$ from $Y_1 \times_S Y_2$ to $X_1 \times_S X_2$ whose components are $\phi_1 \circ \pi_{Y_1}$ and $\phi_2 \circ \pi_{Y_2}$. 

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If the fiber product exists, it is unique up to a unique isomorphism, as is true for solutions to any universal problem. However, it is a good exercise to check this in detail in this specific situation.

**Exercise 16.** Show that if the fiber product exists in the category $\mathcal{C}$, it is unique up to a unique isomorphism.

It is not so hard to come up with examples of categories where fiber products do not exist. For instance, consider the category $\mathcal{C}$ of subsets $X$ of the integers with an even number of elements, and morphisms given by inclusions $Y \subseteq X$. The fiber product of $Y \subseteq X$ and $Z \subseteq X$ over $X$ would be $Y \cap Z \subseteq X$ – however $Y \cap Z$ does not necessarily have an even number of elements!

What is of course much more disappointing is that fiber products fail to exist in our good old categories like manifolds or affine varieties. This shows yet another reason why we need to make the transition from varieties to schemes.

**Exercise 17.** (i) Give an example showing that the fiber product does not always exist in the category of manifolds.

(ii) Give an example showing that the fiber product does not always exist in the category of affine varieties.

### 7.2 Products of affine schemes

The category $\text{Aff}$ of affine schemes is, more or less by definition, equivalent to the category of rings, and in the category of rings we have the tensor product. The tensor product enjoys a universal property *dual* to the one of the fiber product. To be precise, assume $A_1$ and $A_2$ are $B$-algebras, i.e., we have two maps of rings $\alpha_i$

\[
\begin{array}{ccc}
A_1 & \rightarrow & A_2 \\
\alpha_2 & \downarrow & \alpha_1 \\
B & \rightarrow & \\
\end{array}
\]

There are two maps $\beta_i: A_i \rightarrow A_1 \otimes_B A_2$ sending $a_1 \in A_1$ to $a_1 \otimes 1$ and $a_2$ to $1 \otimes a_2$, respectively. These are both ring homomorphisms since $aa' \otimes 1 = (a \otimes 1)(a' \otimes 1)$ respectively $1 \otimes aa' = (1 \otimes a)(1 \otimes a')$, and they fit into the following commutative diagram as $\alpha_1(b) \otimes 1 = 1 \otimes \alpha_2(b)$ by the definition of the tensor product $A_1 \otimes_B A_2$ (this is the significance of the tensor product being taken over
7.2. Products of affine schemes

$B$; one can move elements in $B$ from one side of the $\otimes$-glyph to the other):

$$A_1 \otimes_B A_2$$

Moreover, the tensor product is universal in this respect. Indeed, assume that $\gamma_i: A_i \to C$ are $B$-algebra homomorphisms, i.e., $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$; or said differently, they fit into the commutative diagram analogous to (7.2.1) with the $\beta_i$’s replaced by the $\gamma_i$’s. The association $a_1 \otimes a_2 \to \gamma_1(a_1) \gamma(a_2)$ is $B$ bilinear, and hence it extends to a $B$-algebra homomorphism $\gamma: A_1 \otimes_B A_2 \to C$, that obviously have the property that $\gamma \circ \beta_i = \gamma_i$.

Applying the Spec-functor to all this, we get the diagram

and the affine scheme $\text{Spec}(A_1 \otimes_B A_2)$ enjoys the property of being universal among affine schemes sitting in a diagram like 7.2.2. Hence $\text{Spec}(A_1 \otimes_B A_2)$ equipped with the two projections $\pi_1$ and $\pi_2$ is the fiber product in the category Aff of affine schemes. One even has the stronger statement; it is the fiber product in the bigger category Sch of schemes.

**Proposition 7.1.** Given $\phi_i: \text{Spec } A_i \to \text{Spec } B$. Then $\text{Spec}(A_1 \otimes_B A_2)$ with the two projection $\pi_1$ and $\pi_2$ defined as above is the fiber product of the $\text{Spec } A_i$’s in the category of schemes. That is, if $Z$ is a scheme and $\psi_i: Z \to \text{Spec } A_i$ are morphisms with $\psi_1 \circ \pi_1 = \psi_2 \circ \phi_2$, there exists a unique morphism $\psi: Z \to \text{Spec}(A_1 \otimes_B A_2)$ such that $\pi_i \circ \psi = \psi_i$ for $i = 1, 2$.

**Proof.** We know that the proposition is true whenever $Z$ is an affine scheme; so the salient point is that $Z$ is not necessarily affine. For short, we let $X = \text{Spec}(A_1 \otimes_B A_2)$. The proof is just an application of the gluing lemma for morphisms. One covers $Z$ by open affines $U_\alpha$ and covers the intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by open affine subsets $U_{\alpha\beta\gamma}$ as well. By the affine case of the proposition, for each $U_\alpha$ we get a map $\psi_\alpha: U_\alpha \to X$, such that $\psi_\alpha \circ \pi_i = \psi_i|_{U_\alpha}$.
By the uniqueness part of the affine case, these maps coincide on the open affines $U_{\alpha\beta\gamma}$, and therefore on the intersections $U_{\alpha\beta}$. They can thus be patched together to a map $\psi: Z \to X$, which is unique since the $\psi_{\alpha}$’s are unique.

Example 7.2. Let $k$ be a field and $x$ and $y$ two variables. Consider the tensor product $A = k(x) \otimes_k k(y)$. We can regard this as a localization of $k[x, y]$ where we invert everything in the multiplicative set $S = \{p(x)q(x)|p(x), q(y) \neq 0\}$. Suppose that $m \subseteq A$ is a maximal ideal; it has the form $S^{-1}p$ for some prime ideal $p \subseteq k[x, y]$ which is maximal among the primes that do not intersect $S$. In this case we must have $pk[x] = 0$, since otherwise there will be a non-zero $p(x) \subseteq p \cap S$. Similarly $p \cap k[y] = 0$, which implies that $p$ has height at most 1. Hence either $p = (0)$, of $p = (f)$ for some irreducible polynomial $f \in k[x, y]$ not contained in $k[x] \cup k[y]$. If follows that $A$ has dimension 1, and $A$ has infinitely many maximal ideals.

This example shows how strange the fiber product really is – $\text{Spec } A$ is an infinite set, even though it is the fiber product of two schemes with one-point sets. We will see more examples like this in the end of this chapter.

7.3 A useful lemma

Lemma 7.3. If $X \times_S Y$ exists and $U \subseteq X$ is an open subscheme, then $U \times_S Y$ exists and is (canonically isomorphic to) an open subset of $X \times_S Y$ and projections restrict to projections. Indeed $\pi_X^{-1}(U)$ with the two restrictions $\pi_Y|_{\pi_X^{-1}(U)}$ and $\pi_X|_{\pi_X^{-1}(U)}$ as projections is a fiber product.

Proof. Displayed the situation appears like

```
Y
pi_Y
\pi_X^{-1}(U) \hookrightarrow X \times_S Y
\pi_X
\quad U \quad X,
```

and we are to verify that $\pi_X^{-1}(U)$ together with the restriction of the two projections to $\pi_X^{-1}(U)$ satisfy the universal property. If $Z$ is a scheme and $\phi_X: Z \to U$ and $\phi_Y: Z \to Y$ are two morphisms over $S$ we may consider $\phi_U$ as a map into $X$, and therefore they induce a map of schemes $\phi: Z \to X \times_S Y$ whilst $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. Clearly $\pi_X \circ \phi = \phi_U$ takes values in $U$ and therefore $\phi$ takes values in $\pi_X^{-1}(U)$. It follows immediately that $\phi$ is unique (see the exercise below), and we are through. 

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**Exercise 18.** Assume that $U \subseteq X$ is an open subscheme and let $\iota: U \rightarrow X$ be the inclusion map. Let $\phi_1$ and $\phi_2$ be two maps of schemes from a scheme $Z$ to $U$ and assume that $\iota \circ \phi_1 = \iota \circ \phi_2$. Then $\phi_1 = \phi_2$.

When identifying $\pi^{-1}_X(U)$ with $U \times_S Y$, the inclusion map $\pi^{-1}_X(U) \subseteq X \times_S Y$ will correspond to the map $\iota \times \text{id}_Y$ where $\iota: U \rightarrow X$ is the inclusion, so a reformulation of the lemma is that open immersions stay open immersion under change of basis.

7.4 The gluing process

The following proposition will be basis for all gluing necessary for the construction:

**Proposition 7.4.** Let $\psi_X: X \rightarrow S$ and $\psi_Y: Y \rightarrow S$ be two maps of schemes, and assume that there is an open covering $\{U_i\}_{i \in I}$ of $X$ such that $U_i \times_S Y$ exist for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open covering of $X \times S Y$ and projections restrict to projections.

**Proof.** We need some notation. Let $U_{ij} = U_i \cap U_j$ be the intersections of the $U_i$’s, and let $\pi_i: U_i \times_S Y \rightarrow U_i$ denote the projections. By lemma 7.3 there are isomorphisms $\theta_{ji}: \pi_i^{-1}(U_{ij}) \rightarrow U_{ij} \times_S Y$, and gluing functions we shall use $\tau_{ji} = \theta_{ij}^{-1} \circ \theta_{ji}$ that identifies $\pi_i^{-1}(U_{ij})$ with $\pi_j^{-1}(U_{ij})$. The picture is like this

$$U_i \times_S Y \supseteq \pi_i^{-1}(U_{ij}) \xrightarrow{\theta_{ji}} U_{ij} \times_S Y \xrightarrow{\theta_{ij}^{-1}} \pi_j^{-1}(U_{ij}) \subseteq U_j \times_S Y.$$

The gluing maps $\tau_{ij}$ clearly satisfy the gluing conditions being compositions of that particular form, and the scheme emerging from gluing process is $X \times_S Y$.

The two projections are essential parts of the product and must not be forgotten: The projection onto $Y$ is there all the time since $Y$ is nevener touched during the construction. The projection onto $X$ is obtained by gluing the projections $\pi_i$ along the $\pi_i^{-1}(U_{ij})$. By lemma 7.3 we know that when identifying $\pi_i^{-1}(U_{ij})$ with the product $U_{ij} \times_S Y$, the projection $\pi_{ij}$ onto $U_{ij}$ corresponds to the restriction $\pi_i|_{\pi_i^{-1}(U_{ij})}$. This means that $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_{ij} \circ \theta_{ij}$. Saying that $\pi_i|_{\pi_i^{-1}(U_{ij})}$ and $\pi_j|_{\pi_j^{-1}(U_{ij})}$ become equal after gluing, is to say that $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji}$ (remember that in the gluing process points $x$ and $\tau_{ji}(x)$ are identified), but this holds true since

$$\pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \theta_{ij}^{-1} \circ \theta_{ji} = \pi_{ij} \circ \theta_{ji} = \pi_i|_{\pi_i^{-1}(U_{ij})}.$$

Hence we can glue the $\pi_i$’s together to obtain $\pi_X$.

It is a matter of easy verification that the the glued scheme with the two projections has the universal property.
It is worth while commenting that the product $X \times_S Y$ is not defined as a particular scheme, it is just an isomorphism class of schemes (having the fundamental property that there is a unique isomorphism respecting the projections between any two). In the proof above both $\pi_i^{-1}(U_{ij})$ and $\pi_j^{-1}(U_{ij})$ are products $U_{ij} \times_S Y$, they are however not equal, merely canonically isomorphic. In the construction we could have use any of them, or as we in fact did, any non-specified representative of the isomorphism class, as this makes the situation more symmetric in $i$ and $j$.

An immediate consequence of the gluing proposition 7.4 is that fibre products exist over an affine base $S$.

**Lemma 7.5.** Assume that $S$ is affine, then $X \times_S Y$ exists.

**Proof.** First if $Y$ as well is affine, we are done. Indeed, cover $X$ by open affine sets $U_i$. Then $U_i \times_S Y$ exists by the affine case, and we are in the position to apply proposition 7.4 above. We then cover $Y$ by affine open sets $V_i$. As we just verified, the products $X \times_S V_i$ all exist, and applying proposition 7.4 once more, we can conclude that $X \times_S Y$ exists.

7.5 The final reduction

Let $\{ S_i \}$ be an open affine covering of $S$ and let $U_i = \psi^{-1}(S_i)$ and $V_i = \psi_Y^{-1}(S_i)$. By lemma 7.5 the products $U_i \times_S V_i$ all exist. Using the following lemma and, for the third time, the gluing proposition 7.4 we are trough:

**Lemma 7.6.** With current notation, we have the equality $U_i \times_S V_i = U_i \times_S Y$.

**Proof.** The key diagram is

```
  Z
 / \ /
/   \f/
 U_i   Y
|   g |   \psi_Y
\psi_X|U_i \searrow 
     S
```

where $f$ and $g$ are given maps. If one follows the left path in the diagram, one ends up in $S_i$, and hence the same must hold following the right path. But then, $V_i$ being equal the inverse image $\psi_Y^{-1}(S_i)$, it follows that $g$ necessarily factors through $V_i$, and we are done.
7.6 Notation.

If $S = \text{Spec } A$ one often writes $X \times_A Y$ in short for $X \times_{\text{Spec } A} Y$. If $S = \text{Spec } Z$, one writes $X \times Y$. In case $Y = \text{Spec } B$ the shorthand notation $X \otimes_A B$ is frequently seen as well—it avoids writing $\text{Spec }$ twice.

Diagrams arising from fiber products are frequently called cartesian diagram (kartesiske diagrammer); that is, the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi_1} & X \\
\downarrow{\pi_2} & & \downarrow{\psi_X} \\
Y & \xrightarrow{\psi_Y} & S
\end{array}
\]

is said to be a cartesian diagram if there is an isomorphism $Z \simeq X \times_S Y$ with $\pi_1$ and $\pi_2$ corresponding to the two projections.

**Exercise 19.** Let $X$, $Y$ and $Z$ be three schemes over $S$. Show that $X \times_S Z = X$, that $X \times_S Y \simeq Y \times_S X$ and that $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$. If $T$ is a scheme over $S$, show that $X \times_S T \times_T Y \simeq X \times_S Y$.

**Exercise 20.** Show by using the universal property that if $\phi: X' \to X$ and $g: Y' \to Y$ are morphisms over $S$, then there is morphism $f \times g: X' \times_S Y' \to X \times_S Y$ such that

\[
\begin{array}{ccc}
X' & \xrightarrow{f \times g} & X \times_S Y \\
\downarrow{\pi_{X'}} & & \downarrow{\pi_X} \\
X' & \xrightarrow{\pi_X} & X
\end{array}
\]

and a corresponding diagram involving $Y$ and $Y'$ commute.

7.7 Examples

7.7.1 Varieties versus Schemes

In the important case that $X$ and $Y$ are integral schemes of finite type over the algebraically closed field $k$ the product of the two as varieties coincides with their product as schemes over $k$, with the usual interpretation that the variety associated to the scheme $X$ is the set closed points $X(k)$ with induced topology.

The product $X \times_k Y$ will be a variety (i.e., an integral scheme of finite type over $k$) and the closed points of the product $X \times_k Y$ will be the direct product of the closed points in $X$ and $Y$.

It is not obvious that $A \otimes_k B$ is an integral domain when $A$ and $B$ are, and in fact, in general, even if $k$ is a field, it is by no means true. But it holds true
whenever $A$ and $B$ are of finite type over $k$ and $k$ is an algebraically closed field. The standard reference for this is Zariski and Samuel’s book *Commutative algebra I* which is the Old Covenant for algebraists. It is also implicit in Hartshorne’s book, exercise 3.15 b) on page 22.

However that the tensor product $A \otimes_k B$ is of finite type over when $A$ and $B$ are, is straightforward. If $u_1, \ldots, u_m$ generate $A$ over $k$ and $v_1, \ldots, v_m$ generate $B$ over $k$ the products $u_i \otimes 1$ and $1 \otimes v_j$ generate $A \otimes_k B$.

### 7.7.2 Non-algebraically closed fields

This case the situation is more subtle when one works over fields that are not algebraically closed. To illustrate some of the phenomena that can occur, we study a few basic examples.

**Example 7.7.** A simple but illustrative example is the product $\text{Spec } \mathbb{C} \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$. This scheme has *two* distinct closed points, and it is not integral—it is not even connected!

The example also shows that the underlying set of the fiber product is not necessarily equal to the fiber product of the underlying sets, although this was true for varieties over an algebraically closed field. In the present case the three schemes involved all have just one element and the their fibre product has just one point. So we issue warnings: The product of integral schemes is in general not necessarily integral! The underlying set of the fiber product is not always the fiber product of the underlying sets.

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is in fact isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$ of two copies of the complex field $\mathbb{C}$; indeed, we compute using that $\mathbb{C} = \mathbb{R}[t]/(t^2 + 1)$ and find

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2 + 1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2 + 1) = \mathbb{C}[t]/(t-i)(t+i) = \mathbb{C} \times \mathbb{C}$$

where for the last equation we use the Chinese remainder theorem and that the rings $\mathbb{C}[t]/(t \pm i)$ both are isomorphic to $\mathbb{C}$.

**Example 7.8.** This little example can easily be generalized: Assume that $L$ is a simple, separable field extension of $K$ of degree $d$; that is $L = K(\alpha)$ where the minimal polynomial $f(t)$ of $\alpha$ over $K$ is separable and of degree $d$. Let $\Omega$ be a field extension of $K$ in which the polynomial $f(t)$ splits completely—e.g., a normal extension of $L$ or any algebraically closed field containing $K$—then by an argument completely analogous to the one above one finds that $L \otimes_K \Omega = \Omega \times \ldots \times \Omega$ where the product has $d$ factors. Consequently the product scheme $\text{Spec } L \times_{\text{Spec } k} \text{Spec } \Omega$ has an underlying set with $d$ points, even if the three sets of departure all are prime spectra of fields and thus singletons.
One may push this further and construct examples where \( \text{Spec } K \otimes_{\text{Spec } k} \text{Spec } \Omega \) is not even noetherian and has infinity many points!

**Exercise 21.** With the assumptions of the example above, check the statement that \( L \otimes_K \Omega \simeq \Omega \times \ldots \times \Omega \), the product having \( d \) factors.

**Exercise 22.** Assume that \( A \) is an algebra over the field \( k \) having a countable set \( \{ e_1, e_2, \ldots, e_i, \ldots \} \) of mutually orthogonal idempotents, i.e., \( e_ie_j = 0 \) if \( i \neq j \) and \( e_ie_i = 1 \), and assume that \( e_iA \simeq k \). Assume also that every element is a finite linear combination of the \( e_i \)'s.

- Show that the ideal \( I_j \) generated by the \( e_i \)'s with \( i \neq j \) is a maximal ideal.

**Example 7.9.** In this example we let \( L \in \mathbb{C}[x, y] \) be a linear form that is not real, for example \( L = x + iy + 1 \), and we introduce the real algebra \( A = \mathbb{R}[x, y]/(LL) \). The product \( LL \) of \( L \) and its complex conjugate is a real irreducible quadric; which in the concrete example is \( (x + 1)^2 + y^2 \). The prime spectrum \( \text{Spec } A \) is therefore an integral scheme. However, the fiber product \( \text{Spec } A \times_{\mathbb{R}} \text{Spec } \mathbb{C} \) is not irreducible being the union of the two conjugate lines \( L = 0 \) and \( \overline{L} = 0 \) in \( \text{Spec } \mathbb{C}[x, y] \).

The scheme \( \text{Spec } A \) has just one real point, namely the point \((-1, 0)\) (i.e., corresponding to the maximal ideal \((x + 1, y)\)). The \( \mathbb{C} \)-points however, are plentiful.

They are contained in the \( \mathbb{C} \)-points \( A_R^2(\mathbb{C}) \), which are of orbits \( \{(a, b), (\pi, \overline{b})\} \) of the complex conjugation with \((a, b)\) non-real, and form the subset of those \((a, b)\) such that \( L(a, b) = 0 \).

**Example 7.10.** Another example along same lines as example 7.8 shows that the fiber product \( X \times_S Y \) is not necessarily reduced even if both \( X \) and \( Y \) are; the point being to use an inseparable polynomial \( f(t) \) in stead of a the separable one in 7.8. Let \( k \) be a non-perfect field in characteristic \( p \) which means that there is an element \( a \in k \) not being a \( p \)-th power of any element in \( k \). Let \( L \) be the field extension \( L = k(b) \) where \( b^p = a \). That is, \( L = k[t]/(t^p - a) \), which is a field since \( t^p - a \) is an irreducible polynomial over \( k \). However, upon being tensored by itself over \( k \), it takes the shape

\[
L \otimes_k L = L[t]/(t^p - a) = L[t]/(t^p - b^p) = L[t]/((t - b)^p)
\]

which is not reduced, the non-zero element \( t - b \) being nilpotent. So we issue a third warning: the fiber product of integral schemes is not in general reduced!

One can elaborate these example and construct an example of two noetherian schemes \( X \) and \( Y \) such that \( X \times_S Y \) is not noetherian, even if \( S \) is the spectrum of a field. The next examples shows that if \( X, Y \) and \( S \) are fields, the product \( X \times_S Y \) can even have an uncountable number of elements!
Example 7.11. In this example we take $L$ to the subfield of the field $\overline{\mathbb{Q}}$ of algebraic numbers that is generated by all the elements $a$ such that $a^{2^n} = 2$ for some $n$. The field $L$ is the union of the ascending chain of fields

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt[8]{2}) \subseteq \ldots \subseteq L \subseteq \overline{\mathbb{Q}}$$

We denote $r$-th field in the chain $\mathbb{Q}(\sqrt{2^r})$ by $L_r$. The next field $L_{r+1}$ is the quadratic extension of $L_r$ obtained simply by adjoining $\sqrt{2^{r+1}}$, or, in other words, the square root of $\sqrt{2}$.

We let $A_r = L_r \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. This is a finite algebra of rank $2^r$ over $\mathbb{Q}$. It has a refined algebraic structure given by the smaller algebras $A_s$ for $s \leq r$ which are all subalgebras of $A_r$. The algebra $A_r$ splits as the direct product of two copies of $A_{r-1}$; indeed, one has $L_r \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = L_r \otimes_{L_{r-1}} \overline{\mathbb{Q}} \otimes_{\overline{\mathbb{Q}}} L_{r-1} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = A_{r-1} \oplus A_{r-1}$ since $L_r \otimes_{L_{r-1}} \overline{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q}$.

The two idempotents in $A_r$ that induce this splitting are denoted by $e_{r,0}$ and $e_{r,1}$. They are orthogonal and their sum equals one. Each of the two subalgebras $e_{r,\epsilon}A_r$ are isomorphic to $A_{r-1}$ with $1 \in A_{r-1}$ corresponding to $e_{r,\epsilon}$, they contain the idempotens $e_{r-1,\epsilon}$ which will correspond to the product $e_{r,\epsilon}e_{r-1,\epsilon'}$ in $A_r$. Working our way down in $A_r$, this yields a sequence of idempotents $e_I = e_{r,\epsilon_r}e_{r-1,\epsilon_{r-1}} \ldots e_{1,\epsilon_1}$ where $I = (\epsilon_1, \ldots, \epsilon_r)$ is a sequence of 1’s and 0’s.

Take any sequence $\sigma = (\sigma_i)i \in \mathbb{N}$. Let $I \subseteq A$ be generated by the $e_{i,\epsilon_i}$ with $\epsilon_i \notin \sigma$. Then $I$ is maximal. It contains all product except $\prod_i e_{i,\sigma_i}$. And these all generate the same copy of $\overline{\mathbb{Q}} \subseteq A$!

Then of course $L$ being a field is noetherian as is $\overline{\mathbb{Q}}$, but $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not! Indeed, $\mathbb{Q}(\sqrt[2^r]{2}) \otimes \overline{\mathbb{Q}}$ is isomorphic to the direct product of $2^r$-copies of $\overline{\mathbb{Q}}$ so $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is the union of a sequence of subrings each being a direct product of a steadily increasing number of copies of $\overline{\mathbb{Q}}$.

### 7.8 Base change

The fiber product is in constant use in algebraic geometry, and it is an astonishingly versatile and flexible instrument. In different situations it serves quite different purposes and appears under different names. We shall comment on some of the most frequently encountered applications, and we begin with notion of base change.

In its simples and earliest appearances base change is just extending the field over which one works; e.g., in Galois theory, or even in the theory of real polynomials, when studying an equation with coefficients in a field $k$ one often finds it fruitful to study the equation over a bigger field $K$. Generalizing this to extensions of algebras over which one works, and then to schemes, one arrives naturally at the fiber product.
If $X$ is a scheme over $S$ and $T \to S$ is a map. Considering $T \to S$ as change of base schemes one frequently writes $X_T = X \times_S T$ and says that $X_T$ is obtained from $X$ by base change (basisforandring) or frequently that $X_T$ is the pull back (tilbaketrekningen) of $X$ along $T \to S$. This is a functorial construction, since if $\phi : X \to Y$ is a morphism over $S$, there is induced a morphism $\phi_T = \phi \times \text{id}_T$ from $X_T$ to $Y_T$ over $T$, and one easily checks that $(\phi \circ \phi')_T = \phi_T \circ \phi'_T$. The defining properties of $\phi_T$ are $\pi_Y \circ \phi_T = \phi \circ \pi_X$ and $\pi_T \circ \phi = \pi_T$, as depicted in the diagram:

$$
\begin{array}{ccc}
X_T = X \times_S T & \xrightarrow{\phi_T} & Y \times_S T \\
\downarrow \phi \circ \pi_X & & \downarrow \pi_Y \\
X & \xrightarrow{\phi} & Y
\end{array}
$$

Exercise 23. Verify in detail that $(\phi \circ \phi')_T = \phi_T \circ \phi'_T$.

If $P$ is a property of morphisms, one says that $P$ is stable under base change if for any $T$ over $S$, the map $f_T$ has the property $P$ whenever $f$ has it. For example, another way of phrasing lemma 7.3 on page 109 is to say that being an open immersion is stable under base change.

7.9 Scheme theoretic fibres

In most parts of mathematics, when one studies a map of some sort, a knowledge of what the fibres of the map are, is of great help. This is also true in the theory of schemes.

Suppose that $\phi : X \to Y$ is a map of schemes and that $y \in Y$ is a point. On the level of topological spaces, we are interested in the preimage $\phi^{-1}(y)$, and we aim at giving a scheme theoretic definition of the fiber $\phi^{-1}(y)$. Having the fiber product at our disposal, nothing is more natural than defining the fiber to be the fiber product $\phi^{-1}(Y) = \text{Spec} \ k(y) \times_Y X$, where $\text{Spec} \ k(y) \to Y$ is the map corresponding to the point $y$. Recall that $k(y) = \mathcal{O}_{Y,y}/m_y$ and the map is the composition $\text{Spec} \ k(y) \to \text{Spec} \mathcal{O}_{Y,y} \to Y$. For diagrammoholics, the scheme theoretic fiber of $\phi$ over $y$ fits into the cartesian diagram

$$
\begin{array}{ccc}
\phi^{-1}(y) = X \times_Y \text{Spec} \ k(y) & \xrightarrow{\phi} & X \\
\downarrow & & \downarrow \\
\text{Spec} \ k(y) & \xrightarrow{\phi} & Y
\end{array}
$$

As the next lemma will show, the underlying topological space of $\phi^{-1}(y)$ is the topological fiber, but in addition there is a scheme structure on it. In in
many cases it is not reduced, and this a mostly a good thing since it makes certain continuity results true.

**Proposition 7.12.** The inclusion $X_y \to X$ of the scheme theoretic fiber is a homeomorphism onto the topological fiber $\phi^{-1}(y)$.

**Proof.** We start with the affine case. Obviously $Y$ can always without loss of generality be assumed to be affine, say $Y = \text{Spec } B$, but to begin with we adopt the additional assumption that $X$ is affine as well, let’s say $X = \text{Spec } A$.

The map $\phi$ of affine schemes is induced by a map of rings $\alpha: B \to A$. Let $p \subseteq B$ be a prime ideal. We have the following equality between sets

$$\{ q \subseteq A \mid q \text{ prime ideal}, \alpha^{-1}(q) \supseteq p \} = \{ q \subseteq A \mid q \text{ prime ideal}, q \supseteq pA \}.$$ 

In the particular case that $p$ is a maximal ideal, the inclusion $\alpha^{-1}(q) \supseteq q$ is necessarily an equality, and the sets above describe the fiber set-theoretically:

$$\phi^{-1}(p) = \{ q \subseteq A \mid q \supseteq pA \} \simeq \text{Spec } A/pA.$$ 

But this also describes the good old embedding of $\text{Spec } A/pA$ into $\text{Spec } A$ identifying it with the closed subscheme $V(pA)$, and therefore this yields a homeomorphism between $\text{Spec } A/pA$ and the topological fiber $\phi^{-1}(p)$. On the other hand by standard equalities between tensor products one has

$$A/pA = A \otimes_{B} B/pB = A \otimes_{B} k(y),$$

and so the scheme theoretical fiber $\phi^{-1}(y) = X_y = X \times_Y \text{Spec } k(y) = \text{Spec } A \otimes_{B} k(y)$ is in a canonical way homeomorphic to the topological fiber.

If $p$ is not a maximal ideal, the set $\text{Spec } A/pA$ is strictly bigger than the fiber, the superfluous prime ideals being those for which $\alpha^{-1}q$ strictly bigger than $p$. When localising in the multiplicative system $S = B - p \subseteq B$, these superfluous prime ideals go non-proper, since they all contain elements of the form $\alpha(s)$ with $s \in S$. Hence the points in the fiber correspond to the primes in the localized ring $(A/pA)_p$. Standard formulas for the tensor product gives on the other hand the equality

$$(A/pA)_p = A \otimes_{B} B_p/pB_p = A \otimes_{B} k(y).$$

The topologies coincides as well, since $\text{Spec}(A/pA)_p$ naturally is a subscheme of $\text{Spec } A/pA$; induced topology being the one a prime spectrum.

In the general case, i.e., when $X$ is no longer affine, we cover $X$ by open, affine $U_i$’s. By the universal property of fiber products, we know that $U \cap X_s = U_s$ (see the diagram below). This shows that the scheme theoretical and the topological
fiber coincide as topological spaces.

\[
\begin{array}{c}
U_s \to U \\
\downarrow \\
X_s \to X \\
\downarrow \\
\text{Spec } k(y) \to Y
\end{array}
\]

**Example 7.13.** We take a look at a simple but classic example: The map

\[
\text{Spec } k[x, y]/(x - y^2) \to \text{Spec } k[x]
\]

induced by the injection \( B = k[x] \to k[x, y]/(x - y^2) = A \). Geometrically one would say it is just the projection of the parabola onto the \( x \)-axis.

If \( a \in k \) computing the fiber yields, where \( k(a) \) denotes the field \( k(a) = k[t]/(t - a) \) (which of course is just a copy of \( k \)).

\[
k[x, y]/(x - y^2) \otimes_{k[x]} k[x]/(x - a) \simeq k[y]/(y^2 - a).
\]

(Here we are using the isomorphism \( R/I \otimes_R M \simeq M/IM \) for an ideal \( I \) in an \( R \)-module \( M \).)

Several cases can occur, apart from the characteristic two case being special.

- If \( a \) does not have a square root in \( k \), the fiber is \( \text{Spec } k(\sqrt{a}) \) where \( k(\sqrt{a}) \)
  is a quadratic field extension of \( k \).

- In case \( a \) has a square root in \( K \), say \( b^2 = a \), the polynomial \( y^2 - a \) factors as \((y - b)(y + b)\), and the fiber becomes \( \text{Spec } k[y]/(y - b) \times \text{Spec } k[y]/(y + b) \),
  the disjoint union of two copies of \( \text{Spec } k \).

- Finally, the case appears when \( a = 0 \). The the fiber is not reduced, but
  equals \( \text{Spec } k[y]/y^2 \).

We also notice that the generic fiber is the quadratic extension \( k(x)(\sqrt{x}) \) of the function field \( k(x) \).

Over perfect fields \( k \) of characteristic two, the picture is completely different. Then \( a \) is a square, say \( a = b^2 \) and as \((y^2 - b^2) = (y - b)^2 \) non of the fibers are reduced, they equal \( \text{Spec } k[y]/(y - b)^2 \), except the generic one which is \( k(x)(\sqrt{x}) \). One observes interestingly enough, that all the non-reduced fibers deform into a field!
Example 7.14. A similar example can be obtained from the map

\[ f : \text{Spec } A \to \text{Spec } B \]

where \( A = \text{Spec } k[x, y, z]/(xy - z) \) and \( B = k[z] \) (The map \( k[z] \to k[x, y, z]/(xy - z) \) is the obvious one). As before, we pick an element \( a \in \text{Spec } B \), and consider the fiber

\[ X_a = \text{Spec } (A \otimes_B k(a)) = \text{Spec } k[x, y]/(xy - a) \]

Again, two cases occur. If \( a \neq 0 \), in which case \( xy - a \) is an irreducible polynomial, and so \( X_a \) is an integral scheme. This is intuitive, since it corresponds to the hyperbola \( \{ xy = a \} \). If \( a = 0 \), we are left with \( X_0 = \text{Spec } k[x, y]/(xy) \), which is not irreducible; it has two components corresponding to \( V(x) \) and \( V(y) \). \( X_0 \) is reduced however.

For good measure, we also consider the fiber over the generic point \( \eta \) of \( \text{Spec } B \). This corresponds to

\[ k[x, y, z]/(xy - z) \otimes_{k[z]} k(z) = k[z][x, y]/(xy - z) \]

which is an integral domain. Hence \( X_\eta \) is integral.

7.10 Separated schemes

We have previously seen that the topology on schemes behaves very differently from the usual euclidean topology. In particular, schemes are not Hausdorff, except in trivial cases – the open sets in the Zariski topology are just too large. Still we would like to find an analogous property that can serve as a substitute for this property. The route we take is to impose that the diagonal should be closed.

7.10.1 The diagonal

Let \( X/S \) be a scheme over \( S \). There is a canonical map \( \Delta_{X/S} : X \to X \times_S X \) of schemes over \( S \) called the diagonal map or the diagonal morphism (diagonalavbildningen). The two components maps of \( \Delta \) are both equal to the identity \( \text{id}_X \); that is, the defining properties of \( \Delta_{X/S} \) are \( \pi_i \circ \Delta_{X/S} = \text{id}_X \) for \( i = 1, 2 \) where the \( \pi_i \)'s denote the two projections.

In the case \( X \) and \( S \) are affine schemes, the diagonal has a simple and natural interpretation in terms of algebras; it corresponds to most natural map, the multiplication map:

\[ \mu : A \otimes_B A \to A. \]

It sends \( a \otimes a' \) to the product \( aa' \) and then extends to \( A \otimes_B A \) by linearity. The
projections correspond to the two maps \( \iota_i: A \to A \otimes_B A \) sending \( a \) to \( a \otimes 1 \) respectively to \( 1 \otimes a \). Clearly it holds that \( \mu \circ \iota_i = \text{id}_A \), and on the level of schemes this translates into the defining relations for diagonal map. Moreover \( \mu \) is clearly surjective, so we have established the following:

**Proposition 7.15.** If \( X \) an affine scheme over the affine scheme \( S \), then the diagonal \( \Delta_{X/S}: X \to X \times_S X \) is a closed imbedding.

This is not generally true for schemes and shortly we shall give examples, however from the proposition we just proved, it follows readily that the image \( \Delta_{X/S}(X) \) is locally closed—i.e., the diagonal is locally a closed embedding:

**Proposition 7.16.** The diagonal \( \Delta_{X/S} \) is locally a closed embedding.

**Proof.** Begin with covering \( S \) by open affine subset and subsequently cover each of their inverse images in \( X \) by open affines as well. In this way one obtains a covering of \( X \) by affine open subsets \( U_i \) whose images in \( S \) are contained in affine open subsets \( S_i \). The products \( U_i \times_S U_i = U_i \times_S U_i \) are open and affine, and their union is an open subset containing the image of the diagonal. By proposition 7.15 above the diagonal restricts to a closed embedding of \( U_i \) in \( U_i \times_S U_i \).

**Exercise 24.** In the setting of the previous proof, show that \( \Delta_{X/S}|_{U_i} = \Delta_{U_i/S} \).

### 7.10.2 Separated schemes

**Definition 7.17.** The scheme \( X/S \) is separated (separert) over \( S \), or that the structure map \( X \to S \) is separated if the diagonal map is a closed imbedding. If \( X \) is separated over \( \text{Spec} \mathbb{Z} \) one says for short that it is separated.

**Example 7.18.** The simplest example of a scheme that is not separated is obtained by glueing the prime spectrum of a discrete valuation ring to itself along the generic point.

To give more details let \( R \) be the DVR with fraction field \( K \). Then \( \text{Spec} R = \{ x, \eta \} \) where \( x \) is the closed point corresponding to the maximal ideal, and \( \eta \) is the generic point corresponding to the zero ideal. The generic point \( \eta \) is an open point (the complement of \( \{ \eta \} \) is the closed point \( x \)) and the support of the open subscheme \( \{ \eta \} = \text{Spec} K \). By by the glueing lemma, we may glue one copy of \( \text{Spec} R \) to another copy of \( \text{Spec} R \) by identifying the generic points—that is, the open subschemes \( \text{Spec} K \)—in the two copies.

In this manner we construct a scheme \( Z_R \) together with two open embeddings \( \iota_i: \text{Spec} R \to Z_R \). They send the generic point \( \eta \) to the same point, which is an open point in \( Z_R \), but they differ on the closed point \( x \).
Chapter 7. Fiber products

It follows that the diagonal is not closed. Indeed, the subscheme of $\text{Spec } R \times \text{Spec } R$ where the two maps $\iota_i$ agree is the preimage of the diagonal. But this subscheme has exactly one point which is open.

**Exercise 25.** Show that $Z_R \times Z_R$ is obtained by gluing *four* copies of $\text{Spec } R$ together along their generic points. Show that the diagonal is open and not closed.

**Example 7.19.** The affine line with two origins from Chapter 3 is not separated.

One of the nice properties affine schemes enjoy, is the following:

**Proposition 7.20.** Assume that $X$ is separated and that $U$ and $V$ are to open affine subscheme. Then the intersection $U \cap V$ is affine and the map $\Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V) \to \Gamma(U \cap V, \mathcal{O}_{U \times V})$ is surjective.

**Proof.** The product $U \times V$ is an open and affine subset of $X \times V$, and $U \cap V = \Delta_X(X) \cap (U \times V)$. So if the diagonal is closed, $U \cap V$ is a closed subset of the affine set $U \times V$ hence affine. It is a general fact about products of affine schemes that one has

$$\Gamma(U \times V, \mathcal{O}_{U \times V}) = \Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V),$$

and as $U \cap V$ is a closed subscheme of $U \times V$, the restriction map

$$\Gamma(U \times V, \mathcal{O}_{U \times V}) \to \Gamma(U \cap V, \mathcal{O}_{U \cap V})$$

is surjective.

\[\square\]

### 7.11 The valuative criterion

In some sense, these tiny schemes $Z_R$ together with some of their bigger cousins are always at the root of a non separated scheme. Using the properties of these subschemes, one can prove the following theorem, which gives a convenient description of separated schemes.

**Proposition 7.21.** Assume that $X$ is a quasi-compact scheme. Then $X/S$ is non-separated if and only if it contains a subscheme $Z_R/S$ for some valuation ring $R$. 

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This is usually formulated as follows:

**Theorem 7.22** (Valuative criterion for separateness). A quasi-compact scheme \( X \) is separated if and only if the following condition is satisfied: For any DVR \( R \) with fraction field \( K \), a map \( \text{Spec} \, K \to X \) over \( S \) has at most one extension to \( \text{Spec} \, R \to X \).

In other words, a morphism \( X \to S \) is separated if for every DVR \( R \) with fraction field \( K \) and diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & S
\end{array}
\]

there can be at most one extension \( \text{Spec} \, R \to X \) (the dotted arrow in the diagram) such that everything commutes.

This looks like a technical statement, but it is tremendously powerful for proving theorems about separateness. Here is a sample:

**Corollary 7.23.** (i) Open and closed immersions are separated

(ii) A composition of two separated morphisms is again separated

(iii) Separatedness is stable under base change: If \( f : X \to S \) is separated, and \( S' \to S \) is any morphism, then \( f' : X \times_S S' \to S' \) is separated.

The proof is straightforward using the Valuative Criterion.

For instance, consider item (iii). Taking a base change \( S' \to S \) gives us a diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & S
\end{array}
\]

\[
\begin{array}{ccc}
& X \times_S S' & \\
\downarrow & & \downarrow \\
& S' & \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & S
\end{array}
\]

giving two arrows \( \text{Spec} \, R \Rightarrow X \) (by composition), which have to be equal, by separatedness of \( f \). Now, the two morphisms \( \text{Spec} \, R \Rightarrow X \times_S S' \) have to be equal by the universal property of the fiber product.

It is not true that maps \( \text{Spec} \, K \to X \) like in the criterion always can be extended—schemes with that property are called proper schemes—so the criterion says there can never be two different extensions in case \( X \) is separated.

One doesn’t frequently encounter non-separated scheme in practice, but some very nice properties are only true for separated schemes, and this legitimates the notion. Of course, one needs good criteria to be sure we have a large class of
separated schemes. We have already seen that all affine schemes are separated, and when we come to that point projective schemes will turn out to be separated as well.

7.11.1 Varieties vs schemes again

Finally we can state the definition of a variety:

**Definition 7.24.** A *variety* \(X\) is an integral, separated scheme of finite type over an algebraically closed field.

This definition requires some explanation. Recall that we defined for a field \(k\) a functor

\[ t : \text{Var}/k \to \text{Sch}/k \]

which associates a variety \(V\) to a scheme \(t(V)\) over \(k\), such that \(V\) is homeomorphic to the subset of \(k\)-points of \(t(V)\). We stated that this functor is fully faithful, making the category of varieties \(\text{Var}/k\) equivalent to a full subcategory of \(\text{Sch}/k\). It is a theorem (Hartshorne II.4.10) that this subcategory is exactly the category of schemes satisfying Definition 7.24. For the proof of this statement, see See Hartshorne II.4.10.

**Theorem 7.25.** The functor \(t\) is fully faithful, and gives and equivalence between the category of varieties \(\text{Var}/k\) and the subcategory of \(\text{Sch}/k\) of schemes satisfying Definition 7.24.
Chapter 8
Quasi-coherent sheaves

When you study commutative algebra may be you are primarily interested in
the rings and ideals, but probably you start turning your interest towards the
modules pretty quickly; they are an important part of the world of rings, and to
get the results one wants, one can hardly do without them. The category $\text{Mod}_A$
of $A$-modules is a fundamental invariant of a ring $A$; and in fact, is the principal
object of study in commutative algebra. This is true also with schemes; for
which the so called quasi-coherent $\mathcal{O}_X$-modules form an important attribute of
the scheme, if not a decisive part of the structure. They form a category $\text{QCoh}_X$
with many properties paralling those of the category $\text{Mod}_A$. In fact, in case
the scheme $X$ is affine, i.e., $X = \text{Spec} \ A$, the two categories $\text{Mod}_A$ and $\text{Mod}_X$
are equivalent, as we will prove shortly. Imposing finiteness conditions on the
$\mathcal{O}_X$-modules one arrives at the category $\text{Coh}_X$ of so called coherent $\mathcal{O}_X$-modules
that in the noetherian case parallel the finitely generated $A$-modules.

We start out by describing the much broader concept of an $\mathcal{O}_X$-module. The
theory here is presented for schemes, but the concept is meaningful for any ringed
space.

In the literature one finds different approaches to the quasi-coherent sheaves.
We follow EGA and Hartshorne and introduce the quasi-coherent module first for
affine schemes. If $X = \text{Spec} \ A$ one defines an $\mathcal{O}_X$-module $\tilde{M}$ associated with an
$A$-module $M$, and in the general case these modules serve as the local models for
the quasi-coherent ones. There is a notion of quasi-coherence for $\mathcal{O}_X$-modules on
a general locally ringed space. In some other branches of mathematics they are
important, e.g., analytic geometry, but we concentrate our efforts on schemes.

At the end of the chapter we will discuss locally free sheaves and vector
bundles, which are in many respects the most interesting coherent sheaves on a
A module over a ring is just an additive abelian group equipped with a multiplicative action of $A$. Loosely speaking we can multiply members of the module by elements from the ring, and of course, the usual set of axioms must be satisfied. In a similar way, if $X$ is a scheme, an $\mathcal{O}_X$-module is an abelian sheaf $\mathcal{F}$ whose sections can be multiplied by sections of $\mathcal{O}_X$; the multiplicator and the multiplicand of course being sections over the same open subset.

Formally, an $\mathcal{O}_X$-module structure on the abelian sheaf $\mathcal{F}$ is defined as a family of multiplication maps $\Gamma(U, \mathcal{F}) \times \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{F})$—one for each open subset $U$ of $X$—making the space of sections $\Gamma(U, \mathcal{F})$ into a $\Gamma(U, \mathcal{O}_X)$-module in a way compatible with all restrictions. That is, for every pair of open subsets $V \subseteq U$, the natural diagram below—where vertical arrows are made up of appropriate restrictions, and the horizontal arrows are multiplications—commutes

$$\begin{array}{c}
\Gamma(U, \mathcal{F}) \times \Gamma(U, \mathcal{O}_X) \ar{r} \ar{d} & \Gamma(U, \mathcal{F}) \ar{d} \\
\Gamma(V, \mathcal{F}) \times \Gamma(V, \mathcal{O}_X) \ar{r} & \Gamma(V, \mathcal{F})
\end{array}$$

Maps, or homomorphisms, of $\mathcal{O}_X$-modules are just maps $\alpha: \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules considered as abelian sheaves, respecting the multiplication by sections of $\mathcal{O}_X$. That is, for any open $U$ the map $\alpha_U: \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is a $\Gamma(U, \mathcal{O}_X)$-homomorphism.

We have now explained what $\mathcal{O}_X$-modules are and told what their homomorphisms should be, and this organizes the $\mathcal{O}_X$-modules into a category that we denote by $\text{Mod}_X$.

The category $\text{Mod}_X$ is an additive category: The sum of two $\mathcal{O}_X$-homomorphisms as maps of abelian sheaves is again an $\mathcal{O}_X$-homomorphism. So for all $\mathcal{F}$ and $\mathcal{G}$ the set $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an abelian group, and the compositions maps are bilinear. Moreover, the direct sum of two $\mathcal{O}_X$-modules as abelian sheaves has an obvious $\mathcal{O}_X$-structure, with multiplication being defined componentwise. In fact, this argument works for arbitrary direct sums (or coproducts as they also are called). For any family $\{ \mathcal{F}_i \}$ of $\mathcal{O}_X$-modules, $\bigoplus_{i} \mathcal{F}_i$ is an $\mathcal{O}_X$-module (see Exercise 30 below).

The notions of kernels, cokernels and images of $\mathcal{O}_X$-module homomorphisms now appear naturally. All the three corresponding abelian constructions are invariant under multiplication by sections of $\mathcal{O}_X$, and therefore they have $\mathcal{O}_X$-module structures. The respective defining universal properties (in the category
of $\mathcal{O}_X$-modules) come for free, and one easily checks that this makes $\text{Mod}_X$ an abelian category; i.e., the kernels of the cokernels equal the cokernels of the kernels.

There is tensor product of $\mathcal{O}_X$-modules denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. As in many other cases, the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined by first describing a presheaf that subsequently is sheafified. The sections of the presheaf, temporarily denoted by $\mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G}$, are defined in the natural way by

$$\Gamma(U, \mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G}) = \Gamma(U, \mathcal{F}) \otimes \Gamma(U, \mathcal{O}_X) \Gamma(U, \mathcal{G}).$$

There is also a sheaf of $\mathcal{O}_X$-homomorphisms between $\mathcal{F}$ and $\mathcal{G}$. The definition goes along lines slightly different from the definition of the tensor product. Recall the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ of homomorphisms between the abelian sheaves $\mathcal{F}$ and $\mathcal{G}$ whose sections over an open set $U$ is the group $\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of homomorphisms between the restrictions $\mathcal{F}|_U$ and $\mathcal{G}|_U$. Inside this group one has the subgroup of the maps being $\mathcal{O}_X$-homomorphisms, and these subgroups, for different open sets $U$, are respected by the restriction map. So they form the sections of a presheaf, that turns out to be a sheaf, and that is the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ of $\mathcal{O}_X$-homomorphisms from $\mathcal{F}$ to $\mathcal{G}$.

**Example 8.1** (Modules on spectra of DVR’s). Modules on the prime spectrum of a discrete valuation ring $R$ are particularly easy to describe.

Recall that the scheme $X = \text{Spec} \, R$ has only two non-empty open sets, the whole space $X$ itself and the singleton $\{ \eta \}$ where $\eta$ denotes the generic point. The singleton $\{ \eta \}$ is underlying set of the open subscheme $\text{Spec} \, K$, where $K$ denotes the fraction field of $R$.

We claim that to give an $\mathcal{O}_X$-module is equivalent to giving an $R$-module $M$, a $K$-vector space $N$ and a homomorphism $\rho : M \otimes_R K \to N$.

Indeed, given an $\mathcal{O}_X$ module $\mathcal{F}$, we get an $R$ module $M = \Gamma(X, \mathcal{F})$, and a vector space $N = \Gamma(\{ \eta \}, \mathcal{F})$ over $K$. The homomorphism $\rho$ is just the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{ \eta \})$. Conversely, given the data $M, N, \rho$, we can define $\mathcal{F}(X) = M$ and $\mathcal{F}(\eta) = N$. The map $\rho : \mathcal{F}(X) \to \mathcal{F}(\{ \eta \})$ makes $\mathcal{F}$ into an $\mathcal{O}_X$-module.

Note that the restriction map can be just any $R$-module homomorphism $M \to N$. In particular, it can be the zero homomorphism, and in that case $M$ and $N$ can be completely arbitrary modules. Again this illustrates the versatility of general $\mathcal{O}_X$-modules.

**Example 8.2** (A bunch of wild examples). The $\mathcal{O}_X$-modules (or at least some of them) play a leading role in the theory of schemes, and shortly we shall see a long series of examples. These will all be so-called quasi-coherent sheaves. The examples we now describe are a bunch of wild examples, intended
to show that $\mathcal{O}_X$-modules without any restrictive hypothesis are very general and often unmanageable objects.

Recall the Godement construction from Chapter 1. Given any collection of abelian groups $\{ A_x \}_{x \in X}$ indexed by the points $x$ of $X$. We defined a sheaf $\mathcal{A}$ whose sections over an open subset $U$ was $\prod_{x \in U} A_x$, and whose restriction maps to smaller open subsets were just the projections onto the corresponding smaller products. Requiring that each $A_x$ be a module over the stalk $\mathcal{O}_{X,x}$ makes $\mathcal{A}$ into an $\mathcal{O}_X$-module; indeed, the space of sections $\Gamma(U, \mathcal{A}) = \prod_{x \in U} A_x$ is automatically an $\Gamma(U, \mathcal{O}_X)$-module, the multiplication being defined componentwise with the help of the stalk maps $\Gamma(U, \mathcal{O}_X) \to \mathcal{O}_{X,x}$. Clearly this module structures is compatible with the projections, and thus makes $\mathcal{A}$ into an $\mathcal{O}_X$-module.

**Exercise 26.** Assume that $F$ and $G$ are $\mathcal{O}_X$-modules and that $\alpha: F \to G$ is a map between them. Show that the kernel, cokernel and image of $\alpha$ as a map of abelian sheaves indeed are $\mathcal{O}_X$-modules, and that they respectively are the kernel, cokernel and image of $\alpha$ in the category of $\mathcal{O}_X$-modules as well. Show that a complex of $\mathcal{O}_X$-modules is exact if and only it is exact as a complex of abelian sheaves.

**Exercise 27.** Let $F$ and $G$ be two $\mathcal{O}_X$-modules on the scheme $X$. Show that the stalk $(F \otimes_{\mathcal{O}_X} G)_x$ at the point $x \in X$ is naturally isomorphic to the tensor product $F_x \otimes_{\mathcal{O}_{X,x}} G_x$ of the stalks $F_x$ and $G_x$. Show that the tensor product is right exact in the category of $\mathcal{O}_X$-modules.

**Exercise 28.** Show that the sheaf-hom $\mathcal{H}om_{\mathcal{O}_X}(F, G)$ of two $\mathcal{O}_X$-modules as defined above is a sheaf. Show that $\mathcal{H}om_{\mathcal{O}_X}(F, G)$ is right exact in the second variable and left exact in the first.

**Exercise 29 (Adjunction between Hom and $\otimes$).** Show that for $\mathcal{O}_X$-modules $F, G, \mathcal{H}$

$$\text{Hom}_{\mathcal{O}_X}(F, \mathcal{H} \mathcal{O}m_{\mathcal{O}_X}(G, \mathcal{H})) \simeq \text{Hom}_{\mathcal{O}_X}(F \otimes_{\mathcal{O}_X} G, \mathcal{H})$$

(this is easier than it looks – reduce to the usual tensor product for modules over rings).

**Exercise 30.** Show that the category $\text{Mod}_X$ has arbitrary products and coproducts, by showing that the products and coproducts in the category of abelian sheaves $\text{Sh}_X$ are $\mathcal{O}_X$-modules and are the products, respectively the coproducts, in the category $\text{Mod}_X$.

**Exercise 31.** Assume that $p_1, \ldots, p_r$ is a set of primes, and let $\mathbb{Z}_{(p_i)}$ as usual denote the localization at prime ideal $(p_i)$. Let $X$ be the scheme obtained by gluing the schemes $X_i = \text{Spec} \mathbb{Z}_{(p_i)}$ together along their common open subschemes $\text{Spec} \mathbb{Q}$. Describe the $\mathcal{O}_X$-modules on $X$.

**Exercise 32.** Let $A = \prod_{1 \leq i \leq n} K_i$ be the product of finitely many fields. Describe the category $\text{Mod}_X$. 

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8.1.1 The support of a sheaf

Definition 8.3. Let \((X, \mathcal{O}_X)\) be a ringed space. Let \(\mathcal{F}\) be a sheaf of \(\mathcal{O}_X\)-modules. The **support** of \(\mathcal{F}\), \(\text{Supp}(\mathcal{F})\), is the set of points \(x \in X\) such that \(\mathcal{F}_x \neq 0\). For \(s \in \Gamma(X, \mathcal{F})\) we define the **support** of \(s\) as the set of points \(x \in X\) such that the image \(s_x \in \mathcal{F}_x\) of \(s\) is not zero. We denote this by \(\text{Supp}(s)\).

In general the support of a sheaf of modules is not closed. Indeed, as before, we can get strange sheaves by taking any non-closed \(Z\) set of your favorite ringed space, and define a Godement sheaf \(\mathcal{A}\) by setting \(\mathcal{A}_x \neq 0\) if and only if \(x \in Z\).

However, for sheaves of **rings**, the support always closed: this is because a ring is 0 if and only if 1 = 0, and so the support of a sheaf of rings is the support of the section ‘1’ - which is closed.

8.2 Pushforward and Pullback of \(\mathcal{O}_X\)-modules

8.2.1 Pushforward

Let \(f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) be a morphism of schemes. If \(\mathcal{F}\) is a sheaf on \(X\) the **pushforward** \(f_*\mathcal{F}\) is naturally a \(f_*\mathcal{O}_X\)-module via the addition and multiplication maps

\[ f_*\mathcal{F} \times f_*\mathcal{F} \to f_*\mathcal{F}, \quad f_*\mathcal{O}_X \times f_*\mathcal{F} \to f_*\mathcal{F}. \]

Via \(f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X\) we obtain the structure of a \(\mathcal{O}_Y\)-module on \(f_*\mathcal{F}\).

**Definition 8.4.** The above \(\mathcal{O}_Y\)-module is called the **direct image of \(\mathcal{F}\)** under \(f\).

This condition is clearly functorial in \(\mathcal{F}\) and so we obtain a functor \(f_* : \text{Mod}_{\mathcal{O}_X} \to \text{Mod}_{\mathcal{O}_Y}\). By the properties of \(f_*\) in Chapter 1, we see that this is left exact.

8.2.2 Pullback

The pullback of a sheaf of \(\mathcal{O}_Y\)-modules is a little bit more subtle to define. Recall that we in Chapter 1 defined the inverse image \(f^{-1}\mathcal{G}\) of a sheaf \(\mathcal{G}\) by sheafifying the presheaf given by assigning an open \(U\) to the direct limit of all \(\mathcal{G}(V)\) where \(V\) contains \(f(U)\). This sheaf is naturally a \(f^{-1}\mathcal{O}_Y\)-module. We can make it into an \(\mathcal{O}_X\)-module using the map \(f^{-1}\mathcal{O}_Y \to \mathcal{O}_X\) (which makes \(\mathcal{O}_X\) an \(f^{-1}\mathcal{O}_Y\)-algebra), and taking the tensor product:

\[ f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G} \]

The association \(\mathcal{G} \mapsto f^*\mathcal{G}\) is functorial, so we get a functor \(f^* : \text{Mod}_{\mathcal{O}_Y} \to \text{Mod}_{\mathcal{O}_X}\). The above \(\mathcal{O}_X\)-module is called the **pullback of \(\mathcal{G}\)** under \(f\). Note in particular that \(f^*\mathcal{O}_Y = \mathcal{O}_X\).
8.2. Pushforward and Pullback of $\mathcal{O}_X$-modules

**Proposition 8.5.** For a point $x \in X$ we have the following expression for the stalk 
\[(f^*G)_x = G_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.\]

**Proof.** This follows from the facts that taking stalks commutes with tensor products, and $(f^{-1}G)_x = G_{f(x)}$. \hfill \qed

### 8.2.3 Adjoint properties of $f_*, f^*$

At first sight, the definition of the pullback might seem a bit out of the blue. It is defined from $f^{-1}G$, tensoring with $\mathcal{O}_X$ over $f^{-1}\mathcal{O}_Y$ to rig it into being a $\mathcal{O}_X$-module. However, as like in the case of the inverse image functor $f^{-1}$, the important point is what the sheaf does, rather than how it is explicitly defined.

**Proposition 8.6.** The functors $f_*, f^*$ between $\text{Mod}_{\mathcal{O}_X}, \text{Mod}_{\mathcal{O}_Y}$ are adjoint. In particular, if $F \in \text{Mod}_{\mathcal{O}_X}, G \in \text{Mod}_{\mathcal{O}_Y}$, there is a functorial isomorphism
\[\text{Hom}_{\mathcal{O}_X}(f^*G, F) \simeq \text{Hom}_{\mathcal{O}_Y}(G, f_*F).\]

**Proof.** Suppose $\phi : G \to f_*F$ is an $\mathcal{O}_Y$-module homomorphism. Then by the adjoint property of $f_*$ and $f^{-1}$ (in the categories $\text{Sh}_X$ and $\text{Sh}_Y$), we get a map $f^{-1}G \to F$, which is $f^{-1}\mathcal{O}_Y$-linear. Now $F$ is an $\mathcal{O}_X$-module, so we get a $\mathcal{O}_X$-linear map $f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \to G$ by the universal property of the tensor product.

Conversely, let $\phi : f^*G \to F$ be $\mathcal{O}_X$-linear. There is a map $f^{-1}G \to f^*G$ which is $f^{-1}\mathcal{O}_Y$-linear. Consequently there is a $f^{-1}\mathcal{O}_Y$-linear map $f^{-1}G \to F$. This induces a $\mathcal{O}_Y$-linear map $G \to f_*F$ by the earlier adjointness property of $f_*$ and $f^{-1}$. \hfill \qed

In particular, we the maps $\text{id}_{f^*G}$ and $\text{id}_{f_*F}$ provide us with the canonical maps
\[\eta : G \to f_*f^*G, \quad \nu : f^*f_*F \to F\]

We already saw previously that $f_*$ is a left-exact functor. Now $f^*$ is right-exact, by general properties of adjoint functors.

**Corollary 8.7.** $f^*$ is right exact and $f_*$ is left-exact.

### 8.2.4 Pullback of sections

We can also pull back sections of $\mathcal{G}$. If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, and $s \in \Gamma(V, \mathcal{G})$, then we get a section $f^*(s) = \eta(s) \in \Gamma(f^{-1}(V), f^*\mathcal{G})$ by the map $\eta : \mathcal{G} \to f_*f^*\mathcal{G}.$

If we think of sections of $\mathcal{G}$ as local maps from $X$ into the various stalks, the section $f^*s$ simply associates to each $x \in X$ the germ $s_y \otimes 1 \in \mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_{X,x}$ for $y = f(x)$. 

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8.2.5 Tensor products

From its definition, it is straightforward to check that applying $f^*$ commutes with taking tensor products of sheaves. On the level of presheaves, we have

$$f^*(\mathcal{G} \otimes \mathcal{H}) = f^{-1}(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

$$= (f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{H}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

$$= (f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X^* \otimes (f^{-1}\mathcal{H} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)$$

$$= f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{H}.$$  

and sheafifying, we get an isomorphism of the corresponding sheaves.

However, the pushforward $f_*$ rarely commutes with taking tensor products of sheaves (we will see several examples of this later). There is however, at least, a map $f_*(\mathcal{F}) \otimes f_*(\mathcal{G}) \to f_*(\mathcal{F} \otimes \mathcal{G})$ for for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$: If $U \subseteq Y$ is an open, and $s \in f_*(\mathcal{F}), t \in f_*(\mathcal{G})$, then $s \otimes t$ is an element of $\mathcal{F} \otimes \mathcal{G}$ over $f^{-1}(U)$, and hence $s \otimes t$ defines a section of $f_*(\mathcal{F} \otimes \mathcal{G})$ over $U$.

8.3 Quasi-coherent sheaves

In The Oxford English Dictionary there several nuances of the word coherent are given, but the one at top is:

*That sticks or clings firmly together; esp. united by the force of cohesion. Said of a substance, material, or mass, as well as of separate parts, atoms, etc.*

So the coherence of a sheaf should mean that there are some strong relations between the sections over different open sets, at least over sets from some sufficiently large collection of open sets. In our context the open affine subsets stand out as obvious candidates to form such a collection, and indeed, a quasi-coherent sheaf on the scheme $X$ will turn out$^1$ to have the following coherence property: If $V \subseteq U$ are two affine open sets in $X$ there is a canonical map

$$\Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(V, \mathcal{O}_V) \to \Gamma(V, \mathcal{F}) \quad (8.3.1)$$

sending $s \otimes f$ to $\rho_{UV}(s) \cdot f$, where $\rho_{UV}$ as usual indicates the restriction maps in $\mathcal{O}_X$. The salient point is that for a quasi-coherent sheaf $\mathcal{F}$ this map is an isomorphism. In fact the converse holds true as well, so the map in (8.3.1) being an isomorphism for all pairs $V \subseteq U$ of open affine sets is equivalent to $\mathcal{F}$ being quasi-coherent.

To pin down the quasi-coherent sheaves, one first establishes a collection of model-sheaves on affine schemes. To each $A$-module $M$ one associates an

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$^1$One might have used this coherence property as the definition, but because of obscure reasons we choose another definition.
8.4 Quasi-coherent sheaves on affine schemes

For each $A$-module $M$ we shall exhibit an $\mathcal{O}_X$-module $\tilde{M}$; the construction of which completely parallels what we did when constructing the structure sheaf $\mathcal{O}_X$ on $X = \text{Spec} A$. Letting $\mathcal{B}$ again be the base of the topology of distinguished open subsets on $X$, we can define a $\mathcal{B}$-presheaf $\tilde{M}$ as follows:

$$(\tilde{M})(D(f)) = M_f$$

The restriction maps are defined as before: $D(g) \subseteq D(f)$ we have a canonical localization map $M_f \to M_g$. The same proof as for $\mathcal{O}_X$ shows that this is actually a $\mathcal{B}$-sheaf, and hence gives rise to a unique sheaf $\tilde{M}$ on $X$. By the properties of sheafification, this is functorial in $M$: A homomorphism $\phi : M \to N$ gives rise to a canonical morphism of $\mathcal{O}_X$-modules $\tilde{M} \to \tilde{N}$.

For any $A$-module homomorphism $\alpha : M \to N$ there is an obvious way of obtaining an $\mathcal{O}_X$-module homomorphism $\tilde{\alpha} : \tilde{M} \to \tilde{N}$; indeed, the maps $\alpha \otimes \text{id}_{\Gamma(U, \mathcal{O}_X)}$ are $\Gamma(U, \mathcal{O}_X)$-homomorphisms from $\Gamma(U, \tilde{M})$ to $\Gamma(U, \tilde{N})$ compatible with the restrictions, and thus induce a map between $\tilde{M}$ and $\tilde{N}$. The map $\tilde{\alpha}$ is the associated map between the sheafifications. Clearly one has $\tilde{\phi} \circ \tilde{\psi} = \tilde{\phi} \circ \tilde{\psi}$, and the “tilde-operation” is therefore a functor $\text{Mod}_A \to \text{Mod}_{\mathcal{O}_X}$.

The three main properties of $\tilde{M}$ are listed below. They are completely analogous to the statements in proposition 2.34 on page 58 in Chapter 2 about the structure sheaf $\mathcal{O}_X$; and as well, the proofs are mutatis mutandis the same.

- Stalks: Let $x \in \text{Spec} A$ be a point whose corresponding prime ideal is $p$. The stalk $\tilde{M}_x$ of $M$ at $x \in X$ is $\tilde{M}_x = M_p = M \otimes_A A_p$.

- Sections over distinguished open sets: If $f \in A$ one has $\Gamma(D(f), \tilde{M}) = M_f = M \otimes_A A_f$. In particular it holds true that $\Gamma(X, \tilde{M}) = M$.

- Sections over arbitrary open sets. For any open subset $U$ of $\text{Spec} A$ covered
by the distinguished sets \( \{ D(f_i) \}_{i \in I} \), there is an exact sequence

\[
0 \longrightarrow \Gamma(U, \widetilde{M}) \xrightarrow{\alpha} \prod_i M_{f_i} \xrightarrow{\rho} \prod_{i,j} M_{f_if_j}.
\] (8.4.1)

These statements follow formally from the construction. The first follows since both the stalks \( \widetilde{M}_x \) and the localizations \( M_p \) are direct limits of the same modules over the same inductive system (indexed by the distinguished open subsets \( D(f) \) containing \( x \)); the second follows from the way we defined \( \widetilde{M} \); and the third is just the general exact sequence expressing the space of sections of a sheaf over an open set in terms of the space of sections over members of an open covering.

**Lemma 8.8.** For any two \( A \)-modules \( M \) and \( N \), the association \( \phi \to \widetilde{\phi} \) is an isomorphism \( \text{Hom}_A(M, N) \cong \text{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \).

In the lingo of category theory the lemma is expressed by saying that the tilde-operation is a fully faithful functor (trofast, full funktor), fully meaning the map in the lemma is surjective and faithful that it is injective. Loosely speaking, the "tilde-operation" gives an isomorphism of \( \text{Mod}_A \) with a subcategory of \( \text{Mod}_{\mathcal{O}_X} \). It is a strict subcategory; most of the \( \mathcal{O}_X \)-modules are not of the form \( \widetilde{M} \), but those that are, form the category of quasi-coherent \( \mathcal{O}_X \)-modules, that we shortly return to.

Assume that an \( \mathcal{O}_X \)-module \( \mathcal{F} \) is given on \( X = \text{Spec} \ A \), and let \( M \) denote the global sections of \( \mathcal{F} \); that is, \( M \) is the \( A \)-module \( M = \Gamma(X, \mathcal{F}) \). There is a natural map \( \widetilde{M} \to \mathcal{F} \) of \( \mathcal{O}_X \)-modules. As usual, it suffices to tell what the map does to sections over members of a basis for the topology, e.g., over open distinguished sets. The sheaf \( \mathcal{F} \) being an \( \mathcal{O}_X \)-module, multiplication by \( f^{-1} \) in the space of sections \( \Gamma(D(f), \mathcal{F}) \) has a meaning since \( \Gamma(D(f), \mathcal{O}_X) = A_f \). Hence we may send the section \( mf^{-n} \in M_f \) of \( \widetilde{M} \) to the section of \( \mathcal{F} \) over \( D(f) \) obtained by multiplying the restriction of \( m \) to \( D(f) \) by \( f^{-n} \); i.e., we send \( m \) to \( f^{-n} \cdot m|_{D(f)} \). For later reference we state this observation as a lemma, leaving the task of checking the details to the zealous student:

**Lemma 8.9.** Given a quasi-coherent sheaf \( \mathcal{F} \) on the affine scheme \( X = \text{Spec} \ A \). Then there is a unique \( \mathcal{O}_X \)-module homomorphism

\[
\theta_\mathcal{F}: \Gamma(X, \mathcal{F})^\sim \to \mathcal{F}
\]

inducing the identity on the spaces of global sections. Moreover, it is natural in the sense that if \( \alpha: \mathcal{F} \to \mathcal{G} \) is a map of \( \mathcal{O}_X \)-module inducing the map \( \alpha: \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \) on global sections, one has \( \theta_\mathcal{G} \circ \tilde{\alpha} = \alpha \circ \theta_\mathcal{F} \).

**Exercise 33.** Check that this is a well defined map (there is a choice involved in the definition).
Lemma 8.10. In the canonical identification of the distinguished open subset $D(f)$ with $\text{Spec} \, A_f$, the $\mathcal{O}_X$-module $\widetilde{M}$ restricts to $\widetilde{M}_f$.

Proof. As $\Gamma(D(f), \widetilde{M}) = M_f$, there is a map $\widetilde{M}_f \to \widetilde{M}|_{D(f)}$ that on distinguished open subsets $D(g) \subseteq D(f)$ induces an isomorphism between the two spaces of sections, both being equal to the localization $M_g$.

The “tilde-functor” is really a no-nonsense functor having almost all properties one can desire. It is fully faithful, as we have seen, but it also is an additive as well as an exact functor. Moreover, it takes the tensor product $M \otimes_A N$ of two $A$-modules to the tensor product $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ of the corresponding $\mathcal{O}_X$-modules, and if $M$ is of finite presentation, the $A$-module of homomorphisms $\text{Hom}_A(M, N)$ to the sheaf of homomorphisms $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$. However, this is not true if $M$ is not of finite presentation, the only lacking desirable property of the functor $(\widetilde{-})$.

To verify the first of these claims, assume given an exact sequence of $A$-modules:

$$0 \to M' \to M \to M'' \to 0.$$

That the induced sequence of $\mathcal{O}_X$-modules

$$0 \to \widetilde{M}' \to \widetilde{M} \to \widetilde{M}'' \to 0$$

is exact is a direct consequence of the three following facts. The stalk of a tilde-module $\widetilde{M}$ at the point $x$ with corresponding prime ideal $p$ is $M_p$, localization is an exact functor, and finally, a sequence of abelian sheaves is exact if and only if the sequence of stalks at every point is exact.

For the tensor product, let $T$ denote the presheaf $U \mapsto \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$. We have the map on presheaves $T \to \widetilde{M} \otimes_A \widetilde{N}$ (on $U = D(f)$ it is the isomorphism $M_f \otimes_{A_f} N_f \simeq (M \otimes_A N)_f$; it sends $m/f^a \otimes n/f^b$ to $(m \otimes n)/f^{a+b}$). After sheafifying, we get a the desired map of sheaves

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M} \otimes_A \widetilde{N}.$$

This is an isomorphism, since it is an isomorphism over every stalk.

When it comes to hom’s however, the situation is somehow more subtle. If $M$ is not of finite presentation, it is not true that $\text{Hom}_A(M, N)$ localizes. There is always a canonical map

$$\text{Hom}_A(M, N) \otimes_A A_f \to \text{Hom}_{A_f}(M_f, N_f), \quad (8.4.2)$$

but without some finiteness condition (like being of finite presentation) on $M$, it is not necessarily an isomorphism. Even in the simplest case of a infinitely
generated free module $M = \bigoplus_{i \in I} A e_i$ that map is not surjective. An element in $\text{Hom}_A(M, N) \otimes_A A f$ is of the form $\alpha \otimes f^{-n}$. An element in $\text{Hom}_{A_f}(M_f, N_f)$, however, is given by its values $m_i \otimes f^{-n_i}$ on the free generators $e_i$, and the salient point is that the $n_i$’s may tend to infinity, and no $n$ working for all $i$’s can be found.

**Exercise 34.** Show that the map (8.4.2) above is an isomorphism when $M$ is of finite presentation. Hint: First observe that this holds when $M = A$. Then use a presentation of $M$ to reduce to that case.

If $M$ is of finite presentation, the global sections of the sheaf-hom $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ equals $\text{Hom}_A(M, N)$, and there is a map

$$\text{Hom}_A(M, N) \to \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

In case $M$ is of finite presentation, the maps in (8.4.2) are isomorphisms, and the map above induces isomorphisms between the spaces of sections of the two sides over any distinguished open subset. One concludes that the map is an isomorphism, and one has $\text{Hom}_A(M, N) \simeq \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$.

In short, we have established the important proposition:

**Proposition 8.11.** Assume that $A$ is a ring and let $X = \text{Spec } A$. The functor from the category $\text{Mod}_A$ of $A$-modules to the category $\text{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules given by $M \to \widetilde{M}$ enjoys the following three properties

- It is a fully faithful additive and exact functor.
- One has a canonical isomorphism $(M \otimes_A N) \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.
- If $M$ is of finite presentation, one has a canonical isomorphism $\text{Hom}_A(M, N) \simeq \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$.

**Exercise 35.** Show that the tilde-functor is additive; i.e., takes direct sums of modules to the direct sum and sums of maps to the corresponding sums.

**Example 8.12.** Assume that $A$ is an integral domain and that $K$ is the field of fractions of $A$. Show that the $\mathcal{O}_X$-module $\widetilde{K}$ is a constant sheaf in the strong sense, that is $\Gamma(U, \widetilde{K}) = K$ for any non-empty open $U \subseteq X$ and the restriction maps all equal the identity $\text{id}_K$.

### 8.4.1 Functoriality

Suppose we are given a morphism between the two affine schemes $X$ and $Y$; say $\phi: X \to Y$. We let $X = \text{Spec } A$ and $Y = \text{Spec } B$ and we let $\phi^*: B \to A$ be the map corresponding to $\phi$. 

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If $M$ is an $A$-module, can one describe the sheaf $\phi_\ast \tilde{M}$ on $Y$? The answer is not only yes but the description is very simple. The $A$-module $M$ can be considered a $B$-module via the map $B \to A$ and as such is denoted by $M_B$. Clearly this is a functorial construction in $M$. In this setting one has

**Proposition 8.13.** One has $\phi_\ast \tilde{M} = \tilde{M}_B$.

**Proof.** There is an obvious map $\tilde{M}_B \to \phi_\ast (\tilde{M})$ as $\Gamma(Y, \phi_\ast (\tilde{M})) = \Gamma(X, \tilde{M}) = M$. The argument that follows shows it is an isomorphism, and it is as usual sufficient to verify that sections over distinguished open sets of the two sides coincide. The crucial observation is that $\phi^{-1}D(f) = D(\phi^#(f))$—indeed, a prime ideal $p \subseteq A$ satisfies $f \in \phi^#(p)$ if and only if $\phi^#(f) \in p$. As $f$ acts on $M_B$ as multiplication by $\phi^#(f)$, one clearly has $(M_B)_f = M_{\phi^#(f)} = \Gamma(D(\phi^#(f)), M)$. $\square$

## 8.5 Quasi-coherent sheaves on general schemes

Having established the sheaves on affine space that serve as local models for the quasi-coherent sheaves, we are now ready for the general definition.

**Definition 8.14.** If $X$ is a scheme and $F$ an $\mathcal{O}_X$-module, one says that $F$ is quasi-coherent (kvasikoherent) $\mathcal{O}_X$-module, or quasi-coherent sheaf for short, if there is an open affine covering $\{U_i\}_{i \in I}$ of $X$, say $U_i = \text{Spec } A_i$, and modules $M_i$ over $A_i$ such that $F|_{U_i} \simeq \tilde{M}_i$.

Phrased in slightly different manner, $\mathcal{O}_X$-module $F$ is quasi-coherent if the restriction $F|_{U_i}$ of $F$ to each $U_i$ is of type tilde of an $A_i$-module. In particular, the modules $\tilde{M}$ on affine schemes $\text{Spec } A$ are all quasi-coherent.

The restriction of a quasi-coherent sheaf $F$ to any open set $U \subseteq X$ is quasi-coherent. Indeed, it will suffice to verify this for $X$ an affine scheme, and by lemma 8.10 the restriction of a sheaf of tilde-type to a distinguished open set is of tilde-type. As any open $U$ in an affine scheme is the union of distinguished open subsets, it follows that $F|_U$ is quasi-coherent.

For $F$ to be quasi-coherent, we require that $F$ be locally of tilde-type for just one open affine cover. However, it turns out that this will hold for any open affine cover, or equivalently, that $F|_U$ is of tilde-type for any open affine subset $U \subseteq X$. This is a much stronger than the requirement in the definition, and it is somehow subtle to prove. As a first corollary we arrive at the a priori not obvious conclusion that the modules of the form $\tilde{M}$ are the only quasi-coherent $\mathcal{O}_X$-modules on an affine scheme. We shall also see that quasi-coherent modules enjoy the coherence property (8.3.1) on page 131 that was the point of departure for our discussion.
The story starts with a lemma that establishes the coherence property (8.3.1) in a very particular case; i.e., for sections over distinguished open sets of a quasi-coherent \( \mathcal{O}_X \)-module on an affine scheme \( X = \text{Spec } A \). For any distinguished open set \( D(f) \subseteq X \) it holds that \( \Gamma(D(f), \mathcal{O}_X) = A_f \), and consequently there is for any \( \mathcal{O}_X \)-module a canonical map \( \Gamma(X, \mathcal{F}) \otimes_A A_f \to \Gamma(D(f), \mathcal{F}) \) sending \( s \otimes a f^{-n} \) to \( a f^{-n} \cdot s|_{D(f)} \). It turns out to be an isomorphism whenever \( \mathcal{F} \) is quasi-coherent:

**Lemma 8.15.** Suppose that \( X = \text{Spec } A \) is an affine scheme and that \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_X \)-module. Let \( D(f) \subseteq X \) be a distinguished open set. Then one has

- \( \Gamma(D(f), \mathcal{F}) \simeq \Gamma(X, \mathcal{F}) \otimes_A A_f \).

- Let \( s \in \Gamma(X, \mathcal{F}) \) be a global section of \( \mathcal{F} \) and assume that \( s|_{D(f)} = 0 \), then sufficiently big powers of \( f \) kill \( s \), that is, for sufficiently big integers \( n \) one has \( f^n s = 0 \).

- Let \( s \in \Gamma(D(f), \mathcal{F}) \) be a section. Then for a sufficiently large \( n \), the section \( f^n s \) extends to a global section of \( \mathcal{F} \). That is, there exists an \( n \) and a global section \( t \in \Gamma(X, \mathcal{F}) \) such that \( t|_{D(f)} = f^n s \).

**Proof.** The first statement is by the definition of localization equivalent to the two others.

The sheaf \( \mathcal{F} \) being quasi-coherent by hypothesis, and the affine scheme \( X = \text{Spec } A \) being quasi-compact, there is a finite open affine covering of \( X \) by distinguished sets \( D(g_i) \) such that \( \mathcal{F}|_{D(g_i)} \simeq \tilde{M}_i \) for some \( A_{g_i} \)-modules \( M_i \). The section \( s \) of \( \mathcal{F} \) restricts to sections \( s_i \) of \( \mathcal{F}|_{D(g_i)} \) over \( D(g_i) \), that is, to elements \( s_i \) of \( M_i \).

Further restricting \( \mathcal{F} \) to the intersections \( D(f) \cap D(g_i) = D(fg_i) \) yields the equality \( \mathcal{F}|_{D(fg_i)} = (M_i)_f \), and by hypothesis, the section \( s \) restricts to zero in \( \Gamma(D(fg_i), \mathcal{F}) = (M_i)_f \). This means that the localization map sends \( s \) to zero in \( (M_i)_f \). Hence \( s_i \) is killed by some power of \( f \), and since there is only finitely many \( g_i \)'s, there is an \( n \) with \( f^n s_i = 0 \) for all \( i \); that is, \( (f^n s)|_{D(g_i)} = 0 \) for all \( i \). By the locality axiom for sheaves, it follows that \( f^n s = 0 \).

Assume now a section \( s \in \Gamma(D(f), \mathcal{F}) \) is given. We are to see that \( f^n s \) extends to a global section of \( \mathcal{F} \) for large \( n \). Each restriction \( s|_{D(fg_i)} \in \Gamma(D(fg_i), \mathcal{F}) = (M_i)_f \) is of the form \( f^{-n} s_i \) with \( s_i \in M_i = \Gamma(D(g_i), \mathcal{F}) \), and by the usual finiteness argument, \( n \) can be chosen uniformly for all \( i \). This means that \( s_i = f^n s \) and \( s_j = f^n s \) mach on the intersection \( D(f) \cap D(g_i) \cap D(g_j) \), and by the first part of the lemma applied to \( \text{Spec } A_{g_i g_j} \), one has \( f^N (s_i - s_j) = 0 \) on \( D(g_i) \cap D(g_j) \) for a sufficiently large integers \( N \). Hence the different \( f^N s_i \)'s patch together to give the desired global section \( t \) of \( \mathcal{F} \). \( \square \)
Theorem 8.16. Let $X$ be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for all open affine subsets $U \subseteq X = \text{Spec } A$, the restriction $\mathcal{F}|_U$ is isomorphic to a $\mathcal{O}_X$-module of the form $\tilde{M}$ for an $A$-module $M$.

Proof. As quasi-coherence is conserved when restricting $\mathcal{O}_X$-modules to open sets, we may surely assume that $X$ itself is affine; say $X = \text{Spec } A$. Let $M = \Gamma(X, \mathcal{F})$. We saw in lemma 8.9 on page 133 that there is a natural map $\tilde{M} \to \mathcal{F}$ that on distinguished open sets sends $mf^n$ to $f^{-n}m|_U$, but by the fundamental lemma 8.15 above, this is an isomorphism between the spaces of sections over the distinguished open sets. Hence the two sheaves are isomorphic. \hfill $\square$

Applying this to an affine scheme, yields the important fact that any quasi-coherent sheaf $\mathcal{F}$ on an affine scheme $X = \text{Spec } A$ is of the form $\tilde{M}$ for an $A$-module $M$.

Proposition 8.17. Assume that $X = \text{Spec } A$. The tilde-functor $M \mapsto \tilde{M}$ is an equivalence of categories $\text{Mod}_A$ and $\text{Qcoh}_X$ with the global section functor as an inverse.

When speaking about mutually inverse functors one should be very careful; in most cases such a statement is an abuse of language. Two functors $\mathcal{F}$ and $\mathcal{G}$ are mutually inverses when there are natural transformations, both being an isomorphisms, between the compositions $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ and the appropriate identity functors. In the present case one really has an equality $\Gamma(X, \tilde{M}) = M$, so that $\Gamma \circ (-) = \text{id}_{\text{Mod}_A}$. On the other hand, the natural transformation $\Gamma(X, \mathcal{F}) \to \mathcal{F}$ from lemma 8.9 on page 133 furnishes the required isomorphism of functors.

Theorem 8.18. Let $X$ be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X$. Then $\mathcal{F}$ is quasi-coherent if and only if for any pair $V \subseteq U$ open affine subsets, the natural map

$$\Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(V, \mathcal{O}_X) \to \Gamma(V, \mathcal{F}) \quad (8.5.1)$$

is an isomorphism.

Proof. We may clearly assume that $X$ is affine, say $X = \text{Spec } A$. Assume that the maps (8.5.1) are isomorphisms. We may take $V = D(f)$ and $U = X$ and $M = \Gamma(X, \mathcal{F})$. Then from (8.5.1) it follows that $\Gamma(D(f), \mathcal{F}) = M_f$ which shows that the canonical map $\tilde{M} \to \mathcal{F}$ is an isomorphism over all distinguished open subsets, and therefore an isomorphism. Hence $\mathcal{F}$ is quasi-coherent.

To argue for the implication the other way, assume that $U = \text{Spec } B$ and that $\mathcal{F}$ is quasi-coherent; that is, $\mathcal{F} = \tilde{M}$ for some $A$-module $M$ after theorem 8.18. The restriction of a quasi-coherent sheaf is quasi-coherent, so $\tilde{M}|_U = \tilde{N}$ for some $B$-module $N$, and the map in (8.5.1) is just a map have map $M \otimes_A B \to N$. It
induces an isomorphism over all local rings $A_p = B_p$ (where $p \in U = \text{Spec } B$) since $\tilde{M}|_U \cong \tilde{N}$ and therefore is an isomorphism of $B$-modules.

**Example 8.19.** The example of an discrete valuation ring is always useful to consider, and we continue exploring example 8.1 above. The $\mathcal{O}_X$ module given by the data $M, N, \rho$. We claim that $\mathcal{F}$ is quasi-coherent if and only if $\rho$ is an isomorphism.

If $\mathcal{F}$ is quasi-coherent, then every point has a neighbourhood on which $\mathcal{F}$ is the of some module. The only neighbourhood of the unique closed point is $X$ itself, and so $\mathcal{F} = \tilde{M}$. Therefore, $N = \mathcal{F}(U) = M_{(0)} = M \otimes R K$ and $\rho$ is an isomorphism. Conversely, of $\rho : M \otimes R K \to K$ is an isomorphism, then $\mathcal{F}$ is given by $\mathcal{F}(X) = M$ and $\mathcal{F}(\{ \eta \}) = M \otimes R N$, and so $\mathcal{F} \cong \tilde{M}$ is quasi-coherent.

### 8.6 Coherent sheaves

Let $A$ be a ring and let $M$ be an $A$-module. The module $M$ is of finite presentation if for some integers $n$ and $m$ there is an exact sequence

$$A^n \rightarrow A^m \rightarrow M \rightarrow 0.$$ 

One says that $M$ is coherent if the following two requirements are fulfilled

- $M$ is finitely generated.
- The kernel of every surjection $A^n \rightarrow M$ is finitely presented.

The second statement is equivalent to every finitely generated submodule $N \subseteq M$ being of finite presentation. In the case $A$ is a noetherian ring—which frequently is the case in algebraic geometry—a module $M$ being coherent is equivalent to $M$ being finitely generated. The condition comes from the theory analytic functions where coherent non-noetherian rings are frequent.

When $A$ is noetherian, then the three conditions of coherence, finitely generation and being of finite presentation on an $A$-module $M$ coincide. The key point is that for a finitely generated module $M$ over a noetherian ring $A$, every submodule $N \subseteq M$ is also finitely generated. Indeed, the set $\mathcal{V}$ of finitely generated submodules $N' \subseteq N$ has a maximal element $N^\circ$, which has to equal $N$: If not, there is an $n \in N - N^\circ$, and a finitely generated submodule $N'' = N' + An$ which is strictly bigger than $N^\circ$.

So to show that $M$ is finitely presented, we can take a presentation $A^n \rightarrow M \rightarrow 0$ and let $L$ be the kernel, regarded as a submodule of $A^n$. Then $L$ is finitely generated, so there is a surjection $A^n \rightarrow L$, and we get a presentation $A^n \rightarrow A^m \rightarrow M \rightarrow 0$. Applying this argument to any finitely generated submodule $N \subseteq M$ shows that $M$ is coherent.
**Exercise 36.** Fill in the details of the proof that if $A$ is a noetherian ring, then the three conditions of coherence, finitely generation and being of finite presentation on an $A$-module $M$ coincide.

**Definition 8.20.** On a scheme $X$ a quasi-coherent $\mathcal{O}_X$-module is coherent if there is a covering of $X$ by open affine sets $U_i = \text{Spec } A_i$ such that $\mathcal{F}|_{U_i} = \tilde{M}_i$ with the $M_i$'s being coherent $A_i$-modules. $\mathcal{F}$ is finitely presented if each $M_i$ are finitely presented as $A_i$-modules.

So if $X$ is noetherian (or locally noetherian), the condition that $M$ is coherent is equivalent to the weaker condition that $M$ is finitely generated.

**Remark 8.21.** The definition above is the one that appears in EGA and the Stacks project. However, it differs slightly from the notation used in Hartshorne's book. In that book, one says that $\mathcal{F}$ is coherent if there is a covering by open affines $U_i = \text{Spec } A_i$ such that $\mathcal{F}|_{U_i} = \tilde{M}_i$ with the $M_i$'s being finitely generated $A_i$-modules (that is, the condition about kernels is left out). In the case $X$ is (locally) noetherian, these two notions coincide, but they are not equivalent in general.

The notion of coherent sheaf was actually introduced by Henri Cartan in the theory of holomorphic functions of several variables around 1944. In 1946 Oka proved that $\mathcal{O}_{\mathbb{C}^n}$ is coherent and this is a very difficult theorem (although it seems like a trivial statement if the other definition of ‘coherence’ is used).

One benefit of using coherent modules rather than finitely generated ones is that the category of coherent modules is an abelian category, even in the non-noetherian setting. However, a problem is that coherence is very difficult to check in general and actually for some schemes, even affine ones, the structure sheaf $\mathcal{O}_X$ is not coherent!

### 8.7 Functoriality

**Proposition 8.22.** Suppose that $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent sheaves on the scheme $X$. The kernel, cokernel and the image of $\alpha$ are all quasi-coherent. The category $\text{Qcoh}_X$ is closed under extensions; that is, if

$$
\begin{array}{c}
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0
\end{array}
$$

(8.7.1)

is a short exact sequence of $\mathcal{O}_X$-modules with the two extreme sheaves $M'$ and $M''$ being quasi-coherent, the middle sheaf $M$ is quasi-coherent as well.

**Proof.** If $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent $\mathcal{O}_X$-modules, on any open affine subsets $U = \text{Spec } A$ of $X$ it may be described as $\alpha|_U = \tilde{a}$ where $a : M \to N$ is a $A$-module homomorphism and $M$ and $N$ are $A$-modules with $\mathcal{F}|_U = \tilde{M}$.
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and \( G|_U = \tilde{N} \). Since the tilde-functor is exact, one has \( \ker \alpha|_U = (\ker a)^\sim \). Moreover, by the same reasoning, it holds true that \( \coker \alpha|_U = (\coker a)^\sim \) and \( \text{im} \alpha|_U = (\text{im} a)^\sim \).

Suppose now that an extension like (8.7.1) is given. The leftmost sheaf \( M' \) being quasi-coherent lemma 13.4 entails that the induced sequence of global sections is exact; that is, the upper horizontal sequence in the diagram below. The three vertical maps in the diagram are the natural maps from lemma 8.9 on page 133. Since \( M' \) and \( M'' \) both are quasi-coherent sheaves, the two flanking vertical maps are isomorphisms, and the snake lemma implies that the middle vertical map is an isomorphism as well. Hence \( M \) is quasi-coherent.

\[
\begin{array}{cccccc}
0 & \rightarrow & \Gamma(X, M') & \rightarrow & \Gamma(X, M) & \rightarrow & \Gamma(X, M'') & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0
\end{array}
\]

The category \( \text{QCoh}_X \) is an abelian category with tensor products and internal hom’s.

8.7.1 Quasi-coherence of pullbacks

Recall the notion of pullback of a sheaf via a morphism \( f : X \to Y \). This is a relatively complicated operation, since it involves taking a direct limit, a tensor product, and finally a sheafification. The next result tells us that for \( G \) a quasi-coherent sheaf on \( Y \), we have a much simpler description of the pullback \( f^*G \), which will allow us to do local computations more easily.

**Theorem 8.23.** Let \( f : X \to Y \) be a morphism of schemes.

(i) If \( X = \text{Spec} B, Y = \text{Spec} A \), and \( M \) is an \( A \)-module, then

\[
\text{f}^*(\tilde{M}) = (M \otimes_A B)^\sim
\]

(ii) If \( G \) is a quasi-coherent sheaf on \( Y \), then \( f^*G \) is quasi-coherent on \( X \).

(iii) If \( X \) and \( Y \) are noetherian, then \( f^*G \) is coherent if \( G \) is.

**Proof.** (ii) and (iii) follows by (i), since quasi-coherence is a local property, and since \( M \otimes_A B \) is a finitely generated \( B \) if \( M \) is a finitely generated \( A \)-module. So let us proceed to prove the first point.

First, note that the theorem holds in the special case when \( M = A^I \) is a free module (here the index set \( I \) is allowed to be infinite) – this is simply because
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\[ f^* \mathcal{O}_Y = \mathcal{O}_X \] and \( f^* \) commutes with taking direct sums. To prove it in general, we pick a presentation of \( M \) of the form

\[ A^J \xrightarrow{\gamma} A^I \to M \to 0 \]

Applying \( \sim \) and then \( f^* \) we get a sequence

\[ f^* A^J \xrightarrow{\nu} f^* A^I \to f^* \tilde{M} \to 0 \]

which is exact on the right, since \( f^* \) is right-exact. From this we get that

\[ f^* \tilde{M} = \text{coker} \nu = \text{coker} (\gamma \otimes_A B) = ((\text{coker} \gamma) \otimes_A B) \sim = (M \otimes_A B) \sim. \]

\[ \square \]

It is instructive to see the first point via the adjoint property of \( f_* \) and \( f^* \). If \( \mathcal{F} = \tilde{M} \) for a \( B \)-module \( M \) and \( \mathcal{G} = \tilde{N} \) for an \( A \)-module \( N \), we have the following isomorphisms induced by the adjoint property of Hom and \( \otimes \):

\[
\begin{align*}
\text{Hom}_X(f^*\mathcal{G}, \mathcal{F}) &= \text{Hom}_Y(f^*\tilde{N}, \tilde{M}) \\
&= \text{Hom}((M \otimes_A B) \sim, \tilde{N}) \\
&= \text{Hom}_A(M \otimes_A B, N) \\
&= \text{Hom}_B(M, \text{Hom}_A(B, N)) \\
&= \text{Hom}_B(M, N_B) \\
&= \text{Hom}_Y(\tilde{M}, \tilde{N}_B) \\
&= \text{Hom}_Y(\mathcal{G}, f_*\mathcal{F})
\end{align*}
\]

8.7.2 Quasi-coherence of the direct image

Recall that we showed that for a map \( \phi : \text{Spec} A \to \text{Spec} N \), the pushforward \( \phi_* \mathcal{F} \) is quasi coherent, of \( \mathcal{F} \) is quasi-coherent (since \( \phi_* \tilde{M} = \tilde{M}_B \)). The following result applies to more general morphisms:

**Theorem 8.24.** Let \( \phi : X \to Y \) be a morphism of schemes and that \( \mathcal{F} \) is a quasi-coherent sheaf on \( X \). If \( X \) is noetherian, then the direct image \( \phi_* \mathcal{F} \) is quasi-coherent on \( Y \).

**Proof.** We may assume that \( Y = \text{Spec} A \). Then since \( X \) is quasi-compact, we may cover it by open affines \( U_i \). The intersection \( U_i \cap U_j \) is again quasi-compact, so we can cover it with open affines \( U_{ijk} \).
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For any open $V \subseteq Y$, one has the exact sequence

$$0 \rightarrow \Gamma(\phi^{-1}V, \mathcal{F}) \rightarrow \prod_i \Gamma(U_i \cap \phi^{-1}V, \mathcal{F}) \rightarrow \prod_{i,j,k} \Gamma(U_{ijk} \cap \phi^{-1}V, \mathcal{F}).$$

(8.7.2)

The sequence is compatible with the restriction maps induced from an inclusion $V' \subseteq V$, hence gives rise to the following exact sequence of sheaves on $X$:

$$0 \rightarrow \phi_* \mathcal{F} \rightarrow \prod_i \phi_i_* \mathcal{F}|_{U_i} \rightarrow \prod_{i,j,k} \phi_{ijk}_* \mathcal{F}|_{U_{ijk}}$$

(8.7.3)

where $\phi_i = \phi|_{U_i}$ and $\phi_{ijk} = \phi|_{U_{ijk}}$. Now, each of the sheaves $\phi_i_* \mathcal{F}|_{U_i}$ and $\phi_{ijk}* \mathcal{F}|_{U_{ijk}}$ are quasi-coherent by the affine case of the theorem (proposition 8.13 on page 136). They are finite in number as the covering $U_i$ is finite. Hence $\prod_i \phi_i_* \mathcal{F}|_{U_i}$ and $\prod_{i,j,k} \phi_{ijk}_* \mathcal{F}|_{U_{ijk}}$ are finite products of quasi-coherent $\mathcal{O}_X$-modules and therefore they are quasi-coherent. Now the $\phi_* \mathcal{F}$ is the kernel of a homomorphism between two quasi-coherent sheaves, and so the theorem the follows from proposition 8.22 on page 140.

The following example shows that some of the hypotheses are necessary for this statement to hold:

Example 8.25. Let $X = \bigsqcup_{i \in I} \text{Spec} \mathbb{Z}$ be the disjoint union of countably infinitely many copies of $\text{Spec} \mathbb{Z}$ and let $\phi : X \rightarrow \text{Spec} \mathbb{Z}$ be the morphism that equals the identity on each of the copies of $\text{Spec} \mathbb{Z}$ that constitute $X$. Then $\phi_* \mathcal{O}_X$ is not quasi-coherent. Indeed, the global sections of $\phi_* \mathcal{O}_X$ satisfy

$$\Gamma(\text{Spec} \mathbb{Z}, \phi_* \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i \in I} \mathbb{Z}.$$  

On the other hand if $p$ is any prime, one has

$$\Gamma(D(p), \phi_* \mathcal{O}_X) = \Gamma(\phi^{-1}D(p), \mathcal{O}_X) = \prod_{i \in I} \mathbb{Z}[p^{-1}].$$

It is not true that $\Gamma(D(p), \phi_* \mathcal{O}_X) = \Gamma(\text{Spec} \mathbb{Z}, \phi_* \mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ hence $\phi_* \mathcal{O}_X$ is not quasi-coherent. Indeed, elements in $\prod_{i \in I} \mathbb{Z}[p^{-1}]$ are sequences of the form $(z_i p^{-n_i})_{i \in I}$ where $z_i \in \mathbb{Z}$ and $n_i \in \mathbb{N}$. Such an element lies in $(\prod_{i \in I} \otimes \mathbb{Z}[p^{-1}])$ only if the $n_i$'s form a bounded sequence, which is not the case for general elements of shape $(z_i p^{-n_i})_{i \in I}$ when $I$ is infinite.

Exercise 37. Show that $X = \bigsqcup_{i \in I} \text{Spec} \mathbb{Z}$ is not affine. Show that in the category $\mathbf{Sch}$ of schemes $X$ is the coproduct of the infinitely many copies of $\text{Spec} \mathbb{Z}$ involved. Show that $\text{Spec} \prod_i \mathbb{Z}$ is the coproduct of the copies of $\text{Spec} \mathbb{Z}$ in the category $\mathbf{Aff}$ of affine schemes. Show that these gadgets are substantially different.
8.7.3 Coherence of the direct image

For morphisms of schemes $f : X \to Y$, it is not expected that the pushforward of a coherent sheaf is again coherent, even for 'nice' morphisms $f$. A simple example is the following:

**Example 8.26.** Let $X = \text{Spec} \ k[t]$ and consider the structure morphism $f : X \to \text{Spec} \ k$ (induced by $k \subseteq k[t]$). The sheaf $\mathcal{O}_X$ is of course coherent, but $f_* \mathcal{O}_X$ is not. Indeed, this is $\tilde{k[t]}$, and $k[t]$ is clearly not finitely generated as a $k$-module.

However, for finite morphisms, we have a positive result:

**Lemma 8.27.** Let $f : X \to Y$ be a finite morphism of schemes. If $\mathcal{F}$ be a quasi-coherent sheaf on $X$, then $f_* \mathcal{F}$ is quasi-coherent on $Y$. If $X$ and $Y$ are noetherian, $f_* \mathcal{F}$ is even coherent if $\mathcal{F}$ is.

**Proof.** Since $f$ is finite, we can cover $Y$ by open affines $\text{Spec} \ A$ such that each $f^{-1} \text{Spec} \ A$ = $\text{Spec} \ B$ is also affine, where $B$ is a finite $A$-module. We then have $f_* \mathcal{F}(\text{Spec} \ A) = \mathcal{F}(\text{Spec} \ B)$. Now, since $\mathcal{F}$ is quasicoherent, we have $\mathcal{F}|_{\text{Spec} \ B} = \widetilde{M}$ for some $B$-module, which we can view as an $A$-module via $f$. Hence $f_* \mathcal{F}$ is quasi-coherent. If $X$ and $Y$ are noetherian, and $\mathcal{F}$ is coherent, the module $M$ is finitely generated as an $B$-module, and hence as an $A$-module, since $B$ is a finite $A$-module.

8.8 The categories of coherent and quasi-coherent sheaves

Our work in the previous sections imply the following statement

**Theorem 8.28.** The category $\text{QCoh}(X)$ is an abelian category and is closed under direct limits.

By definition, being an abelian category entails that the hom-sets $\text{Hom}(\mathcal{F}, \mathcal{G})$ are abelian groups; finite direct sums exist; kernels and cokernels of morphisms exist; every monomorphism $\mathcal{F} \to \mathcal{G}$ is the kernel of its cokernel; every epimorphism is the cokernel of its kernel; and every morphism can be factored into an epimorphism followed by a monomorphism. The hard part is thus in the last part of the statement, that any direct limit of quasi-coherent sheaves is again quasi-coherent.

One reason why we prefer the notion of 'coherence' used here (rather than the one in Harthorne) is that the category of coherent sheaves $\text{Coh}(X)$ is also an abelian category, even in the non-noetherian case. Note that it does not contain
all its direct limits, simply because an arbitrary product of coherent \( A \)-modules is typically not coherent (not even finitely generated!)

Coherent sheaves can still be regarded as the building blocks of the category \( \text{QCoh}(X) \). In fact, a common technique is to prove statements about quasi-coherent sheaves by approximating them with coherent sheaves. This is justified by the following

**Theorem 8.29.** Any quasi-coherent sheaf on a noetherian scheme is the direct limit of its coherent subsheaves.

We will not go into details about this statement here, but remark that the proof is not too difficult (see EGA I, Section 6.9).

### 8.9 Closed immersions and subschemes

If \( A \) is a ring and \( I \) is an ideal of \( A \), we have seen that there is a morphism of schemes \( f : \text{Spec}(A/I) \to \text{Spec} A \), inducing an homeomorphism of \( \text{Spec}(A/I) \) onto the closed subset \( V(I) \). In other words, \( f \) is what’s known as a closed immersion of topological spaces. Conversely, such a morphism \( f : \text{Spec} B \to \text{Spec} A \) which is an homeomorphism onto a closed subset, the induced map on rings \( \phi : A \to B \) is surjective and \( \text{Spec} B \) is isomorphic to the scheme \( \text{Spec}(A/\ker \phi) \).

The aim of this section is to globalize these observations for general schemes. This will involve replacing ideals with (quasi-coherent) ideal sheaves.

**Lemma 8.30.** Let \( X \) be a scheme and let \( \mathcal{I} \subseteq \mathcal{O}_X \) be a quasi-coherent sheaf of ideals. Then the ringed space \( Z = (\text{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I}) \) is a scheme with a canonical morphism \( i : Z \to X \).

Indeed, to prove this, we may assume that \( X = \text{Spec} A \) is affine. In this case \( \mathcal{I} \) is the \( \sim \) of some ideal \( I \subseteq A \), and the support of this is exactly the primes \( p \) such that \( (A/I)_p \neq 0 \), or equivalently \( p \in V(I) \). Hence \( Z \) is the closed subset \( V(I) \), which is homeomorphic to \( \text{Spec}(A/I) \). The sheaf of rings on \( \text{Spec}(A/I) \) is the same as \( \mathcal{O}_X/\mathcal{I} \) on \( Z \) and hence \( Z \) is the scheme \( \text{Spec} A/I \). The map \( i \) is just induced by the inclusion \( Z \subseteq X \) and the natural map \( \mathcal{O}_X \to i_* (\mathcal{O}_X/\mathcal{I}) \) is just \( \sim \) of the quotient map \( A \to A/I \).

**Definition 8.31.** A closed subscheme of a scheme \( X \) is a scheme \( (\text{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I}) \) for some quasi-coherent sheaf of ideals \( \mathcal{I} \) on \( X \).

**Definition 8.32.** A morphism of schemes \( f : Y \to X \) is called an closed immersion if there is an isomorphism \( Y \simeq Z \) onto a closed subscheme \( Z = \ldots \)
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(Supp(\(\mathcal{O}_X/\mathcal{I}\)), \(\mathcal{O}_X/\mathcal{I}\)) for some quasi-coherent sheaf of ideals \(\mathcal{I}\) on \(X\), with a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
Z & \xleftarrow{i} & \\
\end{array}
\]

If \(\phi: A \to B\) is a surjective homomorphism of rings, then \(\text{Spec } B \to \text{Spec } A\) is a closed immersion. Conversely, we have the following:

**Lemma 8.33.** Let \(f: Z \to X\) be a closed immersion. If \(X\) is affine, then so is \(Z\) and in fact \(Z \simeq \text{Spec } A/I\) for some ideal \(I\) of \(A\).

Indeed, we can view \(Z\) as a closed subscheme of \(X = \text{Spec } A\), with the locally ringed structure being given by \(\mathcal{O}_X/\mathcal{I}\) for some quasi-coherent sheaf of ideals \(I\) on \(X\). We have seen that this is the sheaf associated to an ideal \(I\) of \(A\).

**Proposition 8.34.** Let \(f: Y \to X\) be a morphism of schemes. Then \(f\) is a closed immersion if and only if \(f\) is a homeomorphism of the topological space of \(Y\) onto a closed subset of \(X\) and \(f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y\) is surjective.

**Proof.** A closed immersion by a quasi-coherent sheaf of ideals clearly satisfies the two conditions, so we need only prove the 'if' direction.

We want to associate to \(f\) a quasi-coherent ideal sheaf \(\mathcal{I}\) on \(X\). Since \(\mathcal{O}_X \to f_*\mathcal{O}_Y\) is surjective, it is natural to take the kernel of this map. However, we have to work a bit to prove that this is actually quasi-coherent.

Let us first show that \(f\) is affine, i.e., that for any \(x \in f(Y)\), there is an open affine neighborhood \(U\) of \(y\) with \(f^{-1}(U)\) affine. Let \(y \in Y\) be the (uniquely defined) preimage of \(x\) and choose affine open subsets \(V \subseteq Y, W \subseteq X\) with \(f(V) \subseteq W\) containing \(y\) and \(x\) respectively. \(f\) is a homeomorphism, so \(f(V)\) can be written as \(W' \cap f(X)\) for some \(W'\) open set \(W' \subseteq W\) containing \(y\). Now choose a section \(s\) of \(\mathcal{O}_X(W)\) such that \(D(s) \subseteq W'\) and contains \(x\). Then the open set \(f^{-1}(D(s))\) is contained in the affine set \(V\) (it is the intersection of \(V\) with \(D(f^*(s))\)). So \(f^{-1}(D(s))\) is also affine, and hence we see that \(f\) is affine.

Now we claim that \(f_*\mathcal{O}_Y\) is quasi-coherent on \(X\). By the above, \(X\) is covered by open affines \(U = \text{Spec } A\) such that \(f^{-1}(U) = \text{Spec } B\) is also affine. In this case, \(f_*\mathcal{O}_Y|_U\) is the \(\sim\) of \(B\) considered as an \(A\)-module, and hence is quasi-coherent.

Now, letting \(\mathcal{I}\) be the kernel of \(f^\#\) (which is quasi-coherent!), we have an exact sequence

\[0 \to \mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0.\]

In particular, we see that \(f_*\mathcal{O}_Y\) is isomorphic to the sheaf \(\mathcal{O}_X/\mathcal{I}\). Moreover, the \(\mathcal{O}_X/\mathcal{I}\) has support on \(f(Y)\). So the map \(f: Y \to X\) is isomorphic to
the inclusion \((f(Y), \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)\), which proves that it is indeed a closed immersion.

\[\square\]

### 8.9.1 Morphisms to a closed subscheme

If \(f : Y \to X\) is a map of schemes, it is natural to ask when it factors through a closed immersion of \(X\). Here we need only work up to isomorphism, so we can assume that the closed immersion is given by \((\text{Supp} \mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)\).

**Proposition 8.35.** Let \(Z\) be a closed subscheme of \(X\) given by sheaf of ideals \(\mathcal{I}\). Suppose \(f : Y \to X\) is a morphism of schemes. Then \(f\) factors through a map \(g : Y \to Z\) if and only if

1. \(f(Y) \subseteq Z\).
2. \(\mathcal{I} \subseteq \ker(\mathcal{O}_X \to f_*(\mathcal{O}_Y))\).

**Proof.** The condition (i) is clearly necessary. If there is a sequence \(Y \to Z \to X\), then there is a sequence of sheaves \(\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to f_*(\mathcal{O}_Y)\), which means that the map \(\mathcal{O}_X \to f_*(\mathcal{O}_Y)\) factors through \(\mathcal{O}_X/\mathcal{I}\), and so also (ii) holds.

Conversely, we define the map \(g\) on topological spaces by the inclusion (i). To define it on sheaves, we use the map \(\mathcal{O}_X \to f_*(\mathcal{O}_Y)\). This annihilates \(\mathcal{I}\), so we thus get a map \(\mathcal{O}_X/\mathcal{I} \to f_*(\mathcal{O}_Y) = g_*(\mathcal{O}_Y)\). This gives us the map \(g : Y \to Z\) factoring \(f\).

**Remark 8.36** (Scheme-theoretic image). Let \(f : Y \to X\) be a morphism of schemes. Then we can define the scheme-theoretic image of \(f\) as a subscheme \(Z \subseteq X\) satisfying the universal property that if \(f\) factors through a subscheme \(Z' \subseteq Z\), then \(Z \subseteq Z'\). To define \(Z\) it is is tempting to use the ideal sheaf \(\mathcal{I} = \ker(\mathcal{O}_X \to f_*(\mathcal{O}_Y))\) – but this may fail to be quasi-coherent for a general morphism \(f\). However, one can show that there is a largest quasi-coherent sheaf of ideals \(\mathcal{J}\) contained in \(\mathcal{I}\), and we then define \(Z\) to be associated to \(\mathcal{J}\).

### 8.9.2 Induced reduced scheme structure

For an open subscheme \(U \subseteq X\), we saw that there was a natural scheme structure on \(U\) induced from that of \(X\). For \(W \subseteq X\) a closed subset, there can be several quasi-coherent ideal sheaves \(\mathcal{I}\) corresponding to \(W\). For instance, \(\text{Spec} k[x]/x\) is naturally a subscheme of \(\text{Spec} k[x]/x^2\), but of course they have the same underlying topological space. So, in contrast with the ‘open subschemes’ the underlying topological space does not determine the scheme structure. However there is one, which is in some sense the ‘smallest’ one:

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Proposition 8.37 (Induced reduced scheme structure). Let $X$ be a scheme. Let $W \subseteq X$ be a closed subset. There exists a unique closed subscheme $Z \subseteq X$ such that

- $Z$ is reduced;
- The underlying topological space of $Z$ is $W$.

Proof. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the sheaf of ideals defined by

$$\mathcal{I}(U) = \{ f \in \mathcal{O}_X(U) \mid f_x \in \mathfrak{m}_x \subseteq \mathcal{O}_{X,x} \text{ for all } x \in W \cap U \}$$

We claim that this is quasi-coherent. On $U = \text{Spec } A$, we have $W \cap U = V(I)$ for a unique radical ideal $I \subseteq R$. Here we have $\mathcal{I}(U) = I$: If $f \in A$ is an element such that $f/1 \in \mathfrak{m}_p \subseteq A_p$ then $f \in p$, and so if $f \in \mathcal{I}(U)$ then $f \in \cap_{p \in V(I)} p = \sqrt{I} = I$. Moreover, for $D(g) \subseteq U$, we have $\mathcal{I}(D(g)) = I_g$ by the same argument, and so $\tilde{I}$ and $\mathcal{I}$ are equal as sheaves on $U$, and hence $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

Now define $Z$ to be the closed subscheme associated with the ideal sheaf $\mathcal{I}$. $Z$ is reduced, and has same underlying topological space as $W$; this holds because it is true on the open affines $U = \text{Spec } A$ as above. Finally, if $Z, Z'$ are two subschemes satisfying the two points, then their ideal sheaves $\mathcal{I}, \mathcal{I}'$ define the same radical ideal $\mathcal{I}(U) = \mathcal{I}'(U) = I$ on $U = \text{Spec } A$, and so they are equal. \qed
Chapter 9

Vector bundles and locally free sheaves

The most important examples of quasi-coherent sheaves are the locally free sheaves.

Definition 9.1. A locally free sheaf $\mathcal{F}$ is an $\mathcal{O}_X$-module such that there exists an open cover $\{U_i\}$ and an index set $I$ such that $\mathcal{F}|_{U_i} \cong \bigoplus_I \mathcal{O}_{U_i}$ for each $i$. We call $U_i$ a trivialization of $\mathcal{F}$. If $I$ is finite, we say that $\mathcal{F}$ has finite rank, and that $r = |I|$ is called the rank of $\mathcal{F}$. A locally free sheaf of rank 1 is called an invertible sheaf.

If $\mathcal{F}$ is a locally free sheaf, the stalk $\mathcal{F}_x$ at a point $x \in X$ is a free $\mathcal{O}_{X,x}$-module, i.e., $\mathcal{F}_x \cong \mathcal{O}_{X,x}^r$ where $r$ is the rank of $\mathcal{F}$. In fact the converse is also true:

Lemma 9.2. Suppose that $X$ is locally noetherian. A coherent sheaf $\mathcal{F}$ having the property that $\mathcal{F}_x \cong \mathcal{O}_{X,x}^r$ for some fixed $r$ is locally free.

Proof. We can assume that $X = \text{Spec}(A)$, where $A$ is noetherian, and $F = \widetilde{M}$, where $M$ is a finitely generated $A$-module. Let $x_1, \ldots, x_n$ be generators for $M$ as an $A$ module. We have $\mathcal{F}_x = M_p$, for the prime ideal $p \subseteq A$ corresponding to $x \in X$. By assumption, $M_p \cong A_p^r$ is free, so let $m_1, \ldots, m_r$ be a basis of $M_p$ as an $A_p$-module. We can write in $M_p$

$$x_i = \sum c_{ij} m_j$$

Clearing denominators, we see that some multiple of $d_i x_i$ (with $d_i \in A - p$) is generated by the elements $m_i$ with coefficients in $A$. Let $s = d_1 \cdots d_r$, and
consider the open set $D(s) \subseteq X$. $s$ is invertible in $M_s$, so there is a surjective map $A^*_s \to M_s$. This is also injective, since any relation between the $m_i$ in $M_s$ must survive in $M_p$ (since $s \not\in p$). Hence $M_s \simeq A^*_s$. It then follows that $F|_{D(s)} \simeq \widetilde{M}_s \simeq \mathcal{O}_X|_{D(s)}$, is free on an open neighbourhood of $x$. 

The coherence condition in this lemma is essential. Indeed, if $X$ is a scheme, and $i_x : x \to X$ denotes the inclusion of a point, the sheaf 

$$
\mathcal{F} = \bigoplus_{x \in X} i_x \mathcal{O}_x
$$

is naturally an $\mathcal{O}_X$-module with $\mathcal{F}_x = \mathcal{O}_x$, but yet it is certainly not locally free.

**Example 9.3.** Here is an example of a locally free sheaf of rank 2. Consider the morphism $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $k[u] \to k[t]$ $u \mapsto t^2$ on $U = \text{Spec } k[t]$, and $k[u^{-1}] \to k[t^{-1}]$ $u^{-1} \mapsto t^{-2}$ on $V = \text{Spec } k[t^{-1}]$. Then $f_* \mathcal{O}_{\mathbb{P}^1}$ is a locally free sheaf. Indeed, on $U$, $f_* \mathcal{O}_{\mathbb{P}^1}|_{U}$ is the $\sim$ of $k[t]$ as a $k[u]$-module, which equals $k[u] \oplus k[u]t$. We get a similar expression on $V = \text{Spec } k[t^{-1}]$. It follows that $f_* \mathcal{O}_{\mathbb{P}^1}$ is locally free of rank 2.

However, the pushforward of a locally free sheaf is not locally free in general. For instance, if $i : Y \to X$ is a closed immersion, then $\mathcal{F} = i_* \mathcal{G} \text{ has } \mathcal{F}_y = \mathcal{G}_{Y,y}$ for $y \in Y$, but zero stalks outside of $Y$.

For pullbacks however, we have the following:

**Lemma 9.4.** Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^* \mathcal{G}$ is a locally free $\mathcal{O}_X$-module.

**Proof.** Let $U_i$ be trivialization of $\mathcal{F}$ on $Y$, such that $\mathcal{F}|_{U_i} \simeq \bigoplus_i \mathcal{O}_{U_i}$. Then, since $f^* \mathcal{O}_Y = \mathcal{O}_X$, we see that $f^{-1}(U_i)$ is a trivialization of $f^* \mathcal{F}$. 

**Example 9.5.** (The tangent bundle of the $n$-sphere) Let $X = \text{Spec } A$ where $A = \mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_n^2 - 1)$, and consider the $A$-module homomorphism $f : A^{n+1} \to A$ given by $f(e_i) = x_i$. Then $M = \ker f$ gives rise to a quasi-coherent sheaf $\mathcal{F} = \mathcal{M}$. Any element in the kernel corresponds to a vector of elements $a = (a_0, \ldots, a_n) \in A^{n+1}$ so that

$$a_0x_0 + \cdots + a_nx_n = 0$$

On $U = D(x_0)$ we may divide by $x_0$, and solve for $a_0$, so $a$ is uniquely determined by the elements $(a_1, \ldots, a_n)$. Conversely, given any such an $n$-tuple of elements in $A$, we may define an element $a \in M_{x_0}$ using the above relation. In particular, $M_{x_0} \simeq A^n$. A similar argument works for the other $x_i$, showing that $\mathcal{F}$ is locally free.

It is a non-trivial theorem that $\mathcal{F} \neq \mathcal{O}_X^n$ if $n \not\in \{0, 1, 3, 7\}$. 

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Chapter 9. Vector bundles and locally free sheaves

9.1 Locally free sheaves and projective modules (work in progress)

On an affine scheme \( X = \text{Spec} \ A \), any quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \) is isomorphic to \( \sim M \) for an \( A \)-module \( M \). In this section we investigate which \( A \)-modules give rise to locally free sheaves. The main result is that \( \mathcal{F} \) is locally free if and only if \( M \) is finitely generated and projective.

Defn. projective module
Defn. Locally free module

**Lemma 9.6.** Let \((A, m)\) be a local ring. If \( M \) is a finitely generated projective \( A \)-module, then \( M \) is free.

Let \( M \) be a projective \( A \)-module, so there is a \( Q \) so that \( M \oplus Q = F \) is finitely generated, and free. Taking localizations, we then see that \( M_p \) is a projective \( A_p \)-module for every prime \( p \). Hence \( M_p \) is free for every \( p \).

**Lemma 9.7.** Let \( M \) be a finitely generated \( A \)-module. then \( M \) is projective if and only if \( M \) is finitely presented and locally free.

**Theorem 9.8.** Let \( M \) be a finitely generated \( A \)-module, and assume \( A \) is a Noetherian ring. Then \( \mathcal{F} = \sim M \) is locally if and only if \( M \) is projective.

**Corollary 9.9.** Let \( X \) be a noetherian scheme and let \( \mathcal{F} \) be a coherent sheaf. Then the following are equivalent:

(i) \( \mathcal{F} \) is locally free

(ii) \( \mathcal{F}_x \) is a free \( \mathcal{O}_{X,x} \)-module for every \( x \in X \).

(iii) For every open affine \( U = \text{Spec} \ A \subseteq X \), we have \( \mathcal{F}|_U \simeq \sim M \), for some finitely generated projective \( A \)-module \( M \).

**Proof.** Suppose \( \mathcal{F}_x \) is free for every \( x \in X \). Fix \( x \in X \) and let \( U = \text{Spec} \ A \) be an open subset containing \( x \). We have \( \mathcal{F}|_U \simeq \sim M \) for some finitely generated \( A \)-module \( M \). Then \( M_q \) is free for every \( q \in \text{Spec} \ A \), and hence \( M \) is projective. Hence \( \sim M \) is locally free, and hence so is \( \mathcal{F} \), since \( x \) was arbitrary.

9.2 Invertible sheaves and the Picard group

An *invertible sheaf* on a scheme \( X \) is a locally free sheaf of rank 1. We usually write \( L \) for such sheaves (for reasons that will become clear later). This means that there exists a covering \( \mathcal{U} = \{ U_i \} \) and isomorphisms \( \phi_i : \mathcal{O}_{U_i} \rightarrow L|_{U_i} \). We say
that \( g_i = (\phi_i)_{U_i}(1) \in L(U_i) \) is a local generator for \( L \) and that \( U_i \) is a trivialization of \( L \). By Lemma 9.2, a coherent \( \mathcal{O}_X \)-module \( L \) is invertible if and only if \( L_x \) is isomorphic to \( \mathcal{O}_{X,x} \) for every \( x \in X \).

**Proposition 9.10.** For \( L, M \) invertible sheaves, we have

(i) \( L \otimes M \) is also an invertible sheaf. If \( g, h \) are local generators for \( L \) and \( M \) respectively, then \( g \otimes h \) a local generator for \( L \otimes_{\mathcal{O}_X} M \).

(ii) \( \mathcal{H}\text{om}(L, \mathcal{O}_X) \) is invertible and \( \mathcal{H}\text{om}(L, \mathcal{O}_X) \otimes L \cong L \). If \( g \) is a local generator for \( L \), then \( \psi_g \) defined by \( \psi_g(a g) = a \) is a local generator for \( \mathcal{H}\text{om}(L, \mathcal{O}_X) \).

(iii) \( \mathcal{H}\text{om}(L, M) \cong \mathcal{H}\text{om}(L, \mathcal{O}_X) \otimes M \)

**Proof.** (i) We can find a common trivialization of \( L \) and \( M \), such that \( X \) is covered by open sets \( U \) where we have isomorphisms \( \phi : \mathcal{O}_U \rightarrow L|_U \) and \( \psi : \mathcal{O}_U \rightarrow M|_U \). Over such a \( U \), we have an isomorphism \( \mathcal{O}_U \cong \mathcal{O}_U \otimes \mathcal{O}_Y \cong L|_U \otimes M|_U \) given by \( 1 \mapsto 1 \otimes 1 \mapsto \phi(1) \otimes \psi(1) \) (all tensor products in this section are over \( \mathcal{O}_X \)). This shows (i).

For (ii), as above the fact that \( \mathcal{H}\text{om}(L, \mathcal{O}_X) \) is invertible can be seen by restricting to an open where \( L|_U \cong \mathcal{O}_U \). The identity for the tensor product follows from (iii) below. Finally, the fact that \( \psi_g \) is a local generator for \( \mathcal{H}\text{om}(L, \mathcal{O}_X) \) follows from the isomorphisms

\[ \mathcal{O}_X|_U \cong \mathcal{H}\text{om}(\mathcal{O}_X, \mathcal{O}_X)|_U \cong \mathcal{H}\text{om}(L|_U, \mathcal{O}_X) \]

given by \( 1 \mapsto \text{id}_{\mathcal{O}_X} \) and \( \alpha \mapsto \alpha \circ \psi^{-1} \). This means that 1 maps to \( \psi^{-1} = \psi_g \).

(iii) By the universal property of the tensor product, we have a canonical isomorphism

\[ \text{Hom}_{\mathcal{O}_X(U)}(L(U), M(U))) \cong \text{Hom}_{\mathcal{O}_X(U)}(\text{Hom}_{\mathcal{O}_X(U)}(L(U), \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} M(U)) \]

This shows (iii) on the level of presheaves. Sheafify to get the result.

This proposition explains the term ‘invertible’. Indeed, the tensor product acts as a sort of binary operation on the set of invertible sheaves; \( L \otimes M \) is invertible if \( L \) and \( M \) are. Tensoring an invertible sheaf by \( \mathcal{O}_X \) does nothing, so \( \mathcal{O}_X \) serves as the identity. Moreover, for an invertible sheaf \( L \) we will define \( L^{-1} = \mathcal{H}\text{om}(L, \mathcal{O}_X) \); by the proposition, \( L^{-1} \) is again invertible, and serves as a multiplicative inverse of \( L \) under \( \otimes \). We can make the following definition:

**Definition 9.11.** Let \( X \) be a scheme. The Picard group \( \text{Pic}(X) \) is the group of invertible sheaves \( L \) modulo isomorphism.
Notice it is the set of isomorphism classes of line bundles that form a group – not the line bundles themselves ($L \otimes L^{-1}$ is isomorphic but strictly speaking not equal to $\mathcal{O}_X$).

Note also that Pic($X$) is also an abelian group, because $L \otimes M$ is canonically isomorphic to $M \otimes L$.

Example 9.12 (Invertible sheaves on the affine line). If $\mathcal{F}$ is a coherent sheaf on $\mathbb{A}^1_k$, then $\mathcal{F} = \tilde{M}$ for some finitely generated $k[t]$-module. The structure theorem of finitely generated modules over a PID, tells us that $M \simeq k[t]^r \oplus T$ where $r \geq 0$ and $T$ is a torsion module (which is in turn a direct sum of modules of the form $k[t]/(t - a)^n$). From this we find

Proposition 9.13. Any invertible sheaf over $\mathbb{A}^1_k$ is trivial; Pic($\mathbb{A}^1_k$) = 0.

More generally, in the correspondence between quasi-coherent sheaves on $X = \text{Spec} A$ and $A$-modules $M$, one can show that an $\mathcal{O}_X$-module is locally free if and only if it is associated to a projective module $M$, that is, there exists $N$ such that $M \oplus N$ is free.

9.2.1 Operations on locally free sheaves

Most constructions for vector spaces and modules globalize and carry over to locally free sheaves. For instance, for a locally free sheaf $\mathcal{F}$ we can define the tensor algebra $T(\mathcal{F})$ as the graded algebra $\bigoplus_{n \geq 0} T^n(\mathcal{F})$ where $T^n(\mathcal{F}) = \mathcal{F} \otimes \mathcal{F}$, and we let $\text{Sym}(\mathcal{F}) = T(\mathcal{F})/I$ where $I$ is the ideal generated by expressions $m \otimes m' - m' \otimes m$ where $m, m' \in \mathcal{F}$. This inherits a $\mathbb{Z}$-grading from $T(\mathcal{F})$, and we let $S^n\mathcal{F}$ denote the $n$-th symmetric power of $\mathcal{F}$.

Similarly, we define $\bigwedge \mathcal{F}$ as the $\mathcal{O}_X$-module $T(\mathcal{F})/J$ where $J$ is the ideal generated by all products $m \otimes m$ where $m \in \mathcal{F}$. This is sometimes called the exterior algebra. This is again graded, and a graded piece $\bigwedge^n \mathcal{F}$ is called the $n$-th wedge product. When $\mathcal{F}$ has rank $r$, $\bigwedge^{r+1} \mathcal{F} = 0$ and $\bigwedge^r \mathcal{F}$ is called the determinant of $\mathcal{F}$ - it is a locally free sheaf of rank 1, i.e. an invertible sheaf.

Proposition 9.14 (Facts about locally free sheaves). (i) Locally free sheaves are closed under direct sums, tensor products, symmetric products, exterior products, duals, and pullbacks.

If $\mathcal{F}$ is locally free of rank $r$ then

- $T^n(\mathcal{F})$ is locally free of rank $r^n$ (spanned by all elements $m_1 \otimes \cdots \otimes m_n$ where $m_i \in \mathcal{F}$)

- $S^n(\mathcal{F})$ is locally free of rank $\binom{n+r-1}{r-1}$ (spanned by all elements $m_1^{n_1} \cdots \otimes m_r$ where $m_i \in \mathcal{F}$ and $\sim n_i = n$)

If $\mathcal{F}$ is locally free of rank $r$ then

- $T^n(\mathcal{F})$ is locally free of rank $r^n$ (spanned by all elements $m_1 \otimes \cdots \otimes m_n$ where $m_i \in \mathcal{F}$)

- $S^n(\mathcal{F})$ is locally free of rank $\binom{n+r-1}{r-1}$ (spanned by all elements $m_1^{n_1} \cdots \otimes m_r$ where $m_i \in \mathcal{F}$ and $\sim n_i = n$)
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- $\bigwedge^n F$ is locally free of rank $\binom{n}{r}$ (spanned by elements $m_{i_1} \wedge \cdots \wedge m_{i_n}$ where $i_1 < i_2 < \cdots < i_n$).

(ii) Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of locally free sheaves of ranks $n', n, n''$ respectively. Then

$$\bigwedge^n F \simeq \bigwedge^n F' \otimes_{\mathcal{O}_X} \bigwedge^n F''$$

9.3 Vector bundles

Locally free sheaves are in many respects the most basic sheaves on a scheme $X$; they are obtained by locally gluing together copies of the structure sheaf $\mathcal{O}_X$ in various ways. What makes these sheaves particularly interesting is the link to the theory of vector bundles.

A vector bundle is essentially a family of vector spaces parameterized over a base scheme $X$. This means that we have a morphism $\pi: E \to X$, such that the scheme theoretic fibers $\pi^{-1}(x)$ are isomorphic to an affine space $\mathbb{A}^r$. The prototype example of a vector bundle is obtained by simply taking the product $E = X \times \mathbb{A}^r$ and letting $\pi$ be the first projection. This is the so called trivial bundle on $X$. A vector bundle is more generally a scheme together with a morphism $\pi: E \to X$ which is locally isomorphic to the trivial bundle:

**Definition 9.15.** Let $X$ be a scheme. A vector bundle $E$ of rank $r$ on $X$ is a scheme with a morphism $\pi: E \to X$ and an open cover $\mathcal{U} = \{U_i\}$ of $X$ such that for each $i$, there is an isomorphism $\phi_i: \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r$ making the following diagram commutative:

$$
\begin{array}{ccc}
\pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{A}^r \\
\downarrow{\pi}_{\pi^{-1}(U_i)} & & \downarrow{pr_1} \\
U_i & & \\
\end{array}
$$

such that for each affine $V = \text{Spec } A \subseteq U_{ij}$ the automorphism $\phi_j \circ \phi_i^{-1}$ of $V \times \mathbb{A}^r$ is linear; i.e., induced by an automorphism $\theta : A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$ such that $\theta(a) = a, \forall a \in A$, and $\theta(x_i) = \sum a_{ij}x_j$ for $a_{ij} \in A$. 

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So a vector bundle is obtained by gluing together trivial bundles $U_i \times \mathbb{A}^r$ via linear gluing maps. A convenient way of rephrasing this is in terms of transition functions: Given a vector bundle $\pi : \mathcal{E} \to X$ defined by the data $(U_i, \phi_i)$ we have for each $i, j$ an element $g_{ij} \in GL_r(\Gamma(U_{ij}, \mathcal{O}_X))$ such that the diagram

$$
\begin{array}{ccc}
\pi^{-1}(U_{ij}) & \simeq & U_i \times \mathbb{A}^r \\
\downarrow \pi & & \downarrow \pi \\
U_{ij} \times \mathbb{A}^r & & U_{ij} \times \mathbb{A}^r
\end{array}
$$

commutes.

**Definition 9.16.** The elements $g_{ij}$ are the *transition functions* of $\mathcal{E}$.

The gluing axioms for $\mathcal{E}$ shows that the transition functions satisfy the following compatibility conditions (or ‘cocycle conditions’)

$$
g_{ik} = g_{ij} \circ g_{jk} \text{ on } U_{ijk} \quad \text{and} \quad g_{ij} = g_{ji}^{-1} \text{ on } U_{ij} \tag{9.3.1}
$$

**Proposition 9.17.** Let $X$ be a scheme with an open cover $\{U_i\}$ and assume that $g_{ij}$ are a collection of elements of $GL_r(\Gamma(U_{ij}, \mathcal{O}_X))$ satisfying the compatibility conditions (9.3.1). Then there is a vector bundle $\pi : \mathcal{E} \to X$, unique up to isomorphism, whose transition functions are the $g_{ij}$.

**Proof.** The compatibility conditions ensure that the maps $(id \times g_{ij})$ glue the affine schemes $U_i \times \mathbb{A}^r$ along $(U_i \cap U_j) \times \mathbb{A}^r$ to a scheme $\mathcal{E}$. Moreover, the projection maps $U_i \times \mathbb{A}^r \to U_i$ glue to give a morphism $\pi : \mathcal{E} \to X$. By construction, the open set of $\mathcal{E}$ corresponding to $U_i \times \mathbb{A}^r$ is identified with $\pi^{-1}(U_i)$, which gives an isomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r$. Hence $\mathcal{E}$ is a vector bundle with transition functions $g_{ij}$. \qed
Example 9.18 (Facts about vector bundles). Let \( \pi : \mathcal{E} \to X \) be a vector bundle or rank \( r \) associated with the transition functions \( g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^r) \) and let \( \mathcal{O} \) denote the trivial rank 1 bundle.

(i) There is a vector bundle \( \mathcal{E}^* \), the *dual bundle* of \( \mathcal{E} \), with corresponding transition functions \( g_{ij}^{-1} = g_{ji} \). \( \mathcal{E}^* = \text{Hom}_{\text{vb}}(\mathcal{E}, \mathcal{O}) \) is also a vector bundle.

(ii) If \( \mathcal{E}, \mathcal{E}' \) are vector bundles with transition functions \( g_{ij}, h_{ij} \) respectively, one can define the *tensor product bundle* \( \mathcal{E} \otimes \mathcal{E}' \) as the vector bundle corresponding to the transition functions \( g_{ij} \otimes h_{ij} \). If \( \mathcal{E} \) and \( \mathcal{E}' \) have ranks \( r, r' \), then \( \mathcal{E} \otimes \mathcal{E}' \) has rank \( rr' \). As a special case of this, we have the *tensor power* \( \mathcal{E} \otimes^n \) which is a vector bundle of rank \( r^n \), with corresponding transition functions \( g_{ij} \otimes g_{ij} \otimes \cdots \otimes g_{ij} \).

9.3.1 The sheaf of sections of a vector bundle

A section of a vector bundle \( \pi : \mathcal{E} \to X \) over \( U \subseteq X \) is a morphism \( s : U \to \mathcal{E} \) such that \( \pi \circ s = \text{id}_U \). So \( s \) picks out a single vector in each fiber \( \pi^{-1}(x) \) for \( x \in X \) a closed point.

Proposition 9.19. Let \( \pi : \mathcal{E} \to X \) be a vector bundle given by the data \((U_i, \phi_i, g_{ij})\). A section \( s : X \to \mathcal{E} \) is determined uniquely by a collection of \( r \)-tuples \( s_i \in \mathcal{O}_X^r \) such that for each \( i, j \)

\[
s_i|_{U_{ij}} = g_{ij}s_j|_{U_{ij}}.
\]

Proof. Given \( s : X \to \mathcal{E}, \phi_i \circ s|_{U_i} \) is a section of \( U_i \times \mathbb{A}^r \to U_i \). Hence \( \phi_i \circ s|_{U_i} = (\text{id}_{U_i} \times s_i) \) for \( s_i \in \mathcal{O}(U_i)^r \). By construction, these satisfy the given compatibility conditions \( s_i|_{U_{ij}} = g_{ij}s_j|_{U_{ij}} \). Conversely, any such section \( s : X \to \mathcal{E} \) defines a set of such sections \( s_i = s|_{U_i} \) satisfying this condition. \( \square \)

Given \( \mathcal{E} \), we can define a sheaf \( \mathcal{O}_X(\mathcal{E}) \) by defining

\[
\mathcal{O}_X(\mathcal{E})(U) = \{ \text{sections } s : U \to \mathcal{E} \}
\]

The above proposition shows that this set is naturally a group: If \( s : U \to \mathcal{E}, \ t : U \to \mathcal{E} \) are two sections given by the data \( s_i \) and \( t_i \) respectively, we can define \( s + t \), by \( s_i + t_i \in \mathcal{O}_X^r \). In fact, by multiplying \( s_i \) with elements of \( \mathcal{O}_X(U) \), we see that \( \mathcal{O}_X(\mathcal{E})(U) \) has the structure of an \( \mathcal{O}_X(U) \)-module. This shows that \( \mathcal{O}_X(\mathcal{E}) \) is a sheaf of \( \mathcal{O}_X \)-modules, locally free of rank \( r \).

9.3.2 Equivalence between vector bundles and locally free sheaves

A vector bundle \( \pi : \mathcal{E} \to X \) gives rise to a locally free sheaf \( \mathcal{F} \) of rank \( r \). In fact, there is a way to reverse this process; that is, to a locally free sheaf \( \mathcal{F} \) one can
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associate a scheme, denoted $\mathcal{V}(\mathcal{F})$ with a morphism $\pi : \mathcal{V}(\mathcal{F}) \to X$ making $\mathcal{V}(\mathcal{F})$ into a vector bundle under which $\mathcal{F}$ is the sheaf of sections. In particular, there is an equivalence between vector bundles and locally free sheaves of finite rank.

The construction goes as follows. If $\mathcal{F}$ is locally free of rank $r$. Choose an open cover $\{U_i\}$ of $X$ such that there is an isomorphism $f_i : \mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}$ for each $i$. If we take two $f_i : \mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}$ and $f_j : \mathcal{F}|_{U_j} \to \mathcal{O}_{U_i}$ and restrict to $U_{ij} = U_i \cap U_j$, we get two different isomorphisms $g_i, g_j : \mathcal{F}|_{U_{ij}} \to \mathcal{O}_{U_{ij}}$. Let $g_{ij} = g_j \circ g_i^{-1}$, which gives an automorphism of $\mathcal{O}_{U_{ij}}$. We can identity this with an $r \times r$ matrix of regular functions on $U_{ij}$.

Now we glue $U_i \times \mathbb{A}^r$ and $U_j \times \mathbb{A}^r$ along $U_{ij} \times \mathbb{A}^r$ by the map sending $(x,v) \in U_{ij} \times \mathbb{A}^r$ to $(x,g_{ij}(v))$. The resulting scheme which we denote by $\mathcal{V}(\mathcal{F})$ comes with a morphism $\pi : \mathcal{V}(\mathcal{F}) \to X$ (obtained by gluing all the first projections $U_i \times \mathbb{A}^r \to U_i$), whose fibers are linear spaces. Over each $U_i \subseteq X$ the maps $g_i$ also give isomorphisms from $\pi^{-1}(U_i)$ to $U_i \times \mathbb{A}^r$. The maps $g_j \circ g_i^{-1}$ are all linear - this follows from the fact that $g_j \circ g_i^{-1}$ is a morphism of $\mathcal{O}_X$-modules. Finally, it is a matter of checking that $\mathcal{F}$ coincides with the sheaf of sections of $\pi$.

In particular, there is a correspondence between vector bundles and locally free sheaves of finite rank. Under this correspondence the trivial bundle $X \times \mathbb{A}^1$ corresponds to the structure sheaf $\mathcal{O}_X$.

**Remark 9.20.** There is also an alternate construction of $\mathcal{V}(\mathcal{F})$ using the relative spectrum of the symmetric algebra of a sheaf. We direct the interested reader to Exc. II.5.16 and II.5.18 in Hartshorne’s book for more details.

### 9.4 Extended example: The tautological bundle on $\mathbb{P}^n_k$

Let $k$ be a field, and let $\mathbb{P}^n_k$ denote the projective $n$-space over $k$. The closed points of $\mathbb{P}^n_k$ parameterizes lines through the origin in $\mathbb{A}^{n+1}_k$. We define the *tautological sheaf*, denoted by $\mathcal{O}(-1)$ as follows. Let $L \subseteq \mathbb{P}^n \times_k \mathbb{A}^{n+1}$ denote the variety of pairs $([l], v)$, where $l \subseteq \mathbb{A}^{n+1}$ is a line, and $v \in l$. In terms of equations $L$ is defined by the $2 \times 2$ minors of the matrix

$$
\begin{pmatrix}
x_0 & x_1 & \cdots & x_n \\
y_0 & y_1 & \cdots & y_n
\end{pmatrix}
$$

where $x_0, \ldots, x_n$ are homogeneous coordinates on $\mathbb{P}^1$ and $y_0, \ldots, y_n$ are affine coordinates on $\mathbb{A}^{n+1}$.

Let $\pi : L \to \mathbb{P}^n_k$ be the projection to the first factor. Then for a closed point $[a] = (a_0 : \ldots, a_n) \in \mathbb{P}^n_k$, the preimage $\pi^{-1}([l])$ is exactly the line $l \subseteq \mathbb{A}^{n+1}$.
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Corresponding to $[a]$. We claim that $L$ is a rank 1 vector bundle on $\mathbb{P}^n$.

On $U_i = D(x_i)$ the equations for $L$ become

$$\frac{x_j}{x_i} y_i = y_j$$

for each $j \neq i$ and hence $(x_0 : \ldots, x_n, y_0, \ldots, y_n) \mapsto (x_0 : \ldots, x_n, y_i)$ defines an isomorphism

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^1$$

In other words, $y_i$ is a local coordinate on the $\mathbb{A}^1$ on the right hand side. On the intersection $U_i \cap U_j$, $y_i$ is related to $y_j$ via

$$\frac{x_i}{x_j} y_j = y_i$$

This means that the transition function

$$U_{ij} \times \mathbb{A}^1 \to U_{ij} \times \mathbb{A}^1$$

is given by $\text{id} U_{ij} \times g_{ij}$ where $g_{ij} = \frac{x_i}{x_j}$. Hence $\pi$ is a vector bundle. We denote the corresponding sheaf of sections by $\mathcal{O}(-1)$.

What are the sections of $\mathcal{O}(-1)$? Note that a section $s : \mathbb{P}^n \to L$ is given by a set of $n + 1$ sections $s_i$, such that for each $i, j$ we have

$$s_i|_{U_{ij}} = g_{ij} s_j|_{U_{ij}} = \frac{x_i}{x_j} s_j|_{U_{ij}} \quad (9.4.1)$$

The functions $s_i : D(x_i) \to L$ can be represented by polynomials in $\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}$. Note that on the right hand side of (9.4.2), the Laurent polynomial $\frac{x_i}{x_j} s_j$ has only $x_j$ in the denominator, whereas $s_i$, on the left, has only $x_i$'s. It follows that the only way this equation in polynomials can hold is that both sides are identically 0. In particular,

**Proposition 9.21.** $\Gamma(\mathbb{P}^n \mathcal{O}(\mathcal{O}(-1)) = 0$.

In particular, the sheaf $\mathcal{O}(-1)$ is invertible, but not isomorphic to $\mathcal{O}_{\mathbb{P}^n}$.

Consider now the vector bundle $\sigma : L^* \to X$ with transition functions

$$g_{ij} = \frac{x_j}{x_i}$$

Note that these transition functions are almost as before, only that we have inverted the $g_{ij}$. The corresponding vector bundle is the dual bundle of $\pi : L \to X$; the fiber $\sigma^{-1}(x)$ is identified with the dual vector space $(\pi^{-1}(x))^*$.

In this case we find that the bundle does in fact have global sections: A
section \( s : \mathbb{P}^n \to \mathbb{L}^* \) is given by a set of sections \( s_i : U_i \to W \) such that

\[
s_i|_{U_{ij}} = g_{ij}s_j|_{U_{ij}} = \frac{x_j}{x_i}s_j|_{U_{ij}} \quad (9.4.2)
\]

As before, the left-hand side is a polynomial in \( x_0, \ldots, x_n \) - in particular it is a Laurent polynomial with \( x_i \) in the denominators. This is ok with respect to the right-hand side, as long as \( s_i \) has degree 1, that is it has a pole of order at most one at \( x_i = 0 \). Conversely, you can start with any section \( s_0 \) which is linear in \( \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \), and use (9.4.2) to define \( s_1, \ldots, s_n \) as Laurent polynomials. These glue together to a global section \( s : \mathbb{P}^n \to \mathbb{L}^* \).

**Proposition 9.22.** The space of global sections of \( \mathbb{L}^* \) can be identified with the vector space of linear forms in \( n + 1 \) variables.
Chapter 10

Sheaves on projective schemes

Projective schemes are to affine schemes what projective varieties are to affine varieties. The construction of the projective spectrum $\text{Proj } R$ was similar to that of $\text{Spec}$: the underlying topological space is defined via prime ideals and the structure sheaf from localizations of $R$. However, there were some fundamental differences between the two: We only consider graded rings $R$, and we only consider homogeneous prime ideals that do not contain the irrelevant ideal $R_+$. As we saw, this reflects the construction of $\text{Proj } R$ as a quotient space

$$\pi : \text{Spec } R - V(R_+) \to \text{Proj } R$$

Given this, we can pull back a quasi-coherent sheaf to $\text{Spec } R - V(R_+)$ and extend it to a sheaf on $\text{Spec } R$ via the inclusion map. Thus, it is natural to expect that quasi-coherent sheaves on $\text{Proj } R$ should be in correspondence with equivariant modules on $\text{Spec } R$, i.e., with graded $R$-modules. The irrelevant subscheme $V(R_+)$ complicates the picture a little bit, so the classification is a little bit more involved than the one for affines schemes. In particular, we will see that different graded $R$-modules can correspond to the same quasi-coherent sheaf on $\text{Proj } R$.

Another important feature of $\text{Proj } R$ is that it comes equipped with a canonical invertible sheaf that we will denote by $\mathcal{O}(1)$. This is the geometric manifestation of the fact that $R$ is graded. Unlike the case of affine schemes, $\text{Proj } S$ can typically not be recovered from the global sections of the structure sheaf. It is the sheaf $\mathcal{O}(1)$, or rather, the various tensor powers $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$, that will play the role of the affine coordinate ring in the affine case.
10.1 The graded \sim-functor

Let $R$ be a graded ring and let $\text{GrMod}_R$ denote the category of graded $R$-modules. Just like in the case of $\text{Spec } A$ we will define a tilde-construction to construct sheaves on $\text{Proj } R$ from graded $S$-modules, giving a functor $\text{GrMod}_R$ to $\text{Mod}_{\mathcal{O}_X}$. However, as we will see, this is not an equivalence of categories.

Recall that for $f, g \in R_+$, such that $D_+(g) \subseteq D_+(f)$, there is a canonical localization homomorphism $\rho_{f,g}: M(f) \to M(g)$ where as before $M(f)$ denotes the degree 0 part of the localization $\{1, f, f^2, \ldots \}^{-1}M$. It follows that we can define a $\mathcal{B}$-presheaf $\tilde{M}$ by defining for each $D_+(f)$,

$$\tilde{M}(D_+(f)) = M(f).$$

Note that $\tilde{M}|_{D(f)} \simeq (\tilde{M}(f))$ via the isomorphism of $D_+(f)$ with $\text{Spec } R(f)$. It follows that this in fact gives a $\mathcal{B}$-sheaf, and hence a sheaf on $\text{Proj } R$. Moreover, as in the Spec case, the assignment $M \mapsto \tilde{M}$ is functorial. The following proposition summarizes the properties of $\tilde{M}$:

**Proposition 10.1.** The contravariant functor $M \mapsto \tilde{M}$ has the following properties:

- $\sim$ is exact, commutes with direct sums and limits.
- The stalks satisfy $\tilde{M}_p = M(p)$ for each $p \in \text{Proj } R$.
- If $R$ is noetherian, and $M$ is finitely generated, then $\tilde{M}$ is coherent.

Proving these properties is straightforward, since most of them can be checked locally on stalks. Using the isomorphisms between $D_+(f)$ and $\text{Spec } R(f)$ we reduce immediately to the affine case.

However, it is important to note that, unlike the affine case, the functor is not faithful, as several modules can correspond to the same module. For instance, take any graded $R$-module $M$ such that $M_d = 0$ for all large $d$. Then $M(f) = 0$ for all $f \in R_+$, and so $\tilde{M} = 0$, even though $M$ is not the 0-module. We will however see shortly that it is only the modules of this form that cause the lack of faithfulness.

The following is useful for working with the localization of $M$. It says essentially that we are allowed to ‘substitute in 1’ when restricting a module to an affine chart $D_+(f) \subseteq \text{Proj } R$.

**Lemma 10.2.** Suppose that $M$ is a graded $R$-module and $f \in R$ homogeneous of degree 1. Then

$$M(f) \simeq M/(f-1)M \simeq M \otimes_R R/(f-1)$$
Proof. Define a map $M \to M_{(f)}$ by $m \mapsto m/f^{\deg m}$. The submodule $(f-1)M$ is annihilated by this map, so we get an induced map $\phi : M/(f-1)M \to M_{(f)}$. We can construct an inverse to this map as follows. Let $m/f^n \in M_{(f)}$ for $m \in M_d$. Map this to the class of $m$ in $M/(f-1)M$. This is well-defined: If $m/f^n = m'/f^n'$, then there is an $N > 0$ with $f^N(mf^n - mf^n) = 0$. so since $f$ gets reduced to 1 modulo $(f-1)$, we find that they have the same images in $M/(f-1)M$.

Let us use this to compare $\tilde{M} \otimes_R N$ with $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$. Let $f \in R$ be a homogeneous element. We have a map $M_{(f)} \times N_{(f)} \to (M \otimes_R N)_{(f)}$ sending $m/f^a \times n/f^b$ to $(m \otimes n)/f^{a+b}$. As this is $R_{(f)}$-bilinear, we get an induced map $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \to (M \otimes_R N)_{(f)}$. Since a map of $\mathcal{B}$-sheaves induces a map of sheaves, we get a natural map

$$\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to \tilde{M} \otimes_R N.$$  \hspace{1cm} (10.1.1)

**Proposition 10.3.** Suppose $R$ is generated in degree 1. Then the natural map (10.1.1) is an isomorphism.

**Proof.** By assumption, $X = \text{Proj } R$ is covered by open affines of the form $D_+(f)$ where $f$ has degree 1. For such an $f$, the functor $M \to M_{(f)}$ is the same as tensoring with $R/(f-1) \simeq R_{(f)}$ by the previous lemma. Furthermore,

$$(M \otimes_R R_{(f)}) \otimes_{R_{(f)}} (N \otimes_R R_{(f)}) \simeq (M \otimes_R N) \otimes_R R_{(f)}$$

This isomorphism provides the inverse to the natural map $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \to (M \otimes_R N)_{(f)}$ defined above. Then, since the map (10.1.1) restricts to an isomorphism on all $D_+(f)$ for $f \in R_1$, it is an isomorphism. 

**10.2 Serre’s twisting sheaf $\mathcal{O}(1)$**

Arguably the most interesting sheaf on $X = \text{Proj } R$ is the so-called *Serre’s twisting sheaf*, denoted by $\mathcal{O}_X(1)$. This is a generalization of the tautological sheaf on $\mathbb{P}^n_k$, and constitutes a geometric manifestation of the fact that $R$ is a graded ring.

For an integer $n$, we will define an $R$-module $R(n)$ as follows: As an underlying module $R(n)$ is just $R$, but with the grading shifted by $n$:

$$R(n)_d = R_{d+n}$$

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This is naturally a graded $R$-module and hence gives rise to a quasi-coherent $\mathcal{O}_X$-module on $X$.

**Definition 10.4.** For an integer $n$, we define

$$\mathcal{O}_X(n) = \widetilde{R(n)}.$$ 

For a sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ on $X$, we define the *twist by $n$* by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

For an element $r \in R_d$, there is a corresponding section in $\Gamma(X, \mathcal{O}_X(d))$. This is because we can think of an element of $\Gamma(X, \mathcal{O}_X(d))$ as a collection of elements $(r_f, D_+(f))$ with $r_f \in (R_f)_d$ matching on the overlaps $D_+(f) \cap D_+(g)$. Hence we can define an $R_0$-module homomorphism

$$R_d \to \Gamma(X, \mathcal{O}_X(d))$$

by $r \mapsto (r/1, D_+(f))$. (On the overlaps $D_+(fg)$ it is clear that the two elements $(r/1, D_+(f))$ and $(r/1, D_+(g))$ become equal, so this defines an actual global section of $\mathcal{O}(d)$.) Abusing notation, we will also denote the section by $r$.

Note that if $f \in R_1$, then $R(n)(f) = f^n R(f)$. Thus, on the affine $D_+(f)$, we have $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_X|_{D_+(f)}$. In particular, $\mathcal{O}_X(n)|_{D_+(f)} \simeq \mathcal{O}_{D_+(f)}$. In other words $\mathcal{O}_X(n)$ is a locally free sheaf of rank 1, that is, an invertible sheaf. By the previous chapter, we know that this gives rise to a line bundle $\mathbb{L} \to \text{Proj } R$.

**Proposition 10.5.** If $R$ is generated in degree 1, then $\mathcal{O}_X(n)$ is an invertible sheaf for every $n$. Moreover, there is a canonical isomorphism

$$\mathcal{O}_X(m + n) \simeq \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

Indeed, if $R$ is generated in degree 1, Proposition 10.3 shows that $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n)$ is the sheaf associated to $R(m) \otimes_R R(n) \simeq R(n + m)$, i.e., $\mathcal{O}_X(n + m)$.

So this is a big difference between affine schemes and projective schemes: Proj $R$ comes equipped with lots of invertible sheaves.

**Example 10.6.** Consider again the example of projective space $\mathbb{P}^n_k = \text{Proj } R$ where $R = k[x_0, \ldots, x_n]$. We use the covering $U_i = D(x_i) \simeq \text{Spec } (k[x_0, \ldots, x_n]_{x_i}) = \text{Spec } (k[x_0, \ldots, x_n]_{x_i})$. Then $R(l)_{x_i} = (k[x_0, \ldots, x_n]_{x_i})_l = x_i^l k[x_0, \ldots, x_n]_{x_i}$, and so

$$\Gamma(U_i, \mathcal{O}_i(l)) = x_i^l k[x_0, \ldots, x_n]_{x_i}.$$ 

On the overlaps, $(R(l)_{x_i})_{x_i}^{x_j} = R(l)_{x_i x_j} = (R(l)_{x_i})_{x_j}$, we find that two regular sections

$$x_i \cdot s_i \left( \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right)$$

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and

\[ x_j^l s_j \left( \frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j} \right) \]

restrict to the same section of \( \mathcal{O} (l) |_{U_i \cap U_j} \) if and only if

\[ s_j \left( \frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j} \right) = \left( \frac{x_i}{x_j} \right)^l s_i \left( \frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right) \]

You should compare this to the descriptions of the transition functions of the tautological bundle in Chapter 7. In particular, we see that the invertible sheaf \( \mathcal{O}(1) \) coincides with the tautological sheaf defined there.

10.3 Extending sections of invertible sheaves

Let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module and let \( x \in X \) be a point. We define the fiber of \( \mathcal{F} \) at \( x \) to be the \( k(x) \)-vector space

\[ \mathcal{F}(x) : \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x) \]

If \( U \) is an open set containing \( x \) and \( s \in \Gamma(U, \mathcal{F}) \), we denote by \( s(x) \) the image of the germ \( s_x \in \mathcal{F}_x \) in \( \mathcal{F}(x) \).

**Definition 10.7.** Let \( L \) be an invertible sheaf and \( f \in \Gamma(X, L) \). We define the open set \( X_s \) as

\[ X_s = \{ x \in X | s(x) \neq 0 \} \]

Equivalently, \( X_f \) is the set of points where \( f \notin \mathfrak{m}_x L_x \).

\( X_s \) is indeed an open set of \( X \): \( L \) is locally free, so through every point there is a neighbourhood \( U \) such that \( L|_U \simeq \mathcal{O}_X|_U \). To show that it is open, we can therefore assume that \( L = \mathcal{O}_X \). If \( x \in X_s \) there exists a \( t_x \in \mathcal{O}_{X,x} \) such that \( s_x t_x = 1 \). Choose an open neighbourhood \( V \) of \( x \) such \( t_x \) is represented by a section \( t \in \Gamma(V, \mathcal{O}_X) \). By shrinking \( V \) we can assume that \( (s|_V)t = 1 \) on \( V \), and so \( V \subseteq X_s \) is open.

**Lemma 10.8.** Suppose \( X \) is a noetherian scheme. Let \( \mathcal{F} \) be a quasi-coherent sheaf on \( X \), and \( L \) an invertible sheaf. Suppose \( f \in \Gamma(X, L) \). Then:

1. If a section \( t \in \Gamma(X, \mathcal{F}) \) restricts to zero on \( X_f \), then there is an integer \( N \) such that \( t \otimes f^N \in \Gamma(X, \mathcal{F} \otimes L^N) \) is zero (on all of \( X \)).

2. Suppose \( t \in \Gamma(X_f, \mathcal{F}) \). Then there is an integer \( N \) such that \( t \otimes f^N \) extends to a global section of \( \mathcal{F} \otimes L^N \).
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**Proof of the lemma.** (i) Suppose $t$ restricts to zero on $X_f$. We may cover $X$ by finitely many open affines $U_i$ such that there is a collection of trivialization isomorphisms $\phi_i : L|_{U_i} \to \mathcal{O}_X|_{U_i}$. Then $t$ maps to zero in $\Gamma(U_i, \mathcal{F})$ because $U_i$ is affine (regarding $f$ as an element of $\mathcal{O}_X(U_i)$ via the isomorphism $\phi_i$). So there is a power of $f$ that annihilates $t$ on $U_i$. In other words, this says that $t \otimes f^N = 0$ on $U_i$. Taking $N = \max N_i$, we find that $t \otimes f^N$ is zero on all of $X$, as desired.

(ii) For each $i$, we know that some $t \otimes f^{N_i}$ extends to a section over all of $U_i$ (because $U_i$ is affine and taking sections over basic open subsets corresponds to localization). Take $M = \max N_i$. Then $t \otimes f^M$ extends to sections $t_i \in \Gamma(U_i, \mathcal{F} \otimes L^M)$. A potential problem is that the $t_i$ might not necessarily glue on $U_i \cap U_j$. However, $t_i = t_j$ on $U_i \cap U_j \cap X_f$ since they are extending something defined on $X_f$ (namely $t$). Since $U_i \cap U_j \cap X_f$ is also noetherian, the first part (i) now implies that there are powers $M_{ij}$ such that $(t_i - t_j) \otimes f^{M_{ij}} = 0 \in \Gamma(U_i \cap U_j, \mathcal{F} \otimes L^{M + M_{ij}})$. Taking $N = M + \max M_{ij}$ now does the trick: the $t_i \otimes f^{\max M_{ij}}$ will glue and extend $t \otimes f^N$. \hfill \Box

**Remark 10.9.** The same proof works also for an invertible sheaf $L$ on a quasi-compact and separated scheme $X$.

### 10.4 The associated graded module

We have associated to a graded $R$-module $M$ a sheaf $\widetilde{M}$ on $X = \text{Proj} R$. To classify quasi-coherent sheaves on $X$ we would, like in the case of affine schemes, give some sort of inverse to this assignment. However, unlike the case for $X = \text{Spec} A$, we can not simply take the global sections functor. Indeed, even for $\mathcal{F} = \mathcal{O}_X$ on $X = \mathbb{P}^1_k$, $\Gamma(X, \mathcal{O}_X) = k$, form which we cannot recover $\mathcal{F}$. However, the remedy is to look at the various Serre twists $\mathcal{F}(m)$ – in fact all of them at once:

**Definition 10.10.** Let $R$ be a graded ring and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X = \text{Proj} R$. We define the **graded $R$-module associated to $\mathcal{F}$**, $\Gamma_*(\mathcal{F})$ as

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

This has the structure of an $R$-module: If $r \in R_d$, we have a corresponding section $r \in \Gamma(X, \mathcal{O}_X(d))$ (abusing notation, as before). So if $\sigma \in \Gamma(X, \mathcal{F}(n))$ then we define $r \cdot \sigma \in \Gamma(\mathcal{F}(n + d))$ by the tensor product $r \otimes \sigma$, and using the isomorphism $\mathcal{F}(n) \otimes \mathcal{O}(d) \simeq \mathcal{F}(n + d)$. In particular, $\Gamma_*(\mathcal{O}_X)$ is a graded ring.
10.4.1 Sections of the structure sheaf

Proposition 10.11. Let $R$ be a graded integral domain, finitely generated in degree 1 by elements $x_1, \ldots, x_n$, and let $X = \text{Proj} R$. Then

1. $\Gamma_*(-(O_X)) = \bigcap_{i=1}^n R(x_i) \subseteq K(R)$

2. If each $x_i$ is a prime element, then $R = \Gamma_*(-(O_X))$.

Proof. Cover $X$ by opens $U_i = D_+(x_i)$. We have, since $\Gamma(D_+(x_i), O(m)) \simeq (R_{x_i})_m$, that the sheaf axiom sequence takes the following form

$$0 \to O(m) \to \bigoplus_{i=0}^n (R_{x_i})_m \to \bigoplus_{i,j} (R_{x_i}x_{x_j})_m$$

Taking directs sums over all $m$, we get

$$0 \to \Gamma_*(-(O_X)) \to \bigoplus_{i=0}^n (R_{x_i}) \to \bigoplus_{i,j} (R_{x_i}x_{x_j})$$

So a section of $\Gamma_*(-(O_X))$ corresponds to an $(n+1)$-tuple $(t_0, \ldots, t_n) \in \bigoplus_{i=0}^n (R_{x_i})$ such that $t_i$ and $t_j$ coincide in $R_{x_i}x_{x_j}$ for each $i \neq j$. Now, the $x_i$ are not zero-divisors in $R$, so the localization maps $R \to R_{x_i}$ are injective. It follows that we can view all the localizations $R_{x_i}$ as subrings of $R_{x_0 \ldots x_n}$, and then $\Gamma_*(-(O_X))$ coincides with the intersection

$$\bigcap_{i=0}^n R_{x_i} \subseteq A[x_0^{\pm 1}, \ldots, x_n^{\pm 1}] .$$

In the case each $x_i$ is prime, this intersection is just $R$. \hfill \Box

Corollary 10.12. Let $X = \mathbb{P}_A^n = \text{Proj} A[x_0, \ldots, x_n]$ for a ring $A$. Then

$$\Gamma_*(O_X) \simeq A[x_0, \ldots, x_n]$$

In particular we can identify $\Gamma(\mathbb{P}_A^n, O(d))$ with the $A$-module generated by homogeneous degree $d$ polynomials.

When $R$ is not a polynomial ring, it can easily happen that $\Gamma_*(-(O_X))$ is different than $R$. Here is an explicit example:

Example 10.13 (A quartic rational curve). Let $k$ be a field and let $R = k[s^4, s^3t, st^3, t^4] \subseteq k[s, t]$. Note that the monomial $s^2t^2$ is missing from the generators of $R$. Define the grading such that $R_1 = k \cdot \{s^4, s^3t, st^3, t^4\}$. 

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We can also think of $R$ as the graded ring

$$R = k[x_0, x_1, x_2, x_3]/\langle x_0^2 x_2 - x_1^3, x_1 x_3^2 - x_2^3, x_0 x_3 - x_1 x_2 \rangle.$$ 

We have a covering $\text{Proj } R = U_0 \cup U_1$, where

$$U_0 = \text{Spec}(R_{(x_0)}) \text{ and } U_1 = \text{Spec}(R_{(x_3)}).$$

Here $R_{(x_0)} = k[t, t_3, t_4]$ and $R_{(x_3)} = k[s, t]$. So $\text{Proj } R$ is simply $\mathbb{P}^1$. We have shown that $X$ embeds as a smooth, rational (degree 4) curve in $\mathbb{P}^3$.

What is $\Gamma(X, \mathcal{O}_X(1))$? On the opens we find $\mathcal{O}_X(1)(U_0) = k[t] \cdot s^4$ and $\mathcal{O}_X(1)(U_1) = k[s, t] \cdot t^4$. So using the sheaf sequence, we get

$$0 \to \Gamma(X, \mathcal{O}_X(1)) \to k[t] s^4 \oplus k[t] t^4 \to k[s, t] u^4$$

Note that the monomial $s^2 t^2$ belongs to both $k[t] s^4$ and $k[t] t^4$, and so defines an element in $\Gamma(X, \mathcal{O}_X(1))$. In fact,

$$\Gamma(X, \mathcal{O}_X(1)) = k\{s^4, s^3 t, s^2 t^2, st^3, t^4\}$$

even though $R_1 = k\{s^4, s^3 t, st^3, t^4\}$.

In this example, the graded ring $\Gamma_*(\mathcal{O}_X)$ is the integral closure of $R$. We will see later that this is not a coincidence.

10.4.2 The homomorphism $\alpha$.

Let $X = \text{Proj } R$, where $R$ be a graded ring and let $M$ be a graded $R$-module. We will define a homomorphism of graded $R$-modules called the saturation map

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

As before, it is useful to think of elements in $\Gamma(X, \widetilde{M}(n))$ as a collection of elements $(m_f, D_+(f))$ for $m \in (M(f))_n$ and $f \in R$ matching on the various overlaps.

Proposition 10.14. When $R$ is generated in degree 1, there is a graded $R$-module homomorphism

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

Indeed, we can define $\alpha$ by sending an element $m \in M_d$ to the collection given by $(m/1, D_+(f))$, where $f$ ranges over $R_1$. On the overlaps $D_+(f) \cap D_+(g) = D_+(fg)$ it is clear that the two elements $(m/1, D_+(f))$ and $(m/1, D_+(g))$ become
equal so this defines an actual global section of $\tilde{M}(n)$. We see that this is a graded homomorphism. Moreover, it is functorial in $M$.

**Lemma 10.15.** If $R$ is a noetherian integral domain generated in degree 1. Then $R' = \Gamma_*(\mathcal{O}_X)$ is integral over $R$.

**Proof.** Let $x_1, \ldots, x_r$ be degree 1 generators of $R$. Let $\alpha : R \to \Gamma_*(\mathcal{O}_X)$, be the map above, and let $s \in R'$ be a homogeneous element of non-negative degree. We can find an $n > 0$, so that $\alpha(x_i^n)s \in \alpha(R)$ for every $i$. $R_m$ is generated by monomials in $x_i$ of degree $m$, so $\alpha(R_m)s \subseteq \alpha(R)$ for $m$ large (e.g., $m \geq rn$). Let $R_{\geq rn}$ be the ideal of $R$ generated by elements of degree $\geq rn$. We have that $\alpha(R_{\geq rn})s \subseteq \alpha(R_{\geq rn})$. Moreover, since $R$ is noetherian, $R_{\geq rn}$ is finitely generated, so applying Cayley–Hamilton, we get that $s$ satisfies an integral equation over $R$. Hence $R'$ is integral over $R$. \qed

### 10.4.3 The map $\beta$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We will define a map of $\mathcal{O}_X$-modules

$$\beta : \Gamma_*(\mathcal{F}) \to \mathcal{F}$$

as follows. Let $f \in R_1$. We will define $\beta$ over $D_+(f)$. A section of $\Gamma_*(\mathcal{F})$ is represented on $D_+(f)$ by a fraction $m/f^d$ where $m \in \Gamma(X, \mathcal{F}(d))$. If we think of $f^{-d}$ as a section in $\mathcal{O}(-d)(D_+(f))$, then we can consider the tensor product $m \otimes f^{-d}$ which is a section of $\mathcal{F}$ via the isomorphism $\mathcal{F}(d) \otimes \mathcal{O}(-d) \simeq \mathcal{F}$. This is compatible with the module structures, so we obtain a homomorphism of $\mathcal{O}_X$-modules

$$\beta : \Gamma_*(\mathcal{F}) \to \mathcal{F}$$

by associating $m/f^d$ to $m \otimes f^{-d}$.

**Proposition 10.16.** Suppose $R$ is a graded ring, finitely generated in degree 1 over $R_0$. Suppose $\mathcal{F}$ is a quasi-coherent sheaf on $\text{Proj} R$. Then the map

$$\beta : \Gamma_*(\mathcal{F}) \to \mathcal{F} \quad (10.4.1)$$

is an isomorphism.

**Proof.** Since $R$ is generated by $R_1$ over $R_0$, the open sets $D_+(f)$ with $f \in R_1$ cover $X$. To show that (10.4.1) is an isomorphism, it is sufficient to prove it on such an open.

Let $f \in R_1$, and consider it as a section of $\Gamma(X, \mathcal{O}(1))$. Then taking $L = \mathcal{O}(1)$ in Lemma 10.8, (i) there says that if an element $s$ of $\Gamma(D_+(f), \mathcal{F})$ is given, we
can find some element $t$ of $\Gamma_*(\mathcal{F})_N$ (for $N$ sufficiently large) that $t \otimes f^{-N} \in \Gamma(D_+(f), \mathcal{F})$ equals $s$. This implies that the map (10.4.1) is surjective.

For injectivity, suppose $s \in \Gamma(X, \mathcal{F}(n))$ is such that $s \otimes f^{-n} = 0$ on $D_+(f)$, i.e. $s/f^n \in \Gamma_*(\mathcal{F})_f$ is in the kernel of (10.4.1) on the $D_+(f)$-sections. Then the lemma implies that there is a power $f^N$ with $s \otimes f^N \in \Gamma(X, \mathcal{F}(n + N)) = 0$. This states that $s/f^n = 0$ in $\Gamma_*(\mathcal{F})_f$ by the definition of localization and so the map is injective.

We have defined two functors

$$\sim: \text{GrMod}_R \to \text{QCoh}_X$$

and

$$\Gamma_*: \text{QCoh}_X \to \text{GrMod}_R$$

Since $\beta: \widehat{\Gamma_*}(\mathcal{F}) \to \mathcal{F}$ is an isomorphism, it follows that $\sim$ is essentially surjective. However, unlike the affine case, the functors do not give mutual inverses. This is because, as we have seen, that $\sim$ is not faithful; the $\sim$ of any module $M$ which is finite over $R_0$ is the zero sheaf.

It turns out that if we restrict to a subcategory of $\text{GrMod}_R$, the category of saturated $R$-modules, we do get an equivalence. Here we say that a graded $R$ module is saturated if $\alpha: M \to \Gamma_*(\widehat{M})$ is an isomorphism. This makes sense, as the module $M = \Gamma_*(\mathcal{F})$ is a saturated $R$-module. We denote this category by $\text{GrMod}^{sat}_R$. We then have a diagram

$$\begin{array}{ccc}
\text{GrMod}_R & \xrightarrow{M \to \widehat{M}} & \text{QCoh}_X \\
\downarrow{\alpha} & & \\
\text{GrMod}^{sat}_R & \xrightarrow{\mathcal{F} \to \Gamma_*} & \Gamma_*(\mathcal{F})
\end{array}$$

We have seen that the map $\beta$ is an isomorphism, so any quasi-coherent sheaf $\mathcal{F}$ is isomorphic to the $\sim$ of some module $\Gamma_*(\mathcal{F})$.

Putting everything together, we find

**Theorem 10.17.** Let $R$ be a graded ring, finitely generated in degree 1 over $R_0$ and let $X = \text{Proj } R$. Then the functors

$$\sim: \text{GrMod}^{sat}_R \to \text{QCoh}_X$$

and

$$\Gamma_*: \text{QCoh}_X \to \text{GrMod}^{sat}_R$$
are mutually quasi-inverse, and establish an equivalence of categories.

So we have a full classification of quasi-coherent sheaves on \( X \) in terms of modules on \( R \).

### 10.5 Closed subschemes of \( \text{Proj} \ R \)

The notion of a saturated module comes from commutative algebra and has a more conceptual definition there. To give some flavour of what it entails, we will explain what this means for a graded ideal \( I \subseteq R \). Here the *saturation* of \( I \) with respect to an ideal \( B \) is defined as the ideal

\[
I : B^\infty := \bigcup_{i \geq 0} I : B^i = \{ r \in R | B^n r \in I \text{ for some } n > 0 \}.
\]

We say that \( I \) is \( B \)-saturated if \( I = I : B^\infty \) and in our case, *saturated* if it is \( R_+ \)-saturated. We will here denote \( I : (R_+)^\infty \) by \( \overline{I} \). Note that the ideal \( \overline{I} \) is homogeneous if \( I \) is.

**Example 10.18.** In \( R = k[x_0, x_1] \), the \((x_0, x_1)\)-saturation of \((x_0^2, x_0x_1)\) is the ideal \((x_0)\). Note that both \((x_0)\) and \((x_0^2, x_0x_1)\) define the same subscheme of \( \mathbb{P}^1_k \), but in some sense the latter ideal is inferior, since it has a component in the irrelevant ideal \((x_0, x_1)\). This example is typical; the saturation is a process which throws away components of \( I \) supported in the irrelevant ideal.

**Proposition 10.19.** Let \( A \) be a ring and let \( R = A[x_0, \ldots, x_n] \).

(i) To each subscheme \( Y \) of \( \mathbb{P}^n_A \), there is a corresponding homogeneous saturated ideal \( I \subseteq R \). Such that \( Y \) corresponds to the subscheme \( \text{Proj}(R/I) \to \text{Proj} R \).

(ii) Two ideals \( I, J \) defined the same subscheme if and only if they have the same saturation.

(iii) If \( Y \subseteq \mathbb{P}^n_A \) is a closed subscheme with ideal sheaf \( \mathcal{I} \), then \( \Gamma_*(\mathcal{I}) \) is a saturated ideal of \( R \). In fact, the ideal \( \Gamma_*(\mathcal{I}) \) is the largest ideal that defines the subscheme \( Y \).

In particular, there is a 1-1 correspondence between closed subschemes \( i : Y \to \mathbb{P}^n_A \) and saturated homogeneous ideals \( I \subseteq R \).

**Proof.** (i) Let \( i : Y \to \mathbb{P}^n_A \) be a subscheme of \( \mathbb{P}^n_A = \text{Proj} R \) and let \( \mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_A} \) denote the ideal sheaf of \( Y \). Using the fact that global sections is left-exact, we have \( \Gamma_*(\mathcal{I}) \subseteq \Gamma_*(\mathcal{O}_{\mathbb{P}^n_A}) \). This is naturally an \( R \)-module, so \( I = \Gamma_*(\mathcal{I}) \) is a (homogeneous) ideal of \( R \).
Any such ideal $I$ gives rise to a closed subscheme $i' : \text{Proj}(R/I) \to \mathbb{P}^n_A$ and hence an ideal sheaf $\mathcal{I}$ satisfying $\widetilde{I} = \mathcal{I}$. By Proposition 10.16, we also have $\widetilde{I} = \mathcal{I}_Y$, so the two quasi-coherent sheaves coincide and $i$ is indeed the same as $i'$. By construction $I = \Gamma_*(\mathcal{I})$, so $I$ is saturated.

(ii) If $I, J$ define the same subscheme, they have the same ideal sheaf $\Gamma_*(\mathcal{I}) = \Gamma_*(\mathcal{J})$ on $\mathbb{P}^n_A$. Let $s \in I_d$, then on $U_i = D_+(x_i)$, the fraction $rx_i^{-d}$ defines an element of $\Gamma(U_i, \widetilde{I}) = \Gamma(U_i, \widetilde{J})$. Since also $\mathcal{I}$ corresponds to $\mathcal{J}$, we have $rx_i^{-d} = t_i x_i^{-d}$ for some $t_i \in J_d$ of degree $d$. Hence there is a power $n_i$ such that $x_i^{n_i}(r - t_i) = 0$ in $R$. This shows that $r$ is in the saturation of $J$. By symmetry, we have $I = J$.

(iii) Let $r \in R$ be such that $x_i^{n_i}r \in \Gamma_*(\mathcal{I})$ (that is $r \in \Gamma_*(\mathcal{I})$). Let $m = \max n_i$. We want to show that $r \in \Gamma_*(\mathcal{I})$ and $\mathcal{I}$ corresponds to $\mathcal{J}$. Indeed, the inverse is given by the map $R(f \otimes g) \to S'$ defined by $r \otimes r' \mapsto \frac{r}{f^s} \otimes \frac{r'}{g^s}$ where $f' = f^{\deg g}$ and $g' = g^{\deg f}$. Indeed, the inverse is given by the map $R(f \otimes g'') \to S'$ defined by $r \otimes r' \mapsto \frac{r}{f^s} \otimes \frac{r'}{g'^s}$.

10.6 The Segre embedding

Theorem 10.20. Let $R, R'$ be graded rings with $R_0 = R'_0 = A$. Let $S = \bigoplus_{n \geq 0}(R_n \otimes R'_n)$. Then

$$\text{Proj } S \simeq \text{Proj } R \times_A \text{Proj } R'$$

Proof. Let $X = \text{Proj } R$, $Y = \text{Proj } R'$ and $Z = \text{Proj } S$. Let $f \in R$ be a homogeneous element. Define

$$Z_f = \bigcup_{g \in R'} \text{Spec } S_{f^{\deg g} \otimes g^{\deg f}}$$

We claim that there is a natural isomorphism

$$S_{f^{\otimes g'}} \to R_{(f)} \otimes_A R'_{(g')},$$

$$r \otimes r' \mapsto \frac{r}{f^s} \otimes \frac{r'}{g'^s}$$

where $f' = f^{\deg g}$ and $g' = g^{\deg f}$. Indeed, the inverse is given by the map $R_{(f)} \otimes_A R'_{(g')} \to S_{f^{\otimes g'}}$ defined by

$$\frac{r}{f^s} \otimes \frac{r'}{g'^s} \mapsto \frac{r}{f^{\deg g}} \otimes \frac{r'}{g'^{\deg f}}$$

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Hence we see that

\[ Z_f = \bigcup_{g \in R'} \text{Spec}(R_{(f)} \otimes_A R'_{(g)}) \]

On the overlaps, we have

\[ \text{Spec}(R_{(f)} \otimes_A R'_{(g)}) \cap \text{Spec}(R_{(f)} \otimes_A R'_{(h)}) = \text{Spec}\left( S_{f \deg g + \
 \deg h} \otimes_{\deg f} \right) \]

\[ = \text{Spec}\left( R_{(f)} \otimes_A R'_{gh} \right) \]

From this is is clear that

\[ Z_f = D_+(f) \times_R Y \]

Moreover, for any other \( f' \in R \) we have \( Z_{f'} = Z_{ff'} = X_{ff'} \times_R Y \). Hence

\[ Z = \bigcup_{f \in R} Z_f = X \times_R Y. \]

\[ \square \]

**Corollary 10.21.** Let \( A \) be a ring and let \( m, n \geq 1 \) be integers. Then there is a closed immersion

\[ \sigma : \mathbb{P}_A^m \times_A \mathbb{P}_A^n \to \mathbb{P}_A^{mn+m+n} \]

**Proof.** Consider the \( A \)-algebra \( S = \bigoplus_{n \geq 0} (R_n \otimes R'_n) \) above, where \( R = A[x_0, \ldots, x_m] \) and \( R' = A[y_0, \ldots, y_n] \) are the polynomial rings. Consider the following morphism of graded \( A \)-algebras.

\[ A[z_{ij}]_{0 \leq i \leq m, 0 \leq j \leq n} \to A[x_0, \ldots, x_m] \otimes A[y_0, \ldots, y_n] \]

\[ z_{ij} \mapsto x_i \otimes y_j. \]

It is clear that \( S \) is generated as an \( R \otimes R' \)-algebra by the products \( x_i \otimes y_j \), so the map is surjective and thus we get the desired closed immersion. \( \square \)

**Example 10.22.** Let \( R = k[x_0, x_1], R' = k[y_0, y_1] \). Then \( u_{ij} = x_i y_i \) defines an isomorphism

\[ S = \bigoplus_{n \geq 0} (R_n \otimes R'_n) \to k[u_{00}, u_{01}, u_{10}, u_{11}]/(u_{00}u_{11} - u_{01}u_{10}) \]

This recovers the usual embedding of \( \mathbb{P}^1_k \times_k \mathbb{P}^1_k \) as a quadric surface in \( \mathbb{P}^3_k \).
10.7 Two important exact sequences

10.7.1 Hypersurfaces

Let $R = k[x_0, \ldots, x_n]$ and $\mathbb{P}^n_k = \text{Proj} R$. Let $F \in R$ denote a homogeneous polynomial of degree $d > 0$. $F$ determines a projective hypersurface $X = V(F)$, which has dimension $n - 1$. Note that $I(X) = (F)$ by the Nullstellensatz. We then have an isomorphism

$$R(-d) \to I(X)$$

given by multiplication with $F$. Note the shift here: The constant ‘1’ gets sent to $F$ should have degree $d$ on both sides! This gives the sequence of $R$-modules

$$0 \to R(-d) \to R \to R/(F) \to 0$$

We have $\tilde{R}(-d) = \mathcal{O}_{\mathbb{P}^n_k}(-d)$ and $\tilde{(R/F)} = i_* \mathcal{O}_X$, where $i : X \to \mathbb{P}^n_k$ is the inclusion, so we get the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \to \mathcal{O}_{\mathbb{P}^n_k} \to i_* \mathcal{O}_X \to 0$$

10.7.2 Complete intersections

Let $F, G$ be two homogeneous polynomials without common factors of degrees $d, e$ respectively. Let $I = (F, G)$ and $X = V(I) \subseteq \mathbb{P}^n_k$. $X$ is called a ‘complete intersection’ – it is the intersection of the two hypersurfaces $V(F)$ and $V(G)$. To study $X$ we have exact sequences

$$0 \to R(-d - e) \xrightarrow{\alpha} R(-d) \oplus R(-e) \xrightarrow{\beta} I \to 0$$

and applying $\sim$:

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d - e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_X \to 0$$
The maps here are defined by $\alpha(h) = (-hG, hF)$ and $\beta(h_1, h_2) = h_1F + h_2G$. These maps preserve the grading. To prove exactness, we start by noting that $\alpha$ is injective (since $R$ is an integral domain) and $\beta$ is surjective (by the definition of $I$). Then if $(h_1, h_2) \in \ker \beta$, we have $h_1F = -h_2G$, which by the coprimality of $F, G$ means that there is an element $h$ so that $h_1 = -hG, h_2 = hF$.

These sequences are fundamental in computing the geometric invariants from $X$. We will see several examples of this later.
Chapter 11

Differentials

So far we have defined schemes and surveyed a few of their basic properties (e.g., how to study sheaves on them). In this chapter, we introduce Kähler differentials, which allow us in some sense to do calculus on schemes. This in turn will allow us to define the most important sheaves in algebraic geometry, namely, the cotangent sheaf, the tangent sheaf, and the sheaves of \( n \)-forms.

Differentials appear prominently throughout many areas of mathematics, e.g., multivariable analysis, manifolds and differential geometry. In algebraic geometry they are introduced algebraically using their formal properties and are usually referred to as Kähler differentials after the German mathematician Erich Kähler (1906–2000).

11.1 Tangent vectors and derivations

Consider a real manifold \( M \) of dimension \( n \). To each point \( p \in M \), we can attach its tangent space of \( T_pM \), which is a real vector space of dimension \( n \). When \( M \) is embedded as a submanifold in some \( \mathbb{R}^N \), we can think of tangent vectors \( v \) as vectors in the ambient space \( \mathbb{R}^N \) that ‘stick out’ from \( p \), and naively define \( T_pM \) as the vector space of vectors that tangentially pass through \( p \). However, we do not like to think of manifolds as embedded in some \( \mathbb{R}^N \), and then giving a precise definition of \( T_pM \) becomes a little bit subtle.

When defining tangent spaces in the abstract setting, there are a few basic properties we want. First of all, each \( \mathbb{R} \)-vector space \( T_pM \) should have dimension \( n \), and the elements should have some sort of geometric interpretation. Moreover, the assignment \( p \to T_pM \) should vary continuously in \( p \), so that we can be able...
11.1. Tangent vectors and derivations

to talk about continuous families of tangent vectors (or vector fields). The most
precise way to phrase this is to say that the collection $TM = \{T_pM\}_{p \in M}$ should
be a vector bundle, i.e., $TM$ has the structure of a $2n$-dimensional manifold,
and there is a projection map $\pi : TM \to M$ sending $(p,v)$ to $p$, which is locally
diffeomorphic to $U \times \mathbb{R}^n$ in a neighbourhood of $p$. This is what is usually known
as the tangent bundle. In this setting a vector field is simply a map $\nu : M \to TM$
of $\pi$ so that $\pi \circ \nu = \text{id}_M$.

There are a number of ways to rigorously define the tangent bundle $TM$.
The simplest, and perhaps most flexible method is using derivations. Here we
think about tangent vectors as ways to differentiate functions defined at $p$.
So we consider the local ring $\mathcal{O}_p$ of $C^\infty$-functions defined at $p$
and think about elements of $T_p$ as linear functionals $d : \mathcal{O}_p \to \mathbb{R}$, satisfying the Leibniz rule

$$d(fg) = fd(g) + gd(f).$$

Note that the set of such derivatives naturally form a real vector space; we can
add them and multiply them with real numbers and the result will be a new
derivation.

In the prototype example, when $M \subseteq \mathbb{R}^N$ is an embedded submanifold, we
can consider the directional derivative

$$\nabla_v f = \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h},$$

which we may think of as a derivation $C^\infty(M, \mathbb{R}) \to \mathbb{R}$. It is not hard to see
that every derivation arises this way, and that conversely a tangent vector $v$
is determined by the functional $\nabla_v$.

An alternative viewpoint to tangent spaces uses the maximal ideal $m \subseteq \mathcal{O}_p$
of functions vanishing at the point $p$. We would like to consider functions $f$
vanishing at $p$, but identify two functions $f_1, f_2$ if they have the same linear
Taylor expansion around $p$, i.e., if $f_1 - f_2 \in m^2$. In this way we can define the
cotangent space $m/m^2$, and then set $T_p^\vee = (m/m^2)^\vee$ (vector space dual). That is,
we can think about elements of $T_p$ as linear functionals

$$v : \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{R}.$$ 

It is not hard to see that these two perspectives give the same vector space. Given a derivation $d : \mathcal{O}_p \to \mathbb{R}$, we must have $d(\mathfrak{m}^2) = 0$, and so $d$ induces via restriction a map $v : \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{R}$. Conversely, any element of $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ arises this way.

So for instance, if $M$ has local coordinates $x_1, \ldots, x_n$ near $p$, we let $dx_1, \ldots, dx_n$ denote their images in $\mathfrak{m}/\mathfrak{m}^2$. Each $x_i$ determines dually a derivation $\frac{\partial}{\partial x_i} : \mathcal{O}_p \to \mathbb{R}$, so that $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ forms a basis $T_p$. Note that by definition, $\frac{\partial}{\partial x_i}(dx_i) = 1$ and $\frac{\partial}{\partial x_i}(dx_j) = 0$ for $i \neq j$.

### 11.2 Regular schemes and the Zariski tangent space

Let $X$ be a scheme and let $x \in X$ be a point. We have the local ring $R = \mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}$. Motivated by the case of manifolds, we will then consider $\mathfrak{m}_x/\mathfrak{m}_x^2$, which is in a natural way a vector space over the residue field $k(x) = R/\mathfrak{m}_x$. This is what we would like to define as the cotangent space of $X$ at $x$.

Intuitively, the cotangent space is what you get by differentiating functions which vanish at that point, but differentiating functions that vanish at $x$ at higher order should give zero.

**Definition 11.1.** The cotangent space of $X$ at $x$ is the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$.

The Zariski tangent space $T_{X,x}$ to $X$ at $x$ is the dual vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$. That is,

$$T_{X,x} = \text{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$$

and an element of $T_{X,x}$ is called a tangent vector; it is a linear functional $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k(x)$.

$X$ is called regular at $x$ if the local ring $\mathcal{O}_{X,x}$ is a regular local ring, i.e.,

$$\dim \mathcal{O}_{X,x} = \dim_{k(x)} \mathfrak{m}/\mathfrak{m}^2.$$ 

$X$ is regular if this condition holds for every $x \in X$.

**Example 11.2.** $\mathbb{A}^1_k$ and $\mathbb{P}^1_k$ are both regular schemes. Indeed, in both cases, the local ring at a point is isomorphic to a localization of a polynomial ring, which is regular.

A non-normal example is $X = \text{Spec} A$ where $A = k[x, y]/(x^2 - y^3)$. In the point $\mathfrak{m} = (x, y)$ corresponding to the origin, the local ring is not normal:
11.2. Regular schemes and the Zariski tangent space

\(x/y\) satisfies a monic equation, \((x/y)^3 - x = 0\), but it is not contained in the localization \(A_p\). In this example, the vector space \(\mathfrak{m}/\mathfrak{m}^2\) is 2-dimensional at the origin – it is spanned by the residue classes of \(x\) and \(y\) (which are linearly independent since all relations in \(A\) happen in degree at least two).

### 11.2.1 Zariski tangent space and the ring of dual numbers

Let \(k\) be a field. The ring \(R = k[\epsilon]/(\epsilon^2)\) is called the *ring of dual numbers* over \(k\). The spectrum of \(R\) is a very simple scheme: Its underlying topological space is a single point. However, the non-reduced structure on \(\text{Spec } R\) shows that it is more interesting than \(\text{Spec } k\). We picture it as follows:

\[\text{Spec } R\]

\[\vdots\]

\[X\]

\[p\]

**Proposition 11.3.** Let \(X\) be a scheme over \(k\). To give a \(k\)-morphism \(\text{Spec}(k[\epsilon]/(\epsilon^2)) \rightarrow X\) is equivalent to giving a point \(x \in X\) which is rational over \(k\) (meaning that \(k(x) = k\)), and an element of \(T_{X,x}\).

**Proof.** Let \(Y = \text{Spec}(k[\epsilon]/(\epsilon^2))\). The map \(Y \rightarrow X\) is a map of schemes over \(k\), and so \(f^\#\) induces a map of \(k\)-algebras:

\[\mathcal{O}_{X,x} \rightarrow k[\epsilon]/\epsilon^2\]

This is a local homomorphism, so we can mod out by the ideals \(m_x\) and \((\epsilon)\) to get

\[k(x) \rightarrow k\]

where the diagonal arrow is an isomorphism; hence \(k(x) = k\) and \(x\) is a \(k\)-rational point. Moreover, restricting \(f_x^\#\) to \(m_x \subseteq \mathcal{O}_{X,x}\) we get a \(k\)-algebra map.
α : m_x → (ε) ≃ k. This has to vanish on m^2_x, and hence induces an element in T_{X,x}.

Conversely, suppose that x ∈ X is a k-rational point and let α : m/m^2 → k be a k-linear map (m is the maximal ideal in O_{X,x}). We want to construct \((f, f^#) : (Y, O_Y) \rightarrow (X, O_X)\). Define \(f((ε)) = x\); this takes care of the map on topological spaces. Now we define \(f^#\). Set for \(U \subseteq X\) not containing \(x\), we define \(f^# : O_X(U) \rightarrow O_Y(∅) = 0\) to be the zero map. For \(U\) containing \(x\), we argue as follows. The point is that each \(O_X(U)\) has a structure of a \(k\)-algebra (ie there is a map \(k \rightarrow O_X(U)\)), and so we can write any element \(a \in O_{X,x}\) as \(a = a(x) + (a - a(x))\) where \(a(x)\) is the image of \(a\) via \(O_{X,x} \rightarrow O_{X,x}/m_x \rightarrow k \rightarrow O_{X,x}\).

Note that the germ of \((a - a(x))\) in \(O_{X,x}\) lies in \(m_x\). Define \(f^# : O_X(U) \rightarrow O_Y(f^{-1}(U)) = k[ε]/ε^2\) by sending \(a\) to the class of \(a(x) + α((a - a(x))|_x + m^2)x_ε\)

One can check that this is a ring homomorphism, using \(k\)-linearity and the fact that the square of \((a - a(x))|_x\) lies in \(m^2\). Everything here is compatible with the inclusions \(V \subseteq U\), and so sets up a map of sheaves \(f^#\).

11.3 Derivations and Kähler differentials

So far we have considered a scheme \(X\) and a point \(x ∈ X\), and the cotangent space \(m/m^2\) at \(x\). We would like to generalize this construction, and instead of fixing \(x\), consider all points at once. That is, we would like to form a sheaf on \(X\) with stalks \(m/m^2\) at each point. This is what will be the cotangent sheaf on \(X\).

We will work over a base ring \(A\). \(B\) will be an \(A\)-algebra and \(M\) an \(B\)-module. The picture to have in mind is that \(A = k\), where \(k\) is a field, and \(X = \text{Spec } B \rightarrow \text{Spec } k\), where \(B\) is a \(k\)-algebra.

**Definition 11.4.** An \(A\)-derivation (from \(B\) with values in \(M\)) is an \(A\)-linear map \(d : B \rightarrow M\) satisfying the product rule:

\[ d(b_1b_2) = b_1 \ d(b_2) + b_2 \ d(b_1). \]

Given that the product rule holds, it is easy to see that \(d\) is \(A\)-linear if and only if \(d\) vanishes on all elements of the form \(a \cdot 1\) where \(a ∈ A\). We can therefore think of the elements in \(B\) of the form \(a \cdot 1\) as the ‘constants’ – however a derivation can also be zero on other elements on \(B\).

**Example 11.5.** A simple example is the classical derivation of polynomial ring
11.3. Derivations and Kähler differentials

Let \( B = k[x] \) over \( k \) to itself given by \( P(x) \mapsto P'(x) \). As usual, \( P(x) \) must be defined formally by \( P(x) = \sum i a_i x^{i-1} \) if \( P(x) = \sum a_i x^i \).

More generally, the differential operators \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \), as well as their linear combinations, define derivations on the polynomial ring \( k[x_1, \ldots, x_n] \).

The set of derivations \( d : B \to M \) is usually denoted with \( \text{Der}_A(B, M) \). This inherits a \( B \)-module structure from \( M \) and it is as such naturally a submodule of \( \text{Hom}_B(B, M) \). This gives rise to a covariant functor \( \text{Der}_A(B, -) \) from \( \text{mod}_B \) to itself. More precisely, if \( \phi : M \to M' \) is a \( B \)-module homomorphism, we can map a derivation \( d \in \text{Der}_A(B, M) \) to \( \phi \circ d : B \to M' \), which is in turn an \( A \)-derivation of \( B \) with values in \( M' \).

The derivation \( \text{Der}_A(B, M) \) is also functorial in the base ring \( A \) and the algebra \( B \); in both of these cases, it is covariant. If \( A \to A' \) is a ring homomorphism, any \( A' \)-derivation \( B \to M \) is in turn an \( A \)-derivation. We therefore obtain an inclusion \( \text{Der}_{A'}(B, M) \subseteq \text{Der}_A(B, M) \).

11.3.1 The module of Kähler differentials

The covariant functor \( \text{Der}_k(A, -) \) on the category of \( A \)-modules is representable. In down-to-earth terms, this just means that there exist a \( B \)-module \( \Omega_{B/A} \) and an isomorphism of functors

\[
\text{Der}_A(B, -) \simeq \text{Hom}_B(\Omega_{B/A}, -).
\]

By formal properties of functors and representability, this condition is equivalent by having a universal derivation \( d_B : B \to \Omega_{B/A} \), having the following property: For any \( A \)-derivation \( d' : B \to M \) there exists a unique \( B \)-module homomorphism \( \alpha : \Omega_{B/A} \to M \) such that \( d' = \alpha \circ d_B \). In terms of diagrams, we have

\[
\begin{array}{ccc}
B & \xrightarrow{d_B} & \Omega_{B/A} \\
\downarrow & & \downarrow \alpha \\
d' & & M
\end{array}
\]

To see directly why such a module exists, we as construct it explicitly as follows. \( \Omega_{B/A} \) is the \( A \)-module defined as the quotient of the free module \( \bigoplus_{b \in B} B db \) by the submodule generated by the expressions \( d(b + b') - db - db' \), \( d(bb') - bdb - b'db \), \( da \), for \( b, b' \in B, a \in A \). Then the universal derivation \( d_B : B \to \Omega_{B/A} \) is given by \( b \mapsto db \).

It is not hard to see that this module indeed satisfies the property above: Given an \( A \)-derivation \( D : B \to M \), we define the homomorphism \( \alpha : \Omega_{B/A} \to M \)
by \( \alpha(db) = D(b) \) (which is well-defined precisely because \( D \) is a derivation!). Conversely, given \( \alpha : \Omega_{B/A} \rightarrow M \), we get a derivation \( D : B \rightarrow M \) defined by \( D(b) = \phi(db) \).

**Definition 11.6.** The elements of the module \( \Omega_{B/A} \) are called the Kähler differentials, or simply differentials of \( B \) over \( A \).

**Example 11.7.** Let \( B = k[t] \) be the polynomial ring over \( k \) in the variable \( t \). Then \( \Omega_{B/k} \) is a free module over \( k \) generated by \( dt \), i.e., \( \Omega_{B/k} = B \cdot dt \). This follows since an element \( m \) in an arbitrary \( A \)-module \( M \) uniquely determines a derivation \( A \rightarrow M \) with \( dt = m \), and so there is an isomorphism of functors \( \text{Der}_k(B, -) = \text{Hom}(B, -) \). By the Yoneda lemma, it follows that \( \Omega_{B/k} = B \cdot dt \).

Alternatively, we can see this from the explicit construction of \( \Omega^1_{A/k} \), because for any \( f \in k[t] \), we have \( df = f'(t)dt \).

More generally we have:

**Proposition 11.8.** Let \( A \) be any ring and let \( B = A[x_1, \ldots, x_n] \). Then \( \Omega_{B/A} \) is the free \( B \)-module generated by \( dx_1, \ldots, dx_n \).

The universal derivation \( d : B \rightarrow \Omega_{B/A} \) is the map \( f \mapsto \sum \frac{\partial f}{\partial x_i} dx_i \). Since \( B \) is generated as an \( A \)-algebra by the \( x_i \), \( \Omega_{B/A} \) is generated as a \( B \)-module by the \( dx_i \) and there is a surjection \( B^r \rightarrow \Omega_{B/A} \) taking the \( i \)th basis vector to \( dx_i \).

On the other hand, the partial derivative \( \partial / \partial x_i \) is an \( A \)-linear derivation from \( B \) to \( B \), and thus induces a \( B \)-module map \( \partial_i : \Omega_{B/A} \rightarrow B \) carrying \( dx_i \) to 1 and all the other \( x_j \) to 0. Putting these maps together we get the inverse map.

**Example 11.9.** If \( B = S^{-1}A \) is a localization of \( A \), then \( \Omega_{B/A} = 0 \). Indeed, take \( b \in B \), and choose \( s \in S \) so that \( sb \in A \), and hence \( sdb = d(sb) = 0 \) This implies that \( db = 0 \), since \( s \) is invertible in \( B \).

**Example 11.10.** If \( B = A/I \), then \( \Omega_{B/A} = 0 \). More generally, if \( \phi : A \rightarrow B \) is surjective, then \( \Omega_{B/A} \). This follows because if \( b = \phi(a) \), then \( db = a \cdot d(1) = 0 \) in \( \Omega_{B/A} \).

**Example 11.11.** Suppose \( k \) is a field of characteristic \( p > 0 \). Let \( B = k[x] \) and let \( A = k[x^p] \). Then \( \Omega_{B/A} \) is the free \( B \)-module of rank 1 generated by \( dx \).

**Example 11.12.** Let \( B = \mathbb{Q}[\sqrt{2}] \) and \( A = \mathbb{Q} \). Then \( 0 = d(2) = d(\sqrt{2} \sqrt{2}) = 2\sqrt{2}d(\sqrt{2}) \) so \( d(\sqrt{2}) = 0 \). Thus \( \Omega_{\mathbb{Q}[\sqrt{2}]/\mathbb{Q}} = 0 \).

### 11.4 Properties of Kähler differentials

There are a few useful ways for computing modules of differentials when changing rings. The proofs of the following propositions are not difficult, and involve only basic commutative algebra.
11.4. Properties of Kähler differentials

11.4.1 Localization

The aim of this section is to show that Kähler differentials behave very well under localization.

**Proposition 11.13.** Let $S \subseteq A$ be a multiplicative subset mapping into the group of units in $B$. Then

$$\Omega_{B|S^{-1}A} = \Omega_{B|A}$$

*Proof.* We have a natural map $A \rightarrow S^{-1}A$, so we get $\text{Der}_{S^{-1}A}(B, M) \subseteq \text{Der}_A(B, M)$. Conversely, if $d : B \rightarrow B$ is an $A$-derivation, then for each $s \in S$, we have

$$0 = d(1) = sd(1/s) + 1/sd(s) = sd(1/s)$$

As $s$ is invertible in $B$, it follows that $d(1/s) = 0$, and hence by the product rule, $d$ is also an $S^{-1}A$-derivation of $B$. \hfill \Box

**Proposition 11.14.** Suppose $S$ is a multiplicative system in $B$ and let $i : B \rightarrow S^{-1}B$ be the localization map. Then

$$di : \Omega_{B/A} \otimes_A S^{-1}B \rightarrow \Omega_{S^{-1}B/A}$$

is an isomorphism.

*Proof.* Using the usual formula of differentiating a fraction, we can define an $A$-derivation $d : S^{-1}B \rightarrow \Omega_{B/A} \otimes_A S^{-1}A$. This will give rise to an $S^{-1}B$-module homomorphism $\Omega_{S^{-1}B/A} \rightarrow \Omega_{B/A} \otimes_A S^{-1}B$ which will turn out to give the inverse to $di$.

Let us first define the derivation $d$. We define

$$d(b/s) = (sd_B(b) - ad_B(s))s^{-2}$$

We need to check that this does not depend on the representative $a/s$. Using the product rule for $d_B$, we have

$$d(b_1b_2/st) = b_1s^{-1}d(b_2/t) + b_2t^{-1}d(b_1/s), \quad (11.4.1)$$

Suppose now that $tb_1 = sb_2$. Using this, the product rule, the identity $td_Bb_1 + b_1d_Bt = sd_Bb_2 + b_2d_Bs$, and solving for $td_Bb_1$ and substituting into (11.4.1), we find

$$t^2(sd_Bb_1 - b_1d_Bs) = ts(sd_Bb_2 + b_2d_Bs - b_1d_Bt) - t^2b_1d_Bs = s^2(td_Bb_2 - b_2d_Bt)$$

where the last equality follows by $tb_1 = sb_2$. This shows that $d(a/s) = d(b/t)$. Finally, if $b_1s^{-1} = b_2t^{-1}$, there is an element $u \in S$ such that $utb_1 = usb_2$. We
then find
\[ d(b_2/t)u^{-1} + b_2t^{-1}d(1/u) = d(b_2/tu) = d(b_1/su) = d(b_1/s)u^{-1} + b_1s^{-1}d(1/u) \]
and since \( b_1s^{-1} = b_2t^{-1} \), it follows that \( d(b_1/s) = d(b_2/t) \), and \( d \) is well defined. Moreover, it is an \( A \)-derivation by (11.4.1).

\[ \square \]

11.4.2 Base change

**Proposition 11.15.** Let \( A, B \) be as above and let \( A' \) be an \( A \)-algebra. Define \( B' = B \otimes_A A' \). Then there is a canonical isomorphism

\[ \Omega_{B'|A'} \cong \Omega_{B|A} \otimes_B B' \]

**Proof.** The universal derivation \( d : B \to \Omega_{B|A} \) induces an \( A' \)-derivation

\[ d' = d \otimes \text{id}_{A'} : B' \to \Omega_{B'|A} \otimes_A A' = \Omega_{B|A} \otimes_B B' \]

One can check that this is the required universal derivation of \( \Omega_{B'|A'} \) which gives the claim.

\[ \square \]

11.4.3 Exact sequences

Let \( A \) be a ring and let \( \rho : B \to C \) be a homomorphism of \( A \)-algebras. Then there are universal homomorphisms of \((C\text{-modules})\)

\[ \alpha : \Omega_{B|A} \otimes_B C \to \Omega_{C|A} \]

defined by \( \alpha(db \otimes c) = cd\rho(b) \). And

\[ \beta : \Omega_{C|A} \to \Omega_{C|B} \]

**Proposition 11.16.** Then there is an exact sequence of \( C \)-modules

\[ \Omega_{B/A} \otimes_B C \overset{\alpha}{\to} \Omega_{C/A} \overset{\beta}{\to} \Omega_{C/B} \to 0. \]

**Proof.** It is sufficient to show that for any \( C \)-module \( N \), the dual sequence

\[ 0 \to \text{Hom}_C(\Omega_{C|B}, N) \to \text{Hom}_C(\Omega_{C|A}, N) \to \text{Hom}_C(\Omega_{B/A} \otimes_B C, N) \]

is exact. Note that \( \text{Hom}_C(\Omega_{B/A} \otimes_B C, N) = \text{Hom}_B(\Omega_{B/A}, N) \), so the sequence can be written as

\[ 0 \to \text{Der}_B(C, N) \to \text{Der}_A(C, N) \to \text{Der}_A(B, N) \]

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The map on the right hand side is the one induced by composition \( A \to B \to C \).

The fact that it is exact on the left is clear: If \( d : C \to N \) maps to the zero derivation, it is the zero homomorphism and hence 0 in \( \text{Der}_B(C, N) \). To check exactness in the middle, note that if \( d : C \to N \) is mapped to the zero derivation in \( \text{Der}_A(B, N) \), then \( d : C \to N \) is in fact \( B \)-linear: This follows by the Leibniz rule \( d(bc) = b \cdot dc + db \cdot c = b \cdot dc + 0 = bdc \). Hence \( d \) is in fact a \( B \)-derivation. □

Applying this to \( C = S^{-1}B \), we obtain a new proof of Proposition 11.14 that

\[
S^{-1}\Omega_{B/A} \simeq \Omega_{S^{-1}B|A}
\]

for the localization in a multiplicative subset \( S \subseteq B \).

**Proposition 11.17** (Conormal sequence). Suppose that \( B \) is an \( A \)-algebra, and \( C = B/I \) for some ideal \( I \subseteq B \). Then there is an exact sequence of \( C \)-modules

\[
I/I^2 \xrightarrow{d} \Omega_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega_{C/A} \to 0
\]

Here the first map sends \([f] \mapsto df \otimes 1\) and the second sends \( c \otimes db \mapsto cdb \).

**Proof.** Note that \( I/I^2 = I \otimes_B C \). As in the previous proposition it suffices to check that for each \( C \)-module \( N \), the dual sequence

\[
0 \to \text{Der}_A(C, N) \to \text{Der}_A(B, N) \to \text{Hom}_C(I/I^2, N) = \text{Hom}(I, N)
\]

is exact. The rightmost map associates to a derivation \( d' : B \to N \) its restriction to \( I \). Note that this is indeed a homorphism of \( C \)-modules since \( IN = 0 \).

That this sequence is exact is a matter of checking, the case of exactness at the left being easy. To check exactness in the middle, we suppose that \( d : B \to N \) restricts to 0 on \( I \). Then \( d \) factors via a homomorphism \( d' : B/I \to N \), which is \( C \)-linear (since \( d \) is \( B \)-linear). This is the required element in \( \text{Der}_A(C, N) \). □

**Corollary 11.18.** Let \( A \) be a ring and let \( B \) be a finitely generated \( A \)-algebra (or a localization of such). Then \( \Omega_{B|A} \) is finitely generated over \( B \).

**Proof.** Write \( B = A[x_1, \ldots, x_n]/I \) for some variables \( x_1, \ldots, x_n \) and apply the above propositions. □

11.4.4 The diagonal and \( \Omega_{B/A} \)

Suppose now that \( B \) is an \( A \)-algebra. There is an exact sequence of \( A \)-modules

\[
0 \to I \to B \otimes_A B \xrightarrow{\Delta} B \to 0
\]

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where $\Delta$ is the diagonal map $b_1 \otimes b_2 \mapsto b_1b_2$ and $I$ is the kernel of $\Delta$. We can make $I/I^2$ into a $B$-module by letting $B$ act on the first factor by $b(x \otimes y) = (bx) \otimes y$ and then define a map $d : B \to I/I^2$ by $db = 1 \otimes b - b \otimes 1$.

**Proposition 11.19.** The module $I/I^2$ along with the map $d$ is the module of differentials for $B$ over $A$. \(\square\)

**Proof.** Let us check that the map $d$ is indeed a derivation: Suppose $b_1, b_2 \in B$, then

$$d(b_1b_2) = 1 \otimes b_1b_2 - b_1b_2 \otimes 1.$$ 

One the other hand,

$$b_1db_2 + b_2db_1 = b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1)$$

$$= b_1 \otimes b_2 - b_1b_2 \otimes 1 + b_2 \otimes b_1 - b_1b_2 \otimes 1$$

The difference between these two elements is

$$1 \otimes b_1b_2 - b_1b_2 \otimes 1 - b_1 \otimes b_2 + b_1b_2 \otimes 1 - b_2 \otimes b_1 + b_1b_2 \otimes 1$$

$$= (1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1) \in I^2$$

Now one has to check that $(I/I^2, d)$ satisfies the required universal property. We leave this as a task for the reader (See [Matsumura p.192] for details). \(\square\)

### 11.5 Some explicit computations

We can use the previous exact sequences to do explicit computations with $\Omega_{B/A}$. If $B$ is a finitely generated $A$-algebra, say $B = A[x_1, \ldots, x_r]/I$ where $I = (f_1, \ldots, f_r)$. Then we have

$$B \otimes_A \Omega_{A[x_1, \ldots, x_r]/A} \cong \bigoplus_{i=1}^n Bdx_i$$

Note that $I/I^2$ is generated by the classes of the $f_1, \ldots, f_r$ modulo $I^2$. This gives a surjection $B^r \to I/I^2$. Now, by the conormal sequence (Proposition 11.17), we have

$$\Omega_{B/A} = \text{coker}(d : I/I^2 \to \bigoplus_{i=1}^n Bdx_i)$$

$$= \text{coker}(d : B^r \to \bigoplus_{i=1}^n Bdx_i)$$

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Explicitly, the map $B^n \to \bigoplus_{i=1}^{n} Bdx_i$ is given by the $n \times r$ Jacobian matrix $J = \left( \frac{\partial f_i}{\partial x_i} \right)$.

**Theorem 11.20.** Let $A$ be a ring, and let $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then

$$\Omega_{B|A} = \frac{\bigoplus_i Bdx_i}{\sum_j B\left(\sum_i \frac{\partial f_j}{\partial x_i} dx_i\right)}$$

together with the $A$-derivation $d_{B|A}f = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

**Example 11.21.** Let $k$ be a field and let $X = \text{Spec } R$ where $R = k[x, y]/(f)$. Let us compute the module of differentials. Let $f_x, f_y$ denote the (images of the) partial derivatives of $f$ in $R$. Then

$$\Omega_{R|A} = \frac{Rdx \oplus Rdy}{(f_x dx + f_y dy)}$$

If $X$ is non-singular, e.g., $V(f, f_x, f_y) = \emptyset$, then $\Omega_{R|A}$ is locally free of rank 1. The bases are given by

$$\frac{dy}{f_x} \text{ on } D(f_x); \quad \text{and} \quad -\frac{dx}{f_y} \text{ on } D(f_y).$$

(Note that on the overlap $D(f_x) \cap D(f_y)$ we have $f_x dx + f_y dy = 0$ in $\Omega_{R|A}$).

**Example 11.22.** The curve $X$ given by $y^2 = x^2(x+1)$ in $\mathbb{A}^2_k$ is the so-called nodal cubic. It has a singular point at the origin $(0, 0)$. Let $B = k[x, y]/(y^2 - x^2(x+1))$. Then

$$\Omega_{B/k} = \frac{Bdx \oplus Bdy}{(2ydy - (3x^2 + 2x)dx)}$$

In this case $\Omega_{X|k}$ has rank 1 for every point $(x, y) \neq (0, 0)$. At the origin, the relation $2ydy - (3x^2 + 2x)dx$ is identically zero, so $\Omega_{X|k}$ has rank two there.

We can also view $B$ as an algebra over $A = k[x]$. In that case, we get

$$\Omega_{B/A} = \frac{Bdy}{(2ydy)}$$

**Example 11.23.** Let $B = k[x, y]/(x^2 + y^2)$. If $k$ has characteristic $\neq 2$, then

$$\Omega_{B|k} = \frac{(Bdx + Bdy)}{(x^2 + y^2)}$$

If $k$ has characteristic 2, then $\Omega_{B|k}$ is the free $B$ module $Bdx + Bdy$. 

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11.6 The sheaf of differentials

For us, the primary motivation for studying $\Omega_{B/A}$ is that it gives us an intrinsic module $\Omega_{B/A}$ associated to a ring homomorphism $A \to B$. By taking $\sim$, we thus get an intrinsic sheaf on $X = \text{Spec } B$ associated to the map of affine schemes $\text{Spec } B \to \text{Spec } A$. We would like to globalize this construction to an arbitrary morphism of schemes $f : X \to S$. This will lead us to form the sheaf of relative differentials $\Omega_{X|S}$ which will be a quasicoherent $\mathcal{O}_X$-module.

This sheaf is locally built out of the various $\Omega_{B/A}$ on local affine charts. These not arbitrary modules that just happen to glue together to a sheaf; each of them come with a universal property of classifying derivations $d : B \to M$. For this reason, we would like to say that the $\Omega_{X|Y}$ should satisfy a similar universal property. We make the following definition:

Definition 11.24. Let $\mathcal{F}$ be an $\mathcal{O}_X$ module. A morphism $d : \mathcal{O}_X \to \mathcal{F}$ of sheaves is an $S$-derivation if for all open affine subsets $V \subseteq S$ and $U \subseteq X$ with $f(U) \subseteq V$, the map $d_{U|V}$ is a $\mathcal{O}_S(V)$-derivation of $\mathcal{O}_X(U)$. The set of all $S$-derivations is denoted by $\text{Der}_S(\mathcal{O}_X, \mathcal{F})$.

Definition 11.25. The sheaf of relative differentials is a pair $(\Omega_{X|S}, d_{X|S})$ of an $\mathcal{O}_X$-module $\Omega_{X|S}$ and a $S$-derivation $d_{X|S} : \mathcal{O}_X \to \Omega_{X|S}$ that satisfies the following universal property: For any $\mathcal{O}_X$-module $\mathcal{F}$, and each $S$-derivation $d : \mathcal{O}_X \to \mathcal{F}$ there exists a unique $\mathcal{O}_X$-linear map $\phi : \Omega_{X|S} \to \mathcal{F}$ such that $d = \phi \circ d_{X|S}$.

When $S = \text{Spec } A$, we sometimes write $\Omega_{X|A}$ for $\Omega_{X|S}$.

In other words, $\Omega_{X|S}$ is a sheaf that represents the sets of $S$-derivations, in the sense that

$$\text{Hom}_{\mathcal{O}_S}(\Omega_{X|S}, \mathcal{F}) = \text{Der}_S(\mathcal{O}_X, \mathcal{F}).$$

As usual, the universal property gives that this sheaf is unique up isomorphism, if it exists.

Theorem 11.26. Let $f : X \to S$ be a morphism of schemes. Then there is a sheaf of relative differentials $\Omega_{X|S}$, which is a quasicoherent sheaf on $X$.

Moreover, $\Omega_{X|S}$ has the property that for each open affine open $V = \text{Spec } A$, and open affine $U = \text{Spec } B \subseteq f^{-1}(V)$,

$$\Omega_{X|S}|_U \simeq \Omega_{B|A}$$

Also for each $x \in X$,

$$(\Omega_{X|S})_x \simeq \Omega_{\mathcal{O}_{X,x}|\mathcal{O}_{S,f(x)}}$$
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Proof. Fix an open subset $V = \text{Spec} \ A$ of $S$, and let $U = \text{Spec} \ B$ be an affine open subset in $X$ so that $f(U) \subseteq V$. For these two, we define

$$\Omega_{U|V} = \tilde{\Omega}_{B|A}$$

on $U$. We first show that the $\Omega_{U|V}$ glue to a $\mathcal{O}_{f^{-1}V}$-module $\Omega_{f^{-1}(V)|V}$. This comes down to showing that if $U' = \text{Spec} \ B'$ is a distinguished open affine subset of $U$, then

$$\Omega_{U|V|U'} \simeq \Omega_{U'|V}.$$ 

But this follows from Proposition 11.14, as $B'$ is a localization of $B$.

Then we show that the sheaves $\Omega_{f^{-1}V|V}$ for all affine opens $V \subseteq S$ glue to a $\mathcal{O}_{X}$-module $\Omega_{X|S}$. This amounts to showing that for each distinguished open $V' = \text{Spec} \ A' \subseteq V$, and all open $U = \text{Spec} \ B$ of $f^{-1}(V')$, we have

$$\Omega_{U|V} = \Omega_{U|V'}$$

But this follows from Proposition 11.13, as $A'$ is a localization of $A$ in a single element (which maps to an invertible element in $B$).

This means that we get an $\mathcal{O}_{X}$-module $\Omega_{X|S}$. Let us check that it satisfies the above universal property. So we need to define the universal derivation $d_{X|S} : \mathcal{O}_{X} \to \Omega_{X|S}$.

Let $V = \text{Spec} \ A \subseteq S$ and $U = \text{Spec} \ B \subseteq X$ be an affine open subset such that $f(U) \subseteq V$. Define $d_{X|S}(U) = d_{B|A}$. By the gluing construction above, this map does not depend on the chosen affine open $V$, and it can be checked that the rule is compatible with restriction maps. Hence this gives a morphism of sheaves $d_{X|S} : \mathcal{O}_{X} \to \Omega_{X|S}$, which by construction is a $S$-derivation.

To check that this is universal, we again work locally. Let $d : \mathcal{O}_{X} \to \mathcal{F}$ be a $S$-derivation, where $\mathcal{F}$ is an $\mathcal{O}_{X}$-module. Let $U = \text{Spec} \ A \subseteq S$ and $V = \text{Spec} \ B \subseteq X$ so that $f(U) \subseteq V$. By the universal property of $\Omega_{B|A}$, we get an $A$-derivation $d(V) : B \to \mathcal{F}(V)$, and hence a unique $B$-linear map $\phi(V) : \Omega_{X|S}(V) = \Omega_{B|A} \to \mathcal{F}$ such that $d(V) = \phi(V) \circ d_{X|V}(V)$. One has to check that these maps are compatible with restriction maps (use the universal property of $\Omega_{B|A}$), but after that, we obtain a unique $\mathcal{O}_{X}$-linear map $\phi : \Omega_{X|S} \to \mathcal{F}$ so that $d = \phi \circ d_{X|S}$.

Note that the sheaf $\Omega_{X|S}$ is always quasi-coherent (it is by definition locally of the form $\tilde{M}$ for some module). Moreover, when $X$ is of finite type over a field, $\Omega_{B|A}$ is finitely generated, and so $\Omega_{X|k}$ is even coherent.

Example 11.27. Let $A$ be a ring and let $X = A_{S}^{n} = \text{Spec} \ S[x_{1}, \ldots, x_{n}]$ be affine $n$-space over $S = \text{Spec} \ A$. Then $\Omega_{X|S} \simeq \mathcal{O}_{X}^{n}$ is the free $\mathcal{O}_{X}$-module generated by $dx_{1}, \ldots, dx_{n}$.
Remark 11.28. If $X$ is a separated scheme over $S$ then one could also define $\Omega_{X/S}$ as follows. Let $\Delta : X \to X \times_S X$ be the diagonal morphism and let $\mathcal{I}_\Delta$ be the ideal sheaf of the image of $\Delta$. Then $\Omega_{X/S} = \Delta^*(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)$. This does in fact give the sheaf above, since these two definitions coincide when $X$ and $S$ are both affine (Section 11.4.4). This definition gives a quick way of obtaining the sheaf $\Omega_{X/S}$, but it is not very well suited for computations.

The properties of $\Omega_{B/A}$ translate into the following results for $\Omega_{X/Y}$:

Proposition 11.29 (Base change). Let $f : X \to S$ be a morphism of schemes and let $S'$ be a $S$-scheme. Let $X' = X \times_S S'$ and let $p : X' \to X$ be the projection. Then

$$\Omega_{X'/S'} \cong p^*\Omega_{X/S}$$

Proposition 11.30. Let $X$, $Y$, and $Z$ be schemes along with maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then there is an exact sequence of $\mathcal{O}_X$-modules

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Proposition 11.31 (Conormal bundle sequence). Let $Y$ be a closed subscheme of a scheme $X$ over $S$. Let $\mathcal{I}_Y$ be the ideal sheaf of $Y$ on $X$. Then there is an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_{X/S} \otimes \mathcal{O}_Y \to \Omega_{Y/S} \to 0.$$

11.7 The sheaf of differentials of $\mathbb{P}^n_A$

We have seen that the cotangent sheaf of $\mathbb{A}^n$ is trivial, i.e., isomorphic to $\mathcal{O}^n_{\mathbb{A}^n}$. In this sections we give a concrete description of the cotangent bundle of projective space, suitable for explicit computations.

Theorem 11.32. Let $X = \mathbb{P}^n_A$, then there is an exact sequence

$$0 \to \Omega_{X/A} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

Proof. Let $R = A[x_0, \ldots, x_n]$ so that $X = \text{Proj} R$. Let $T$ be the $R$-module $R(-1)^{n+1}$ with basis elements $e_0, \ldots, e_n$. Define the map $\alpha : T \to R$ by sending $e_i \mapsto x_i$ and let $M = \ker \alpha$. We have an exact sequence

$$0 \to M \to T \xrightarrow{\alpha} R$$

Taking $\sim$ this gives

$$0 \to \widetilde{M} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$
which is surjective on the right, because the map $\alpha$ is surjective in degrees $\gg 0$ (the cokernel $C$ of $\alpha$ is a sheaf with $C_d = 0$ for large $d$, and so it has $\tilde{C} = 0$). We claim that $\tilde{M} = \Omega_{X/A}$.

Let $U_i = D(x_i) = \text{Spec } R(x_i)$ denote the standard open affines. We have

$$\Omega_{X/A}|_{U_i} = \Omega_{U_i/A} = \Omega_{R(x_i)/A}$$

and $\Omega_{R(x_i)/A}$ is generated as an $R(x_i)$-module by the expressions $d\left(\frac{x_i}{x_j}\right)$ for $i \neq j$. Now define the maps

$$\phi_i : \Omega_{X/Y}|_{U_i} \rightarrow \tilde{M}|_{U_i}$$

$$d\left(\frac{x_i}{x_j}\right) \mapsto \frac{1}{x_i}(x_i e_j - x_j e_i)$$

Note that $x_i e_j - x_j e_i \in \ker \alpha$. The factor $\frac{1}{x_i}$ is included to obtain an element of degree zero. This is an isomorphism on $U_i$, so we are done if we can make sure that the $\phi_i$ glue. Now, on the overlaps $U_i \cap U_j$, we have $\frac{x_k}{x_i} = \frac{x_k}{x_j} \frac{x_j}{x_i}$, which gives

$$d\left(\frac{x_k}{x_i}\right) = \frac{x_k}{x_j} d\left(\frac{x_j}{x_i}\right) + \frac{x_j}{x_i} \frac{x_k}{x_j}$$

or, in other words,

$$\frac{x_j}{x_i} d\left(\frac{x_k}{x_i}\right) = d\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j} d\left(\frac{x_j}{x_i}\right)$$

Applying $\phi_i$ to both sides, gives the same answer $\left(\frac{1}{x_i x_j} (x_j e_k - x_k e_j)\right)$, so the maps $\phi_i$ are compatible.

Since $\Omega_{\mathbb{P}^n_A}$ injects into $\mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1}$ (which has no global sections), we get:

**Corollary 11.33.** $\Gamma(\mathbb{P}^n_A, \Omega_{\mathbb{P}^n_A}) = 0$

### 11.8 Relation with the Zariski tangent space

Let $X$ be a scheme over an algebraically closed field $k$. Recall that for a point $x \in X$, the Zariski tangent space is defined as $m/m^2$ where $m$ is the maximal ideal in the local ring $B = \mathcal{O}_{X,x}$. In this section we relate this to the module of differentials $\Omega_{R/k}$. Note that since $k$ is algebraically closed, the Nullstellensatz gives that $B/m = k$.

**Proposition 11.34.** Suppose $(B, m)$ is a local ring with residue field $k = B/m$
and there is an injection $k \hookrightarrow B$. Then there is an exact sequence

$$\frac{m}{m^2} \xrightarrow{d} \Omega_{B/k} \otimes_B k \rightarrow 0$$

and $d$ is actually an isomorphism.

**Proof.** The exact sequence is obtained from the conormal sequence by letting $A = C = k$.

The above map sends $x \in m$ to $dx$. We construct an inverse $\psi : \Omega_{B|k} \otimes_B k \rightarrow \frac{m}{m^2}$ as follows. Constructing such a map is equivalent to constructing a map of $B$-modules $\Omega_{B|k} \rightarrow \frac{m}{m^2}$, or equivalently, a derivation $\partial : B \rightarrow \frac{m}{m^2}$. We define $\partial$ by

$$\partial(a + x) = x$$

for $a \in k$ and $x \in m$. (This is well-defined since $k + m = B$). To complete the proof we need only prove that $\partial$ is a derivation:

$$\partial((a + x)(a' + x')) = \partial(aa + (ax' + a'x) + x'x)$$

$$= \partial(aa) + \partial(ax' + a'x) + \partial(x'x)$$

$$= ax' + a'x$$

We get the same answer when we expand

$$(a' + x')\partial(a + x) + (a + x)\partial(a' + x')$$

(since $xx' \in m^2$). Hence the map is a derivation, and we get a map $\psi$. This is indeed an inverse to the map $d$, since via the identification $\text{Der}_A(B, M) = \text{Hom}_B(\Omega_{B|A}, A)$, $\psi$ sends $dx$ to $x$.

**Corollary 11.35.** Let $(B, m)$ be as above. Then $B$ is a regular local ring if and only if

$$\dim B_m = \dim_k \Omega_{B|k} \otimes_B k$$

**Definition 11.36.** Let $X$ be a scheme over a field $k$ and let $x \in X$ be a point. We say that $X$ is smooth at $x$ if $\Omega_{X|k}$ is locally free at $x$. $X$ is called smooth if it is smooth at every point.

**Theorem 11.37.** Let $X$ be a variety over an (algebraically closed) field $k$ of characteristic $0$ and let $x \in X$. Then the following are equivalent:

(i) $X$ is smooth at $x$

(ii) $(\Omega_{X|k})_x$ is free of rank $\dim_x X$

(iii) $X$ is non-singular at $x$.  

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Proof. (i) $\iff$ (ii) is clear.

(ii) $\implies$ (iii). If $\Omega_{B/k}$ is free of rank $n = \dim R$, then by the above proposition, we have $\dim_{k(x)} m/m^2 = n$ and so $B_m$ is a regular local ring.

(iii) $\implies$ (i). This direction requires some commutative algebra of $\Omega_{B/k}$ we haven’t covered. See [Matsumura p. 191] or [Hartshorne p. 174] for details.

**Definition 11.38.** When $X$ is smooth and $\Omega_{X|k}$ is a locally free sheaf on $X$, and we refer to it as the cotangent bundle or cotangent sheaf of $X$.

The sheaf of $p$-forms is defined as

$$\Omega^p_X = \bigwedge^p \Omega_{X|k}$$

In particular, if $X$ has dimension $n$, $\omega_X = \Omega^n_X$ is called the canonical bundle of $X$. As $\Omega_{X|k}$ has rank $n$, $\omega_X$ is locally free of rank 1, i.e., an invertible sheaf.

The tangent sheaf, or tangent bundle is the sheaf

$$\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X|k}, \mathcal{O}_X)$$

Thus for $X = \mathbb{P}^1$, we get $\Omega_{\mathbb{P}^1} = \mathcal{O}(-2)$ and $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$.

**Proposition 11.39.** Let $X = \mathbb{P}^n$. Then $\Omega^n_X = \mathcal{O}(-n - 1)$.

Proof. Consider the Euler sequence for the cotangent bundle of $\mathbb{P}^n$

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0$$

In general, if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, we have $\bigwedge^e \mathcal{E} = \bigwedge^{e'} \mathcal{E}' \otimes \bigwedge^{e''} \mathcal{E}''$. Hence we get

$$\bigwedge^n \Omega_{\mathbb{P}^n} = \bigwedge^{n+1} \mathcal{O}(-1) = \mathcal{O}(-n - 1).$$

Note by the way that the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ fits into the following sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to \mathcal{T}_X \to 0$$

where the left-most map sends 1 to the vector $(x_0, \ldots, x_n)$. \(\square\)

**Example 11.40.** Let $A$ be a ring and let $X = \mathbb{P}^1_A$. Then we have $\Omega_{X|A} \simeq \mathcal{O}_{\mathbb{P}^1}(−2)$. For this, we can use the standard covering of $\mathbb{P}^1 = \text{Proj} A[x_0, x_1]$, given by $U_i = D_+(x_i)$. On $U_0$, we have

$$\Omega_{U_0|A} \simeq A[\frac{x_1}{x_0}]d\left(\frac{x_1}{x_0}\right)$$

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, and similarly on $U_1$. On the intersection $D_+(x_0) \cap D_+(x_1)$, we have

$$x_0^2 d\left(\frac{x_1}{x_0}\right) = -x_1^2 d\left(\frac{x_0}{x_1}\right)$$

This gives a non-vanishing section of $\Omega_X(2) \otimes \Omega_X|_A$ and furthermore an isomorphism $\Omega_X|_A \simeq \mathcal{O}_X(-2)$.

11.8.1 Smooth morphisms

We can also use the sheaves of differentials to define a notion of smoothness for morphisms. In short, we say that a morphism $f : X \to S$ at a point $x \in X$ if smooth if $\Omega_X|_S$ is locally free of rank $\dim_x X - \dim_{f(x)} S$ there. If this is not the case, we say that $f$ is ramified at $x$, and that $x$ is a ramification point. $f$ is smooth if it is smooth at every point.

Example 11.41. Let $A = k[x]$ and let $B = k[x, y]/(x - y^2) \simeq k[y]$ where $k$ is a field of characteristic not equal to 2. Let $X = \text{Spec } B$ and let $Y = \text{Spec } A$. Let $f : X \to Y$ be the morphism induced by the inclusion $A \hookrightarrow B$ (thus $x \mapsto y^2$).

![Diagram](image)

Since $B \simeq k[y]$ it follows that $\Omega_X = Bdy$, the free $B$-module generated by $dy$. Similarly $\Omega_Y = Adx$. The conormal sequence gives us

$$\rightarrow \Omega_Y \otimes_A B \rightarrow \Omega_X \rightarrow \Omega_{X/Y} \rightarrow 0$$

$$\begin{array}{c}
Bdx \\
\| \| \| \\
Bdy \\
\| \|
\end{array}
\rightarrow
\begin{array}{c}
Bdy \rightarrow Bdy/B(2ydy)
\end{array}$$

The point is that $\Omega_{X/Y} = (k[y]/(2y))dy$ is a torsion sheaf supported on the ramification locus of the map $f : X \to Y$. (The only ramification point is above 0.) Note that $\Omega_{X/Y}$ is the quotient of $\Omega_X$ by the submodule generated by the image of $dx$ in $\Omega_X = Bdy$. The image of $dx$ is $2ydy$.  

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11.9 The conormal sheaf

Let \( Y \subseteq X \) be a closed subscheme defined by an ideal sheaf \( \mathcal{I} \). Then \( \mathcal{I}/\mathcal{I}^2 \) is naturally a \( \mathcal{O}_Y \)-module via \( \mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes \mathcal{O}_X/\mathcal{I} = \mathcal{I} \otimes \mathcal{O}_Y \). We call \( \mathcal{I}/\mathcal{I}^2 \) the conormal sheaf of \( Y \). Its dual, \( \mathcal{N}_Y = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \) is the normal sheaf of \( Y \) in \( X \).

**Proposition 11.42.** When \( X \) and \( Y \) are non-singular schemes over a field \( k \), the sheaves \( \mathcal{I}/\mathcal{I}^2 \) and \( \mathcal{N}_Y \) are locally free of rank \( r = \text{codim}(Y, X) \).

In this case, they fit into the exact sequences

\[
0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_X|k|Y \to \mathcal{O}_Y(k) \to 0
\]

\[
0 \to \mathcal{T}_Y \to \mathcal{T}_X|Y \to \mathcal{N}_Y \to 0
\]

(This is the conormal bundle sequence and its dual respectively. The first sequence is exact on the left, because \( \mathcal{I}/\mathcal{I}^2 \) is locally free).

Taking \( \bigwedge \) of these sequences we get the following result, which is very useful for computing canonical bundles of subvarieties:

**Proposition 11.43** (Adjunction formula). If \( X \) and \( Y \) are non-singular, we have

\[
\omega_Y = \omega_X \otimes \bigwedge^r \mathcal{N}_Y = \omega_X \otimes \det \mathcal{N}_Y
\]

In particular, if \( Y \subseteq X \) is a smooth divisor, we get

\[
\omega_Y = \omega_X \otimes \mathcal{O}_X(Y)|_Y
\]

**Example 11.44.** Let \( X \subseteq \mathbb{P}^2 \) be a non-singular plane curve of degree \( d \). Then \( \mathcal{I}_X = \mathcal{O}_{\mathbb{P}^2}(-d) \), and so \( \mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^2}/\mathcal{I} = \mathcal{O}_Y(-d) \). Here \( \bigwedge^1 \mathcal{I}/\mathcal{I}^2 = \mathcal{O}_Y(-d) \) so the previous proposition shows that

\[
\omega_Y \simeq \mathcal{O}_Y(d - 3)
\]

For \( d = 1 \), this gives our previous computation that \( \Omega_{\mathbb{P}^1|k} = \mathcal{O}_{\mathbb{P}^1}(-2) \).

Also for \( d = 2 \), \( Y \simeq \mathbb{P}^1 \), and \( \Omega_Y = \mathcal{O}_{\mathbb{P}^2}(-1)|_Y \). This is consistent with the previous example, because \( \mathcal{O}_{\mathbb{P}^2}(1)|_Y \simeq \mathcal{O}_{\mathbb{P}^1}(2) \).

For \( d \geq 3 \), the invertible sheaf \( \omega_Y \) has many global sections. In particular, we recover the fact that a non-singular plane curve of degree \( d \geq 3 \) is not rational (i.e., not isomorphic to \( \mathbb{P}^1 \)). In fact, it will follow from the results of the next chapter that \( \Gamma(Y, \omega_Y) \) has dimension exactly \( \frac{1}{2}(d - 1)(d - 2) \). So for \( d \geq 3 \), no two smooth curves of different degrees can be isomorphic.
Chapter 12
First steps in sheaf cohomology

We have seen several examples of the global sections functor $\Gamma$ failing to be exact when applied to a short exact sequence. On the other hand, we have a sequence

\[ 0 \to \Gamma(X, F') \to \Gamma(X, F) \to \Gamma(X, F'') \]

which is exact at each stage except on the right. In many situations in algebraic geometry, knowing that $\Gamma(X, F) \to \Gamma(X, F'')$ is surjective of fundamental importance. We will see several examples of this phenomenon later in this chapter. Essentially, cohomology is a tool that allows us to continue this sequence, so that the sequence can be continued as follows:

\[ 0 \to \Gamma(X, F') \to \Gamma(X, F) \to \Gamma(X, F'') \]

\[ \to H^1(X, F') \to H^1(X, F) \to H^1(X, F'') \]

\[ \to H^2(X, F') \to H^2(X, F) \to H^2(X, F'') \to \cdots \]

In other words, the failure of surjectivity of the above is controlled by the group $H^1(X, F')$, and the other groups in the sequence.

Cohomology groups can be defined in a completely general setting, for any topological space $X$ and a sheaf $F$ on it. With that as input, we define the cohomology groups $H^k(X, F)$, which will capture the main geometric invariants of $F$. These should also be functorial in $F$, which means that we want to construct
12.1. Some homological algebra

functors

\[ H^q(X, -): \text{AbSh}_X \to \text{Ab} \]
\[ \mathcal{F} \mapsto H^q(X, \mathcal{F}) \]

satisfying the following properties:

(I) \( H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X) \),

(II) A morphism of sheaves \( \phi: \mathcal{F} \to \mathcal{G} \) induces maps \( H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G}) \) which are functorial and take the identity to the identity (e.g., \( H^i(X, -) \) is a functor).

(III) Every short exact sequence \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) gives a long exact sequence as above.

There are several ways to define these groups. The modern approach, and the one summarized in Hartshorne Chapter III, takes the approach of using derived functors. This is in most respects this is the ‘right way’ define the groups in general, but going through the whole machinery of derived functors and homological algebra would take us too far astray. We take a more down-to-earth approach using Čech cohomology and the Godement resolution. The Godement resolution has the advantage that it is completely canonical, and we can prove the main theorems we need by hand. On the other hand, the Čech resolution, which depends on the choice of a covering of \( X \), but is better suited for computations. Of course, the two notions of cohomology of course turn out to be the same, see section XX.

12.1 Some homological algebra

A complex of abelian groups \( A^\bullet \) is a sequence of groups \( A^i \) together with maps between them

\[ \cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots \]

such that \( d^{i+1} \circ d^i = 0 \) for each \( i \). A morphism of two complexes \( A^\bullet \xrightarrow{f_n} B^\bullet \) is a collection of maps \( f_n: A^n \to B^n \) so that the following diagram commutes:

\[ \cdots \xrightarrow{f_{i-1}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots \]
\[ \xrightarrow{f_i} \cdots \xrightarrow{f_{i+1}} B^{i-1} \xrightarrow{d^{i-1}} B^i \xrightarrow{d^i} B^{i+1} \xrightarrow{d^{i+1}} \cdots \]

In this way, we can talk about kernels, images, cokernels, exact sequences of complexes, etc.
Let $Z^n A^\bullet = \ker d^n$ and $B^i A^\bullet = \operatorname{im} d^{i-1}$. From this condition on the $d^i$, we have $Z^n A^\bullet \supseteq B^n A^\bullet$. We define the cohomology groups of $A^\bullet$, denoted $H^p A^\bullet$ as

$$H^p A^\bullet = \frac{Z^n (A^\bullet)}{B^n (A^\bullet)} = \frac{\ker d^n}{\operatorname{im} d^{n-1}}$$

A resolution of a group $G$ is a complex of the form

$$0 \to G \to A^0 \to A^1 \to A^2 \to \ldots$$

where the maps are exact at each $A^i$, i.e., $H^p A^\bullet = 0$ for each $p \geq 0$.

**Proposition 12.1.** Suppose that $0 \to F^\bullet \xrightarrow{f} G^\bullet \xrightarrow{g} H^\bullet \to 0$ is an exact sequence of complexes. Then there is a long exact sequence

$$\cdots \to H^n F^\bullet \to H^n G^\bullet \to H^n H^\bullet \to H^{n+1} F^\bullet \to H^{n+1} G^\bullet \to H^{n+1} H^\bullet \to \cdots$$

**Proof.** Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \to & F^n & \xrightarrow{f_n} & G^n & \xrightarrow{g_n} & H^n & \to & 0 \\
& & d^n & \downarrow & d^n & \downarrow & d^n & \\
0 & \to & F^{n+1} & \xrightarrow{f_{n+1}} & G^{n+1} & \xrightarrow{g_{n+1}} & H^{n+1} & \to & 0
\end{array}$$

where the rows are exact. By the snake lemma, we obtain a sequence

$$0 \to Z^n (F^\bullet) \xrightarrow{f_n} Z^n (G^\bullet) \xrightarrow{g_n} Z^n (H^\bullet) \xrightarrow{\delta} F^{n+1} / B^n (F^\bullet) \xrightarrow{f_{n+1}} G^{n+1} / B^n (G^\bullet) \xrightarrow{g_{n+1}} H^{n+1} / B^n (H^\bullet) \to 0$$

Consider now the diagram

$$\begin{array}{ccccccc}
0 & \to & F^n / B^n (F^\bullet) & \xrightarrow{f_n} & G^n / B^n (G^\bullet) & \xrightarrow{g_n} & H^n / B^n (H^\bullet) & \to & 0 \\
& & d^n & \downarrow & d^n & \downarrow & d^n & \\
0 & \to & Z^{n+1} (F^\bullet) & \xrightarrow{f_{n+1}} & Z^{n+1} (G^\bullet) & \xrightarrow{g_{n+1}} & Z^{n+1} (H^\bullet)
\end{array}$$

where the rows are exact. Applying the snake lemma one more time, we get the desired exact sequence. \qed

A complex of sheaves $F^\bullet$ is a sequence of sheaves with maps between them

$$\cdots \to F_{i-1} \xrightarrow{d_{i-2}} F_i \xrightarrow{d_{i-1}} F_{i+1} \xrightarrow{d^i} \ldots$$

such that $d^{i+1} \circ d^i = 0$ for each $i$. Given such a complex, we define the cohomology sheaves $H^p F^\bullet$ as $\ker d^i / \operatorname{im} d^{i-1}$. A resolution of a sheaf $F$ is a complex of the
12.2 Čech cohomology

Let \( X \) be a topological space, and let \( \mathcal{F} \) be a sheaf on it. Let \( U = \{ U_i \} \) be an open cover of \( X \), indexed by an ordered set \( I \). As we saw previously, the sheaf axioms give that the following sequence is exact:

\[
0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j).
\]

The Čech complex is essentially the continuation of this sequence; it is a complex obtained by adjoining all the groups \( \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \) over all intersections \( U_{i_1} \cap \cdots \cap U_{i_r} \).

**Definition 12.2.** For a sheaf \( \mathcal{F} \) on \( X \), define the Čech complex of \( \mathcal{F} \) (with respect to \( U \)) as

\[
C^\bullet(U, \mathcal{F}) : C^0(U, \mathcal{F}) \xrightarrow{d^0} C^1(U, \mathcal{F}) \xrightarrow{d^1} C^2(U, \mathcal{F}) \xrightarrow{d^2} \cdots
\]

where

\[
C^0(U, \mathcal{F}) = \prod_i \mathcal{F}(U_i),
\]

\[
C^1(U, \mathcal{F}) = \prod_{i_0 < i_1} \mathcal{F}(U_{i_0} \cap U_{i_1}),
\]

\[
\vdots
\]

\[
C^p(U, \mathcal{F}) = \prod_{|I|=p+1} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}).
\]

and the coboundary map \( d : C^p(U, \mathcal{F}) \to C^{p+1}(U, \mathcal{F}) \) by

\[
(d^p \sigma)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0, \ldots, \hat{i}_j, \ldots, i_{p+1}}|_{U_{i_0} \cap \cdots \cap U_{i_p}}
\]

where \( i_0, \ldots, \hat{i}_j, \ldots, i_{p+1} \) means \( i_0, \ldots, i_{p+1} \) with the index \( i_j \) omitted.

**Example 12.3.** Let us look at the first few maps in the complex:
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In degree 0: \( d^0 : C^0(U, \mathcal{F}) \to C^1(U, \mathcal{F}) \). If \( \sigma = (\sigma_i)_i \), then

\[
(d^0 \sigma)_{ij} = \sigma_j - \sigma_i
\]

In degree 1: \( d^1 : C^1(U, \mathcal{F}) \to C^2(U, \mathcal{F}) \). If \( \sigma = (\sigma_{ij})_i \), then

\[
(d^1 \sigma)_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}
\]

Notice that in the example that \( d^1 \circ d^0 = 0 \) (all the \( \sigma_{ij} \) cancel). This happens also in higher degrees by a basic computation using the definition of \( d^p \).

**Lemma 12.4.** \( d^{p+1} \circ d^p = 0 \).

In particular, the \( C^\bullet(U, \mathcal{F}) \) forms a complex of abelian groups. We say that an element \( \sigma \in C^p \) is a cocycle if \( d^p \sigma = 0 \), and a coboundary if \( \sigma = d^{p-1} \tau \). By the lemma, all coboundaries are cocycles. The cohomology groups of \( \mathcal{F} \) are set up to measure the difference between these two notions:

**Definition 12.5.** The \( p \)-th Čech cohomology of \( \mathcal{F} \) with respect to \( U \) is defined as

\[
H^p(U, \mathcal{F}) = (\ker d^p)/\text{im} d^{p-1}
\]

One can show that a sheaf homomorphism \( \mathcal{F} \to \mathcal{G} \) induces a mapping of Čech cohomology groups, so we obtain functors \( \mathcal{F} \to H^p(U, \mathcal{F}) \).

**Example 12.6.** Again it is instructive to consider the group \( H^0(U, \mathcal{F}) \). Here the map \( d^0 : C^0(U, \mathcal{F}) \to C^1(U, \mathcal{F}) \), which is simply the usual map

\[
\prod F(U_i) \to \prod F(U_i \cap U_j)
\]

which has kernel \( F(X) \) by the sheaf axioms. It follows that \( H^0(U, \mathcal{F}) = \mathcal{F}(X) \).

**Example 12.7.** \( X = S^1, \mathcal{U} = \{U, V\} \) the standard covering of \( S^1 \) (intersecting in two intervals) and let \( \mathcal{F} = \mathbb{Z}_X \) (the constant sheaf).

Here we have

\[
C^0(U, \mathcal{F}) = \mathbb{Z}_U \times \mathbb{Z}_V \quad C^1(U, \mathbb{Z}) = \mathbb{Z}_{U \cap V} = \mathbb{Z} \times \mathbb{Z}
\]

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The map $C^0 \to C^1$ is the map $d^0 : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $d^0(a,b) = (b-a, b-a)$. Hence

$$H^0(U, \mathbb{Z}_X) = \ker d = \mathbb{Z}(1,1) \simeq \mathbb{Z}$$

and

$$H^1(U, \mathbb{Z}_X) = \text{coker } d = \mathbb{Z} \times \mathbb{Z}/\mathbb{Z}(1,1) \simeq \mathbb{Z}$$

Students familiar with algebraic topology, might recognize that this is gives the same answer as singular cohomology.

**Exercise 38.** Let $X = S^1$ and let $\mathcal{U}$ be the covering of $X$ with three pairwise intersecting open sets with empty triple intersection. Show that the Čech -complex is of the form

$$\mathbb{Z}^3 \xrightarrow{d^0} \mathbb{Z}^3 \to 0$$

Compute the map $d^0$ and use it to verify again that $H^i(U, \mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 1$ as above.

**Exercise 39.** Let $X = S^2$ and let $\mathcal{U}$ be the covering of $X$ with four pairwise intersecting open sets with empty triple intersection. Show that the Čech -complex takes the form

$$\mathbb{Z}^4 \xrightarrow{d^0} \mathbb{Z}^6 \xrightarrow{d^1} \mathbb{Z}^4 \to 0$$

Compute the matrices $d^0, d^1$ and show that $H^i(U, \mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 2$ and $H^i(U, \mathbb{Z}_X) = 0$ for $i \neq 0, 2$.

The following proposition shows that constant sheaves are not so interesting for our purposes.

**Proposition 12.8.** Let $X$ be an irreducible topological space (so that any open set $U \subseteq X$ is connected). Then for any covering $\mathcal{U}$ of $X$ we have for a constant sheaf $A_X$

$$H^i(U, A_X) = 0$$

for $i > 0$.

**Proof.**

At this point, there are good news and bad news. As seen in the examples above, the groups $H^p(U, \mathcal{F})$ are easily computable, if one is given a nice cover of $X$. Indeed, the maps in the Čech complex are completely explicit, and computing their kernels and cokernels are usually just a computation in linear algebra.

On the other hand, the definition is a little bit unsatisfactory for various reasons. First of all, the groups $H^p(U, \mathcal{F})$ depend on the cover $\mathcal{U}$, whereas we want something canonical that only depends on $\mathcal{F}$. More importantly, it is not clear that the definition above should capture enough of the desired information.
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of $\mathcal{F}$. For instance, $\mathcal{U}$ could consist of the single open set $X$, and so $H^i(\mathcal{U}, \mathcal{F}) = 0$ for $i \geq 1$! Finally, it is not at all clear if these groups satisfy the requirements mentioned in the introduction.

There is a fix for these problems, which involves passing to finer and finer ‘refinements’ of the covering. We say that a covering $\mathcal{V}$ is a refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ so that $V \subseteq U$. This defines an ordering on the coverings which we denote by $\mathcal{V} \leq \mathcal{U}$. One can then define a group $H^p(\mathcal{X}, \mathcal{F})$ to be the direct limit of all $H^p(\mathcal{U}, \mathcal{F})$ as $\mathcal{U}$ runs through all possible open covers $\mathcal{U}$ ordered by $\leq$. The resulting groups are indeed canonical, and turn out to give the right answer:

**Definition 12.9.** The groups $H^p(\mathcal{X}, \mathcal{F})$ are called the **cohomology groups of $\mathcal{F}$**. In symbols,

$$H^p(\mathcal{X}, \mathcal{F}) = \lim_{\to} H^p(\mathcal{U}, \mathcal{F})$$

The main properties of Čech cohomology are summarized in the following theorem:

**Theorem 12.10.** Let $\mathcal{X}$ be a topological space and let $\mathcal{F}$ be a sheaf on $\mathcal{X}$.

- The Čech cohomology groups are functors $H^i(\mathcal{X}, -) : \text{Sh}_X \to \text{Groups}$.
- $H^0(\mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{X}, \mathcal{F}) = \mathcal{F}(\mathcal{X})$,
- Short exact sequences of sheaves induce long exact sequences of cohomology.
- (Leray’s theorem) If $\mathcal{F}$ is a sheaf and $\mathcal{U}$ is a covering such that $H^i(U_{i_1} \cap \cdots U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$ and multiindexes $i_1 < \cdots < i_p$, then

$$H^i(\mathcal{X}, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F}).$$

We have proved the first two of these properties. The next two require more work, but the arguments are mostly formal. We will nevertheless postpone the proof until Appendix 13.

Note that the last statement shows that it suffices to compute the Čech cohomology groups on a covering which is ‘sufficiently fine’ in the sense that the higher groups $H^i(U_{i_1} \cap \cdots U_{i_p}, \mathcal{F}) = 0$ vanish for $i > 0$.

**Remark 12.11.** Let us at least say a few words about the first boundary map $\delta$ in the long-exact sequence above. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves. We would like to construct the boundary map $\delta$ which is a map

$$\delta : H^0(\mathcal{X}, \mathcal{F}'') \to H^1(\mathcal{X}, \mathcal{F}')$$
If each restriction map
\[ \Gamma(U_i, \mathcal{F}) \to \Gamma(U_i, \mathcal{F}') \] (12.2.1)
is surjective, then we can explicitly describe \( \delta \) as follows: Represent a class \( \sigma \in H^0(X, \mathcal{F}'') \) by a collection of elements \( f_i \in \Gamma(U_i, \mathcal{F}'') \) so that \( f_i = f_j \) on \( U_i \cap U_j \). If the above restriction map is surjective, we can choose lifts \( g_i \in \Gamma(U_i, \mathcal{F}) \) and define \( f_{ij}' = g_i - g_j \). The \( f_{ij}' \) then form a 2-cocycle, since \( f_{ij}' - f_{ik}' + f_{jk}' = 0 \). So we get a well defined class \( \delta(\sigma) \in H^1(X, \mathcal{F}') \) (of course one needs to check that this is independent of the choice of lift, but this is straightforward). Hence we get a sequence
\[
0 \to H^0(\mathcal{F}') \to H^0(\mathcal{F}) \xrightarrow{\delta} H^0(\mathcal{F}) \to H^1(\mathcal{F}) \to H^1(\mathcal{F}'')
\]
which, by another diagram chase is exact.

In general, it can certainly happen that the restriction map (12.2.1) is not surjective – one can for instance take the open covering of \( X \) with just one open set \( X \). This explains why the \( \check{\text{C}}ech \) cohomology groups \( H^i(\mathcal{U}, \mathcal{F}) \) do not give long exact sequences in general. However, by passing to smaller refinements \( \mathcal{V} \leq \mathcal{U} \), we can arrange that any section lifts and we can use the above to construct \( \delta \).

## 12.3 Cohomology of sheaves on affine schemes

**Theorem 12.12.** Let \( X = \text{Spec} \, A \) and let \( \mathcal{F} \) be a quasi-coherent sheaf on \( X \). Then
\[
H^p(X, \mathcal{F}) = 0 \text{ for all } p > 0.
\]

**Proof.** Recall that we defined the groups \( H^i(X, \mathcal{F}) \) by taking the direct limit of \( H^i(\mathcal{U}, \mathcal{F}) \) finer and finer coverings \( \mathcal{U} \) of \( X \). Since the distinguished opens subsets form a basis for the topology on \( X \). It suffices to prove that
\[
H^p(\mathcal{U}, \mathcal{F}) = 0 \text{ for all } p > 0.
\]
for a covering \( \mathcal{U} = \{ D(g_i) \} \) where the \( g_i \) are finitely many elements of \( A \) generating the ring. (We can choose finitely many \( g_i \), by the quasi-compactness of \( X \)).

As \( \mathcal{F} \) is quasi-coherent, \( \mathcal{F} = \check{\mathcal{M}} \) for some \( A \)-module \( \mathcal{M} \), and \( \mathcal{M} = \Gamma(\check{\mathcal{X}}, \mathcal{F}) \). With this setup, the fact that \( \check{\text{C}}ech \) cohomology vanishes of a quasi-coherent module on an affine scheme corresponds to the observation that for some commutative ring \( A \), some finite sequence of elements \( (g_i)_{i \in I} \) of \( A \) generating \( A \) as an ideal, and some \( A \)-module \( M \), the following sequence is exact:
\[
0 \to M \to \prod_{i \in I} M_{g_i} \to \prod_{i,j \in I} M_{g_i g_j} \to \prod_{i,j,k \in I} M_{g_i g_j g_k} \to \ldots
\]
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Here, the boundary maps are given as alternating sums of localization maps. For example,

\[ d : \prod_{i,j \in I} M_{g_i g_j} \to \prod_{i,j,k \in I} M_{g_i g_j g_k} \]

maps \((\sigma_{ij})_{i,j} \in M_{g_i g_j}\) to \((\sigma_{jk} - \sigma_{ik} + \sigma_{ij})_{i,j,k}\).

Notice that the beginning of the exact sequence

\[ 0 \to M \to \prod_i M_{g_i} \to \prod_{i,j} M_{g_i g_j} \]

appeared in the construction of the quasi-coherent module \(\tilde{M}\). The proof for the exactness of this sequence is similar to the general case.

To prove that the cohomology groups vanish, we must to each cocycle \(\sigma\) (such that \(d\sigma = 0\)) find an element \(\tau\) such making \(\sigma = d\tau\) a boundary. The proof is a direct calculation; one constructs an element \(\tau\) by hand.

To see how this can be done, let us for simplicity consider the case \(p = 1\) first. Let \(\sigma \in \prod_{i,j} M_{g_i g_j}\) be in the kernel of \(d\). We may write

\[ \sigma_{ij} = \frac{m_{ij}}{(g_i g_j)^r} \text{ where } m_{ij} \in M \]

for some \(r\) (since the index set \(I\) is finite, we may choose this independent of \(i, j\)). The relation \(d\sigma = 0\) gives the relation

\[ \frac{m_{jk}}{(g_j g_k)^r} - \frac{m_{ik}}{(g_i g_k)^r} + \frac{m_{ij}}{(g_i g_j)^r} = 0 \]

or in other words, we have, in \(M_{g_i g_k}\),

\[ \frac{g_i^{r+l} m_{jk}}{(g_j g_k)^r} = \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \tag{12.3.1} \]

for some \(l \geq 0\). As the sets \(D(g_i) = D(g_i^{r+l})\) cover \(X\), we have a relation

\[ 1 = \sum_{i \in I} h_i g_i^{r+l} \]

where \(h_i \in A\). Now, define \(\tau \in \prod M_{g_i}\) by

\[ \tau_j = \sum_{i \in I} h_i \frac{g_i^l m_{ij}}{g_j^r} \]
12.3. Cohomology of sheaves on affine schemes

In \( \prod M_{g_j g_k} \), we may write it as

\[
\tau_j = \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r}
\]

We want to show that \( d\tau = \sigma \). This is a basic computation, using the relation (12.3.1) above:

\[
(d\tau)_{jk} = \tau_k - \tau_j = \sum_{i \in I} h_i \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r}
\]

\[
= \sum_{i \in I} h_i \left( \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \right)
\]

\[
= \sum_{i \in I} h_i \frac{g_i^{r+l} m_{jk}}{(g_j g_k)^r} = \frac{m_{jk}}{(g_j g_k)^r} \sum_{i \in I} h_i g_i^{r+l}
\]

\[
= \frac{m_{jk}}{(g_j g_k)^r} = \sigma_{jk}
\]

As desired. Hence \( H^1(\mathcal{U}, \mathcal{F}) = 0 \).

In the general case, let \( \sigma \in \prod_{i_0, \ldots, i_p} M_{\substack{g_{i_0} \cdots g_{i_p}}} \) be in the kernel of \( d \). We may write

\[
\sigma_{i_0, \ldots, i_p} = \frac{m_{i_0, \ldots, i_p}}{(g_{i_0} \cdots g_{i_p})^r} \text{ where } m_{i_0, \ldots, i_p} \in M
\]

for some \( r \) (since the index set \( I \) is finite, we may choose this independent of \( i, j \)). The relation \( d\sigma = 0 \) gives the relation in \( M_{\substack{g_{i_0} \cdots g_{i_p}}} \):

\[
\frac{g_i^{r+l} m_{i_0, \ldots, i_p}}{(g_{i_0} \cdots g_{i_p})^r} = \sum_{k=0}^p (-1)^k g_i^{l_k} g_{i_k}^r m_{i_0, \ldots, i_k \cdots i_p} \quad (12.3.2)
\]

As the sets \( D(g_i) = D(g_i^{r+l}) \) cover \( X \), we have a relation

\[
1 = \sum_{i \in I} h_i g_i^{r+l}
\]

where \( h_i \in A \). Now, define \( \tau \in \prod M_{\substack{g_{i_0} \cdots g_{i_{p-1}}}} \) by

\[
\tau_{i_0, \ldots, i_{p-1}} = \sum_{i \in I} h_i g_i^{l} \frac{m_{i_0, \ldots, i_p}}{(g_{i_0} \cdots g_{i_{p-1}})^r}
\]

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Localizing to $\prod M_{g_{i_0},\ldots,g_{i_p}}$, we may write this as
\[
\tau_{i_0,\ldots,i_p} = \sum_{i \in I} h_i g_i^r m_{i_0,\ldots,i_p} (g_{i_0} \cdots g_{i_p})^r
\]
We want to show that $d\tau = \sigma$. As before, we can check this using the relation (12.3.2) above:
\[
(d\tau)_{i_0,\ldots,i_p} = \sum_{k=0}^{p} (-1)^k \tau_{i_0,\ldots,i_k,\ldots,i_p}
= \sum_{i \in I} h_i g_i^{r+1} \sigma_{i_0,\ldots,i_p} = \sigma_{i_0,\ldots,i_p}
\]
This completes the proof. $\square$

12.3.1 Čech cohomology and affine coverings

As a Corollary of the previous theorem, we see that affine coverings of schemes satisfy the conditions of Leray’s theorem (see Theorem 12.10). This implies

**Corollary 12.13.** Let $X$ be a noetherian scheme and let $U = \{U_i\}$ be an affine covering such that all intersections $U_{i_0} \cap \cdots \cap U_{i_s}$ are affine. Then
\[
H^i(X,F) = H^i(U,F)
\]
In particular, the theorem applies to *any* affine covering on a noetherian separated scheme.

12.4 Cohomology and dimension

**Lemma 12.14.** Let $X$ be a topological space and let $Z \subseteq X$ be a closed subset. Then for any abelian sheaf $\mathcal{F}$ on $Z$ we have $H^p(Z,F) = H^p(X,i_*\mathcal{F})$.

**Theorem 12.15.** Let $X$ be a quasi-projective scheme of dimension $n$. Then $X$ admits an open cover $U$ consisting of at most $n+1$ affine open subsets.

This implies that
\[
H^i(X,F) = 0 \text{ for } i > n
\]
for any quasi-coherent sheaf $F$ on $X$.

**Proof.** Let $X$ be a quasi-projective scheme, i.e., $X = Y - W$, where $Y,W \subseteq \mathbb{P}^n_A$ are (closed) projective schemes. Consider the irreducible decomposition $Y = \cup_i Y_i$ and observe that $I_W \not\subseteq I_Y$, where $I_T \subseteq A[x_0,\ldots,x_N]$ denotes the ideal of the set
$T \subseteq \mathbb{P}^N$. Pick a homogenous polynomial $f$ of degree $d$ such that $f \in I_W - (\cup_i I_{Y_i})$. Let $H = Z(f)$. Then $\mathbb{P}^n - H$ is affine and hence so is $Y - H$.

By construction $Y - H \subseteq Y - W = X$ and $H \not\supseteq Y_i$ for any $i$ by the choice of $f$. Therefore $\dim(Y_i \cap H) < \dim Y_i$ so we may use induction on $\dim X$. Notice that the affine subset is obtained as an affine subset of the ambient projective space intersected with our scheme, so the affine schemes obtained subsequently are restrictions of affine subschemes of the original $X$. This shows the first claim.

For the second, note that for $\leq n + 1$ affines, there are at most $n$ terms $C^i(X, \mathcal{F})$ in the Čech complex. From this it follows that $H^i(U, \mathcal{F}) = 0$ for any $\mathcal{F}$ and $i > n$.

In fact, this theorem holds in much greater generality:

**Theorem 12.16.** Let $X$ be a topological space of dimension $n$, and let $\mathcal{F}$ be a sheaf on $X$. Then

$$H^p(X, \mathcal{F}) = 0$$

for all $p > n$.

### 12.5 Cohomology of sheaves on projective space

On projective space $\mathbb{P}^n_A$, we can use the Čech complex associated with the standard covering to compute the cohomology of any invertible sheaf on $\mathbb{P}^n_A$.

**Theorem 12.17.** Let $X = \mathbb{P}^n_A = \text{Proj } R$ where $R = A[x_0, \ldots, x_n]$ where $A$ is a noetherian ring.

(i) $$H^0(X, \mathcal{O}_X(m)) = \begin{cases} R_m & \text{for } m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) $$H^n(X, \mathcal{O}_X(m)) = \begin{cases} A & \text{for } m = -n - 1 \\ 0 & \text{for } m > -n - 1 \end{cases}$$

(iii) For $m \geq 0$, there is a perfect pairing\(^1\) of $A$-modules

$$H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m - n - 1)) \to H^n(X, \mathcal{O}_X(-n - 1)) \cong A$$

---

\(^1\)Recall that a bilinear map $M \times N \to A$ is a perfect pairing if the induced map $M \to \text{Hom}_A(N, A)$ is an isomorphism
(iv) For $0 < i < n$ and all $n \in \mathbb{Z}$, we have
\[ H^i(X, \mathcal{O}_X(m)) = 0 \]

Proof. Consider the $\mathcal{O}_X$-module $\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}(m)$. We would like to show for instance that $H^1(X, \mathcal{F}) = 0$ for $i \neq 0, n$; this is equivalent to $H^i(X, \mathcal{O}_X(m)) = 0$ for all $m$, but $\mathcal{F}$ has the advantage that it is a graded $\mathcal{O}_X$-algebra. Consider the Cech complex associated with the standard covering $\mathcal{U} = \{U_i\}$ where $U_i = D_+(x_i) = \text{Spec } R(x_i)$. This is simply
\[
\prod_i R_{x_i} \xrightarrow{d^0} \prod_{i,j} R_{x_i x_j} \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} R_{x_0 \ldots x_n}
\]

We have a graded isomorphism of $R$-modules:
\[
H^0(X, \mathcal{F}) = \ker d^0 = \{(r_i)_{i \in I} | r_i \in R_{x_i}, r_i = r_j \in R_{x_i x_j}\} \cong R.
\]

This isomorphism preserves the grading, so we get (i).

For (ii): Note that $R_{x_0 \ldots x_n}$ is a free graded $A$-module spanned by monomials of the form
\[
x_0^{a_0} \ldots x_n^{a_n}
\]
for multidegrees $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$. The image of $d^{n-1}$ is spanned by such monomials where at least one $a_i$ is non-negative. Hence
\[
H^n(X, \mathcal{F}) = \text{coker } d^{n-1} = A \{ x_0^{a_0} \ldots x_n^{a_n} | a_i < 0 \forall i \} \subseteq R_{x_0 \ldots x_n}
\]

Hence
\[
H^n(X, \mathcal{O}_X(m)) = H^n(X, \mathcal{F})_m = A \{ x_0^{a_0} \ldots x_n^{a_n} | a_i < 0 \forall i, \sum a_i = m \} \subseteq R_{x_0 \ldots x_n}
\]
In degree $-n-1$ there is only one such monomial, namely $x_0^{-1} \ldots x_n^{-1}$.

(iii): If we identify $H^n(X, \mathcal{O}_X(-m-n-1))$ with
\[
A \{ x_0^{a_0} \ldots x_n^{a_n} : a_i < 0 \forall i, \sum a_i = m \}
\]
and $H^0(X, \mathcal{O}(m)) = R_m$, we can define the pairing via multiplication of Laurent
polynomials:

\[ H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m - n - 1)) \to R_{x_0 \cdots x_n} \]

\[ (x_0^{m_0} \cdots x_n^{m_n}) \times (x_0^{a_0} \cdots x_n^{a_n}) \mapsto x_0^{a_0 + m_0} \cdots x_n^{a_n + m_n} \]

Here the exponents satisfy \( m_i \geq 0, a_i < 0 \sum a_i = -m - n - 1, \sum m_i = m \). This gives a map

\[ H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m - n - 1)) \to H^n(X, \mathcal{O}_X(-n - 1)) = Ax_0^{-1} \cdots x_n^{-1} \]

sending \((x_0^{m_0} \cdots x_n^{m_n}) \times (x_0^{a_0} \cdots x_n^{a_n})\) to zero if \( m_i + a_i \geq 0 \) for some \( i \). This pairing is perfect: The dual of a monomial \((x_0^{m_0} \cdots x_n^{m_n})\) is represented by \((x_0^{-m_0 - 1} \cdots x_n^{-m_n - 1})\).

(iv) This point is more involved, and we proceed by induction on \( n \). For \( n = 1 \), there is nothing to prove. For \( n > 1 \), let \( H = V(x_n) \simeq \mathbb{P}^{n-1} \) be the hyperplane determined by \( x_n \). We have an exact sequence

\[ 0 \to R(-1) \xrightarrow{x_n} R \to R/\langle x_n \rangle \to 0 \quad (12.5.1) \]

Applying \( \sim \), we find

\[ 0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_* \mathcal{O}_H \to 0 \]

where \( i : H \to X \) is the inclusion. If we take the direct sum of all the twists of this sequence, we get

\[ 0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_* \mathcal{F}_H \to 0 \]

where \( \mathcal{F}_H = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_H(m) \). By induction on \( n \), we have for \( 0 < i < n - 1 \) and all \( m \in \mathbb{Z} \): \( H^i(X, i_* \mathcal{O}_H(m)) = H^i(H, \mathcal{O}_H(m)) = 0 \). So taking the long exact sequence of cohomology, we get isomorphisms

\[ H^i(X, \mathcal{F}(-1)) \xrightarrow{x_n} H^i(X, \mathcal{F}) \]

for \( 1 < i < n - 1 \). We claim that we have isomorphisms also for \( i = 1 \) and \( i = n - 1 \). For \( i = 1 \), this follows because the sequence

\[ 0 \to H^0(X, \mathcal{F}(-1)) \to H^0(X, \mathcal{F}) \to H^0(X, i_* \mathcal{F}_H) \to 0 \]

is exact (this is the same sequence as (12.5.1)).

For \( i = n - 1 \) we need to show that

\[ 0 \to H^{n-1}(X, i_* \mathcal{F}_H) \xrightarrow{\delta} H^n(X, \mathcal{F}(-1)) \xrightarrow{x_n} H^n(X, \mathcal{F}) \]
is exact. The kernel of \( \cdot x_n \) is generated by monomials \( x_0^{a_0} \cdots x_n^{a_n} \) with \( a_i < 0 \) for all \( i \). So it suffices to show that the connecting map \( \delta \) is just multiplication by \( x_n^{-1} \). Define \( R' = R/x_n \). Writing the arrows in the Čech complex, vertically we get the diagram

\[
\begin{array}{c}
\text{0} \longrightarrow \prod_i R(-1)_{x_0 \cdots x_i \cdots x_n} \xrightarrow{-x_n} \prod_i R_{x_0 \cdots x_i \cdots x_n} \longrightarrow R'_{x_0 \cdots x_{n-1}} \longrightarrow 0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{0} \longrightarrow R_{x_0 \cdots x_n}(-1) \xrightarrow{-x_n} R_{x_0 \cdots x_n} \longrightarrow 0 
\end{array}
\]

If \( x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} \) is a monomial in \( H^{n-1}(H, \mathcal{F}_H) \) with \( a_i < 0 \) for all \( 1 \leq i \leq n - 1 \), then it comes from an \((n+1)\)-tuple in \( \prod_i R_{x_0 \cdots x_i \cdots x_n} \) which maps to \( \pm x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} \) in \( R_{x_0 \cdots x_n} \), which in turn mapped onto by the monomial \( x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} \) in \( R_{x_0 \cdots x_n}(-1) \). So \( \delta(x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}) \) is represented by the monomial \( x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} x_n^{-1} \) in \( H^n(X, \mathcal{F}(-1)) \).

Now we claim that we have an isomorphism \( H^*(X, \mathcal{F})_{x_n} = H^*(U_n, \mathcal{F}|_{U_n}) \). Indeed, the Čech complex of \( \mathcal{F}|_{U_n} \) with respect to the covering \( U_i \cap U_n \) is just the localization of \( C^*(X, \mathcal{F}) \) at \( x_n \). Localization is exact, so it preserves cohomology, which gives the claim.

We know that \( H^i(X, \mathcal{F})_{x_n} = H^i(U_n, \mathcal{F}|_{U_n}) = 0 \) for all \( i > 0 \), since \( U_n \) is affine. Hence for \( l \gg 0 \), \( x_n^l H^i(X, \mathcal{F}) = 0 \) as an \( A \)-module. However, we have shown that \( \cdot x_n \) gives an isomorphism of \( H^i(X, \mathcal{F}) \) for \( 0 < i < n \). This implies that \( H^i(X, \mathcal{F}) = 0 \).

In the above proof, we used the following lemma:

**Lemma 12.18.** Let \( X = \mathbb{P}^n_A \) over a noetherian ring \( A \) and let \( \mathcal{F} \) be a quasi-coherent sheaf on \( X \). Then \( H^i(X, \mathcal{F}) = 0 \) for all \( i > n \).

**Proof.** \( \mathbb{P}^n_A \) can be covered with \( n+1 \) open affine schemes, all of whose intersections are affine. Hence the Čech complex associated with \( \mathcal{F} \) has length \( n+1 \), and so there is no cohomology in degrees higher than \( n \). \( \Box \)

### 12.6 Extended example: Plane curves

Let \( X = V(f) \) denote an integral subvariety of \( \mathbb{P}^2 \), defined by an irreducible homogeneous polynomial \( f \) of degree \( d \).

Let \( X \subseteq \mathbb{P}^2 \) be a plane curve, defined by a homogeneous polynomial \( f(x_0,1,x_2) \) of degree \( d \). Let us compute \( H^i(X, \mathcal{O}_X) \). We have the ideal sheaf sequence

\[
0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^2} \to i_* \mathcal{O}_X \to 0
\]

```
12.7. Extended example: The twisted cubic in $\mathbb{P}^3$

where the ideal sheaf $I_X$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X$. By Section 10.7.1, $\mathcal{O}_{\mathbb{P}^2}(-X) \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$. More explicitly, on Spec $k[x_0, x_1, x_2, x_3]$, we see $I_X$ is a $k[x_0, x_1, x_2, x_3]$-module generated by $f(x_0, x_1, x_2, 1)$, and $\mathcal{O}_{\mathbb{P}^2}(-d)$ is a $k[x_0, x_1, x_2, x_3]$-module generated by $x_0^{-d}$, and this isomorphism sends generator to generator. The above exact sequence can thus be rewritten as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$ 

From the short exact sequence, we get the long exact sequence as follows:

$$0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(-d)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow$$

$$H^1(\mathbb{P}^2, \mathcal{O}(-d)) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow$$

$$H^2(\mathbb{P}^2, \mathcal{O}(-d)) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0.$$ 

Using the results on cohomology of line bundles on $\mathbb{P}^2$, we deduce that $H^0(X, \mathcal{O}_X) \simeq k$ and 

$$H^1(X, \mathcal{O}_X) \simeq k^{(d-1)}.$$ 

12.7 Extended example: The twisted cubic in $\mathbb{P}^3$

Let $k$ be a field and consider $\mathbb{P}^3 = \text{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will consider the twisted cubic curve $C = V(I)$ where $I \subseteq R$ is the ideal generated by the $2 \times 2$-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix}$$

i.e., $I = (q_0, q_1, q_2) = (x_1^2 - x_0x_2, x_0x_3 - x_1x_2, -x_2^2 + x_1x_3)$. Let us by hand compute the group $H^1(X, \mathcal{O}_X)$. Of course we know what the answer should be, since $X \simeq \mathbb{P}^1$, and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Indeed, $\mathbb{P}^1 = \text{Proj} \ k[s, t]$ is isomorphic to the Veronese embedding Proj $k[s, t]^{(3)}$, where $k[s, t]^{(3)} = k[s^3, s^2t, st^2, t^3]$, and the latter ring is isomorphic to $S = R/I$.

Now, to compute $H^1(X, \mathcal{O}_X)$ on $X$, it is convenient to relate it to a cohomology group on $\mathbb{P}^3$. We have $H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$ where $i : X \to \mathbb{P}^3$ is the inclusion. This fits into the ideal sheaf sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_X \to 0.$$
where $\mathcal{I}$ is the ideal sheaf of $X$ in $\mathbb{P}^3$. Applying the long exact sequence in cohomology we get

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_X \to 0.$$ 

From the short exact sequence, we get the long exact sequence as follows:

$$\cdots \to H^1(\mathbb{P}^3, \mathcal{I}) \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(\mathbb{P}^3, i_*\mathcal{O}_X) \to H^2(\mathbb{P}^3, \mathcal{I}) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to \cdots$$

By our description of sheaf cohomology on $\mathbb{P}^3$, $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$, we find that $H^1(X, \mathcal{O}_X) = H^2(\mathbb{P}^3, \mathcal{I})$. We can compute the latter cohomology group as follows. Note that $\mathcal{I} = \widetilde{\mathcal{I}}$. Consider the map $R^3 \to \mathcal{I} \to 0$ sending $e_i \mapsto q_i$. Let us consider the kernel of this map, that is, relations of the form $a_0q_0 + a_1q_1 + a_2q_2 = 0$ for $a_i \in R$. There are two obvious relations of this form, i.e., the ones we get from expanding the determinants of the two matrices

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}, \quad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

(So first matrix gives $x_0q_2 - x_1q_1 + x_2q_2 = 0$ for instance). These give a map $R^2 \xrightarrow{\cdot \, M} R^3$, where $M$ is the matrix above. This map is injective, and there is an exact sequence

$$0 \to R^2 \xrightarrow{\cdot \, M} R^3 \to I \to 0$$

If we want to be completely precise, and consider these as graded modules, we must shift the degrees according to the degrees of the maps above

$$0 \to R(−3)^2 \xrightarrow{\cdot \, M} R(−2)^3 \to I \to 0$$

Applying $\sim$, we get an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(−3)^2 \to \mathcal{O}_{\mathbb{P}^3}(−2)^3 \to \mathcal{I} \to 0$$

Now, this we can use to compute $H^2(X, \mathcal{I})$; taking the long exact sequence we get

$$\cdots \to H^2(\mathbb{P}^3, \mathcal{O}(−3)^2) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(−2)^3) \to H^2(X, \mathcal{I}) \to H^3(\mathbb{P}^3, \mathcal{O}(−3)^2) \to H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(−2)^3) \to H^3(X, \mathcal{I})$$

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Here $H^2(\mathbb{P}^3, \mathcal{O}(-2)) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(-3)) = 0$ (!) by our previous computations. Hence by exactness, we find $H^2(X, \mathcal{I}) = 0$, as expected.

**Exercise 40.** Using the sequences above, show that

- $H^0(\mathbb{P}^3, \mathcal{I}(2)) = k^3$ (find a basis!)
- $H^1(\mathbb{P}^3, \mathcal{I}(m)) = 0$ for all $m \in \mathbb{Z}$.
- $H^2(\mathbb{P}^3, \mathcal{I}(-1)) = k$. 


Chapter 13

More on sheaf cohomology

13.1 Flasque sheaves

A sheaf $\mathcal{F}$ on a topological space $X$ is said to be \textit{flasque} if the restriction maps

\[ \mathcal{F}(X) \to \mathcal{F}(U) \]

for every open subset $U \subseteq X$.

\textbf{Example 13.1.} If $X$ is irreducible, then any constant sheaf is flasque.

\textbf{Example 13.2.} The skyskraper sheaves $A(x)$ defined in Chapter 1 are flasque.

\textbf{Example 13.3.} $\mathcal{O}_{\mathbb{P}^1_k}$ is not flasque, since $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_k}) = k$, whereas $\Gamma(U, \mathcal{O}_{\mathbb{P}^1_k})$ is typically much larger (e.g., for $U = \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ it is an infinite dimensional $k$-vector space).

Recall that the global section functor $\Gamma$ is left exact. The following lemma says that it is also exact on the right, given that the left-most term in the sequence is flasque:

\textbf{Lemma 13.4.} Given an exact sequence

\[ 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \]

of sheaves $\mathcal{F}$, $\mathcal{G}$ where $\mathcal{F}$ is flasque. Then the corresponding sequence of global sections

\[ 0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0 \]

is exact for every open set $U \subseteq X$. 

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13.1. Flasque sheaves

Proof. By restricting \( F, G, H \) to \( U \), it suffices to prove the statement for \( U = X \). We only need to check that the sequence is exact on the right. Let \( s \in H(X) \) and let \( V \) be the family of open subsets \( U \subseteq X \) such that \( s|_U \) is represented by an element of \( G(X) \). We need to show that \( X \in V \).

Since \( V \) contains a covering of \( X \) and \( X \) is quasi-compact, it suffices to show that \( V \) is closed under finite unions. So let \( U_1, U_2 \in V \) and let \( t_i \in \Gamma(U_i, F) \) represent \( s \) on \( U_i \), \( i = 1, 2 \). Note that by flasque property of \( F \), \( t_1 - t_2 \) on \( U_1 \cap U_2 \) lifts to a global section \( u \) of \( F \). Moreover, by adding \( u|_{U_2} \) to \( t_2 \), we may assume that \( t_1, t_2 \) agree on \( U_1 \cap U_2 \), so by gluing we get a section \( t \in G(X) \) which represents \( s \), and hence \( U_1 \cup U_2 \in V \).

Lemma 13.5. Suppose we are given an exact sequence of sheaves

\[
0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.
\]

If \( F \) and \( G \) are flasque, then so is \( H \).

Proof. Let \( U \subseteq X \) be a subset of \( X \). Then any section \( h \in H(U) \) is represented by a section \( g \in G(U) \) by the previous lemma. Since \( G \) is flasque, this can be extended to an element \( \tilde{g} \) of \( G(X) \). Then \( \tilde{g} \) maps to an element \( \tilde{h} \in H(X) \) which has the property that \( \tilde{h}|_U = h \).

Lemma 13.6. Suppose we are given an exact sequence of flasque sheaves

\[
0 \longrightarrow E \longrightarrow F^0 \rightarrow F^1 \rightarrow F^2 \rightarrow \cdots, \quad (13.1.1)
\]

then for each open set \( U \subseteq X \), the sequence

\[
0 \longrightarrow E(U) \rightarrow F^0(U) \rightarrow F^1(U) \rightarrow F^2(U) \rightarrow \cdots, \quad (13.1.2)
\]

is exact.

Proof. We chop (13.1.2) into short exact sequences as follows:

\[
0 \rightarrow E \rightarrow F^0 \rightarrow Q^0 \rightarrow 0
\]
\[
0 \rightarrow Q^0 \rightarrow F^1 \rightarrow Q^1 \rightarrow 0
\]
\[
\cdots
\]

By induction, it follows that the sheaves \( Q^i \) are flasque, being quotients of flasque sheaves. Applying \( \Gamma \) we then get exact sequences

\[
0 \rightarrow E(U) \rightarrow F^0(U) \rightarrow Q^0(U) \rightarrow 0
\]
\[
0 \rightarrow Q^0(U) \rightarrow F^1(U) \rightarrow Q^1(U) \rightarrow 0
\]
\[
\cdots
\]

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If we connect these together, we obtain the exact sequence (13.1.2).

\section{The Godement resolution}

We will define the cohomology groups \( H^p(X, \mathcal{F}) \) using a special resolution of \( \mathcal{F} \) known as the Godement resolution. To explain how this works, we again recall the canonical map \( \kappa : \mathcal{F} \to \Pi(\mathcal{F}) \), which over an open set \( U \subseteq X \) sends a section \( s \in \mathcal{F}(U) \) to the element \( (s_x)_{x \in U} \in \Pi(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x \). Defining \( \mathcal{C}^0 \mathcal{F} = \Pi(\mathcal{F}) \) and \( \mathcal{Z}^1 \mathcal{F} \) as the cokernel of \( \kappa \), we get a canonical exact sequence

\[ 0 \to \mathcal{F} \to \mathcal{C}^0 \mathcal{F} \to \mathcal{Z}^1 \mathcal{F} \to 0. \]

Taking global sections, we get

\[ 0 \to \mathcal{F}(X) \to \mathcal{C}^0 \mathcal{F}(X) \to \mathcal{Z}^1 \mathcal{F}(X). \]

As we have seen, we cannot expect to be exact on the right. In fact, this failure of exactness is how we will define cohomology groups: We define \( H^1(X, \mathcal{F}) \) as the cokernel of the right most map \( \mathcal{C}^0 \mathcal{F}(X) \to \mathcal{Z}^1 \mathcal{F}(X) \), i.e.,

\[ H^1(X, \mathcal{F}) = \mathcal{Z}^1(X)/\mathcal{B}^0(X), \]

where \( \mathcal{B} = \text{im}(\mathcal{C}^0 \mathcal{F}(X) \to \mathcal{Z}^1(X)) \).

We note that the assignments \( \mathcal{F} \mapsto \mathcal{C}^0 \mathcal{F}, \mathcal{F} \mapsto \mathcal{Z}^1 \mathcal{F} \) and \( \mathcal{F} \mapsto H^1(X, \mathcal{F}) \) are indeed functors. Indeed, given a map \( \mathcal{F} \to \mathcal{G} \) of sheaves, we get canonical maps between the two sequences

\[
\begin{array}{cccccc}
0 & \to & \mathcal{F}(X) & \to & \mathcal{C}^0 \mathcal{F}(X) & \to & \mathcal{Z}^1 \mathcal{F}(X) & \to & H^1(X, \mathcal{F}) & \to & 0 \\
0 & \to & \mathcal{G}(X) & \to & \mathcal{C}^0 \mathcal{G}(X) & \to & \mathcal{Z}^1 \mathcal{G}(X) & \to & H^1(X, \mathcal{G}) & \to & 0 \\
\end{array}
\]

and from this we see that we have a canonical map \( H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \).

What about the higher cohomology groups? Well, we can continue, replacing \( \mathcal{F} \) with the new sheaf \( \mathcal{Z}^1 \). We again have a canonical map \( \kappa : \mathcal{Z}^1 \to \Pi(\mathcal{Z}^1) \), and we can define \( \mathcal{C}^1 \mathcal{F} = \mathcal{C}^0 \mathcal{Z}^1 \) and \( \mathcal{Z}^2 \mathcal{F} = \text{coker}(\mathcal{Z}^1 \mathcal{F} \to \mathcal{C}^1 \mathcal{F}) \). We then have an exact sequence

\[ 0 \to \mathcal{Z}^1 \mathcal{F} \to \mathcal{C}^1 \mathcal{F} \to \mathcal{Z}^2 \mathcal{F} \to 0 \]

and we can define \( H^2(X, \mathcal{F}) \) to be the cokernel of the map on global sections

\[ \mathcal{C}^1 \mathcal{F}(X) \to \mathcal{Z}^2 \mathcal{F}(X). \]
13.2. The Godement resolution

So to define cohomology groups in general, we define inductively the sheaves $\mathcal{C}^i$

\[
\begin{align*}
\mathcal{C}^0 F &= \Pi(F) \\
\mathcal{Z}^1 F &= \text{coker}(\kappa : F \to \mathcal{C}^0 F) \\
\mathcal{C}^n F &= \mathcal{C}^0 \mathcal{Z}^n \text{ for } n \geq 1 \\
\mathcal{Z}^{n+1} F &= \mathcal{Z}^1 \mathcal{Z}^n = \text{coker}(\kappa : \mathcal{Z}^n \to \mathcal{C}^n) \text{ for } n \geq 1
\end{align*}
\]

The way to visualise these is as follows:

\[
\begin{array}{cccccc}
0 & \to & F & \xrightarrow{\kappa} & \mathcal{C}^0 F & \xrightarrow{d^0} & \mathcal{C}^1 F & \xrightarrow{d^1} & \mathcal{C}^2 F & \xrightarrow{d^2} & \cdots \\
& & \downarrow{\kappa} & & \downarrow{\kappa} & & \downarrow{\kappa} & & & & \\
& & \mathcal{Z}^1 F & & \mathcal{Z}^2 F & & & & & & \\
\end{array}
\]

(13.2.1)

Here the map $d^q : \mathcal{C}^q \to \mathcal{C}^{q+1}$ is defined by the composition

\[
\mathcal{C}^q F \to \mathcal{Z}^{q+1} F \hookrightarrow \mathcal{C}^0 \mathcal{Z}^{q+1} = \mathcal{C}^{q+1} F
\]

The second map is injective, which means that the complex in (13.2.1) is exact. It follows that the complex $0 \to F \to \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \cdots$ is actually a resolution, and we call it the Godement resolution of $F$.

13.2.1 Functoriality

Given a morphism of sheaves $\phi : F \to G$, we have a canonical morphism $\Pi(F) \to \Pi(G)$ which is compatible with the two maps $\kappa_F : F \to \Pi(F)$ and $\kappa_G : G \to \Pi(G)$. Therefore, we get a morphism of quotient sheaves

\[
\mathcal{C}^0 F / F \to \mathcal{C}^0 G / G
\]

or in other words, a morphism $\mathcal{Z}^1 F \to \mathcal{Z}^1 G$. Taking again Godement sheaves, we get

\[
\mathcal{C}^0 \mathcal{Z}^1 F \to \mathcal{C}^0 \mathcal{Z}^1 G
\]

or, a morphism $\mathcal{C}^1 F \to \mathcal{C}^1 G$. Thus, inductively, we construct canonical sheaf maps

\[
\mathcal{C}^k \phi : \mathcal{C}^k F \to \mathcal{C}^k G
\]

These $\mathcal{C}^k()$ define the Godement functors; they are functors from $\text{Sh}_X$ to $\text{Sh}_X$. The Godement resolution is also functorial, in the sense that we have a canonical morphism of complexes $\mathcal{C}^* F \to \mathcal{C}^* G$:
Chapter 13. More on sheaf cohomology

13.2.2 Sheaf cohomology

If we take global sections in the Godement resolution $\mathcal{C}^\bullet \mathcal{F}$, we get a sequence

$$
0 \to \mathcal{F}(X) \to \mathcal{C}^0 \mathcal{F}(X) \xrightarrow{d^0(X)} \mathcal{C}^1 \mathcal{F}(X) \xrightarrow{d^1(X)} \mathcal{C}^2 \mathcal{F}(X) \xrightarrow{d^2(X)} \cdots
$$

(13.2.2)

This may or may not be exact, but since $d^{i+1}(X) \circ d^i(X) = 0$ we still have a complex of abelian groups, and we can talk about its cohomology.

**Definition 13.7.** The $p$-th cohomology group $H^p(X, \mathcal{F})$ is defined to be the $p$-th cohomology group of the complex of abelian groups (13.2.2), i.e.,

$$
H^p(X, \mathcal{F}) = (\ker d^p(X))/(\text{im } d^{p-1}(X))
$$

Equivalently,

$$
H^p(X, \mathcal{F}) = \mathcal{Z}^p(X)/\mathcal{B}^p(X)
$$

where $\mathcal{B}^p(X) = (\text{im}(\mathcal{C}^{p-1}(X) \to \mathcal{Z}^p(X)))$.

So what is $H^0(X, \mathcal{F})$? By definition this group is the kernel of the map $d^0$, which is defined as the composition

$$
\mathcal{C}^0 \mathcal{F}(X) \to \mathcal{Z}^1 \mathcal{F}(X) \hookrightarrow \mathcal{C}^1 \mathcal{F}(X)
$$

Hence $H^0(X, \mathcal{F}) = \ker(\mathcal{C}^0 \mathcal{F}(X) \to \mathcal{Z}^1 \mathcal{F}(X))$. On the other hand, we have the sequence $0 \to \mathcal{F} \to \mathcal{C}^0 \to \mathcal{C}^1$; taking global sections, we get the exact sequence

$$
0 \to \mathcal{F}(X) \to \mathcal{C}^0(X) \xrightarrow{d^0(X)} \mathcal{Z}^1 \mathcal{F}(X)
$$

from which we see that $H^0(X, \mathcal{F}) = \ker d^0(X) = \Gamma(X, \mathcal{F})$.

By the recursive definition of the sheaves $\mathcal{C}^p \mathcal{F}$ and $\mathcal{Z}^p$, we only need to know how to define $H^1$ of a sheaf in order to see the higher cohomology groups. Indeed, from the complex (13.2.2), it is clear that the $\mathcal{C}^0$ also give a resolution of $\mathcal{Z}^1$, so that

$$
H^i(X, \mathcal{F}) = \begin{cases} 
\Gamma(X, \mathcal{F}) & \text{if } i = 0 \\
\ker(\mathcal{C}^0(X) \to \mathcal{Z}^1(X)) & \text{if } i = 1 \\
H^1(X, \mathcal{Z}^{i-1}) & \text{if } i \geq 2
\end{cases}
$$

The geometric meaning of the elements of these higher cohomology groups
$H^p(X, \mathcal{F})$ is on the other hand far from obvious, and unfortunately this is a general tendency when defining cohomology groups. What is important is what sort of properties they have: at the moment these groups are *completely canonical* in their definition, and again since $\kappa$ and $\Pi(\mathcal{F})$ are functorial, we find that the assignments $\mathcal{F} \mapsto H^i(X, \mathcal{F})$ define functors from sheaves to groups. This takes care of the two first properties of sheaf cohomology.

We now turn to prove the third property, that is, that we have a long exact sequence. First we prove that the assignments $\mathcal{F} \mapsto C^i\mathcal{F}$ and $\mathcal{F} \mapsto Z^i\mathcal{F}$ are in fact exact functors.

**Lemma 13.8.** Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of sheaves. Then we have exact sequences

$$0 \to C^i\mathcal{F}_1 \to C^i\mathcal{F}_2 \to C^i\mathcal{F}_3 \to 0$$

and

$$0 \to Z^i\mathcal{F}_1 \to Z^i\mathcal{F}_2 \to Z^i\mathcal{F}_3 \to 0$$

**Proof.** By definition we have $C^0_i(U) = \prod_{x \in U} (\mathcal{F}_i)_x$ for an open set $U \subseteq X$. Since the above sequence is exact on stalks, we get also that the sequence

$$0 \to C^0\mathcal{F}_1 \to C^0\mathcal{F}_2 \to C^0\mathcal{F}_3 \to 0$$

is also exact. (In fact this observation is one of main reasons why we use the Godement sheaves in the definition of cohomology in the first place.) This row fits into the following diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & C^0\mathcal{F}_1 & C^0\mathcal{F}_2 & C^0\mathcal{F}_3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & Z^1\mathcal{F}_1 & Z^1\mathcal{F}_2 & Z^1\mathcal{F}_3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

(13.2.3)

Since the two top rows are exact, the snake lemma implies that also the lower row is exact. Now using the fact that the composition of exact functors is still exact, and the fact that $\mathcal{C}^i\mathcal{F} = C^0\mathcal{F}^i$ and $Z^{i+1}\mathcal{F} = Z^1Z^i\mathcal{F}$, we conclude by induction on $i$. \qed
Lemma 13.9. The sheaves $C^i \mathcal{F}$ and $Z^i \mathcal{F}$ are flasque for every $i \geq 0$.

Proof. Note first that $\Pi(\mathcal{F})$ is flasque: For any open set $U \subseteq X$ it is clear that

$$\prod_{x \in X} \mathcal{F}_x \to \prod_{x \in U} \mathcal{F}_x$$

is surjective. Hence $C^0$ is flasque. This implies that $Z^1$ is flasque as well, being a quotient of $C^0$ by a subsheaf. Now we can conclude by induction, using $C^i \mathcal{F} = C^0 Z^i$ and $Z^{i+1} \mathcal{F} = Z^1 Z^i \mathcal{F}$.

Corollary 13.10. If $\mathcal{F}$ is flasque, then for every $U \subseteq X$ we have

$$H^i(U, \mathcal{F}) = 0$$

In particular, $\mathcal{F}$ is acyclic.

Proof. We take the Godement resolution $0 \to \mathcal{F} \to C^0 \mathcal{F} \to C^1 \mathcal{F} \to \cdots$. Since $\mathcal{F}$ and each $C^i \mathcal{F}$ are flasque, we get

$$0 \to \mathcal{F}(U) \to C^0(U) \to C^1(U) \to C^2(U) \to \cdots$$

Since this complex is exact, we obtain that $H^i(X, \mathcal{F}) = 0$ by the definition of cohomology.

We are now ready to prove property (iii), namely the long exact sequence of cohomology.

Theorem 13.11. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of sheaves. Then there is a long exact sequence

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to \delta \to H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_3) \to \cdots$$

Proof. First of all, by the previous lemma, there are exact sequences

$$C^i : \quad 0 \to \mathcal{C}^i \mathcal{F}_1 \to \mathcal{C}^i \mathcal{F}_2 \to \mathcal{C}^i \mathcal{F}_3 \to 0$$

Since each $C^i \mathcal{F}$ is flasque, we get also that

$$0 \to \mathcal{C}^i \mathcal{F}_1(X) \to \mathcal{C}^i \mathcal{F}_2(X) \to \mathcal{C}^i \mathcal{F}_3(X) \to 0$$

is exact. Hence we get an exact sequence of complexes of abelian groups

$$0 \to \mathcal{C}^i_{\mathcal{F}_1}(X) \to \mathcal{C}^i_{\mathcal{F}_2}(X) \to \mathcal{C}^i_{\mathcal{F}_3}(X) \to 0$$

Then, since $H^p(X, \mathcal{F}_i)$ is defined as the cohomology of this complex, we conclude by Proposition 12.1 that we have the above long exact sequence. 

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Exercise 41. Let $X$ be a topological space and let $i: Z \to X$ be the inclusion of a subset. Show that for a sheaf $\mathcal{F}$ on $Z$,

$$H^i(X, i_*\mathcal{F}) = H^i(Z, \mathcal{F})$$  \hspace{1cm} (13.2.4)

for all $i$.

13.2.3 Acyclic sheaves

Definition 13.12. A sheaf $\mathcal{F}$ is called acyclic if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Given a resolution

$$0 \to \mathcal{F} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \cdots$$

of $\mathcal{F}$ (which by definition means that the sequence is exact), the resulting sequence

$$\mathcal{C}^\bullet(X) : \mathcal{C}^0(X) \to \mathcal{C}^1(X) \to \mathcal{C}^2(X) \to \cdots,$$

may fail to be exact, but is still a complex of abelian groups, so it makes sense to ask about its cohomology.

Lemma 13.13. If the sheaves $\mathcal{C}^i$ are acyclic, then there is a natural isomorphism

$$H^i(X, \mathcal{F}) \simeq H^i(\mathcal{C}^\bullet(X))$$

Proof. Define $K^{-1} = \mathcal{F}$, and $K^i = \ker(\mathcal{C}^{i+1} \to \mathcal{C}^{i+2})$ for $i \geq 0$. By exactness of $\mathcal{C}^\bullet$, we have for each $i \geq 0$ and exact sequence

$$0 \to K^{i-1} \to \mathcal{C}^i \to K^i \to 0$$

Taking the long exact sequence, gives

$$0 \to H^0(K^{i-1}) \to H^0(\mathcal{C}^i) \to H^0(K^i) \to H^1(K^{i-1}) \to H^1(\mathcal{C}^i) = 0$$  \hspace{1cm} (13.2.5)

where the right-most group is zero because $\mathcal{C}^i$ is acyclic. Also, the same sequence shows that $H^p(K^i) = H^{p+1}(K^{i-1})$ for every $p \geq 1$. The maps in these sequences fit into the diagram

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From this, we see that
\[ \text{im} \left( H^0(C^i) \to H^0(K^i) \right) = \text{im} \left( H^0(C^{i+1}) \to H^0(C^i) \right) . \]

We furthermore have
\[ H^0(K^{i-1}) = \ker \left( H^0(C^i) \to H^0(C^{i+1}) \right) . \]

Indeed, if \( \sigma \in H^0(C^i) \) maps to zero in \( H^0(F^{i+1}) \), then it maps to zero in \( H^0(K^i) \), and hence it lies in the image of \( H^0(K^{i-1}) \). Hence
\[ H^0(F) = \ker \left( H^0(C^i) \to H^0(C^{i+1}) \right) \]

In particular,
\[ H^0(F) = \ker \left( H^0(C^0) \to H^0(C^1) \right) = H^0(C^\cdot(X)) \]
and the theorem holds in degree \( p = 0 \). By the same token, we have
\[ H^0(K^i) = \ker \left( H^0(C^{i+1}) \to H^0(C^{i+2}) \right) . \]

From (13.2.5), and the isomorphisms \( H^p(K^i) \simeq H^{p+1}(K^{i-1}) \) we therefore get
\[
H^{i+1}(C^\cdot(X)) = \ker \left( H^0(C^{i+1}) \to H^0(C^{i+2}) \right) / \text{im} \left( H^0(C^i) \to H^0(C^{i+1}) \right)
= H^0(K^i) / \text{im} \left( H^0(C^{i+1}) \to H^0(K^i) \right)
= H^1(K^{i-1})
= H^2(K^{i-2})
= \ldots
= H^{i+1}(K^{-1})
= H^{i+1}(F)
\]
13.2. The Čech resolution

We also have the following sheafified version of the Čech complex. Given a sheaf $F$ on $X$ and an open cover $U$, we set

$$\tilde{C}^p(U, F) = \prod_{i_0 < \cdots < i_p} i_\ast F|_{U_{i_0} \cap \cdots \cap U_{i_p}}$$

where $i : U_{i_0} \cap \cdots \cap U_{i_p} \to X$ denotes the inclusion.

We will now show that the sheaves $C^p(U, F)$ give a resolution of $F$. So we have an exact sequence

$$\tilde{C}^\bullet(U, F) : 0 \to F \to \tilde{C}^0(U, F) \xrightarrow{d^0} \tilde{C}^1(U, F) \xrightarrow{d^1} \tilde{C}^2(U, F) \to \cdots \quad (13.2.6)$$

In particular, if we apply $\Gamma$, we get back the Čech complex from earlier.

We define the maps in the complex as follows. The first map $F \to \tilde{C}^0(U, F)$ is takes a section and restricts it to $U_i$. The differentials $d^i$ are constructed in the same way as in the Čech complex. Thus if we apply the global sections functor $\Gamma$ to $\tilde{C}^p(U, F)$ we get the earlier defined Čech complex.

The main difference here is that the new complex is exact. We can check exactness on stalks and the nicest way to do that is with a homotopy operator $k : \tilde{C}^p(U, F) \to \tilde{C}^{p-1}(U, F)$ so that $(dk + kd)(\alpha) = \alpha$: If we are given such a $k$, every cocycle is a coboundary and so $H^p(C^\bullet(U, F)) = 0$.

For each $x \in X$, and open set $V \ni x$, refine $V$ to $V \cap U_j$ for some $U_j \ni x$. So now, $V \subseteq U_j$. Define $k$ by

$$(k\alpha)_{i_0, \ldots, i_{p-1}} = \alpha_{j, i_0, \ldots, i_{p-1}}$$

Then it is easy to check that $(dk + fd)(\alpha) = \alpha$.

**Proposition 13.14.** If $F$ is flasque, then so are the sheaves $\tilde{C}^p(U, F)$ for $p > 0$. Hence (13.2.6) is an acyclic resolution for $F$ and

$$H^p(X, F) = H^p(\tilde{C}^\bullet(U, F))$$

**Proof.** If $F$ is flasque, then so is each restriction to each $U_{i_0} \cap \cdots \cap U_{i_p}$, and products of flasque sheaves are flasque, so $\prod_{i_0 < \cdots < i_p} i_\ast F|_{U_{i_0} \cap \cdots \cap U_{i_p}}$ is flasque. □
13.3 Godement vs. Čech

It remains to see why these two definitions are equivalent. So let \( U = \{U_i\} \) be a covering for \( \mathcal{F} \). We will assume that this is Leray in the sense that \( H^i(U_I, \mathcal{F}) = 0 \) for all multi-indices \( I \) and \( i > 0 \). We claim that there is a natural isomorphism

\[
H^i(X, \mathcal{F}) \simeq \check{H}^i(U, \mathcal{F}),
\]

where we, in order to avoid confusion, let \( \check{H}^i(U, \mathcal{F}) \) denote the Čech cohomology group. The statement is clearly true for \( i = 0 \), since both coincide with \( \Gamma(X, \mathcal{F}) \).

Lemma 13.15. Let \( 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \) be an exact sequence. If \( U \) is Leray, there is a long exact sequence

\[
0 \to \check{H}^0(U, \mathcal{F}) \to \check{H}^0(U, \mathcal{G}) \to \check{H}^0(U, \mathcal{H}) \to \check{H}^1(U, \mathcal{F}) \to \check{H}^1(U, \mathcal{G}) \to \check{H}^1(U, \mathcal{H}) \to \cdots
\]

Proof. Since \( U \) is Leray, we have \( H^1(U_I, \mathcal{F}) = 0 \) for all multi-indexes \( I \) (in fact, this is the only property we need from the covering \( U \)). Hence the following sequences are exact:

\[
0 \to \mathcal{F}(U_I) \to \mathcal{G}(U_I) \to \mathcal{H}(U_I) \to 0
\]

Then applying the Čech complex, we get an exact sequence of complexes

\[
0 \to \check{C}^\bullet(U, \mathcal{F}) \to \check{C}^\bullet(U, \mathcal{G}) \to \check{C}^\bullet(U, \mathcal{H}) \to 0
\]

Now the claim follows from Proposition 12.1. \( \square \)

Hence we get our desired theorem:

Theorem 13.16 (Leray). Suppose \( U \) is a cover of \( X \) and \( H^q(U_1 \cap \cdots \cap U_p, \mathcal{F}) = 0 \) for all \( p, q > 0 \) and all \( U_1, \ldots, U_p \in U \). Then there is a natural isomorphism between cohomology and Čech cohomology:

\[
H^p(X, \mathcal{F}) \simeq \check{H}^p(U, \mathcal{F})
\]

Proof. We use induction on \( p \). For \( p = 0 \), the claim is clear. Note that we have the exact sequence

\[
0 \to \mathcal{F} \to \Pi(\mathcal{F}) \to Z^1 \to 0
\]

and \( H^1(X, \mathcal{F}) = \text{coker}(\Gamma(X, \Pi(\mathcal{F})) \to \Gamma(X, Z^1 \mathcal{F})) \), and

\[
H^p(X, \mathcal{F}) = H^{p-1}(X, Z^1 \mathcal{F})
\]
for $p \geq 2$. On the other hand, we also have the corresponding result for Čech cohomology:

\[
\begin{aligned}
0 \to \check{H}^0(U, \mathcal{F}) &\to \check{H}^0(U, \Pi(\mathcal{F})) \to \check{H}^0(U, \mathcal{Z}^1) \\
&\to \check{H}^1(U, \mathcal{F}) \to \check{H}^1(U, \Pi(\mathcal{F})) = 0
\end{aligned}
\]

where $\check{H}^1(U, \mathcal{F}) = 0$ by Lemma 13.13, since $\Pi(\mathcal{F})$ is acyclic. Hence also

\[
\check{H}^1(X, \mathcal{F}) = \text{coker}(\Gamma(X, \Pi(\mathcal{F})) \to \Gamma(X, \mathcal{Z}^1)) = H^1(X, \mathcal{F})
\]

Hence the theorem also holds for $p = 1$.

We continue by induction on $p$. Since also $\check{H}^i(U, \Pi(\mathcal{F})) = 0$ for all $i > 0$, same long exact sequence of Čech cohomology also shows that $\check{H}^p(U, \mathcal{F}) = \check{H}^{p-1}(U, \mathcal{Z}^1)$. Moreover, the cover $\mathcal{U}$ is also Leray with respect to $\mathcal{Z}^1$: $H^i(U_1, \mathcal{Z}^1) = H^{i+1}(U_1, \mathcal{F}) = 0$. Hence replacing $\mathcal{F}$ with $\mathcal{Z}^1$, we get the desired conclusion.
Chapter 14

Divisors and linear systems

When studying a scheme it is natural to ask about its closed subschemes. We have seen that such subschemes are in correspondence with quasi-coherent ideal sheaves. For (integral) subschemes of codimension 1, these ideal sheaves tend to be much simpler than for higher codimension. This is essentially because of Krull’s Hauptidealsatz, which says roughly that (under certain hypotheses), this is generated locally by one element. This is essentially the prototype of a Cartier divisor; a subscheme which is locally cut out by one equation.

The prototype of a divisor is a hypersurface of projective space, that is, an integral subscheme of $\mathbb{P}^n$ of codimension one. Each such subscheme is defined by a homogeneous ideal $I$ of $k[x_0, \ldots, x_n]$, and since the codimension is assumed to be one, Krull’s theorem implies that $I$ is principal, i.e., $I = (f)$ for some degree $d$ polynomial $f \in k[x_0, \ldots, x_n]$. To give a concrete example, consider the case of $\mathbb{P}^2$, and the curve

$$D_0 = V(x^3 + y^3 + z^3) \subseteq \mathbb{P}^2_k$$

This is clearly integral, since the defining equation is irreducible. Similarly, we can consider the subscheme

$$D_1 = V(xy)$$

This subscheme is reduced, but not irreducible: $D$ has three irreducible components $V(x), V(y), V(z)$.

The main feature of divisors to talk about sums and differences of such subschemes, thereby turning them into a group. This is illustrated in the example above, by writing $D_1 = V(x) + V(y) + V(z)$. The sum here is completely formal – it is an element in the free group on integral subschemes of codimension one. Such a sum is by definition, a Weil divisor.
There is an equivalence relation defined on such objects, designed to capture when two divisors belong to the same family. In the example $D$ and $D'$ belong to the same family, namely

$$D_{[s:t]} = V(sxyz + t(x^3 + y^3 + z^3))$$

More precisely, there is a closed subscheme $D \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ defined by the above equation, so that the fiber over a closed point $[s : t] \in \mathbb{P}^1$ is exactly the curve $D_{[s:t]}$ of $\mathbb{P}^2_k$. Geometrically, we have a family of ‘moving divisors’, parameterized over the projective line. We say the two divisors are linearly equivalent. The key feature is that there is this morphism $f : D \to \mathbb{P}^1$, and $f^{-1}([0 : 1]) = D_0$ and $f^{-1}([1 : 0]) = D_1$. Moreover, the quotient

$$g = \frac{x^3 + y^3 + z^3}{xyz}$$

defines a rational function on $\mathbb{P}^2$. This is the pullback of the rational function $s/t$ on $\mathbb{P}^1$: $g = f^*(s/t)$.

There is a second approach to divisors, which is perhaps more algebraic in nature, namely Cartier divisors. The definition is motivated by the fact that integral subschemes of codimension one are typically defined by a single equation locally. Note that in the above example, the special properties of projective space imply that $D_0$ and $D_1$ are defined globally by a single equation. It is not hard to come up with examples of schemes with subschemes of codimension one that are not. In fact, for most schemes, the concept of a ‘globally defined equation’ does not make sense, since we do not have global coordinates to work with. However, locally this concept makes sense: We can consider subschemes $Y \subseteq X$ so that the ideal sheaf $I_Y$ is locally generated by a single element $f_i \in A_i$, on some affine covering $X = \bigcup \text{Spec } A_i$. In other words, the ideal sheaf $I_Y$ is an invertible sheaf.

While Weil divisors are conceptual and more geometric, Cartier divisors do have some advantages. For instance, they are very closely related to invertible sheaves and line bundles, which in turn makes computations with them easier. For instance, given a morphism $f : X \to Y$, we would like to define a ‘pullback’ of a divisor on $Y$ to a divisor on $X$ – this turns out to only be possible for Cartier divisors. There are also other settings where the special properties of Cartier divisors are essential, for instance defining intersection products.

That being said, for a smooth scheme we will see that the various notions of a divisor are equivalent, and there are natural ways to switch between the four
categories in the diagram

\[
\begin{array}{cccc}
\text{Weil divisors} & \text{line bundles} \\
\downarrow & \downarrow \\
\text{Cartier divisors} & \text{invertible sheaves}
\end{array}
\]

The main interest in these concepts is of course that they give us tools to classify varieties and schemes. So given \( X \), perhaps as an abstract scheme, we can first look for a divisor \( D \) on it. If the corresponding invertible sheaf \( L = \mathcal{O}_X(D) \) is globally generated, we have a morphism to projective space \( f : X \to \mathbb{P}^n \) and we can use this to study the geometry of \( X \): We can ask about the fibers of \( f \), whether it is an closed immersion, and if so, what the equations of the image is.

We will in this chapter assume that

\textit{All schemes are integral and noetherian.}

Noetherianness is somewhat important, since we want to talk about the decomposition of a closed subschemes into its irreducible components. The assumption of integrality, especially irreducibility, is not essential for most of the statements, but it makes the definitions more transparent, and more importantly, the proofs considerably simpler (e.g., not having to worry about nilpotent elements allows us to work with meromorphic functions as elements in a fraction field, rather some more obscure localization). Still, we remark that the theory of Cartier divisors work in a completely general setting, although there are some subtle points one has to take into account. If you are curious about general background on meromorphic functions on arbitrary schemes, see EGA IV\(_4\), section 20.

### 14.1 Weil divisors

In this section \( X \) will denote an integral, normal noetherian scheme. Recall that this means that each local ring \( \mathcal{O}_{X,x} \) is an integral domain, which is integrally closed in its function field \( K = K(X) \).

**Definition 14.1.** A prime divisor on \( X \) is a closed integral subscheme \( Y \) of codimension 1. A Weil divisor on \( X \) is a finite formal sum

\[
D = \sum n_i Y_i
\]

(14.1.1)

where \( n_i \in \mathbb{Z} \) and \( Y_i \) are prime divisors.

We say \( D \) is effective if all the \( n_i \) are non-negative in (14.1.1).

We call \( \bigcup_i Y_i \) the support of \( D \).

We denote by \( \text{Div}(X) \) the group of Weil divisors; this is the free abelian group on prime divisors.
14.1. Weil divisors

If \( Y \) is a prime divisor on \( X \) and \( U \subseteq X \) is an open set, then \( Y \cap U \) is naturally a prime divisor on \( U \). It follows that we obtain a presheaf \( U \mapsto \text{Div}(U) \). This is in fact a sheaf; it coincides with

\[
\bigoplus_{x \in X, \text{codim } x=1} j_x^*(\mathbb{Z}_{\{x\}})
\]

where \( j_x : \{x\} \to X \) is the inclusion of a point \( x \) of codimension 1 in \( X \). We will denote this sheaf by \( \text{Div} \).

14.1.1 The divisor of a rational function

The main reason for the normality assumption is that if \( X \) is normal, then it is regular in codimension one. This implies that for each point \( \eta \in X \) with \( \text{codim } \eta = 1 \), each local ring \( \mathcal{O}_{X,\eta} \) (which has Krull dimension 1) is a discrete valuation ring, with a corresponding valuation \( v_Y : K^\times \mathbb{Z} \). The concept of a valuation is a generalization of the ‘order’ of a zero or a pole of a meromorphic function in complex analysis. Intuitively, an element \( f \in K^\times \) has positive valuation \( N \) if it vanishes to order \( N \) along \( Y \), and negative valuation \( -N \) if it has a pole of order \( N \) there.

To say what \( v_Y \) is explicitly, we define for an element \( f \in \mathcal{O}_{X,\eta} \),

\[
v_Y(f) = n
\]

where \( n \) is the integer so that \( f \in m^n - m^{n-1} \). In the function field, an element \( f \) is represented by a fraction \( g/h \) and we define \( v_Y(f) = v_Y(g) - v_Y(h) \). (Check that this is independent of the chosen representative). This means that \( \mathcal{O}_{X,\eta} = v_Y^{-1}(\mathbb{Z}_{\geq 0}) \), and the maximal ideal \( m = v_Y^{-1}(\mathbb{Z}_{\geq 1}) \).

**Definition 14.2.** If \( f \in K^\times \), we define its divisor as

\[
\text{div}(f) = \sum v_Y(f)Y
\]

Divisors of the form \( \text{div}(f) \) are called principal divisors, and they form a subgroup \( \text{Div}_0(X) \subseteq \text{Div}(X) \).

This is well defined by the following lemma:

**Lemma 14.3.** Suppose that \( X \) is an integral normal noetherian scheme with fraction field \( K \) and let \( f \in K \). Then \( v_Y(f) = 0 \) for all but finitely many prime divisors \( Y \).

**Proof.** We first reduce to the case when \( X \) is affine. Let \( U = \text{Spec } A \) be an open affine subset such that \( f|_U \in \Gamma(U, \mathcal{O}_X) \). Since \( X \) is noetherian, the complement
Z = X − U is a closed subset of X which has finitely many irreducible components; in particular, only finitely many prime divisors Y are supported in Z. So we reduce to the affine case X = Spec A and f ∈ Γ(X, O_X), by ignoring these finitely many components. Then v_Y(f) ≥ 0 and v_Y(f) > 0 if and only if Y is contained in V(f); and since V(f) has only finitely many irreducible components of codimension 1, we are done.

Proposition 14.4. Let f ∈ K^×, and let Y ⊆ X be a prime divisor with generic point η. Then

1. v_Y(f) ≥ 0 if and only if f ∈ O_{X, η}
2. v_Y(f) = 0 if and only if f ∈ O^×_{X, η}.

If X is projective, then v_Y(f) ≥ 0 for all Y if and only if v_Y(f) = 0 for all Y if and only if f ∈ k^×.

Example 14.5. Consider X = A^1_k = Spec k[t] and let K = k(t). Here prime divisors in X correspond to closed points p ∈ X. Let f = \frac{x^2(x-1)}{x+1}. Then v_p(f) = 0 for all Y except when p = 0, ±1, where we have

v_0(f) = 2, \quad v_1(f) = 1, \quad v_{-1}(f) = -1

Hence the divisor of f is 2V(t) + V(t−1) − V(t+1).

Example 14.6. Consider X = P^1_k = Proj k[u, v] and let K be the fraction field. Let f = \frac{u^2-v^2}{(u-2v)^2}. On D(u) we use the coordinate t = v/u so that K = k(t). Here f = \frac{1-t^2}{(1-2t)^2}, and the only non-zero valuations are

v_1(f) = 1, \quad v_{-1}(f) = 1, \quad v_{1/2}(f) = -2

The rational function f does not have any poles or zeroes at the point [0 : 1], so the divisor is [1 : 1] + [1 : -1] − 2[2 : 1].

Example 14.7. Let X be the curve V(y^2 - x^3 - 1). Then div x = (0, 1) + (0, -1) and div y = (-1, 0) + (ω, 0) + (ω^2, 0) where ω is a primitive third root of −1, and

\text{div}(x^2/y) = 2(0, -1) + 2(0, 1) - (-1, 0) - (ω, 0) - (ω^2, 0)

14.1.2 The divisor of a section of an invertible sheaf

Let L be an invertible sheaf and let s be a rational section (a section defined over an open set). We can define a divisor div s as follows. For Y a prime divisor, let y be the generic point. Let U ⊆ X be a neighbourhood of y such that there is a trivialization φ : L|_U → O_X|_U. On U, φ(s) defines a rational section of O_X(U),
and hence an element of $K$. We define $v_Y(s) = v_Y(f)$. It is not hard to see that this is independent of $U$.

**Example 14.8.** Let $L = \mathcal{O}(2)$ on $X = \mathbb{P}^1_k = \text{Proj} k[u,v]$ and let $K$ be the fraction field. Let $f = \frac{v^2}{u^2 + v}$. On $D(u)$ we use the coordinate $t = v/u$ so that $K = k(t)$. Here $f = \frac{t^2}{1+t}$, and the only non-zero valuations are

$v_0(f) = 2, \quad v_{-1}(f) = -1.$

So on $D(u)$ we get the divisor $2[1 : 0] - [1 : -1]$. On $D(v)$, we can use the coordinate $t = u/v$, where $f$ becomes $f = \frac{1}{1+t}$. The non-zero valuations are the following:

$v_{t=0}(f) = 1, \quad v_{t=-1}(f) = -1.$

So we get $[0 : 1] - [1 : 1] = [0 : 1] - [1 : -1]$. On $X = D(u) \cup D(v)$, we then find the divisor is

$2[1 : 0] - [1 : -1] + [0 : 1]$.

### 14.1.3 The class group

**Definition 14.9.** We define the *class group* of $X$ as

$$\text{Cl}(X) = \text{Div}(X) / \text{Div}_0(X)$$

Two Weil divisors $D, D'$ are *linearly equivalent* (written $D \sim D'$) if they have the same image in $\text{Cl}(X)$, or equivalently, that $D - D'$ is principal.

**Example 14.10.** In fact, any divisor on $\mathbb{A}^1_k$ for an algebraically closed field $k$ is principal. Indeed, if $D = \sum_{i=1}^n n_i[p_i]$ where $n_i \in \mathbb{Z}$ and $p_i \in k$, then

$$f = \prod_{i=1}^n (t - p_i)^{n_i}$$

is an element of $k(t)^\times$ with $\text{div}(f) = D$. It follows that $\text{Cl}(\mathbb{A}^1_k) = 0$ in this case.

### 14.1.4 Projective space

**Proposition 14.11.** $\text{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$.

Write $\mathbb{P}^n_k = \text{Proj} R, \quad R = k[x_0, \ldots, x_n]$. Prime divisors on $\mathbb{P}^n_k$ correspond to height one prime ideals in $R$, i.e., $p = (g)$ where $g$ is a homogeneous irreducible polynomial. We can use this to define the *degree* of a divisor, by taking the
degrees of the corresponding polynomials:

$$\text{Div}(\mathbb{P}^n) \rightarrow \mathbb{Z}$$

$$\sum n_i V(g_i) \mapsto \sum n_i \deg g_i$$

If \( f \in K(\mathbb{P}^n) \) is a rational function, then \( f \) can be written as a quotient of two homogeneous polynomials of the same degree. (Indeed, \( K = K(\mathbb{P}^n) = K(U) = K(R(x_0)) \) and, so every rational non-zero function on \( \mathbb{P}^n \) gives rise to an element of \( F \) of degree 0, via the inclusion \( K(R(x_0))^* \rightarrow K \). Conversely, any fraction in \( K \) of degree 0 gives an element in \( R(x_0) \).) Now, if \( f \in K(\mathbb{P}^n) \) is a rational function, we can write it as

$$f = \prod_i f_i^{n_i}$$

where the \( f_i \) are irreducible coprime polynomials in \( R \) and \( n_i \in \mathbb{Z} \). Let us first show that

$$\text{div} f = \sum n_i [V(f_i)]$$

If \( Y \subseteq X \) is a prime divisor, let \( y \in X \) be the generic point. \( Y = V(g) \) for some irreducible polynomial \( g \) of degree \( d \). For any other polynomial \( h \) of degree \( d \), the fraction \( g/h \) is a generator of \( m_y \mathcal{O}_{X,y} \). We can write \( f = (g/h)^r f' \) with \( r = n_i \) if \( f_i \) divides \( g \) (and 0 if no \( f_i \) divides \( g \)) and \( f' \) a rational function not containing \( g \) in the numerator or the denominator. Hence \( v_Y(f) = r \), and hence

$$\text{deg div} f = \sum n_i \deg f_i = 0$$

because \( f \) is a rational function. It follows that the degree map descends to a map

$$\text{deg} : \text{Cl}(\mathbb{P}^n) \rightarrow \mathbb{Z}$$

$$\sum n_i V(g_i) \mapsto \sum n_i \deg g_i$$

We claim that this is an isomorphism. Note that \( \text{deg} H = 1 \), where \( H = V(x_0) \) is a hyperplane, so \( \text{deg} \) is surjective. Now, any \( Z = \sum n_i [V(f_i)] \) in the kernel of \( \text{deg} \), must have \( \sum n_i \deg f_i = 0 \). Consider the element \( f = \prod_i f_i^{n_i} \), which now defines an element of \( K \). Using the formula for \( \text{div} f \) above, we see that \( Z = \text{div} f \), and hence \( Z \) is a principal divisor, and so \( \text{deg} \) is injective.

**Example 14.12.** Consider the curve \( X \) as in Figure , given by \( X = V(y^2 x - x^3 - xz^2) \subseteq \mathbb{P}^2 \). By the above discussion any two lines \( L, L' \) in \( \mathbb{P}^2 \) are linearly equivalent. Since \( X \) is integral, then there are well-defined restrictions \( L|_X \) and \( L'|_X \), which are linearly equivalent divisors on \( X \). The figure below shows one example where \( L|_X = P + Q + R \) and \( L' = 2S + T \).
14.1. Weil divisors

14.1.5 Weil divisors and invertible sheaves

Let $D$ be a Weil divisor on $X$. We define the sheaf $\mathcal{O}_X(D)$ as follows:

$$\mathcal{O}_X(D)(U) = \{ f \in K \mid \text{div } f + D \geq 0 \text{ on } U \} \cup \{ 0 \}$$

This is a quasi-coherent sheaf on $X$. (We will see soon that it is invertible if and only if $D$ is a Cartier divisor).

Example 14.13. Let $X$ be the projective line $\mathbb{P}^1_k = \text{Proj } k[x_0, x_1]$ over $k$ and let $D = V(x_0) = [1 : 0]$. Let $U_0 = \text{Spec } k[x_1/x_0] = \text{Spec } k[t], U_1 = \text{Spec } k[x_0/x_1] = \text{Spec } k[s]$ be the standard covering of $\mathbb{P}^1_k$ (so $s = t^{-1}$ on $U_0 \cap U_1$). Note that the point $[1 : 0]$ does not lie in $U_0 = D_+(x_0)$. This means that a rational function $f \in K$ such that $f + D$ is effective on $U_0$ must be regular on $U_0$, i.e.,

$$\Gamma(U_0, \mathcal{O}_X(D)) = k[t]$$

On $U_1$, we are looking at elements $f \in k(s)$ having valuation $\geq -1$ at $s = 0$. This implies that

$$\Gamma(U_1, \mathcal{O}_X(D)) = k[s] \oplus k[s]s^{-1}$$

Now we think of elements in $\Gamma(X, \mathcal{O}_X(D))$ as pairs $(f, g)$ with $f, g \in K$ sections of $\mathcal{O}_X(D)$ over $U_0$ and $U_1$ respectively, so that $f = g$ on $U_0 \cap U_1$. Here $f = f(t)$ is a polynomial in $t$, and

$$g(s) = p(s) + q(s)s^{-1} = p(t^{-1}) + q(t^{-1})t.$$
If $f = g$ in $k[t, t^{-1}]$ we must $\deg p, q \leq 1$. This implies that
\[ \Gamma(X, \mathcal{O}_X(D)) = k \oplus kt. \]

### 14.1.6 The class group and unique factorization domains

The term ‘class group’ comes from algebraic number theory and its origins be traced back Kummer’s work on Fermat’s last theorem. If $A$ is a Dedekind domain then $\text{Cl}(\text{Spec} A)$ coincides with the class group of $A$, which measures how far $A$ is from being a unique factorization domain.

**Proposition 14.14.** Let $A$ be a noetherian integral domain and let $X = \text{Spec} A$. Then $A$ is a UFD if and only if $\text{Cl}(\text{Spec} A) = 0$ and $X$ is normal.

**Proof.** If $A$ is a UFD, then it is integrally closed, and hence normal. Moreover, $A$ is a UFD if and only if every prime ideal $p$ of height 1 is principal. We want to show that this condition is equivalent to $\text{Cl}(X) = 0$.

$\Rightarrow$: If $Y \subseteq X$ is a prime divisor, then $Y = V(p)$ for $p$ of height 1. Hence $Y = V(f) = \text{div} f$ for some $f$, and so $\text{Cl}(X) = 0$.

$\Leftarrow$: If $\text{Cl}(X) = 0$, let $p$ be a prime of height 1, and let $Y = V(p) \subseteq X$. By assumption, there is a $f \in K^\times$ such that $\text{div}(f) = Y$. We want to show that $f \in A$ and that $p = (f)$.

$v_Y(f) = 1$, so $f \in A_p$ and $f$ generates $pA_p$. If $q \subseteq A$ is any prime ideal not equal to $p$, then $Y' = V(q)$ is a prime divisor not equal to $Y$. $v_{Y'}(f) = 0$, since $(f) = Y$, and so $f \in A_q$. So in all, we have
\[ f \in \bigcap_{q \in \text{Spec} A} A_q = A \]

where the equality comes from the fact that $A$ is an integrally closed domain and the ‘algebraic Hartog’s theorem’: A function which is regular in codimension 1 is regular (see Matsumura p. 124). Hence $f \in A$, and in fact $f \in A \cap pA_p = p$.

We show that $f$ generates $p$: Take any $f \in p$. Then $v_Y(g) \geq 1$ and $v_{Y'}(g) \geq 0$ for all $Y' \neq Y$. It follows that $v_{Y'}(g/f) \geq 0$ for all prime divisors $Y'$. Hence $g/f \in A_q$ for all $q$ prime of height 1, and hence $g/f \in A$, by the above. It follows that $g \in fA$ and so $p = fA$ is principal.

### 14.1.7 A useful exact sequence

Given a scheme $X$ and an open subset $U$, the restriction of a prime divisor on $X$ is a prime divisor on $U$, so it is natural to ask how the two class groups are related. The answer is given by the following theorem:
Theorem 14.15. Let \( X \) be a normal, integral scheme, let \( Z \subseteq X \) be a closed subscheme and let \( U = X - Z \). If \( Z_1, \ldots, Z_r \) are the prime divisors corresponding to the codimension 1 components of \( Z \), then there is an exact sequence

\[
\bigoplus_{i=1}^r ZZ_i \to \text{Cl}(X) \to \text{Cl}(U) \to 0 \quad (14.1.2)
\]

where the map \( \text{Cl}(X) \to \text{Cl}(U) \) is defined by \([Y] \mapsto [Y \cap U]\).

Here we regard each \( Z_i \) as a subscheme of \( X \) with the reduced scheme structure (cf. Proposition 8.37)

Proof. If \( Y \) is a prime divisor on \( U \), the closure in \( X \) is a prime divisor in \( X \), so the map is surjective.

We just need to check exactness in the middle. Suppose \( D \) is a prime divisor which is principal on \( U \). Then \( D|_U = \text{div} f \) for some \( f \in K(U) = K = K(X) \). Now \( \text{div} f \) is a divisor on \( X \) such that \( D|_U = \text{div} (f)|_U \). Hence \( D - \text{div}(f) \) is supported in \( X - U \), and hence it is a linear combination of the \( Z_i \). \( \square \)

As a special case, we see that removing a codimension 2 subset does not change the group of Weil divisors. So for instance \( \text{Cl}(\mathbb{A}^2 - 0) = \text{Cl}(\mathbb{A}^2) \).

Example 14.16. Consider the projective line \( \mathbb{P}^1_k \) over a field \( k \), and let \( p \) be a point. We have the exact sequence

\[
\mathbb{Z}[p] \to \text{Cl}(\mathbb{P}^1) \to \text{Cl}(\mathbb{A}^1) \to 0
\]

We saw that \( \text{Cl}(\mathbb{A}^1) = 0 \), so the map \( \mathbb{Z} \to \text{Cl}(\mathbb{P}^1) \) is surjective. It is also injective: if \([nP] = 0\) in \( \text{Cl}(\mathbb{P}^1) \) for some \( n \). Then \( nP = \text{div} f \) for some \( f \in k(\mathbb{P}^1) \), so if \( \text{div} f|_{\mathbb{A}^1} = 0 \), we have \( f \in \Gamma(\mathbb{A}^1, \mathcal{O}) = k^* \). Hence \( f \) is constant, and so \( n = 0 \). It follows that \( \text{Cl}(\mathbb{P}^1) = \mathbb{Z} \).

14.2 Cartier divisors

Let \( X \) be a noetherian integral scheme with function field \( K \). Let \( \mathcal{K}_X \) denote the constant sheaf with value \( K \). The constant sheaf \( \mathcal{K}_X^* \) with value \( K^* \) (the group of non-zero elements of \( K \)) is a subsheaf, and contains \( \mathcal{O}_X^* \), the sheaf of units in \( \mathcal{O}_X^* \).

Definition 14.17. A Cartier divisor \( D \) on \( X \) is an element of \( \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \).

This definition of a Cartier divisor is pretty obscure, and one rarely thinks of a divisor in this way. We can shed some light on the definition by considering an
open cover $U_i$ of $X$. The sheaf axiom sequence takes the following form

$$0 \rightarrow \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \rightarrow \prod_i \Gamma(U_i, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \rightarrow \prod_{i,j} \Gamma(U_{ij}, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$$

From this, we see that a Cartier divisor is therefore given as a set of sections of $\mathcal{K}_X^\times / \mathcal{O}_X^\times$ over the open sets $U_i$ that agree on the overlaps $U_{ij}$.

If we shrink the $U_i$ we may assume that $s_i$ are induced by a section $f_i$ of $\mathcal{K}_X^\times$ over $U_i$, that is, an element of $K^\times$. To see what it means that two such data $U_i, f_i$ agree on $U_{ij}$ we keep in mind that they are sections of $\mathcal{K}_X^\times / \mathcal{O}_X^\times$. The condition $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ translates into the statement that $f_i|_{U_{ij}} f_j^{-1}|_{U_{ij}}$ is a section of $\mathcal{O}_X^\times$ over $U_{ij}$. In other words, there are units $c_{ij} \in \mathcal{O}_X|_{U_{ij}}$ such that $f_i = c_{ij} f_j$ over $U_{ij}$. In $K^\times$ we have $c_{ij} = \frac{f_i}{f_j}$, which implies that the $c_{ij}$ satisfy the ‘cycyle conditions’

$$c_{ik} = c_{ij} c_{jk}; \quad c_{ji} = c_{ij}^{-1}; \quad c_{ii} = 1$$

The idea here is that the $f_i$ give the local equations for a subscheme of $X$. On an affine open set $U$ where $f_i$ is regular, the zero-set $V(f_i)$ defines a codimension 1 closed subset of $U$. If both $f_j$ and $f_i$ are regular on $U$, their zero-sets must be the same since they differ by a unit.

**Definition 14.18.** The pairs $(U_i, f_i)$ are called the local defining data or the local equations for the divisor $D$ (with respect to the covering $U_i$).

Such defining data are not unique: $\{(U_i, f_i)\}_{i \in I}$ and $\{(V_j, g_j)\}_{j \in I}$ give the same Cartier divisor if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathcal{O}_X^\times)$ for all $i, j$.

Now, the set of Cartier divisors naturally forms an abelian group: Given $D$ and $D'$ represented by the data $\{(U_i, f_i)\}_{i \in I}$ and $\{(V_i, g_i)\}_{i \in I}$, we can define $D + D'$ as the Cartier divisor associated to the data

$$\{(U_i \cap V_j, f_i g_j)\}_{i,j}$$

Moreover, the inverse $-D$ will be defined as $\{(U_i, f_i^{-1})\}_{i \in I}$, and the identity is defined by the data $\{(U_i, f_i)\}_{i \in I}$ where $f_i \in \Gamma(U_i, \mathcal{O}_X^\times)$. We denote the group of Cartier divisors by $\text{CaDiv}(X) = \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$.

**Definition 14.19.** We say that a Cartier divisor is principal if it is equal (as an element of $\text{CaDiv}(X)$) to $(X, f)$ where $f \in K^\times$.

The set of principal Cartier divisors form a subgroup of $\text{CaDiv}(X)$, which is typically much smaller than $\text{CaDiv}(X)$. However, by definition, any Cartier divisor is ‘locally principal’.

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We define $\text{CaCl}(X)$ to be the group of Cartier divisors modulo principal divisors:

$$\text{CaCl}(X) = \text{CaDiv}(X)/\{(X, f)|f \in K^\times\}$$

We say that two Cartier divisors $D, D'$ are linearly equivalent, if $D - D' = (X, f)$ for some principal divisor $(X, f)$, or equivalently, $[D] = [D']$ in $\text{CaCl}(X)$.

14.2.1 Cartier divisors and invertible sheaves

Let $D$ be a Cartier divisor on a scheme $X$ given by the data $(U_i, f_i)$. We can associate to it an invertible sheaf, which we denote by $\mathcal{O}_X(D)$. On $U_i$ we take the the subsheaf $f_i^{-1}\mathcal{O}_{U_i} \subseteq \mathcal{K}_{U_i}$ of $\mathcal{K}_X$. This subsheaf is isomorphic to $\mathcal{O}_{U_i}$ and has $f_i^{-1}$ as a local generator. So over an affine subset $U = \text{Spec}A \subseteq U_i$, the sheaf is the sheaf associated to $f_i^{-1}A \subseteq K(A)$.

On the intersection, $U_i \cap U_j$, we have $f_i = c_{ij}f_j$ where $c_{ij}$ is an invertible section of $\mathcal{O}_{U_{ij}}$. This means that $f_i^{-1}\mathcal{O}_{U_{ij}} = f_j^{-1}\mathcal{O}_{U_{ij}}$ as subsheaves of $\mathcal{K}_{U_{ij}}$. We have therefore, constructed a sheaf on each $U_i$, and the elements coincide on the intersections $U_{ij}$. In order to be able to glue to a sheaf, there is a cocycle condition that has to be satisfied. But since these sheaves are all subsheaves of a fixed sheaf $\mathcal{K}$, the gluing maps are actually identity maps, and the cocycle condition is automatically satisfied. It follows that the sheaves $f_i^{-1}\mathcal{O}_{U_i}$ glue to a sheaf $\mathcal{O}_X(D)$ defined on all of $X$. It is by construction invertible, since it is invertible on each $U_i$.

Explicitly, $\mathcal{O}_X(D)$ is defined as the subsheaf of $\mathcal{K}_X$ given by

$$\Gamma(V, \mathcal{O}_X(D)) = \{f \in K|f_i \in \Gamma(U_i \cap V, \mathcal{O}_X)\forall i\}$$

Two different data $(U_i, f_i)$ and $(V_j, g_j)$ for the same divisor $D$ give rise to the same invertible sheaf. This is because over $U_i \cap V_j$, we have $f_i = d_{ij}g_j$, for some sections $d_{ij} \in \mathcal{O}_X(U_i \cap V_j)^\times$. This means that $f_i^{-1}\mathcal{O}_{U_i \cap V_j} = g_j^{-1}\mathcal{O}_{U_i \cap V_j}$, and so the sheaf is uniquely determined.

Moreover, by how we defined the sum $D + D'$ we have

**Proposition 14.20.** Let $X$ be an integral noetherian scheme and let $D$ and $D'$ be two Cartier divisors.

(i) $\mathcal{O}_X(D + D') \simeq \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$

(ii) $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ if and only if $D$ and $D'$ are linearly equivalent.

**Proof.** We can pick a covering $U_i$ so that both $D$ and $D'$ are both represented by data $(U_i, f_i)$, $(U_i, f'_i)$. Then $D + D'$ is defined by $(U_i, f_i f'_i)$. Locally, over $U_i$ the sheaf $\mathcal{O}_X(D + D')$ is defined as the subsheaf of $\mathcal{K}$ given by $(f_i f'_i)^{-1}\mathcal{O}_{U_i} =$
During the discussion of Cartier divisors, we see that for an isomorphism \( f : X \to Y \), we get a Cartier divisor \( D \) which sends \( D \) to the class of \( \mathcal{O}_X(D) \) isomorphic to \( \mathcal{O}_Y(f^*D) \). For the second claim, it suffices (by point (i)) to show that \( \mathcal{O}_X(D) \cong \mathcal{O}_X \) if and only if \( D \) is a principal Cartier divisor. So suppose that \( \mathcal{O}_X(D) \subseteq \mathcal{K}_X \) is a sub \( \mathcal{O}_X \)-module which is isomorphic to \( \mathcal{O}_X \). Then the image of 1 in \( \mathcal{O}_X \) will be a section of \( \mathcal{O}_X(D) \) which generates \( \mathcal{O}_X(D) \) everywhere. This means that \( (X, f) \) is the local defining data, and \( D \) is a principal Cartier divisor.

Conversely, if \( D = (X, f) \), then \( \mathcal{O}_X(D) = f^{-1} \mathcal{O}_X \), and multiplication by \( f \) gives an isomorphism \( \mathcal{O}_X(D) \cong \mathcal{O}_X \).

**14.2.2 \( \text{CaCl}(X) \) and \( \text{Pic}(X) \)**

By the item (ii) in Proposition 14.20. We see that the map \( \text{CaDiv}(X) \to \text{Pic}(X) \), which sends \( D \) to the class of \( \mathcal{O}_X(D) \) in \( \text{Pic}(X) \) is additive and has the subgroup of principal divisors \( \text{CaDiv}_0(X) \) as its kernel. This means that the induced map \( \rho : \text{CaCl}(X) \to \text{Pic}(X) \) is injective. In this section we will show that this map is also surjective, so that \( \text{CaCl}(X) = \text{Pic}(X) \).

**Proposition 14.21.** When \( X \) is integral, the map \( \rho : \text{CaCl}(X) \to \text{Pic}(X) \) is an isomorphism.

**Proof.** We need to show that \( \rho \) is surjective. It suffices to show that any invertible \( \mathcal{O}_X \)-module \( L \) is a submodule of \( \mathcal{K}_X \): If \( L \subseteq \mathcal{K} \), let \( U_i \) be a trivializing cover of \( L \) and let \( g_i \) be its local generators. Then we have \( L|_{U_i} = g_i \mathcal{O}_{U_i} \subseteq \mathcal{K}_{U_i} \) and the \( g_i \) are rational functions on \( U_i \). On \( U_{ij} = U_i \cap U_j \), we have \( g_i \mathcal{O}_{U_{ij}} = L|_{U_{ij}} = g_j \mathcal{O}_{U_{ij}} \), and it follows that \( g_i = c_{ij} g_j \) for units \( c_{ij} \in \mathcal{O}_{U_{ij}}^\times \). Consequently, \( (U_i, g_i^{-1}) \) forms a set of local defining data for a Cartier divisor \( D \), and of course we have \( L = \mathcal{O}_X(D) \).

Let \( L \) be an invertible sheaf and consider the sheaf \( L \otimes_{\mathcal{O}_X} \mathcal{K}_X \). Let \( U_i \subseteq X \) be a set of open subsets such that \( L|_{U_i} = \mathcal{O}_X|_{U_i} \). Note that the restriction of \( L \otimes_{\mathcal{O}_X} \mathcal{K}_X \) to each \( U_i \) is a constant sheaf. Since \( X \) is irreducible, any sheaf whose restriction to opens in a covering is constant, is in fact a constant sheaf, and therefore \( L \otimes_{\mathcal{O}_X} \mathcal{K}_X \cong \mathcal{K}_X \) as sheaves on \( X \). Now we can regard \( L \) as a rank 1 subsheaf of \( \mathcal{K}_X \) using the composition \( L \to L \otimes \mathcal{K}_X \cong \mathcal{K}_X \). Hence we get a Cartier divisor \( D \) on \( X \) such that \( L = \mathcal{O}_X(D) \).

**14.2.3 From Cartier to Weil**

Let \( D \) be a Cartier divisor given by the data \((U_i, f_i)\). If \( Y \) is a prime divisor on \( X \), with generic point \( \eta \), then since \( U_i \) is a cover, \( \eta \) lies in some \( U_i \). We can then define

\[
v_Y(D) = v_Y(f_i)
\]
This is independent of the choice of $U_i$: If $\eta \in U_i \cap U_j$, then $f_i f_j^{-1}$ is an element of $\mathcal{O}^\times_X(U_i \cap U_j)$, and so $v_Y(f_i f_j^{-1}) = 0$, and hence $v_Y(f_i) = v_Y(f_j)$.

This gives a map $\phi : \text{CaDiv}(X) \rightarrow \text{Div}(X)$ defined by

$$\phi(D) = \sum_Y v_Y(D)Y$$

**Proposition 14.22.** $\phi$ is injective.

Indeed, if $\phi(D) = 0$, then all $v_Y(f_i) = 0$ for all $i$ (so that $\eta_Y \in U_i$). Then the Hartogs’ lemma implies that $f_i \in \mathcal{O}^\times(U_i)$, so that $D = 0$ (as a Cartier divisor).

Note that by the definition of $(f)$, a principal Cartier divisor $(X, f)$ gives rise to a principal Weil divisor. It follows that we get an induced map $\text{CaCl}(X) \rightarrow \text{Cl}(X)$

**Proposition 14.23.** $\text{CaCl}(X) = \text{Div}(X)$ if and only if $X$ is locally factorial (all the local rings $\mathcal{O}_{X,x}$ are UFDs).

**Proof.** The inclusion $\text{CaDiv}(X) \subseteq \text{Div}(X)$ is an isomorphism if and only if the map of sheaves $\mathcal{K}_X^\times \rightarrow \mathcal{Z}^1$ is surjective, i.e., the map is surjective for every stalk. However, at the stalk at $x \in X$, this map is simply

$$K^\times \rightarrow \text{Div}(\text{Spec} \mathcal{O}_{X,x})$$

which sends $f$ to the sum $\sum v_p(f)V(p)$ where $p$ is a height 1 prime ideal of $\mathcal{O}_{X,x}$. This is surjective if and only if every $p \subseteq \mathcal{O}_{X,x}$ is a principal ideal. This happens if and only if $\mathcal{O}_{X,x}$ is a UFD.

**Corollary 14.24.** On a non-singular variety $X$, there is a bijective correspondence between

(i) Line bundles
(ii) Invertible sheaves
(iii) Cartier divisors
(iv) Weil divisors

up to isomorphism.

From this, we get the following theorem:

**Theorem 14.25.** Let $k$ be a field. Then $\text{Pic}(\mathbb{A}^n) = \text{Cl}(\mathbb{A}^n) = \text{CaCl}(\mathbb{A}^n) = 0$.

**Proof.** The polynomial ring is a UFD. \qed
14.2.4 Projective space

Let us take a closer look at the projective space \( \mathbb{P}^n_k \) over a field \( k \). Write \( \mathbb{P}^n_k = \text{Proj} \, R \) where \( R = k[x_0, \ldots, x_n] \). Consider the standard covering \( U_i = D_+(x_i) \) of \( \mathbb{P}^n_k \).

Let \( F(x_0, \ldots, x_n) \in R_d \) denote a homogeneous polynomial of degree \( d \). Then the function \( F(x/x_i) = F(x_0/x_i, \ldots, x_{i-1}/x_i, 1, x_{i+1}/x_i, \ldots, x_n/x_i) \) defines a non-zero regular function on \( U_i \), and the collection

\[
(U_i, F(x_j/x_i))
\]

forms a Cartier divisor on \( X \). Indeed, on the overlap \( U_i \cap U_j \) we have the relation

\[
F(x/x_i) = (x_j/x_i)^n F(x/x_j)
\]

and \( x_j/x_i \) is a regular, and invertible function. The corresponding invertible sheaf is \( \mathcal{O}_{\mathbb{P}^n_k}(d) \). Two homogeneous polynomials \( F, G \) of the same degree \( d \) give linearly equivalent divisors, because the quotient \( F(x)/G(x) \) is a global rational function on \( \mathbb{P}^n_k \). In particular, this applies to \( F \) and \( x_0^d \), and we get \( \mathcal{O}_{\mathbb{P}^n_k}(n) = \mathcal{O}_{\mathbb{P}^1_k}(1)^\otimes n \).

**Corollary 14.26.** On \( \mathbb{P}^n_k \) any invertible sheaf is isomorphic to some \( \mathcal{O}(m) \).

In the case of \( \mathbb{P}^1 \) over a field \( k \), the above result is not so difficult to prove directly. Indeed, recall that \( \mathbb{P}^1 \) is obtained via gluing \( U_0 = \text{Spec} \, k[s] \) and \( U_1 = \text{Spec} \, k[t] \) along \( U_{01} = \text{Spec} \, k[s, s^{-1}] = \text{Spec} \, k[t, t^{-1}] \) using the identification \( t = s^{-1} \). So given an invertible sheaf \( L \) on \( \mathbb{P}^1 \), the restriction of it to each open must be trivial (since \( \text{Pic}(\mathbb{A}^1_k) = 0 \)), we must have isomorphisms \( \phi_i : L|_{U_i} \to \mathcal{O}_{U_i} \). In particular, over \( U_{012} \) we obtain two isomorphisms \( \phi_i|_{U_{01}} : L|_{U_{01}} \to \mathcal{O}_{U_{01}} \). In particular, the map \( \psi : \phi_1 \circ \phi_0^{-1} : \mathcal{O}_{U_{01}} \to \mathcal{O}_{U_{01}} \) is an isomorphism. This is induced by a map \( k[s, s^{-1}] \to k[s, s^{-1}] \) which is just multiplication by some polynomial \( p(s, s^{-1}) \). Note that \( p(s, s^{-1}) \) cannot vanish at any point on \( U_{01} \) (otherwise \( \psi \) would not be an isomorphism over the stalk at that point). Hence \( p(s, s^{-1}) = s^n \) for some \( n \in \mathbb{Z} \). But that implies that \( L \simeq \mathcal{O}_{\mathbb{P}^1}(n) \).

14.3 Effective divisors

We say that a Cartier divisor \( D \) is effective (and write \( D \geq 0 \)) if \( D \) can be represented by local data \((U_i, f_i)\) where the rational functions (which a priori are just assumed to be sections of \( \mathcal{K}_X \) over \( U_i \)) actually lie in \( \mathcal{O}_X(U_i) \). It is straightforward to verify that if this is true for one set of data, then it holds for any other set as well.

We will write \( D \geq D' \) if \( D - D' \geq 0 \), i.e., the difference \( D - D' \) is effective. Note also that if \( D \) and \( D' \) are both effective, then so is \( D + D' \). This means
14.3. Effective divisors

that Pic\((X)\) is an ordered group.

Effective divisors are of paramount importance in algebraic geometry; they carry lots of essential geometric information. With the set up above, \(D\) being effective is equivalent to the statement that \(\mathcal{O}_X(-D)\) (regarded as a subsheaf of \(\mathcal{K}_X\)) is contained in \(\mathcal{O}_X\). The \(\mathcal{O}_X\)-module \(\mathcal{O}_X(-D)\) is a coherent sheaf of ideals which is locally generated by one element (in other words, it is locally principal). We will usually denote the corresponding closed subscheme also by \(D\) (this is a somewhat bad example of abusing the notation, but it has its advantages).

The inclusion \(\mathcal{O}_X(-D) \subseteq \mathcal{O}_X\) induces an exact sequence

\[
0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0
\]

where the right hand side can be interpreted both as the cokernel of the left-most map and as the structure sheaf of the subscheme associated to \(D\). Moreover, the support of the \(\mathcal{O}_D\) (i.e., the set of points \(x \in X\), such that \(\mathcal{O}_{D,x} \neq 0\)) coincides with underlying topological space of \(D\).

So effective divisors give rise to closed subschemes of codimension one. The converse is also true: If \(D\) is a closed subscheme which is locally defined by one equation (which is not a zero-divisor), then we can find, for any \(x \in X\), an open affine neighbourhood \(U_x\) and an element \(f_x \in A = \Gamma(U_x, \mathcal{O}_X)\) so that \(D = V(f_x) = \text{Spec } A/(f_x A)\). Two such elements \(f_x, f'_x\) generate the same principal ideal, so there must be a relation \(f_x = c_x f'_x\) for some \(c_x \in A^\times\). From this, we see that \((U_x, f_x)\) form the defining data for a Cartier divisor, which we will also denote by \(D\). Indeed, on an overlap \(U_{xy} = U_x \cap U_y\) we have \(f_x|_{U_{xy}} = c_{xy} f_y|_{U_{xy}}\) for a section \(c_{xy}\) of \(\mathcal{O}_X^{\times}\). We have therefore proved the following theorem:

**Theorem 14.27.** Let \(X\) be an noetherian, integral scheme. Then there is a one-to-one correspondence between closed subschemes \(D \subseteq X\) locally defined by a principal ideal, and effective Cartier divisors on \(X\).

The inclusion \(\mathcal{O}_X(-D) \subseteq \mathcal{O}_X\) can be dualized, i.e., we apply the functor \(\text{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)\), and obtain a map \(\alpha : \mathcal{O}_X \to \mathcal{O}_X(D)\). Such a map is uniquely determined by its value on 1, i.e., the global section \(\sigma = \alpha(1) \in \Gamma(X, \mathcal{O}_X(D))\).

Conversely, given a global section \(\Gamma(X, \mathcal{O}_X(D))\), we dually a map \(\mathcal{O}_X(-D) \to \mathcal{O}_X\). When \(X\) is integral, then this gives us a divisor, which is of course the effective divisor \(D\). We have therefore proved:

**Theorem 14.28.** Let \(X\) be an noetherian, integral scheme and let \(D\) be a Cartier divisor on \(X\). Then \(D\) is effective if and only if \(\mathcal{O}_X(D)\) has a non-zero global section. For each section \(\sigma\), we get a divisor denoted by \(\text{div } \sigma\). Two such sections \(\sigma, \sigma'\) give rise to the same divisor if and only if \(\sigma = c \sigma'\) where \(c \in \Gamma(X, \mathcal{O}_X)^\times\).
14.3.1 Linear systems

Definition 14.29. The set of effective divisors $D'$ linearly equivalent to $D$ is denoted by $|D|$. This is called the complete linear system of $D$.

The name ‘linear system’ comes from the special case when $X$ is a projective variety $X$ over a field $k$ (thus $X$ is integral, separated of finite type over $k$). In this case, we have $\Gamma(X, \mathcal{O}_X)^\times = k^\times$, and the previous discussion shows that the linear system $|D|$ is given by

$$|D| = \{ D' | D' \geq 0 \text{ and } D' \sim D \}$$

$$= (\Gamma(X, \mathcal{O}_X(D)) - 0) / k^\times$$

$$= \mathbb{P} \Gamma(X, \mathcal{O}_X(D))$$

So the linear system $|D|$ is (as a set) a projective space. When $X$ is projective, the cohomology groups $H^0(X, \mathcal{O}_X(D))$ is a finite dimensional (we will prove this fact in Chapter 17), so the set of divisors $D'$ linearly equivalent to $D$ is parameterized by a projective space $\mathbb{P}^n_k$.

Definition 14.30. A linear system of divisors is a linear subspace of a complete linear system $|D|$.

Example 14.31. Consider the case $X = \mathbb{P}^n_k$ and $D = dH$, where $H$ is the hyperplane divisor (so $H$ is a Cartier divisor with $\mathcal{O}_X(H) \simeq \mathcal{O}_X(1)$). In this case the linear system of $D$ associated to $\mathcal{O}_X(dH)$ is given by

$$|D| = \mathbb{P} R_d \simeq \mathbb{P}^N$$

where $N = \binom{n+d}{d} - 1$. The points of this projective space correspond to degree $d$ hypersurfaces.

14.4 Examples

14.4.1 The quadric surface

Let $k$ be a field, and let $Q = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Recall that $Q$ embeds as a quadric surface in $\mathbb{P}^3_k$ via the Segre embedding:
14.4. Examples

So we can view $Q$ both as a fiber product $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric $V(xy - zw) \subseteq \mathbb{P}^3$.

Weil divisors on $Q$

Since $Q$ is a product of two $\mathbb{P}^1$s there are natural ways of constructing Weil divisors on $Q$ from those on $\mathbb{P}^1$. For instance, we can let

$$L_1 = [0 : 1] \times \mathbb{P}^1 \subseteq Q,$$

which is a prime divisor on $Q$ corresponding to the ‘vertical fiber’ of $Q$. Similarly, $L_2 = \mathbb{P}^1 \times [0 : 1]$ is a Weil divisor on $Q$.

$$\mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \to \text{Cl}(Q) \to \text{Cl}(Q - L_1 - L_2) \to 0$$

We claim that the first map is injective, and in fact that

$$\text{Cl}(Q) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2.$$

For this, we need to analyse the two divisors $L_1, L_2$ a bit closer.

Cartier divisors on $Q$

To study Cartier divisors on $Q$, we first need a covering. $Q$ is covered by four affines

$$U_{00} = \text{Spec } \mathbb{k}[x, y] \quad U_{10} = \text{Spec } \mathbb{k}[x^{-1}, y]$$
$$U_{01} = \text{Spec } \mathbb{k}[x, y^{-1}] \quad U_{11} = \text{Spec } \mathbb{k}[x^{-1}, y^{-1}]$$

Consider $\mathbb{P}^1_k = W_0 \cap W_1$, where $W_0 = \text{Spec } \mathbb{k}[t], W_1 = \text{Spec } \mathbb{k}[t^{-1}]$. The first projection $p_1 : Q \to \mathbb{P}^1_k$ is induced by the ring maps

$$k[t] \to k[x, y] \quad k[t^{-1}] \to k[x^{-1}, y]$$
$$t \mapsto x \quad t^{-1} \mapsto x^{-1}$$
$$k[t^{-1}] \to k[x, y^{-1}] \quad k[t^{-1}] \to k[x^{-1}, y^{-1}]$$
$$t \mapsto x \quad t^{-1} \mapsto x^{-1}$$

Let $p = [0 : 1]$ be the Weil divisor on $\mathbb{P}^1$. The Cartier data is given by $(W_0, t), (W_1, 1)$. The pullback $D = p^*_1(p)$ is a Cartier divisor on $Q$, corresponding to the Weil divisor $[0 : 1] \times \mathbb{P}^1$. Writing $K = k(x, y)$, the Cartier data is given by

$$(U_{00}, x), \quad (U_{10}, 1)$$
$$(U_{01}, x), \quad (U_{11}, 1)$$

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Let \( L_1 = [0 : 1] \times \mathbb{P}^1_k \) and \( L_2 = \mathbb{P}^1_k \times [0 : 1] \). Consider the restriction of \( D \) to \( L_2 \). \( L_2 \) is covered by the two open subsets \( V_0 = U_{00} \cap L_2 = \text{Spec} \ k[x, y]/y, V_1 = U_{10} \cap L_2 = \text{Spec} \ k[x^{-1}, y]/y \). In terms of these opens, \( D|_{L_2} \) has Cartier data
\[
(V_0, x), (V_1, 1)
\]
obtained by restricting the data above. In particular, identifying \( L_2 \cong \mathbb{P}^1 \), we see that \( \mathcal{O}_Q(D)|_{L_1} \cong \mathcal{O}_{\mathbb{P}^1}(1) \). In particular, since \( \text{Cl}(\mathbb{P}^1) = \mathbb{Z} \), no multiple \( nD \) is equivalent to 0 in \( \text{Cl}(Q) \) – if that were the case, we would have \( \mathcal{O}_Q(nD) \cong \mathcal{O}_Q, \) and hence \( \mathcal{O}_Q(nD)|_{L_2} \cong \mathcal{O}_Q|_{L_2} \cong \mathcal{O}_{\mathbb{P}^1}, \) a contradiction.

In all, this shows that the first map in the exact sequence (14.4.1) is injective. We also have that \( \text{Cl}(Q - L_1 - L_2) = 0 \): Indeed \( Q - L_1 - L_2 = U_{11} = \text{Spec} \ k[x^{-1}, y^{-1}] \) which is the Spec of a UFD. Hence
\[
\text{Cl}(Q) \cong \mathbb{Z}L_1 \oplus \mathbb{Z}L_2
\]

If \( D \) is a divisor on \( Q \), \( D \sim aL_1 + bL_2 \) and we call \( (a, b) \) the ‘type’ of \( D \).

**The canonical divisor of \( Q \)**

We saw in the previous section that \( \text{Cl}(Q) \cong \mathbb{Z}^2 \). Thus it makes sense to ask how to represent the canonical divisor of \( Q \) in terms of \( L_1, L_2 \). Since \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( K_{\mathbb{P}^1} = -2p \), it should perhaps not come as a big surprise that
\[
K_Q = -2L_1 - 2L_2
\]

In fact, \( \Omega_Q = p_1^*\Omega_{\mathbb{P}^1} \oplus p_2^*\Omega_{\mathbb{P}^1} \). So taking \( \wedge^2 \), we get
\[
\omega_Q = \wedge^2 (p_1^*\Omega_{\mathbb{P}^1} \oplus p_2^*\Omega_{\mathbb{P}^1}) \\
= p_1^*\Omega_{\mathbb{P}^1} \otimes p_2^*\Omega_{\mathbb{P}^1} \\
= \mathcal{O}_Q(-2, 0) \otimes \mathcal{O}_Q(0, -2) = \mathcal{O}_Q(-2, -2)
\]

A second way of seeing this, is to use the fact that \( Q \) is embedded by the Segre embedding \( Q \subseteq \mathbb{P}^3 \) as a smooth quadric surface. Then the Adjunction formula of Proposition 11.43 on page 196 tells us that
\[
\Omega_Q = \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(Q)|_Q = \mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(2)|_Q = \mathcal{O}_Q(-2)
\]

This gives the same answer as before, since \( \mathcal{O}_{\mathbb{P}^3}(1)|_Q = \mathcal{O}_Q(L_1 + L_2) \).
14.4.2 The quadric cone

Let $X = \text{Spec } R$ where $R = k[x, y, z]/(xy - z^2)$, and $k$ has characteristic $\neq 2$. Let $Z = V(x, z)$ be the closed subscheme corresponding to the line $\{x = z = 0\}$. Note that $Z \simeq \text{Spec } k[x, y, z]/(xy - z^2, x, z) = \text{Spec } k[y]$ is integral of codimension 1.

![A singular quadric surface](image)

Note that $X - Z = \text{Spec } k[x, y, y^{-1}, z]/(xy - z^2) = \text{Spec } k[y][t, u]/(t - u^2) = \text{Spec } k[y][u]$ which is a a UFD. It follows that $\text{Cl}(X - Z) = 0$. Recall now the sequence

$$Z \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(X - Z) \rightarrow 0$$

where the first map sends 1 to $[Z]$. Hence $\text{Cl}(X)$ is generated by $[Z]$.

We first show that $2Z = 0$ in $\text{Cl}(X)$. This is because we can consider the divisor of $x$. This can only be non-zero along $Z$. The valuation here is 2: The local ring is

$$(k[x, y, z]/(xy - z^2))(x, z)$$

since $y$ is invertible here, we see that $x \in (z^2)$ and that $z$ is the uniformizer.

Now we show that $Z$ is not a principal divisor. It suffices to prove that this is not principal in $\text{Spec } \mathcal{O}_{X,x}$ where $x \in X$ is the singular point of $X$. The local ring here is

$$(k[x, y, z]/(xy - z^2))(x, y, z)$$

In this ring $p = (x, z)$ is a height 1 prime ideal, but it is not principal: Let $m \subseteq \mathcal{O}_{X,x}$ be the maximal ideal. Note that $x, y \in m$, since $x, y$ are not units. However, we can compute that the vector space $m/m^2$ (which is the Zariski cotangent space at $x$) is 3-dimensional, spanned by $\{x, y, z\}$. Then $I = (y, z) \subseteq m$, and $\pi, \eta$ give a 2-dimensional subspace $I/m^2$ of $m/m^2$. Hence, since $\pi$ and $\eta$ are linearly independent here, there couldn’t be an element $f \in \mathcal{O}_{X,x}$ for which $x = af, y = bf$. Hence

$$\text{Cl}(X) = \mathbb{Z}/2.$$  

Note that $X - \{0, 0, 0\}$ is factorial. Hence removing a codimension 2 subset has an effect on $\text{CaCl}(X)$. Recall however, that the class group of Weil divisors $\text{Cl}(X)$ stays unchanged under removing a codimension 2 subset.
Chapter 14. Divisors and linear systems

Projective quadric cone

Let $X = \text{Proj} \, R$ where $R = k[x, y, z, w]/(xy - z^2)$. Let $H = V(w)$ be the hyperplane determined by $w$. We have

$$0 \rightarrow \mathbb{Z} H \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(X - H) \rightarrow 0$$

(Here $H$ is an ample divisor, hence non-torsion in $\text{Cl}(X)$, so the first map is injective). $X - H$ is isomorphic to the affine quadric cone from before, hence $\text{Cl}(X - H) = \mathbb{Z}/2$. Using this sequence, we see that $\text{Cl}(X) = \mathbb{Z}$, generated by $\frac{1}{2}H$.

14.4.3 Quadric hypersurfaces in higher dimension

Theorem 14.32 (Nagata’s lemma). Let $A$ be a noetherian integral domain, and let $x \in A - 0$. Suppose that $(x)$ is prime, and that $A_x$ is a UFD. Then $A$ is a UFD.

Proof. We first show that $A$ is normal. Of course $A_x$ is normal, being a UFD. So if $t \in K(A)$ is integral in $A$, it lies in $A_x$. We need to check that if $a/x^n \in A_x$ is integral over $A$ and $x \nmid a$, then $n = 0$. If we have an integral relation

$$(a/x^n)^N + b_1(a/x^n)^{N-1} + \cdots + b_N = 0$$

Multiplying by $x^{nN}$ give get $a^N \in xA$, so $x|a$, because $A$ is an integral domain. This shows that the divisor $D = \text{div} \, x$ is an effective divisor and so there is an exact sequence

$$\mathbb{Z} D \rightarrow \text{Cl}(\text{Spec} \, A) \rightarrow \text{Cl}(\text{Spec} \, A_x) = 0 \rightarrow 0$$

The image of the left-most map is 0, so $\text{Cl}(A) = 0$, and so $A$ is a UFD. \qed

Let $A = k[x_1, \ldots, x_n, y, z]/(x_1^2 + \cdots + x_n^2 - yz)$. We will prove that $A$ is a UFD for $m \geq 3$. $A$ is a domain, since the defining ideal is prime. Apply Nagata’s lemma with the element $y:

$$A_y = k[x_1, \ldots, x_n, y, y^{-1}, z]/(y^{-1}(x_1^2 + \cdots + x_n^2) - z) \simeq k[x_1, \ldots, x_n, y, y^{-1}, z]$$

which is a UFD. We show that $y$ is prime: Taking the quotient we get

$$A/y = k[x_1, \ldots, x_n, x]/(x_1^2 + \cdots + x_n^2)$$

which is an integral domain, because $x_1^2 + \ldots, x_n^2$ is irreducible (for $m \geq 3$).

Note that for $m = 2$, we get the quadric cone, which we have seen is not a UFD.
Applying a change of variables, we find the following:

**Proposition 14.33.** Let $k$ be a field containing $\sqrt{-1}$ and let $X = V(x_0^2 + \cdots + x_m^2) \subseteq \mathbb{A}^{n+1}_k = \text{Spec } k[x_0, \ldots, x_n]$.

1. $m = 2$, $\text{Cl}(X) = \mathbb{Z}/2$
2. $m = 3$, $\text{Cl}(X) = \mathbb{Z}$
3. $m \geq 4$, $\text{Cl}(X) = 0$

**Proposition 14.34.** Let $X = V(x_0^2 + \cdots + x_m^2) \subseteq \mathbb{P}^n = \text{Proj } k[x_0, \ldots, x_n]$.

1. $m = 2$, $\text{Cl}(X) = \mathbb{Z}$
2. $m = 3$, $\text{Cl}(X) = \mathbb{Z}^2$
3. $m \geq 4$, $\text{Cl}(X) = \mathbb{Z}$

### 14.4.4 Hirzebruch surfaces

Let $r \geq 0$ be an integer and consider the scheme $X$ which is glued together by the four affine scheme charts

\[
\begin{align*}
U_{00} &= \text{Spec } k[x, y] \\
U_{01} &= \text{Spec } k[x, y^{-1}] \\
U_{10} &= \text{Spec } k[x^{-1}, x^r y] \\
U_{11} &= \text{Spec } k[x^{-1}, x^{-r} y^{-1}]
\end{align*}
\]

This is a non-singular, integral 2-dimensional scheme over $k$. When $k = \mathbb{C}$, this are the so-called $r$-th Hirzebruch surface. In many ways, these behave as the 'mobius strips' in algebraic geometry.

Note in particular, when $r = 0$, we get $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

**Divisors**

Let us define two divisors $D_1, D_2$ by the following Cartier divisors:

Writing $K = \text{k}(x, y)$, the Cartier data is given by

\[
\begin{align*}
D_1 &= \left[ (U_{00}, x), (U_{01}, x) \right] \\
&\quad \left[ (U_{10}, 1), (U_{11}, 1) \right] \quad , \quad D_2 = \left[ (U_{00}, y), (U_{01}, 1) \right] \\
&\quad \left[ (U_{10}, y), (U_{11}, 1) \right]
\end{align*}
\]

We will show that $D_1, D_2$ generate $\text{Pic}(X)$. Note that both of these divisors are effective, since they are defined by rational functions which are regular on the
Let $D_1$ be the Cartier divisor

$$D_1' = \left[ \frac{(U_{00}, 1), (U_{01}, 1)}{(U_{10}, x^{-1}), (U_{11}, x^{-1})} \right].$$

We can compute that $D_1'|_{D_1}$ the divisor $(V_0, 1), (V_1, 1)$ which is principal. In fact,

$$\text{div } x = D_1 - D_1'$$

So, $D_1 = D_1'$ in $\text{Cl}(X)$. This shows that $D_1$ and $D_2$ are independent in $\text{Cl}(X) - D_1'$ restricts to 0 on $D_1$ by $\mathcal{O}(1)$ on $D_2$.

Now let $U = X - D_1 - D_2$. This is isomorphic to $U_{11}$ which is the spectrum of $k[x^{-1}, x^{-r}y^{-1}]$ which is a UFD. The exact sequence

$$\mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \to \text{Cl}(X) \to 0$$

and the previous analysis shows that $\text{Cl}(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$.

Sheaf cohomology

We want to compute $H^i(X, \mathcal{O}_X(D))$ for a divisor $D = aD_1 + bD_2$. As usual, we utilize a Čech complex.

We want to mimick the computation for $\mathbb{P}^n$. In that proof, the polynomial ring $k[x_0, \ldots, x_n]$ played an important role – the groups in the Čech complex corresponded to degree 0 localizations of it. For $X$ there is no such $\mathbb{Z}$-graded ring lying around, but we can get by by introducing the bigraded ring

$$R = k[x_0, x_1, y_0, y_1]$$

where the degrees of the variables are defined by

$$\deg x_0 = (1, 0), \deg x_1 = (1, 0), \deg x_2 = (0, 1), \deg x_3 = (-r, 1).$$

By identifying $x = \frac{x_1}{x_0}, y = \frac{x_0y_1}{y_0}$, we find that the Čech complex of $U_{ij}$ can be
written

\[
\begin{align*}
R_{[x_0,x_1]} & \oplus R_{[x_0x_1y_0]} \\
R_{[x_0x_1]} & \oplus R_{[x_0x_1y_0]} \\
R_{[x_0x_1]} & \rightarrow R_{[x_0x_1y_0]} \\
R_{[x_0x_1]} & \oplus R_{[x_0x_1y_0]}
\end{align*}
\]

where the bracket means that we take the \((0,0)\)-part of the localization. So for instance,

\[
R_{[x_0y_0]} = R \left[ \frac{x_1}{x_0}, \frac{x_0^ry_1}{y_0} \right]
\]

As in the \(\mathbb{P}^n\) case, we now have a bigraded isomorphism

\[
\bigoplus_{a,b \in \mathbb{Z}} H^0(X, \mathcal{O}_X(aD_1 + bD_2)) \simeq \bigcap_{i,j} R_{[x_1y_j]} = R
\]

In particular, \(H^0(X, \mathcal{O}_X(aD_1 + bD_2))\) can be identified with polynomials of bidegree \((a,b)\) in \(R\). So for example \(H^0(X, \mathcal{O}_X(D_1))\) corresponds to degree \((1,0)\)-polynomials, e.g., polynomials in \(x_0, x_1\). These two sections corresponds to the sections 1, \(x\) above. Similarly, \(H^0(X, \mathcal{O}_X(D_2))\) is 1-dimensional.

Perhaps the most interesting divisor is \(E = D_2 - rD_1\), which is effective. This has \(H^0(X, \mathcal{O}_X(nE)) = k\) for every \(n \geq 0\), by \(E\) is not the trivial divisor. Therefore \(E\) couldn’t possibly by globally generated. In fact, \(E \simeq \mathbb{P}^1\) and

\[
\mathcal{O}_X(E)|_E \simeq \mathcal{O}_{\mathbb{P}^1}(-r)
\]

Map to projective space

Note that \(\mathcal{O}_X(D_1)\) is globally generated by the sections \(x_0, x_1\). These define a map

\[
f : X \rightarrow \mathbb{P}^1
\]

This is an example of a \(\mathbb{P}^1\)-fibration – the closed fibers here are all isomorphic to \(\mathbb{P}^1\). In fact, one can show that any such fibration over \(\mathbb{P}^1\) is isomorphic to this example for some choice of \(r\).
Chapter 15

Vector bundles on the projective line

As an instructive and not too complicated example of how cohomology groups can give important results, we give a proof that all locally free sheaves on the projective line \( \mathbb{P}^1_k \) decompose as direct sums of invertible sheaves, and as all invertible sheaves are of the form \( \mathcal{O}_{\mathbb{P}^1_k}(n) \) for some integer \( n \), we have the following theorem:

**Theorem 15.1.** Assume that \( k \) is a field and that \( E \) is a locally free sheaf of rank \( r \) on the projective line \( \mathbb{P}^1_k \). Then there is an isomorphism

\[
E \cong \mathcal{O}_{\mathbb{P}^1_k}(\alpha_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(\alpha_r),
\]

where the \( \alpha_i \) are integers, uniquely defined up to ordering.

We need some preparatory results about zeros of sections of locally free sheaves on curves. Assume for a moment that \( C \) is a non-singular curve and that \( E \) is locally free sheaf of finite rank on \( C \). Let \( \sigma: \mathcal{O}_C \to E \) be an injective map of sheaves on \( C \); such maps correspond bijectively to sections of \( E \), that we as well denote by \( \sigma \) (the correspondence is \( \sigma \leftrightarrow \sigma(1) \)). The dualized the map \( \sigma^\vee \), sits in an exact sequence

\[
E^* \to \mathcal{O}_C \to \mathcal{O}_{Z(\sigma)} \to 0 \quad (15.0.1)
\]

where \( Z(\sigma) \) is a closed subscheme of \( C \). Generically \( \sigma \) is injective and therefore \( \sigma^\vee \) is generically surjective, and hence \( Z \) is not equal to the whole curve \( C \).
It follows that \( Z(\sigma) \) is finite \( C \) being a curve. The scheme \( Z(\sigma) \) is called the zero-scheme of \( \phi \).

As \( C \) is non-singular, any non-zero coherent ideal \( I \subseteq \mathcal{O}_C \) is an invertible sheaf. The ideal sheaf of \( Z(\sigma) \) – or the image of \( \sigma^\vee \), if you want – is therefore an invertible sheaf \( L_\sigma \), and we may tensor the sequence (15.0.1) by \( L_\sigma^{-1} \) to obtain the following factorization of \( \sigma^\vee \)

\[
E^* \otimes L_\sigma^{-1} \xrightarrow{\sigma^\vee} \mathcal{O}_C \xrightarrow{} L_\sigma^{-1}.
\]

For later references we formulate this as a proposition:

**Proposition 15.2.** Let \( C \) be a non-singular curve over a field \( k \) and \( E \) a locally free sheaf of finite rank on \( C \). Assume that \( \sigma \) is a non-zero section of \( E \). Then there is a finite subscheme \( Z_\sigma \) in \( C \) and a surjective map \( \mathcal{O}_C \xrightarrow{} L_\sigma^{-1} \sigma_Z \) where \( L_\sigma \) denotes the invertible sheaf being the ideal of \( Z(\sigma) \), and a factorization \( \sigma^\vee = (\iota \otimes L_\sigma^{-1}) \circ \sigma_\mathcal{O}_C \) where \( \iota \) denotes the inclusion of the ideal \( L_\sigma \) in \( \mathcal{O}_C \).

**Proof of the theorem.** The first step in the proof relies on two fundamental facts about locally free sheaves of finite rank on projective spaces. Firstly, sufficiently high positive twists have global sections; stated slightly differently \( \Gamma(\mathbb{P}^1_k, E(\alpha)) \neq 0 \) if \( \alpha > 0 \). Secondly, sufficiently negative twists do not have global sections, i.e., it holds true that \( \Gamma(\mathbb{P}^1_k, E(-\alpha)) = 0 \) for all \( \alpha > 0 \). The third salient point is that any invertible sheaf on the projective line is isomorphic to \( \mathcal{O}_{\mathbb{P}^1_k}(\alpha) \) for some integer \( \alpha \), in other words, the theorem is true in rank one. The proof goes by induction on the rank of \( E \), and this will be the start of the induction.

We now let \( n_0 \) be the greatest integer such that \( \Gamma(\mathbb{P}^1_k, E(-n_0)) \neq 0 \), and after having replaced \( E \) by \( E(-n_0) \), we may thus assume that \( \Gamma(\mathbb{P}^1_k, E) \neq 0 \), but \( \Gamma(\mathbb{P}^1_k, E(-\alpha)) = 0 \) for all \( \alpha > 0 \). Let \( \sigma \) be a global section of \( E \) (it is called a minimal section).

**Lemma 15.3.** The minimal section \( \sigma \) does not vanish anywhere.

**Proof.** Assume that \( Z_\sigma \neq 0 \); then its sheaf of ideals equals \( \mathcal{O}_{\mathbb{P}^1_k}(-\alpha) \) for some \( \alpha > 0 \), and by (15.2) there is a surjective map \( E^*(\alpha) \rightarrow \mathcal{O}_{\mathbb{P}^1_k} \), whose dual is a section of \( E(-\alpha) \). This contradicts the fact that \( \sigma \) is a minimal section.

To prove the theorem we proceed as announced by induction on the rank \( r \) of \( E \), the \( r = 1 \) being taken care of by the fact that every invertible sheaf on \( \mathbb{P}^1_k \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^1_k}(\alpha) \) for some \( \alpha \in \mathbb{Z} \). Choose a minimal section \( \sigma \) of \( E \). Then there is an exact sequence

\[
0 \longrightarrow F \longrightarrow E^* \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0 \quad (15.0.2)
\]
where the kernel $F$ is locally free of rank $r-1$. By induction $F \simeq \bigoplus_{1 \leq i \leq r-1} \mathcal{O}_{\mathbb{P}^1_k}(\alpha_i)$.

**Lemma 15.4.** $H^1(\mathbb{P}^1_k, F) = 0$

**Proof.** Taking the dual of (15.0.2) and twisting by $\mathcal{O}(-1)$ gives:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1) \longrightarrow E(-1) \longrightarrow F^*(-1) \longrightarrow 0$$

Then the long exact sequence of cohomology shows $H^0(F^*(-1)) = 0$. But then also $H^0(F^*(-2)) = 0 = H^0(F^*\Omega_{\mathbb{P}^1_k})$, so $H^1(F) = 0$ by Serre duality. (Note that we only need Serre duality for a direct sum of invertible sheaves of the form $\mathcal{O}_{\mathbb{P}^1_k}(\alpha_i)$ in this proof). \qed

Since $H^1(\mathbb{P}^1_k, F) = 0$, the long exact sequence associated to 15.0.2 shows that we have a surjection

$$\Gamma(\mathbb{P}^1_k, E^*) \longrightarrow \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow 0$$

and the section 1 can be lifted to a section $\tau$ of $E^*$. This is translates into the diagram

$$\begin{array}{ccc}
0 & \longrightarrow & F \\
& \longrightarrow & E^* \\
& \tau \uparrow & \mathcal{O}_{\mathbb{P}^1_k} \\
& \longrightarrow & 0 \\
\end{array}$$

It follows that the sequence (15.0.2) is split, and as a consequence we have

$$E^* \simeq \mathcal{O}_{\mathbb{P}^1_k} \oplus F,$$

that is $E \simeq \mathcal{O}_{\mathbb{P}^1_k} \oplus \bigoplus_{1 \leq i \leq r-1} \mathcal{O}_{\mathbb{P}^1_k}(-\alpha_i)$ and the proof is complete. \qed

### 15.1 Non-split vector bundles on $\mathbb{P}^n$

A locally free sheaf is said to be **split** if it is isomorphic to a direct sum of invertible sheaves. For instance, we showed before that any locally free sheaf on on $\mathbb{P}^1_k$ is split. Let us give one example of a non-split locally free sheaf. We consider again the sheaf $\mathcal{E}$ of rank $n$ on $\mathbb{P}^n_k$ sitting in the exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^n_k}^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n_k}(1) \rightarrow 0.$$ 

Let us show that $\mathcal{E}$ is not split, i.e., $\mathcal{E}$ is not isomorphic to a direct sum of invertible sheaves. Since $\text{Pic}(\mathbb{P}^n_k)$ is generated by the class of $\mathcal{O}(1)$, this would mean that $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \mathcal{O}_{\mathbb{P}^n_k}(a_n)$ for some integers $a_1, \ldots, a_n$. 

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15.1. Non-split vector bundles on $\mathbb{P}^n$

Recall that for $n \geq 2$, we have $H^1(\mathbb{P}^n_k, \mathcal{O}(m)) = 0$ for any $m \in \mathbb{Z}$. So if we could show that $H^1(\mathbb{P}^n_k, \mathcal{E}) \neq 0$, we would have a contradiction. Actually, $H^1(\mathbb{P}^n_k, \mathcal{E}) = 0$, but we can instead consider $\mathcal{E}(-1)$, which fits into

$$0 \to \mathcal{E}(-1) \to \mathcal{O}_{\mathbb{P}^n_k}^{n+1}(-1) \to \mathcal{O}_{\mathbb{P}^n_k} \to 0.$$ 

Taking the long exact sequence in cohomology, we get

$$\cdots \to H^0(\mathcal{E}(\mathcal{O}(-1)^{n+1}) \to H^0(\mathcal{O}_{\mathbb{P}^n_k}) \to H^1(\mathcal{E}) \to H^1(\mathcal{O}(-1)^{n+1}) \to \cdots$$

Here the two outer cohomology groups are zero, since $\mathcal{O}(-1)$ does not have any global sections, and $H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)) = 0$. Hence, by exactness, we find that $H^1(\mathcal{E}(-1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^n_k}) = k$. This implies that $\mathcal{E}(-1)$, and hence $\mathcal{E}$ cannot be a sum of invertible sheaves, and we are done.

However, coming up with examples of non-split vector bundles of low rank is a notoriously difficult problem, even for projective space. In fact, a famous conjecture of Hartshorne says that any rank 2 vector bundle on $\mathbb{P}^n$ for $n \geq 5$ is split. (On $\mathbb{P}^4$ this statement does not hold, as shown by the so-called Horrocks–Mumford bundle). This is related to the conjecture that any smooth codimension 2 subvariety of $\mathbb{P}^n$ is a complete intersection.

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Chapter 16
Maps to projective space

Given a scheme $X$ it is natural to ask when there is a morphism to a projective space

$$f : X \to \mathbb{P}^n,$$

or when there is a closed immersion $X \hookrightarrow \mathbb{P}^n$. Given such a morphism, we get geometric information about $X$ using this map, e.g., using the fibers $f^{-1}(y)$ or pulling back sheaves from $\mathbb{P}^n$.

The corresponding question for $\mathbb{A}^n$ has already been answered. Morphisms $X \to \mathbb{A}^n$ are in one-to-one correspondence with elements of $\Gamma(X, \mathcal{O}_X)^n$. In fancy terms, $\mathbb{A}^n$ represents the functor

$$\text{AffSch} \to \text{Sets}$$

$$X \mapsto \Gamma(X, \mathcal{O}_X)$$

So which functor does projective space represent? Intuitively we would like to associate to each morphism $f : X \to \mathbb{P}^n$ to a set of data on $X$. As we have seen, there is not so much information $\Gamma(X, \mathcal{O}_X)$ in the context of projective schemes. However, we do have something canonical associated to $\mathbb{P}^n$, namely sections of $\mathcal{O}_{\mathbb{P}^n}(1)$. Taking the usual global sections $x_0, \ldots, x_n$ on $\mathcal{O}_{\mathbb{P}^n}(1)$, we can pull back via the map $f^*$ and get $n + 1$ sections $s_i = f^*x_i$ of the invertible sheaf $f^*\mathcal{O}(1)$. Note that there is no point of $\mathbb{P}^n$ where the $x_i$ simultaneously vanish. More precisely, for every $y \in \mathbb{P}^n$, the stalk $\mathcal{O}_{\mathbb{P}^n}(1)_y$ is generated by one of the $x_i$ (as an $\mathcal{O}_{\mathbb{P}^n}$-module). So my the properties of the pullback, we see that the same statement holds for the $s_i$ and $X$.

In this chapter we will see that there is a way to reverse this process, i.e., that from a given invertible sheaf $L$ and $n + 1$ global sections $s_i \in \Gamma(X, L)$ with the
above property, we can uniquely reconstruct a morphism \( f : X \to \mathbb{P}^n \) so that 
\[ f^*\mathcal{O}_{\mathbb{P}^n}(1) = L \text{ and } f^*x_i = s_i. \]

## 16.1 Globally generated sheaves

**Definition 16.1.** Let \( X \) be a scheme and let \( \mathcal{F} \) be an \( \mathcal{O}_X \)-module. We say \( \mathcal{F} \) is *globally generated* (or generated by global sections) if there is a family of sections \( s_i \in \mathcal{F}(x), i \in I \), such that the images of \( s_x \) generate \( \mathcal{F}_x \) as an \( \mathcal{O}_{X,x} \)-module. Equivalently, \( \mathcal{F} \) is globally generated if there is a surjection

\[ \mathcal{O}_X^I \to \mathcal{F} \to 0 \]

for some index set \( I \).

Let us consider a few examples:

**Example 16.2.** On an affine scheme any quasi-coherent sheaf is globally generated. Indeed, if \( X = \text{Spec} \ A, \mathcal{F} = \widetilde{M} \), for some \( A \)-module \( M \), then picking any presentation \( A^I \to M \to 0 \) for \( M \) shows that \( \mathcal{F} \) is globally generated.

**Example 16.3.** \( X = \text{Proj} \ R \). Then \( \mathcal{F} = \mathcal{O}(1) \) is globally generated if \( R \) is generated in degree 1. Indeed, if this is the case, we have a surjection \( R \otimes R_1 \to R(1). \) Taking \( \sim \), we get \( \mathcal{O}_X \otimes R_1 \to \mathcal{O}_X(1) \to 0 \), which shows that \( \mathcal{O}(1) \) is globally generated.

**Example 16.4.** The tangent bundle of \( \mathbb{P}^n \) is globally generated. To see why, we recall the *Euler sequence* on \( \mathbb{P}^n *:

\[ 0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0 \]

Here the leftmost map is given by multiplication by the vector \( (x_0, \ldots, x_n) \), and the rightmost sends \( x_i \) to \( \frac{\partial}{\partial x_i} \). So in fact, even \( T_{\mathbb{P}^n}(-1) \) is globally generated.

### 16.1.1 Pullbacks and pushforwards

Let us recall a few things about pulling back sheaves and sections. If \( f : X \to Y \) is a morphism of schemes, and \( \mathcal{G} \) is an \( \mathcal{O}_Y \)-module, we define \( f^*\mathcal{G} \) as the tensor product \( f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \). The stalk of \( f^*\mathcal{G} \) at a point \( x \in X \) is isomorphic to \( \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_X,x \).

If \( \mathcal{G} \) is an invertible sheaf, the pullback \( f^*\mathcal{G} \) is also invertible. Indeed, if \( U \subseteq X \) and \( V = f(U) \subseteq Y \) are open affine subsets such that \( f(U) \subseteq V \), and \( L|_U \simeq \mathcal{O}_U \), then \( f^*L|_V \simeq \mathcal{O}_V \) (recall that a morphism pulls back the structure sheaf of the target gives to the structure sheaf of the source).
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We can pull back sections \( s \in \Gamma(Y, \mathcal{G}) \) as well. Let us consider the affine case first, where \( U = \text{Spec} B \) and \( V = \text{Spec} A \). In this case, \( \mathcal{G} = \tilde{M} \) for some \( A \)-module, and \( f^* \mathcal{G} = \tilde{M} \otimes_A B \). So finally, if \( s \in \Gamma(U, \mathcal{G}) = M \), then the pullback is given by \( f^*(s) = s \otimes 1 \in M \otimes_A B \).

**Proposition 16.5.** Let \( f : X \to Y \) be a morphism of schemes. If \( \mathcal{F} \) is an \( \mathcal{O}_Y \)-module which is globally generated by sections \( \{ s_i \}_{i \in I} \), then the pullbacks \( t_i = f^*(s_i) \) generate \( f^* \mathcal{F} \).

**Proof.** The \( s_i \) determine a surjection \( \mathcal{O}_I \to \mathcal{F} \to 0 \), and the pullback \( \mathcal{O}_I \Xrightarrow{f} \mathcal{F} \to 0 \) (which is surjective since \( f^* \) is right-exact) is the map determined by the \( t_i \). Hence \( f^* \mathcal{F} \) is globally generated by the \( t_i \).

**Remark 16.6.** For the pushforward, \( f_* \mathcal{F} \) is typically not globally generated even when \( \mathcal{F} \) is the structure sheaf. For example, if \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) is the map induced by \( k[u, v] \subseteq k[u^3, u^2v, uv^2, v^3] \), then \( f_* \mathcal{O}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \). The latter is not globally generated: If it were, we would in particular get a surjection \( \mathcal{O}_1 \to \mathcal{O}(-1) \to 0 \), however \( \mathcal{O}(-1) \) is clearly not globally generated, since it has no global sections at all.

### 16.2 Morphisms to projective space

#### 16.2.1 Example: The twisted cubic

Consider the ring \( R = k[u^3, u^2v, uv^2, v^3] \) with the grading such that the monomials have degree 1. We have seen that Proj \( R = \mathbb{P}^1_k \) (since \( R \) is the Veronese subring \( k[u, v]^{(3)} \) of \( k[u, v] \)). \( R \) is isomorphic to the ring

\[
k[x_0, x_1, x_2, x_3]/(x_1^2 - x_0x_2, x_0x_3 - x_1x_2, x_2^2 - x_1x_3).
\]

This shows that Proj \( R \) embeds as a cubic curve in \( \mathbb{P}^3_k \).

Intuitively, we would like to say that this morphism \( f : X \to \mathbb{P}^3 \) is given by something like

\[
[u, v] \mapsto [u^3, u^2v, uv^2, v^3].
\]

However, here \( u, v \) are not regular functions (they however define rational functions), so to define an actual morphism of schemes, we need to be a little bit more rigorous. Note, however, that on \( D_+(u) = \text{Spec} R(u) = \text{Spec} k[x_1, x_2, x_3] \), the ratio \( t = \frac{v}{u} \) is a regular section \( \Gamma(D_+(u), \mathcal{O}) \). Moreover, the ring homomorphism

\[
\phi : k[x_1, x_2, x_3] \to k[t]
\]

\[
\begin{align*}
x_1 &\mapsto t \\
x_2 &\mapsto t^2 \\
x_3 &\mapsto t^2
\end{align*}
\]

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A morphism of affine schemes

\[ D_+(u) \to D_+(x_0) \subseteq \mathbb{P}^3_k \]

Note that this corresponds to the usual affine twisted cubic \( \mathbb{A}^1 \to \mathbb{A}^3, t \mapsto (t, t^2, t^3) \). Similarly, on \( D_+(v) \), the ratio \( s = \frac{u}{v} \) defines a morphism \( D_+(x_0) \to D_+(x_3) \). On the overlaps, these maps are compatible with the standard gluing construction of \( \mathbb{P}_k^1 \), and so we get finally a morphism \( \mathbb{P}_k^1 \to \mathbb{P}_k^3 \).

16.2.2 Morphisms and globally generated invertible sheaves

The morale of the previous example is the morphism is not specified using a set of regular functions, but rather, sections of an invertible sheaf. Using the same type of gluing argument, we will prove the following general result:

**Theorem 16.7.** Let \( X \) be a scheme over a ring \( A \).

(i) If \( f: X \to \mathbb{P}_A^n \) is a morphism over \( A \), then \( f^*(\mathcal{O}(1)) \) is an invertible sheaf which is globally generated by the sections \( s_i = f^*(x_i) \) for \( i = 0, \ldots, n \).

(ii) If \( L \) is an invertible sheaf on \( X \) which is globally generated by \( n+1 \) sections \( s_0, \ldots, s_n \), then there exist a unique morphism \( f: X \to \mathbb{P}_A^n \) such that \( L \cong f^*(\mathcal{O}(1)) \) and \( s_i = f^*(x_i) \) under this isomorphism.

**Proof.** (i) This is a consequence of Proposition 16.5.

(ii) Given \( n+1 \) sections \( s_0, \ldots, s_n \) of an invertible sheaf \( L \), we will construct a morphism to \( \mathbb{P}_A^n \), such that \( s_i \) is the pullback of \( x_i \). Let

\[ X_i = X_{s_i} = \{ x \in X | s_i(x) \neq 0 \} \]

(Recall that \( s(x) \) is the residue class of \( s \) in \( L_x/\mathfrak{m}_xL_x \).) As we have seen, this is the complement of the support of the section \( s_i \), and is therefore an open set. By the assumption that these sections generate \( L \), we see that \( X \) is covered by the sets \( X_i \).

Now, we have given local isomorphisms \( \phi_i : L|_{U_i} \to \mathcal{O}_X|_{U_i} \), which we can use to view the fraction \( \frac{s_j}{s_i} \) can be regarded as a regular section on \( X_i \), that is, an element of \( \Gamma(X_i, \mathcal{O}_X) \).

Write \( R = A[x_0, \ldots, x_n] \) and consider the ring homomorphism given by

\[ R_{(x_i)} \to \Gamma(X_i, \mathcal{O}_X) \]

\[ \frac{x_j}{x_i} \mapsto \frac{s_j}{s_i} \]

By the correspondence between ring homomorphisms and maps into affine schemes, we obtain a morphism of schemes \( f_i : X_i \to D_+(x_i) \). It is not so hard
to see that on the overlaps $X_i \cap X_j$, two maps are given by the same ratios of sections, and so they glue to a morphism $f : X \to \mathbb{P}^n$. By construction of the map $f$, we have $f^* x_i = s_i$.

Abusing notation, we will refer to a morphism $\phi : X \to \mathbb{P}^n_A$ as ‘given by the data $(L, s_0, \ldots, s_n)$ and write

$$X \to \mathbb{P}^n_A$$

$$x \mapsto [s_0(x) : \cdots : s_n(x)]$$

One should still keep in mind that the sections $s_i$ are sections of $L$, rather than regular functions.

Given a scheme $X$ with $s_0, \ldots, s_n$ of a line bundle $L$, there is a maximal open subset $U$ such that the sections generate $L$ for all points in $U$. We then get a rational map $\phi : X \dashrightarrow \mathbb{P}^n_A$, that is $\phi$ is a morphism on the smaller open set $U$.

**Example 16.8.** Let $X = \mathbb{P}^1_k = \text{Proj} \ k[s, t]$ and $L = \mathcal{O}(2)$. Then $L$ is globally generated by $s^2, st, t^2$ and the corresponding morphism

$$X \to \mathbb{P}^2_k$$

$$[s : t] \mapsto [s^2 : st : t^2]$$

has image $V(x_0x_2 - x_1^2)$ which is a smooth conic.

**Example 16.9** (Cuspidal cubic). Let $X = \mathbb{A}^1_k$ and $L = \mathcal{O}_X$. Then, $\Gamma(X, L) = k[t]$ is infinite dimensional over $k$. Choosing the three sections $1, t^2, t^3$, we get a map of schemes

$$X \to \mathbb{P}^2_k$$

$$t \mapsto [1 : t^2 : t^3]$$

whose image in $\mathbb{P}^2$ is the cuspidal cubic minus the point at $\infty$.

**Example 16.10** ($\mathbb{P}^1$ as a quotient space). Let $X = \mathbb{A}^2_k$ and $L = \mathcal{O}_X$. Then, $\Gamma(X, \mathcal{L}) = k[x, y]$. If we take the sections $x, y$, then they generate $L$ outside $V(x, y)$. Hence we get a morphism of schemes

$$\mathbb{A}^2_k - V(x, y) \to \mathbb{P}^1_k$$

$$(x, y) \mapsto [x : y]$$

which we is the quotient map used to define $\mathbb{P}^1$.

**Example 16.11** (Projection from a point). Consider the projective space $X = \mathbb{P}^n_A$ and sections $x_1, \ldots, x_n$ of $\mathcal{O}(1)$, then these sections generate $\mathcal{O}(1)$ outside
the point \( p \) corresponding to \( I = (x_1, \ldots, x_n) \) (that is, what we would classically write as the point \( p = (1 : : 0 : \cdots : 0) \)). The induced morphism \( \mathbb{P}_A^n - V(I) \rightarrow \mathbb{P}_A^{n-1} \) is the projection from \( p \).

**Example 16.12** (Cremona transformation). Consider the projective space \( X = \mathbb{P}^2_A \) and sections \( x_0, x_1, x_2 \) of \( \mathcal{O}(1) \), then the sections \( x_0x_1, x_0x_2, x_1x_2 \) generate \( \mathcal{O}(2) \) outside \( V(x_0x_1, x_0x_2, x_1x_2) \) corresponding to the three points \( (1,0,0), (0,1,0), (0,0,1) \). The induced rational map \( \mathbb{P}_A^2 - \rightarrow \mathbb{P}_A^2 \) is the Cremona transformation.

**Example 16.13** (The Veronese surface). Consider \( X = \mathbb{P}^2 \), and \( L = \mathcal{O}_{\mathbb{P}^2}(2) \). If \( x_0, x_1, x_2 \) are projective coordinates on \( X \), then the quadratic monomials

\[
x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2
\]

form a basis for \( H^0(X, L) \), and generate \( L \) at every point. The corresponding map \( X \rightarrow \mathbb{P}^5 \) is an embedding; the image is the Veronese surface. It is a classical fact that the image is defined by the \( 2 \times 2 \) minors of the matrix

\[
\begin{pmatrix}
u_0 & u_1 & u_2 \\
u_1 & u_3 & u_4 \\
u_2 & u_4 & u_5
\end{pmatrix}
\]

**Example 16.14.** Let us consider again the case \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \). Keeping the notation from Section 14.4.1, we have two divisors, \( L_1 = [0 : 1] \times \mathbb{P}^1 \), \( L_2 = \mathbb{P}^1 \times [0 : 1] \). Note that each \( L_i \) is globally generated (being the pullback of a base point free divisor on \( \mathbb{P}^1 \)). The corresponding map is of course the \( i \)-th projection map \( p_i : Q \rightarrow \mathbb{P}^1 \).

If \( x_0, x_1 \) is a basis for \( \Gamma(X, L_1) \), and \( y_0, y_1 \) is a basis for \( \Gamma(X, L_2) \), we find that \( \Gamma(X, L_1 + L_2) \) is spanned by the sections

\[
s_0 = x_0y_0, s_1 = x_0y_1, s_2 = x_1y_0, s_3 = x_1y_1
\]

Moreover, these sections generate \( \mathcal{O}_Q(D) \) everywhere, and so we get a map

\[
Q \rightarrow \mathbb{P}^3
\]

This is of course nothing but the Segre embedding; note the quadratic relation between the four sections \( s_0s_3 - s_1s_2 = 0 \).

### 16.3 Application: Automorphisms of \( \mathbb{P}^n \)

If \( k \) is a field, then any invertible \( (n+1) \times (n+1) \) matrix \( m \) gives an automorphism \( \mathbb{P}^n_k \rightarrow \mathbb{P}^n_k \) (since it induces an automorphism of \( k[x_0, \ldots, x_n] \)). Moreover, two
matrices \( m \) and \( m' \) determine the same automorphism if and only if \( m = \lambda m' \) for some non-zero scalar \( \lambda \in k^* \). So we are led to consider the projective linear group

\[
PGL_n(k) = GL_n(k)/k^*
\]

**Theorem 16.15.** \( \text{Aut}_k(\mathbb{P}^n) = PGL_n(k) \).

**Proof.** The above shows that there is a containment \( \supseteq \). To show the reverse inclusion, let \( \phi : \mathbb{P}^n_k \to \mathbb{P}^n_k \) be any automorphism. Then we get an induced map

\[
\phi^* : \text{Pic}(\mathbb{P}^n) \to \text{Pic}(\mathbb{P}^n)
\]

which is an isomorphism. Since \( \text{Pic}(\mathbb{P}^n) = \mathbb{Z} \), we must have either \( \phi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^n}(1) \) or \( \phi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}(1) \). The latter case is impossible, since \( \phi^* (\mathcal{O}_{\mathbb{P}^n}(1)) \) has a lot of global sections, whereas \( \mathcal{O}_{\mathbb{P}^n}(1) \) has none. So \( \phi^* (\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^n}(1) \).

In particular, \( \phi^* \) gives rise to a map

\[
\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)),
\]

which is an isomorphism of \( k \)-vector spaces. However, these vector spaces can be identified with \( \langle x_0, \ldots, x_n \rangle \), and so \( \phi^* \) gives rise to an invertible \((n + 1) \times (n + 1)\)-matrix \( m \). By construction \( m \) induces the map \( \phi \), and so \( \phi \) comes from an element of \( PGL_n(k) \). \( \square \)

### 16.4 Projective embeddings

We have seen how morphisms \( X \to \mathbb{P}^n \) corresponds to invertible sheaves \( L \) plus \( n + 1 \) global sections \( s_0, \ldots, s_n \) that generate it. Given this it is natural to ask which data \((L, s_i)\) that corresponds to special types of morphisms, e.g., closed immersions. The following criterion tells us how to check this locally:

**Proposition 16.16.** Let \( \phi : X \to \mathbb{P}^n_A \) be a morphism corresponding to the data \((L, s_0, \ldots, s_n)\). Then \( \phi \) is a closed immersion if and only if for each \( i = 0, \ldots, n \), \( X_i = X_{s_i} \) is affine, and the homomorphism \( A[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_X) \) \( y_j \mapsto \frac{s_j}{s_i} \) is surjective.

**Proof.** \( \Rightarrow \): If \( \phi \) is a closed immersion, we may view \( X \) as a closed subscheme of \( \mathbb{P}^n_A \) (so it is given by some ideal in \( A[y_0, \ldots, y_n] \)). Note that in this case \( X_i = X \cap D_+(x_i) \) is a closed subscheme of the affine scheme \( D_+(x_i) \). By our results on closed subschemes of affine schemes, \( X_i \) corresponds to an ideal in \( A[y_0, \ldots, y_n]_{(x_0)} \), and the map \( A[y_0, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_X) \) \( y_j \mapsto \frac{s_j}{s_i} \) is surjective.

\( \Leftarrow \): Since \( X_i \) is affine, and the map to the right is surjective, we see that \( X_i \) corresponds to a closed subscheme of \( D_+(x_i) \). As \( X \) is glued together by the \( X_i \) we see that \( X \) is a closed subscheme of \( \mathbb{P}^n_A \) as well. \( \square \)
In practice however, this condition is not so hard to verify (that is, not much easier than checking directly that the induced map is an embedding). To get a better criterion we first introduce some more notation.

**Definition 16.17.** Let $X$ be a scheme over a field $k$.

1. A linear series $V$ is a subspace $V \subseteq \Gamma(X, L)$ where $L$ is an invertible sheaf on $X$.

2. A linear series $V$ separates points if for any two points $x, y \in X$ there is a section $s \in V$ such that $s(x) = 0$ but $s(y) \neq 0$.

3. A linear series $V$ separates tangent vectors if for any $x \in X$ the set \( \{ s \in V | s(x) \neq 0 \} \) spans $m_x L_x / m_x^2$.

The definitions here come from the following theorem

**Theorem 16.18.** Let $X$ be a projective scheme over an algebraically closed field $k$ and let $\phi : X \rightarrow \mathbb{P}^n_k$ be a morphism over $k$ corresponding to $(L, s_0, \ldots, s_n)$. Then $\phi$ is a closed immersion if and only if $V = \text{span}\{s_0, \ldots, s_n\}$ separates points and tangent vectors.

We will not prove this theorem here, but let us at least say a few words about it.

If $\phi$ is a closed immersion, we can consider $X \subseteq \mathbb{P}^n_k$ as closed subscheme, and the $s_i$ are simply restrictions of the $x_i$ to $X$. Now the theorem is essentially a consequence of the easy geometric fact that the $x_0, \ldots, x_n$ separate points and tangent vectors on $\mathbb{P}^n$. Indeed, we can pick a linear form $l = a_0 x_0 + \cdots + a_n x_n$ such that the hyperplane $H = V(l)$ meets $x$ but not $y$ (here we are using that $k$ is algebraically closed!). $l|_X$ is a linear combination of the $s_i$, and so $V$ separates points. Similarly, linear forms on $\mathbb{P}^n$ separate tangent vectors, so they also separate tangent vectors on the subscheme $X$.

The hard part is showing that the converse holds. If $V$ is a linear series separating points and tangent vectors, then the morphism $\phi$ is injective (since you can use linear forms for distinguishing the images of two points). The main ingredient we need is that $\phi$ is proper, which implies that $\phi$ is closed, and furthermore that $\phi$ gives a homeomorphism onto a closed set in $\mathbb{P}^n$. Given this, the rest of the proof is relatively straightforward: We need only check that $\mathcal{O}_{\mathbb{P}^n} \rightarrow \phi_* \mathcal{O}_X$ is surjective, which can be done on stalks.

What makes this theorem more powerful than the previous is that the conditions on sections to separate points and tangent vectors can be turned in to statements about restriction maps of certain sheaves being surjective. The latter task is something we can approach using the machinery of cohomology. And indeed, using the long exact sequence in cohomology we can turn this into simple, computable numerical criteria that guarantee that the corresponding map is an embedding.
16.5 Projective space as a functor

Recall that we defined a functor $F : \text{Sch}^{\text{op}} \to \text{Sets}$ to be representable if there is a scheme $X$ and an isomorphism of functors $\Phi : h_X \simeq F$; i.e., for each $S \in \text{Sch}^{\text{op}}$ a bijection $\Phi(S) : \text{Hom}(S, X) \to F(S)$.

We saw in Chapter 6 that affine space $A^n = \text{Spec} \, \mathbb{Z}[x_1, \ldots, x_n]$ represented the functor $F$ taking a scheme $S$ to $\Gamma(X, \mathcal{O}_X)$; this just amounts to saying that a morphism $X \to A^n$ is determined by $n$ regular functions. More generally Corollary 3.7 says that an affine scheme Spec $A$ is represented by the functor

$$F(S) = \text{Hom}_{\text{Rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

So which functor does projective space represent? Intuitively we would like to associate to each morphism $f : X \to \mathbb{P}^n$ to a set of data on $X$. As we have seen, there is not so much information $\Gamma(X, \mathcal{O}_X)$ in the context of projective schemes. However, we do have something canonical associated to $\mathbb{P}^n$, namely sections of $\mathcal{O}(1)$.

As a first question, we should ask what $F(\text{Spec} \, k)$ should correspond to. Motivated by the very definition of projective space over a field, we would like to say that the value of $F$ on closed points should correspond to lines in a vector space. More precisely, morphisms $\text{Spec} \, k \to \mathbb{P}^n$ should be in correspondence with lines $l \subseteq k^{n+1}$ through the origin. This assignment should also be continuous, e.g., for each open $U \subseteq \mathbb{P}^n$, we should have a sub-line bundle $\mathbb{L}_U$ of the trivial bundle $\mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$. By the correspondence between vector bundles and invertible sheaves, this is equivalent to a subsheaf $L \subseteq \mathcal{O}_{\mathbb{P}^n}^{n+1}$ which is an invertible sheaf.

This motivates the following definition:

**Definition 16.19.** Define a functor $F : \text{Sch}^{\text{op}} \to \text{Sets}$ by defining for a scheme $X$

$$F(X) = \{\text{invertible subsheaves } L \subseteq \mathcal{O}_X^{n+1}\} = \{\text{invertible sheaf quotients } \mathcal{O}_X^{n+1} \to L \to 0\} / \sim$$

where the $\sim$ says that quotients $\mathcal{O}_X^{n+1} \to L \to 0$, $\mathcal{O}_X^{n+1} \to M \to 0$ are equivalent if have the same kernel.

Note that on projective space $\mathbb{P}^n = \text{Proj} \, \mathbb{Z}[x_0, \ldots, x_n]$, we do have such a quotient

$$\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0.$$ 

Indeed, on a $D_+(f)$ we can define it by sending an element $(t_0, \ldots, t_m) \in R(f)$ to $x_0 t_0 + \cdots + x_n t_n \in R(1)(f)$. Of course this quotient is not canonical, as it depends on the choice $x_0, \ldots, x_n$. 

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Theorem 16.20. The functor $F$ is represented by the scheme $\mathbb{P}^n$.

Proof. We need to define natural transformations of the two functors $h_{\mathbb{P}^n}$ to $F$. For a scheme $X$, we define

$$\Phi(X) : \text{Hom}(X, \mathbb{P}^n) \rightarrow F(X)$$

by sending a morphism $f : X \rightarrow \mathbb{P}^n$ to the (equivalence class of the) quotient

$$\mathcal{O}^{n+1}_X \rightarrow f^* \mathcal{O}(1) \rightarrow 0$$

which is the pullback of $\mathcal{O}^{n+1} \rightarrow \mathcal{O}(1) \rightarrow 0$ on $\mathbb{P}^n$. It is clear that this assignment is functorial, i.e., it gives a natural transformation. Moreover, by definition $\Phi$ sends the identity map $\text{id}_{\mathbb{P}^n}$ to the equivalence class of the sequence above.

To prove the theorem, we need to construct an inverse $\Psi$ to $\Phi$. In other words, to each quotient $\mathcal{O}^{n+1}_X \rightarrow L \rightarrow 0$, we need to produce a morphism of schemes $f : X \rightarrow \mathbb{P}^n$ so that $\mathcal{O}^{n+1}_{\mathbb{P}^n} \rightarrow \mathcal{O}(1) \rightarrow 0$ pulls back to $\mathcal{O}^{n+1}_X \rightarrow L \rightarrow 0$. However, from the quotient $\mathcal{O}^{n+1}_X \rightarrow L \rightarrow 0$ we obtain $n + 1$ sections $s_0, \ldots, s_n$ by taking the $n + 1$ maps $\mathcal{O}_X \rightarrow \mathcal{O}^{n+1}_X \rightarrow L$ (where the first map corresponds to a ‘coordinate inclusion’). We then define $\Psi$ by associating $\mathcal{O}^{n+1}_X \rightarrow L \rightarrow 0$ to the morphism $f : X \rightarrow \mathbb{P}^n \in \text{Hom}(S, \mathbb{P}^n)$. One can check that two choices of quotients induce the same map $f : X \rightarrow \mathbb{P}^n$, so this is well-defined.

Finally, in the construction of the morphisms in Theorem 16.7, we have $L = f^* \mathcal{O}(1)$ and $s_i = f^* x_i$, and so $\Psi$ is the inverse to $\Phi$.

Why should one care about such a statement? There are several good reasons, but perhaps the most basic is that it tells us how to think about points of the scheme projective space as corresponding to something geometric. Indeed, a priori, the only thing we know about $\mathbb{P}^n$ is that it is glued together by $n + 1$ coordinate affine $n$-spaces. The theorem above shows that in fact does what it is supposed to do; the $k$-points $\mathbb{P}^n(k)$ correspond to lines in the vector space $k^{n+1}$ for any field $k$.

The case of $\mathbb{P}^1$ is particularly vivid. The functor of points of $\mathbb{A}^1$ shows that a morphism $X \rightarrow \mathbb{A}^1$ is equivalent to giving an element of $\Gamma(X, \mathcal{O}_X)$. In less precise terms this is what we think of as a ‘regular function’ on $X$. The corresponding statement for $\mathbb{P}^1$ is the following. A map $X \rightarrow \mathbb{P}^1$ corresponds to a section $s$ of a line bundle $L$ on $X$, or in other words a ‘meromorphic function’, which is only required to be regular on $X_s$. .

16.5.1 Generalizations

If we modify of the functor of points of $\mathbb{P}^n$, we obtain other interesting examples of schemes parameterizing geometric objects. The following examples are only
meant as basic illustrations of this point - they will not appear later in the notes.

Example 16.21 (The Grassmannian). Consider the functor

$$F(S) = \{\text{rank } r \text{ locally free quotients } \mathcal{E}^n \to Q \to 0\} / \sim$$

where two quotients again are defined to be equivalent if they have the same kernel. Then $F$ is represented by a scheme, known as the Grassmannian $Gr(r, n)$. This scheme can realized as the projective scheme in $\mathbb{P}^{(n+r-1)}_\mathbb{Z} = \text{Proj } \mathbb{Z}[p_{i_1,\ldots,i_r}]$ where $0 \leq i_1 < i_2 < \ldots i_r \leq n$. Explicitly, $Gr(r, n)$ is given by the Plücker equations

$$\sum_{k=0}^{r+1} (-1)^k p_{i_1, i_2, \ldots, i_{r-1}, j_k} \cdot p_{i_1, \ldots, j_k, \ldots, j_{r+1}} = 0$$

(over all sequences of indices $i_1, \ldots, i_r$ and $j_1, \ldots, j_{r+1}$). We can define the map $Gr(k, n) \to \mathbb{P}^{(n+r-1)}_\mathbb{Z}$ using the Yoneda lemma, by giving a natural transformation between the functors. If $S$ is a scheme, this transformation takes an $S$-valued point of $Gr(k, n)$, that is, a quotient $\mathcal{E}^n \to Q \to 0$, to $\wedge^r\mathcal{E}^n \to \wedge^r Q \to 0$, which since $\wedge^r Q$ has rank 1, defines a point in $\mathbb{P}^{(n+r-1)}_\mathbb{Z}(S)$.

Proving that this scheme actually represents $Gr(k, n)$ is similar to what we did for projective $n$-space, although it is slightly more involved, due to the Plücker equations.

Example 16.22 (Projective bundles). Let $\mathcal{E}$ be a locally free sheaf on a scheme $X$. Consider the functor on $\text{Sch}/X$ given by

$$F(h : Y \to X) = \{\text{invertible sheaf quotients } h^*\mathcal{E} \to L \to 0\}$$

Then $F$ is represented by a scheme $\mathbb{P}(\mathcal{E}) \to X$ over $X$ called the projectivization of $\mathcal{E}$. One can think of closed points of this scheme as hyperplanes in the fibers of $\mathcal{E}$.

Example 16.23 (Proj of an $\mathcal{O}_X$-module). Let $X$ be a scheme and let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Consider the functor on $\text{Sch}/X$ given by

$$F(h : Y \to X) = \{\text{invertible sheaf quotients } h^*\mathcal{F} \to L \to 0\}$$

Then $F$ is represented by a scheme $\text{Proj}(\mathcal{F})$. Many geometric constructions can be formulated as such schemes. For instance, if $\mathcal{I}$ is a sheaf of ideals on $X$, then $\text{Proj}(\mathcal{F})$ can be identified with the blow-up of $X$ along $\mathcal{I}$ (see Hartshorne II.7).
Chapter 17

More on projective schemes

17.1 Ample invertible sheaves and Serre’s theorems

The prototype of an invertible sheaf is the sheaf $O(1)$ on $X = \text{Proj } R$, which is globally generated if $R$ is generated in degree 1. As we have seen, this corresponds to a morphism $f : X \to \mathbb{P}^n$, with the property that $f^* O_{\mathbb{P}^n}(1) = O_{\text{Proj } R}(1)$. In the case where $f$ is a closed immersion, we can regard $X$ as a subscheme of $\mathbb{P}^n$, and $O(1)$ is simply the restriction $O_{\mathbb{P}^n}(1)|_X$. This restricted invertible sheaf has a special property - $R$ and hence $X$ is completely recovered by the global sections of it and the polynomial relations between them. This motivates the following definition:

**Definition 17.1.** Let $X$ be a scheme over $S$. A invertible sheaf $L$ on $X$ is said to be very ample over $S$ if there is a closed immersion $i : X \to \mathbb{P}^n_S$ such that $L \cong i^* O_{\mathbb{P}^n}(1)$. $L$ is ample if $L^\otimes N$ is very ample for some $N > 0$.

**Theorem 17.2** (Serre). Let $X \subseteq \mathbb{P}^n_A$ be a projective scheme over a noetherian ring $A$ and let $L$ be an ample invertible sheaf. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then there is an integer $m_0$ such that $\mathcal{F} \otimes L^m$ is globally generated (by a finite set of global sections) for all $m \geq m_0$.

**Proof.** By replacing $L$ with a multiple, we may assume that $L$ is very ample and even that $L = i^* O(1)$ for a projective embedding $i : X \to \mathbb{P}^n$. Since $X$ is noetherian, $i$ is finite and $\mathcal{F}$ is coherent, we have that $i_* \mathcal{F}$ is coherent on $\mathbb{P}^n$. Moreover, $(i_* \mathcal{F})(m) = i_*(\mathcal{F} \otimes L^m)$, so $\mathcal{F} \otimes L^m$ is globally generated if and only if $(i_* \mathcal{F})(m)$ is. Hence we may assume that $X = \mathbb{P}^n_A$. 

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17.1. Ample invertible sheaves and Serre’s theorems

Write $\mathbb{P}^n_A = \text{Proj } R$ where $R = \text{Proj } A[x_0, \ldots, x_n]$. We can cover $X$ by open sets $D_+(x_i)$ where $i = 0, \ldots, n$, such that $\mathcal{F}|_{D(x_i)} = M_i$ some finitely generated $R_{(x_i)}$-module. For each $i$ we can choose finitely many elements $s_{ij}$ generating $M_i$. Regarding $s_{ij}$ as sections of the sheaf $M_i$, we see that there are integers $m_{ij}$ such that $x_m s_{ij}$ extend to global sections of $\mathcal{F}(m_{ij})$. Take $m = \max_{ij} m_{ij}$, then for each $i,j$ we have a section $t_{ij} \in \Gamma(X, \mathcal{F}(m))$.

We claim that the global sections $t_{ij}$ generate $\mathcal{F}(m)$ locally. It is sufficient to prove this on the opens $D_+(x_i)$ over which $\mathcal{F}(m)$ is isomorphic to the $R_{(x_i)}$-module $M_i(m) \cong M_i \otimes_R R(m)$, which is isomorphic to $x^m M_i$ as a graded $R_{(x_i)}$-module. Since $s_{ij}$ generates $M_i$, we see that $x^m s_{ij}$ generate $\mathcal{F}(m)|_{D_+(x_i)}$.

This shows that $\mathcal{F}(m_0)$ is globally generated for the chosen $m_0 = m$ above. Then also $\mathcal{F}(m)$ is globally generated for $m \geq m_0$, by multiplying the generating sections of $\mathcal{F}(m_0)$ by the various monomials in the $x_i$.

In particular, we see that any coherent sheaf can be written as a quotient of a direct sum of invertible sheaves of the form $\mathcal{O}(-m)$: Take any $\mathcal{F}(m)$ which is globally generated; then there is a surjection $\mathcal{O}^I \to \mathcal{F}(m) \to 0$ (with $I$ finite!), which becomes

$$\mathcal{O}(-m)^I \to \mathcal{F} \to 0$$ (17.1.1)

after tensoring by $\mathcal{O}(-n)$.

**Theorem 17.3 (Serre).** Let $X = \text{Proj } R$ be a noetherian projective scheme over Spec $A$, where $A$ is a noetherian ring, and let $\mathcal{F}$ be a coherence sheaf on $X$. Then:

(i) The cohomology groups $H^i(X, \mathcal{F})$ are finitely generated $A$-modules for every $i \geq 0$.

(ii) There exists an $n_0 > 0$ such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_0$.

**Proof.** As in the above proof, we immediately reduce to the case where $X = \mathbb{P}^n_A$.

Note that both of the conclusions hold for the twisting sheaves $\mathcal{O}_X(n)$. To prove it for any coherent $\mathcal{F}$, take a quotient of the form (17.1.1) and let $\mathcal{K}$ be the kernel, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E} = \mathcal{O}(-n)^I$, with $I$ finite. Note that $\mathcal{K}$ is again coherent.
(i): Take the l.e.s of cohomology to get
\[ H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{K}) \]

We can now prove the theorem by downwards induction on \( i \): \( H^{i+1}(X, \mathcal{K}) \) and \( H^i(X, \mathcal{E}) \) are both finitely generated, and hence so is \( H^i(X, \mathcal{F}) \), since \( A \) is noetherian.

(ii): Twist the above sequence by \( \mathcal{O}_X(m) \) and take the long exact sequence in cohomology to get
\[ H^i(X, \mathcal{E}(m)) \to H^i(X, \mathcal{F}(m)) \to H^{i+1}(X, \mathcal{K}(m)) \]

By downward induction on \( i \), and the fact that \( H^i(X, \mathcal{E}(m)) \) for any \( m \), we find that \( H^i(X, \mathcal{F}(m)) = 0 \).

17.1.1 Euler characteristic

Let \( X \) be a noetherian scheme over a field \( k \) and let \( \mathcal{F} \) be a coherent sheaf on \( X \). By Serre’s theorem, the cohomology groups \( H^i(X, \mathcal{F}) \) are finite dimensional \( k \)-vector spaces, so it makes sense to ask about their dimensions. It turns out that the alternating sum of these dimensions is the right thing to look at:

**Definition 17.4.** Let \( X \) be a projective scheme of dimension \( n \). We define the **Euler characteristic** of \( \mathcal{F} \) as
\[ \chi(\mathcal{F}) = \sum_{k=0}^{n} (-1)^k \dim H^k(X, \mathcal{F}) \]

**Proposition 17.5.** Let \( X \) be a projective scheme and let \( \mathcal{O}(1) \) be the very ample invertible sheaf. Then the function
\[ P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m)) \]

is a polynomial in \( m \)

This polynomial is called the **Hilbert polynomial** of \( \mathcal{F} \). When \( \mathcal{F} = \tilde{M} \), this coincides with the Hilbert polynomial of \( M \) as defined in commutative algebra.

17.2 Proper and projective schemes

Let \( S \) be a scheme and let \( X \) be a scheme over \( S \). We say call \( X \) is **projective** over \( S \) (or that the structure morphism \( f : X \to S \) is projective) if \( f : X \to S \)
factors as $f = \pi \circ i$ where $i : X \to \mathbb{P}^n_S$ is a closed immersion and $\pi : \mathbb{P}^n_S \to S$ is the projection.

The primary examples are of course $X = \text{Proj} R$ where $R$ is a graded $R_0$-algebra generated in degree 1 and $S = \text{Spec} R_0$. In this case, we can define the projective immersion $i$ by taking a surjection $R_0[x_0, \ldots, x_n] \to R$, which upon taking Proj, a closed immersion $X \to \mathbb{P}^n_{R_0}$.

**Theorem 17.6.** A projective morphism $X \to S$ is proper.
Chapter 18

Curves

18.1 Curves

We will in this chapter assume that $k$ is an algebraically closed field. Recall that a variety over $k$ is a separated, integral scheme of finite type over $k$. Throughout this chapter, a curve will mean a 1-dimensional variety over $k$.

18.2 The genus of a curve

Definition 18.1. The arithmetic genus of $X$ is defined as

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

The geometric genus of $X$ is defined as

$$p_g(X) = \dim_k H^0(X, \Omega_X)$$

Note that $\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$, so we get $p_a(X) = 1 - \chi(\mathcal{O}_X)$.

At first sight, these numbers are defined using different sheaves, and there is no a priori reason to expect that they should have anything to do with each other. However, we shall see later in the chapter that there is a strong relation between them: $p_a = p_g$. For the time being we will still refer to $p_a$ as the genus of $X$.

Example 18.2. When $X = \mathbb{P}^1$, we have $H^1(\mathbb{P}^1, \mathcal{O}) = 0$ and $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, so both genera are zero.
Example 18.3. Let $X \subseteq \mathbb{P}^2$ be a plane curve, defined by a homogeneous polynomial $f(x_{0,1}, x_2)$ of degree $d$. In Chapter 12, we computed that $H^1(X, \mathcal{O}_X) \simeq k^{(d-1)/2}$. Hence the genus of $X$ is $\frac{(d-1)(d-2)}{2}$.

18.3 Divisors on Curves

Let $X$ be a non-singular curve. Since $X$ has dimension 1, a Weil divisor $D$ on $X$ is a finite formal combination of points on $X$,

$$D = \sum n_i p_i$$

where $p_i \in X$. We say that $D$ is effective if each $n_i \geq 0$.

The degree of $D$ is defined as the sum $\deg D = \sum n_i$. Equivalently, if we view $D$ as a Cartier divisor given by the data $(U_i, f_i)$, then

$$\deg D = \sum_{x \in X} v_x(D)$$

where $v_x(D) = v_x(f_i)$ as before.

Recall that each Weil divisor determines an invertible sheaf $\mathcal{O}_X(D)$, which over an open set $U$ takes the value

$$\mathcal{O}_X(D)(U) = \{ f \in K | (\text{div } f + D)|_U \geq 0 \}$$

Then $D$ is effective if and only if $\Gamma(\mathcal{O}_X(D)) \neq 0$. In particular, if $\Gamma(X, \mathcal{O}_X(D))$ has dimension at least 2, there is a second effective divisor $D' = \sum m_i q_i$ such that $D$ and $D'$ are linearly equivalent.

18.3.1 The canonical divisor

When $X$ is a non-singular curve, the sheaf of differentials $\Omega_X$ is a locally free sheaf of rank 1, i.e., an invertible sheaf. Thus $\Omega_X$ gives rise to a divisor, which we denote by $K_X$. We can describe $K_X$ as a Cartier divisor as follows. A local section of $\Omega_X$ is of the form $\omega = f_U dx$. We can then define a Cartier divisor with the data

$$(U, f_U)$$

This is well-defined, because on the overlaps $U \cap V$, two sections $f_U dx, f_V dy$ are related by $f_U = \frac{dy}{dx} f_V$, and $\frac{dy}{dx}$ is a unit in $\mathcal{O}_X(U \cap V)$.

Any other section $\omega'$ of $\Omega_X$ has the form $f \omega$ for some $f \in K^\times$. Therefore,

$$\text{div } (\omega') = \text{div } (\omega) + \text{div } (f)$$
so the divisor class \([K_X] := \text{div} (\omega) \in \text{Cl}(X)\) associated to \(\omega\) is independent of the chosen \(\omega\). We call this the \textit{canonical class} of \(X\). Also, any divisor \(D\) so that \([D] = [K_X]\) is called a \textit{canonical divisor} and it is denoted by \(K_X\).

**Definition 18.4.** A \textit{canonical divisor} is a divisor \(K_X\), so that \(O_X(K_X) \cong \Omega_X\).

**Example 18.5.** Let \(X = \mathbb{P}^1_k\), and consider the open set \(U_0 = D_+(x_0)\). On \(U_0\) we have a local coordinate \(t = \frac{x_1}{x_0}\), and we consider the differential form \(dt = d\left(\frac{x_1}{x_0}\right)\). On the overlap \(U_0 \cap U_1\), we have

\[
d\left(\frac{x_0}{x_1}\right) = d(t^{-1}) = -t^{-2} dt = -\left(\frac{x_0}{x_1}\right)^2 d\left(\frac{x_1}{x_0}\right)
\]

It follows that \(\Omega_X \cong \mathcal{O}_{\mathbb{P}^1}(-2)\).

### 18.4 Morphisms of curves

Let \(f : X \to Y\) be a morphism of schemes. Recall that we say that \(f\) is said to be \textit{finite} if the following two conditions are satisfied:

(i) \(f\) is affine, i.e., if \(U \subseteq Y\) is open and affine, then \(f^{-1}(U) \subseteq X\) is affine.

(ii) If \(U = \text{Spec } A\) and \(V = f^{-1}(U) = \text{Spec } B\), then \(B\) is a finite \(A\)-algebra (via \(f^\#\)).

For such morphisms, the pushforward \(f_* \mathcal{O}_X\) is a coherent \(\mathcal{O}_Y\)-module.

If \(y \in Y\) is a closed point, corresponding to the sheaf of ideals \(m_y \subseteq \mathcal{O}_Y\), then the scheme theoretic fibre \(X_y\) is defined in \(X\) by the ideal sheaf \(f^* m_y \cdot \mathcal{O}_X \subseteq \mathcal{O}_X\). Note that in the case \(X = \text{Spec } B\), and \(Y = \text{Spec } A\), the scheme theoretic fibre over \(y\) is naturally isomorphic to \(\text{Spec}(B/m_y B)\) where \(m_y\) is the maximal ideal in the point \(y\). If \(k(y) = \mathcal{O}_{Y,y}/m_y \mathcal{O}_{Y,y} = A/m\) denotes the residue field, then we have \(f^{-1}(x) \cong \text{Spec}(B \otimes_A k(y))\). Note that \(B \otimes_A k(y)\) is a \textit{finite} algebra over the field \(k(y)\). Such an algebra has only finitely many prime ideals (and all of these are maximal). In particular, a finite morphism can only have a finite number of preimages over a closed point \(y \in Y\).

The converse of this statement is of course not true. If \(f : U = \text{Spec } A_f \to X = \text{Spec } A\) is an open immersion, e.g., \(f : \text{Spec } k[x,y]/(xy - 1) \to \text{Spec } k[x]\), the number of preimages are of course finite (0 or 1 in fact). However, \(A_f\) is usually not finite as an \(A\)-algebra.

However if \(f : X \to Y\) is a \textit{projective} morphisms (or more and its fibers are finite, then \(f\) is finite. We will prove this in the next section for curves.

We can be a little bit more precise when talking about the ‘number of preimages’ of \(f\). Let \(Y = \text{Spec } A\) be an integral noetherian scheme; so that \(A\) is
a noetherian integral domain. Let \( f : X \to Y \) be a finite morphism. Then \( X \) is also affine, say \( X = \text{Spec} \, B \) for some finite \( A \)-algebra \( B \). We finally suppose that \( f \) is dominant. This means that the image of each component of \( X \) is dense in \( X \). As \( A \) is an integral domain, this is equivalent to \( A \subseteq B \), and \( q \cap A = 0 \) for each associated prime ideal \( q \subseteq B \). It follows that we can extend the inclusion \( A \subseteq B \) to an inclusion \( K(A) \subseteq K(B) \). Since \( B \) is a finite \( A \)-algebra, this extension is finite. We define the degree of \( f \) as the degree of the field extension

\[
[K(B) : K(A)] = \dim_{K(A)} K(B).
\]

**Example 18.6.** Let \( Y = \text{Spec} \, k[x] \), \( X = \text{Spec} \, k[x, y]/(x-y^2) \) and let \( f : X \to Y \) denote the projection map (induced by \( k[x] \subseteq k[x, y]/(x-y^2) \)). This is a morphism of degree two, since \( K(X) = k(y) \) is a degree two extension of \( K(Y) = k(x) \) (since \( x = y^2 \)). Note that the fiber of \( f \) over the point \( 0 \in Y \) has set-theoretically only one preimage, namely the point \( (0,0) \in Y \) corresponding to \( m = (x,y) \). However, the scheme theoretic fiber here has a non-reduced structure that reflects the number 2:

\[
X_y = \text{Spec} \, k[x, y]/(x-y^2) \otimes_{k[x]} k(x) \simeq \text{Spec} \, k[y]/y^2
\]

Let us now specialize to the case of curves. For a closed point \( x \in X \), the local ring \( O_{X,x} \) is a discrete valuation ring and there is an element \( t \in O_{X,x} \) generating the maximal ideal \( m \). We say that \( t \) is a local parameter or a uniformizing parameter if \( v_x(t) = 1 \). Note that we can always normalize the valuation so that a generator of \( m \) has valuation 1.

**Proposition 18.7.** Let \( X \) be a projective non-singular curve over \( k \) and let \( Y \) be any curve. Let \( f : X \to Y \) be a morphism. Then either

(i) \( f(X) = \) a point in \( Y \); or

(ii) \( f(X) = Y \);

In the case (ii) \( K(X) \supseteq K(Y) \) is a finite extension, \( f \) is a finite morphism and \( Y \) is also projective.

**Proof.** Since \( X \) is projective, the image \( f(X) \) must be closed in \( Y \) (Add this theorem). On the other hand \( X \) is irreducible, and hence so is \( f(X) \). Hence \( f(X) \) is either a point or all of \( Y \).

If \( f(X) = Y \), then \( f \) is dominant, so we get an inclusion of function fields \( K(Y) \subseteq K(X) \). Both of these fields are finitely generated over \( k \) of the same transcendence degree (i.e., 1), so the extension is finite algebraic.

Let \( V = \text{Spec} \, B \subseteq Y \) be an open set of \( Y \) and let \( A \) denote the integral closure of \( B \) in \( K(X) \). Then by [Hartshorne I.6.7] we have \( \text{Spec} \, A = f^{-1}(V) \). So since \( A \) is a finite \( B \)-module, we have that \( f \) is finite. \( \square \)
Chapter 18. Curves

We also have the following theorem of [Hartshorne Ch. I. 6.12]:

**Theorem 18.8.** There is an equivalence of categories

1. Non-singular projective curves, and dominant morphisms
2. Function fields $K$ of dimension 1 over $k$ and $k$-homomorphisms.

### 18.4.1 Pullbacks of divisors

Given $f : X \to Y$ (non-constant) we have a well-defined pullback map

$$f^* : \text{Pic}(Y) \to \text{Pic}(X)$$

sending the class of an invertible sheaf $L$ on $X$ to that of $f^*L$. If $L = \mathcal{O}_X(D)$ for some divisor, we can make this pullback a bit more explicit as follows.

Let $t \in \mathcal{O}_{Y,y}$ denote the uniformizing parameter. We consider $t$ as an element of $K(X)$ via the field extension $K(X) \supseteq K(Y)$. In particular, we can talk about the valuations of $t$ in each local ring $\mathcal{O}_{X,x}$ and define the Weil divisor

$$f^*(y) = \sum_{x \in f^{-1}(y)} v_x(t)x$$

(Since $f$ is finite, there are only finitely many preimages of $y$, and so the sum is finite). The expression is also independent of the choice of $t$: If $t'$ is a different uniformizing parameter, we can in any case write $t' = ut$ where $u$ is a unit, so that $v_x(t) = v_x(t')$.

Thus we get a well defined map

$$f^* : \text{Div}(Y) \to \text{Div}(X)$$

which descends to the above map $f^* : \text{Pic}(Y) \to \text{Pic}(X)$.

**Lemma 18.9.** We have $\deg f^*D = \deg f \cdot \deg D$.

**Proof.** TODO.

**Definition 18.10.** For a morphism $f : X \to Y$, we call the number $e_x = v_x(t)$ the **ramification index** of $f$ at $x$. If $e_x > 1$, we say that $f$ is **ramified** at $x$.

**Example 18.11.** Let $A = k[x]$ and let $B = k[x, y]/(x - y^2) \simeq k[y]$ where $k$ is a field. Let $X = \text{Spec } B$ and let $Y = \text{Spec } A$. Let $f : X \to Y$ be the morphism induced by the inclusion $A \hookrightarrow B$ (thus $x \mapsto y^2$).

The morphism $f$ is ramified only at the origin $(0, 0)$, and here the ramification index is two. Indeed, $x$ is a uniformizing parameter of $\mathcal{O}_{Y,y} = k[x]_{(x)}$, while $y$ is the uniformizer of $\mathcal{O}_{X,x} = B_{(x,y)}$. Then we have $v_{(0,0)}(x) = v_{(0,0)}(y^2) = 2.$
18.5 Hyperelliptic curves

The reader might notice a resemblance between the previous example and Example 11.41, where ramification was defined in terms of the relative sheaf of differentials $\Omega_{X|Y}$. In that example, $\Omega_{X|Y}$ was a torsion sheaf supported on the single point $(0,0)$. This correspondence between the two notions of ramification is a general fact (at least in characteristic 0), and we have the useful formula for the ramification indexes of curves:

$$e_p = \text{length}(\Omega_{X|Y})_p + 1$$

18.4.2 Morphisms to $\mathbb{P}^1$

Let $X$ be a non-singular curve, and let $f \in K$. Then $f$ induces a morphism $\phi : X \to \mathbb{P}^1$ in the following way. Let $U = X - \text{Supp} f^{-1}(\infty)$ and $V = X - f^{-1}(0)$, so that $f \in \mathcal{O}_X(U)$ and $1/f \in \mathcal{O}_X(V)$.

Write $\mathbb{P}^1 = \text{Proj } k[x_0, x_1]$. On $D_+(x_0)$, the map $k[x_1/x_0] \to \mathcal{O}_X(U)$ sending $x_1/x_0 \mapsto f$, induces a map $\phi_U : U \to D_+(x_0) \subseteq \mathbb{P}^1$. Similarly, on $D_+(x_1)$, we get a map $\phi_V : V \to \mathbb{P}^1$. These maps coincide on $U \cap V$, and therefore we get a morphism

$$\phi : X \to \mathbb{P}^1$$

This morphism is non-constant, hence finite.

18.5 Hyperelliptic curves

Let us recall the hyperelliptic curves defined in Chapter 3. For an integer $g \geq 1$ we consider the scheme $X$ glued together by the affine schemes $U = \text{Spec } A$ and $V = \text{Spec } B$, where

$$A = \frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1x)} \quad \text{and} \quad B = \frac{k[u,v]}{(-v^2 + a_{2g+1}u + \cdots + a_1u^{2g+1})}$$

As before, we glue $D(x)$ to $D(u)$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

There is a morphism $f : X \to \mathbb{P}^1$ coming from the inclusions $k[x] \subseteq A$ and
Chapter 18. Curves

$k[u] \subseteq B$; note that $u = x^{-1}$ gives the standard gluing of $\mathbb{P}^1$ by the two affine schemes $\text{Spec } k[x]$ and $\text{Spec } k[u]$. The corresponding morphism $X \to \mathbb{P}^1$ is finite, of degree 2, as we are adjoining a square root of an element of $k[x]$.

Let us compute the genus of $X$ using Čech cohomology. We will use the covering $\mathcal{U} = \{U, V\}$ above. Viewing the first ring as a $k[x]$-module, we can write

$$k[x, y] / (-y^2 + a_{2g+1} x^{2g+1} + \cdots + a_1 x) = k[x] \oplus k[x]y$$

and similarly $k[u] \oplus k[u]v$ as a $k[u]$-module for the ring on the right.

The first map in the Čech complex of $\mathcal{O}_X$ has the following form:

$$d^0 : (k[x] \oplus k[x]y) \oplus (k[x^{-1}] \oplus k[x^{-1}]x^{-g-1}y) \to k[x^{\pm 1}] \oplus k[x^{\pm 1}]y$$

$$(p(x) + q(x)y, r(x^{-1}) + s(x^{-1})x^{-g-1}y) \mapsto p(x) - r(x^{-1}) + (q(x) - s(x^{-1})x^{-g-1})y$$

From this we can read off $H^0(X, \mathcal{O}) = \ker d^0 = k$ and $H^1(X, \mathcal{O}) \simeq k^g$, with basis $yx^{-1}, yx^{-2}, \ldots, yx^{-g}$. In particular, $g$ is exactly the arithmetic genus of the curve.

For $g = 2$, we get a particularly interesting curve—a smooth projective curve which cannot be embedded in $\mathbb{P}^2$. Indeed, any smooth curve in $\mathbb{P}^2$ has a degree $d$ and corresponding arithmetic genus $\frac{1}{2}(d-1)(d-2)$. However, there is no integer solution to $\frac{1}{2}(d-1)(d-2) = 2!$

**Proposition 18.12.** There exist non-singular projective curves which cannot be embedded in $\mathbb{P}^2$.

To prove this, we need to verify that the curve is in fact projective. That is, we need to find some projective embedding of it. To do this, we will need to work out the cohomology groups $H^0(X, \mathcal{O}_X(nP))$ for a point $p \in X$.

Let us for simplicity assume that $a_{2g+1} = 1$. Let $p$ be the unique point in $\{u = v = 0\}$ in $X$. In the local ring at $p$, we have

$$u = v^2(1 + a_2 u + \cdots + a_1 u^{2g})^{-1} = v^2(\text{unit}),$$

and hence $v\mathcal{O}_p$ generates $\mathfrak{m}_p$. We compute the some valuations of elements in $\mathcal{O}_p$.

$$\nu_p(v) = 1, \quad \nu_p(u) = 2, \quad \nu_p(x) = -2, \quad \nu_p(y) = 1 + (g+1)(-2) = -(2g+1)$$
18.5. Hyperelliptic curves

We’ve seen that $\Gamma(X, \mathcal{O}_X) = k$, which agrees with our expectation that there are no non-constant regular function on a projective curve. Let us consider the case where the functions are allowed to have poles at $p$ (and only at $p$). In other words, we are interested in elements $s \in \Gamma(X, \mathcal{O}_X(p))$. Note that the point $p$ does not lie in $U$; this means that $s$ is regular there, and hence can be viewed as a polynomial in $x, y$. Now, as $A = k[x] + k[x]y$ as a $k[x]$-module, we will express any element $s$ can be expressed as $f(x) + h(x)y$. We then can calculate

$$\nu_p(f(x) + h(x)y) = \min\{\nu_p(f(x)), \nu_p(h(x))\nu_p(y)\}$$

$$= \min\{-2 \deg f, -(2 \deg h + 2g + 1)\}$$

Thus, even if we allow a pole or order 1 at $p$, we still have $\Gamma(X, \mathcal{O}_X(p)) = k$.

On the other hand for $2p$ we gain an extra section, corresponding to $f(x) = x$:

$$\Gamma(X, \mathcal{O}_X(2p)) = k\{1, x\}$$

Note that $\mathcal{O}(2p)_p = \mathcal{O}_p \cdot x$. The section $x \in \Gamma(X, \mathcal{O}_X(2p))$ is nonvanishing at $p$, while the section $1 \in \Gamma(X, \mathcal{O}_X(2p))$ is vanishing at $p$, since $1 = u \cdot x$ and $u \in \mathfrak{m} \subseteq \mathcal{O}_p$. Note that the linear series generated by $1, x$ generates $\mathcal{O}_X(2p)$ everywhere, inducing the morphism

$$X \xrightarrow{\phi} \mathbb{P}^1$$

$$(x, y) \mapsto [1 : x]$$

which is the same morphism as the double cover above.

We can allow even higher order poles at $p$. The computation above shows that

$$\Gamma(X, \mathcal{O}(3p)) = \begin{cases} 
  k\{1, x, y\} & \text{if } g = 1 \\
  k\{1, x\} & \text{if } g > 1
\end{cases}$$

Using the embedding criterion of Chapter 10, one can show that in the case $g = 1$, the sections $x_0 = 1, x_1 = x, x_2 = y$ give an embedding

$$X \hookrightarrow \mathbb{P}^2_k$$

$$(x, y) \mapsto [1 : x : y]$$

The image is even seen to be a non-singular cubic curve: One computes that $\Gamma(X, \mathcal{O}(6p))$ is 6-dimensional, but we have 7 global sections: $1, x, y, x^2, xy, x^3, y^2$. That means that there must be some relation between them. But of course this is the relation

$$y^2 = a_3x^3 + a_2x^2 + a_1x$$
Chapter 18. Curves

giving the following relation in $\mathbb{P}^2$:

$$x_2^2x_0 = a_3x_1^3 + a_2x_0x_1^2 + a_1x_0^2x_1$$

For $g = 2$, this map is not an embedding, since the image is $\mathbb{P}^1$. However, the map given by $5p$ is: We obtain

$$\Gamma(X, \mathcal{O}_X(5p)) = k\{1, x, x^2, y\}$$

These sections generate $\mathcal{O}_X(5p)$, so we obtain a morphism

$$\phi : X \to \mathbb{P}^3$$

given by $u_0 = 1, u_1 = x, u_2 = x^2, u_3 = y$. Notice that $u_0u_2 - u_1^2 = 0$, so $X$ lies on a quadric surface. In fact, the image of $\phi$ is precisely the relations between the sections:

$$C = V\left(u_1^2 - u_0u_2, u_2^2 - u_1u_3, u_2^3 - u_1u_2^2 - u_0u_2^2 - u_0u_3^2\right) \subseteq \mathbb{P}_k^3$$

This shows that $X$ is projective.
Chapter 19

The Riemann–Roch theorem

When $X$ is a projective curve over a field $k$, the cohomology groups $H^i(X, F)$ are finite-dimensional $k$-vector spaces and we define

$$h^i(X, F) := \dim_k H^i(X, F)$$

Note that in this case, $h^i(X, F) = 0$ for all $i \geq 2$, so we have two cohomology groups $h^0(X, F)$ and $h^1(X, F)$ to work with.

Recall, that we defined for a sheaf $F$, the Euler characteristic $\chi(F)$ as the alternating sum of the $h^i(X, F)$. One useful property of $\chi(X, -)$ is that it is additive on exact sequences:

**Lemma 19.1.** Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. Then

$$\chi(F) = \chi(F') + \chi(F'')$$

This follows because if $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of $k$-vector spaces, then $\sum_i (-1)^i \dim_k V = 0$. Applying this to the long exact sequence in cohomology gives the claim.

The most important sequence we will encounter is the ideal sheaf sequence of a point $p \in X$, which takes the form

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to k(p) \to 0 \quad (19.0.1)$$

where the first map is the inclusion and the second is evaluation at $p$. Here we have identified the ideal sheaf $\mathfrak{m}_p \subseteq \mathcal{O}_X$ by the invertible sheaf $\mathcal{O}_X(-p)$, and the sheaf $i_*\mathcal{O}_p$ with the skyscraper sheaf with value $k(p)$ at $p$. If $L$ is an invertible
sheaf, we can tensor (19.0.1) by \(L\) and get

\[
0 \to L(-p) \to L \to k(p) \to 0
\]  

(19.0.2)

where \(L(-p)\) is the invertible sheaf of sections of \(L\) vanishing at \(p\). Taking \(\chi\), we get

\[
\chi(L(-p)) = \chi(L) - \chi(k(-p)) = \chi(L) - 1.
\]

**Theorem 19.2** (Easy Riemann–Roch). Let \(X\) be a smooth projective curve of genus \(g\) and let \(D\) be a Cartier divisor on \(X\). Then

\[
\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g
\]

**Proof.** Let \(p \in X\) be a point and consider the sequence (19.0.2) with \(L = \mathcal{O}_X(D + p)\). Then, as we just saw, \(\chi(\mathcal{O}_X(D + p)) = \chi(\mathcal{O}_X(D)) + 1\). Also the right-hand side of the equation above increases by 1 by adding \(p\) to \(D\) (since \(\deg(D + p) = \deg D + 1\)). This means that the theorem holds for a divisor \(D\) if and only if it holds for \(D + p\) for any closed point \(p\). So by adding and subtracting points, we can reduce to the case when \(D = 0\). But in that case, the left hand side of the formula is by definition \(\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g\), which equals the right hand side. \qed

The formula above is extremely useful because the right hand side is so easy to compute. The number we are really after is the number \(h^0(X, \mathcal{O}_X(D))\), since this is the dimension of global sections of \(\mathcal{O}_X(D)\). This in turn would help us to study \(X\) geometrically, since we could use sections of \(\mathcal{O}_X(D)\) to define rational maps \(X \dasharrow \mathbb{P}^n\). So if we, for some reason, could argue that say, \(H^1(X, \mathcal{O}_X(D)) = 0\), we would have a formula for the dimension of the space of global sections of \(\mathcal{O}_X(D)\).

In any case, we can certainly say that \(h^1(X, \mathcal{O}_X(D)) \geq 0\), so we get the following bound on \(h^0(X, \mathcal{O}_X(D))\):

**Corollary 19.3.** \(h^0(X, \mathcal{O}_X(D)) \geq \deg D + 1 - g\)

**Example 19.4.** A typical feature is that \(H^1(X, \mathcal{O}_X(D)) = 0\) provided that the degree \(\deg D\) is large enough. This is essentially a consequence of Serre’s theorem.

To give an example, consider again the case where \(X\) is a hyperelliptic curve of genus 2, as in the introduction. We have the following table of the various cohomology groups \(H^i(X, \mathcal{O}_X(np))\) for the point \(p = (u, v)\):

<table>
<thead>
<tr>
<th>(D)</th>
<th>0</th>
<th>1p</th>
<th>2p</th>
<th>3p</th>
<th>4p</th>
<th>5p</th>
<th>6p</th>
<th>7p</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi(\mathcal{O}_X(D)))</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>(H^0(X, \mathcal{O}_X(D)))</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>(H^1(X, \mathcal{O}_X(D)))</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

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and it is not so hard to prove directly using the Čech complex that \( H^1(X, \mathcal{O}_X(np)) = 0 \) for all \( n \geq 4 \).

Fortunately, there are more general results which tell us when \( H^1(X, \mathcal{O}_X(D)) = 0 \). This is due to the following fundamental theorem:

**Theorem 19.5 (Serre duality).** Let \( X \) be a smooth projective variety of dimension \( n \) and let \( D \) be a Cartier divisor on \( X \). Then for each \( 0 \leq p \leq n \),

\[
\dim_k H^p(X, \mathcal{O}_X(D)) = \dim_k H^{n-p}(X, \mathcal{O}_X(K_X - D))
\]

So if \( X \) is a curve, we get that \( H^1(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(K_X - D)) \) and the Riemann–Roch theorem takes the following form:

**Theorem 19.6 (Riemann–Roch).** Let \( X \) be a projective curve of genus \( g \) and let \( D \in \text{Div}(X) \) be a divisor. Then

\[
h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - g
\]

This is a much stronger statement than the Riemann–Roch formula we had before, since group \( H^0(X, \mathcal{O}_X(K_X - D)) \) is easier to interpret: it is the space of global sections of the sheaf associated to the divisor \( K_X - D \), or equivalently \( \Omega_X(-D) \). It is usually easier to argue that there can be no such global sections of this divisor. For instance, in the case \( \deg D < \dim K_X \) then \( K_X - D \) cannot be effective: if \( s \in H^0(X, \mathcal{O}_X(K_X - D)) \), then the divisor of \( s \) defines an effective divisor on \( X \), that is, a sum \( \text{div}(s) = \sum n_i p_i \) where \( n_i \geq 0 \), contradicting our assumption \( \deg K_X < \dim D \).

So what is this degree of the canonical divisor \( K_X \)? From Serre duality, we get that \( H^0(X, \mathcal{O}_X(K_X)) \) and \( H^1(X, \mathcal{O}_X) \) have the same dimension, so the geometric genus and arithmetic genus agree:

\[ p_g = p_a = g. \]

Then applying the Riemann–Roch formula to \( D = K_X \), we get

\[ g - 1 = \dim_k H^0(X, \mathcal{O}_X(K)) - \dim_k H^0(X, \mathcal{O}_X(K_X - K)) = \deg K + 1 - g \]

and so \( \deg K_X = 2g - 2 \). This observation gives us

**Corollary 19.7.** Suppose that \( D \) is a Cartier divisor of degree \( > 2g - 2 \). Then \( H^1(X, \mathcal{O}_X(D)) = 0 \), and

\[ \dim_k H^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g \]

Moreover, if \( \deg D = 2g - 2 \), then \( H^1(X, \mathcal{O}_X(D)) \neq 0 \) only if \( D \sim K_X \).
19.1 Proof of Serre duality for curves

Riemann–Roch and Serre duality are two of the very deep theorems in mathematics. The statement holds true in any dimension and they have numerous consequences.

In this section we will outline the proof of Theorem 19.5 for curves. We follow the proof in of Serre’s Groupes algébriques et corps de classes, with some changes to make the proof as self-contained as possible.

So let $X$ be a smooth projective curve over an algebraically closed field $k$ and let $K$ denote the function field of $X$.

The product $\prod_{x \in X} K$ will play an important role in the following proof. It has a natural structure of a (big!) $k$-vector space. An element $r = (r_x) \in \prod_{x \in X} K$ is a rational function $r_x \in K$ for each closed point $x \in X$. These rational functions can a priori be chosen in a completely arbitrary way, but or course for special elements $r$, the $r_x$ can be linked together more closely. For instance, the function field $K$ is ‘diagonally embedded’ into $\prod_{x \in X} K$: For each $f \in K$ we get the element $r(f)$ by $r_x = f$ for all $x \in X$.

We follow Serre and call an element $r = (r_x) \in \prod_{x \in X} K$ a repartition if $r_x \in \mathcal{O}_{X,x}$ for all but finitely many $x$. Thus $r$ consists of infinitely many rational functions $r_x$, indexed by the closed points in $X$, but all but finitely many of them are regular near $x$ for each $x$. We denote the set of repartitions by $R$. This is clearly a subring of $\prod_{x \in X} K$, as $r_x s_x$ is regular near $x$ if both $r_x$ are $s_x$ are. Also $R$ contains the ‘diagonally embedded’ $K \subseteq \prod_{x \in X} K$ as a subring: If $f \in K$ then $f$ lies in $\mathcal{O}_{X,x}$ for almost all $x$ and hence $r(f)$ is a repartition.

Consider now a divisor $D = \sum n_p p$ on $X$. This defines an $\mathcal{O}_X$-module $\mathcal{O}_X(D) \subseteq \mathcal{K}$ (where $\mathcal{K}$ is the constant sheaf with value $K$ on $X$). This is similar to the definition of $\mathcal{O}_X(D)$, but we treat each point and each rational function $r_x$ separately. The stalk of $\mathcal{O}_X(D)$ at a point $x$ consists of all elements $f \in K$ such that $v_x(f) \geq -v_x(D)$.

Let $R(D) \subseteq R$ denote the additive subgroup of repartitions $r \in R$ so that $v_x(r_x) \geq -v_x(D)$. Equivalently, we pick the $r = (r_x)$ so that each $r_x$ lies in the stalk $\mathcal{O}_X(D)_x \subseteq K$. Then $R(D)$ has the structure of a sub-$k$-vector space of $R$.

If $D \leq D'$ (or in other words, $D' - D$ is effective), then clearly $R(D) \subseteq R(D')$, as $-v_x(D') \leq -v_x(D)$. Note also that $R(0)$ coincides with the repartitions $r = (r_x)$ so that $r_x \in \mathcal{O}_{X,x}$ is regular for every $x \in X$.

The quotient $R/R(D)$ is naturally contained in the product $\prod_{x \in X} R/\mathcal{O}_X(D)_x$, and it can be identified with the direct sum $\bigoplus_{x \in X} R/\mathcal{O}_X(D)_x$ since the elements of $R$ are contained in $\mathcal{O}_{X,x}$ for almost all $x$.

Proposition 19.8. We have a canonical isomorphism

$$H^1(X, \mathcal{O}_X(D)) = R/(R(D) + K)$$
We will from now on define $I(D) := R/(R(D) + K)$.

**Proof.** We start out with the short exact sequence

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{K} \rightarrow \mathcal{S} \rightarrow 0$$

where $\mathcal{S} = \mathcal{K}/\mathcal{O}_X(D)$ is the quotient sheaf. The sheaf $\mathcal{K}$ is a constant sheaf, and hence all of its higher cohomology groups vanish; in particular $H^1(X, \mathcal{K}) = 0$. Taking the long exact sequence in cohomology, we get

$$K \rightarrow H^0(X, \mathcal{S}) \rightarrow H^1(X, \mathcal{O}_X(D)) \rightarrow 0$$

So it remains to identify the global sections of $\mathcal{S}$.

For this, we need a simpler description of $\mathcal{S}$. At a point $x \in X$, we have

$$\mathcal{S}_x = (\mathcal{K}/\mathcal{O}_X(D))_x = \{f \in K | v_x(f) + v_x(D) < 0\}$$

and hence $R/R(D) = \bigoplus_{x \in X} \mathcal{S}_x$.

We would like to show that $\mathcal{S}$ equals the the direct sum its skyscraper sheaves, that is, $\mathcal{S} = \bigoplus \mathcal{S}_x$ and so sections of $\mathcal{S}$ coincides with a selection of elements of $\mathcal{S}_x$ for $x \in X$, almost all of which are zero. Given this, the space of global sections in cohomology, we get

$$K \rightarrow H^0(X, \mathcal{S}) \rightarrow H^1(X, \mathcal{O}_X(D)) \rightarrow 0$$

In any case, we recall that an element of $\mathcal{S}_x$ is represented by an element $(r,U)$, where $r \in \mathcal{S}(U)$ and $U \subseteq X$ is an open set containing $x$. Since $\mathcal{K} \rightarrow \mathcal{S}$ is surjective, we may assume that $r$ is represented by an element of $\mathcal{K}$.

The element $r$ may or not satisfy $v_x(r) < -v_x(D)$ (i.e., $r \notin \mathcal{O}_X(D)_x$): This happens if the class $[r] \in \mathcal{S}_x$ is non-zero. Note that $v_y(r) = 0$ for $y$ in a neighbourhood $V$ of $U - x$: This is because the rational function is regular in a neighbourhood of $x$ (here we use that $X$ is a curve), and we can assume that it is has no zero there. We also have that $v_x(D) = 0$ in a neighbourhood $W$ of $U - x$. Hence $[r]$ vanishes in the stalks $\mathcal{S}_y$ for $y \in V \cap W$.

Now we conclude the proof using the following lemma:

**Lemma 19.9.** Let $\mathcal{F}$ be a sheaf on a quasicompact topological space $X$ so that the closed points are dense in $x$. Suppose that for each point $x \in X$ and each element $s \in \mathcal{F}_x$ there exists an open set $U \ni x$ and a section $\sigma \in \mathcal{F}(U)$ which induces $s$ in $\mathcal{F}_x$ and vanishes on $U - x$. Then

$$\mathcal{F}(X) = \bigoplus_{x \in X} \mathcal{F}_x$$

where the sum is over the closed point of $X$. 

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Proof. We have an injective map

$$\mathcal{F}(X) \to \bigoplus_{x \in X} \mathcal{F}_x$$

which sends a section $\sigma \in F(X)$ to $\sigma_x$ in each stalk. We will show that the image is the whole direct sum of the stalks $\mathcal{F}_x$ (i.e., that a global section of $\mathcal{F}$ is zero for all but a finite number of points $x$, and there are no conditions on the points $x$ and the ‘values’ $s_x$).

Let $\sigma \in \mathcal{F}(X)$, and choose an open set $U_x$ of each point $x \in X$ so that $\sigma|_{U_x-x}$ vanishes. Since $X$ is quasicompact, $X$ can be covered by finitely many of the sets $U_x$, say $U_{x_1}, \ldots, U_{x_r}$. So $\sigma$ is zero outside of the points $x_1, \ldots, x_r$.

Conversely, let $x_1, \ldots, x_r$ be points in $X$ and suppose we are given an element $s_i \in \mathcal{F}_{x_i}$ for each $i = 1, \ldots, r$. Each $s_i$ can be extended to a section $\sigma_i$ of $\mathcal{F}$ in some neighbourhood $U_i$ of $x_i$ which vanishes on $U_i-x_i$. If we let $V_i = U_i-\{x_j\}_{j \neq i}$, then the $V_i$ form an open cover of $X$, which is such that $V_i \cap V_j$ contains none of the points $x_1, \ldots, x_r$. Therefore, $\sigma|_{V_i \cap V_j} = 0 = \sigma_j|_{V_i \cap V_j}$ and the $\sigma_i$ glue to a global section of $\mathcal{F}$ with the desired property.

The take-away from this is that we have an explicit description of $H^1(X, \mathcal{O}_X(D))$ in terms of rational functions. This is already interesting in the case $D = 0$. In this case, $H^1(X, \mathcal{O}_X) = 0$ means that for any collection of rational functions $r_x$ which are allowed to have poles at a finite number of points in $X$, there is a rational function $f \in K$ so that $r_x - f$ is regular for every $x \in X$.

19.1.1 Laurent tails

In this section $A$ will be a discrete valuation ring, which is an algebra over its residue field $k = A/m$, where of course $m$ is the maximal ideal of $A$. Our main example will be when $A = \mathcal{O}_{X,x}$ where $X$ is a curve over an algebraically closed field and $x$ is a closed point, or more generally $A = \Omega_{X,\eta}$ where $\eta$ is a point of codimension 1.

Let $K$ be the fraction field of $A$ and let $v : K \to \mathbb{Z}$ denote the normalized valuation. The field $K$ has a decreasing filtration (infinite in both directions) of $A$-modules $K_i$ given by

$$K_i = \{ f \in K | v(f) \geq i \}$$

Note that $K_0 = A$, and $K_i = m^i$ for $i \geq 0$. Moreover, the quotient $K_i/K_{i+1}$ has the structure of a $k$-vector space of dimension 1: it is spanned by $[t^i]$ where $t$ is a uniformizing parameter of $A$. 286
Lemma 19.10. Any element \( f \in K \) can be written as
\[
f = \sum_{i>0} a_{-i} t^{-i} + g
\]
where \( g \in A \).

Proof. This is obvious if \( f \in A \). More generally, take \( f \in F \) and suppose \( n = v(f) < 0 \). Then \( f \in K_n \) and the class \([f]\) in \( K_n/K_{n+1} \) is a scalar multiple of the generator \([t^n]\), say \([f] = \alpha [t^n] \) where \( \alpha \in k \). In particular, \( f - \alpha t^n \) maps to zero in \( K_n/K_{n+1} \), and hence \( v(f - \alpha t^n) \geq n + 1 \). So by induction on \( |v(f)| \) we can write \( f = \sum_{i>0} \alpha_{-i} t^{-i} + g \) and we are done. \( \square \)

19.1.2 Residues on \( A \)

Let \( t \) be a uniformizing parameter in \( A \), and consider a differential \( \omega = f dt \) which lies in \( M = \Omega_{K|k} \). We define the residue of \( f \) (relative to \( t \)) as the coefficient \( a_{-1} \) in the above decomposition of \( f \). This is denoted by \( \text{Res}_t f \), \( \text{Res}_t df \) or \( \text{Res}_t \omega \).

We have
\[
fdt = a_{-n} t^{-n} dt + \cdots + a_{-2} t^{-2} dt + \text{Res}_t f \cdot t^{-1} dt + g dt
\]
An important point, which we will address below, is that this definition does not depend on the choice of uniformizing parameter \( t \).

Here are some standard properties of residues

Lemma 19.11. (i) \( \text{Res}_t \) is a \( k \)-linear map \( \text{Res}_t : \Omega_{A|k} \rightarrow k \).

(ii) \( \text{Res}_t(fd) = 0 \) if \( f \in A \)

(iii) \( \text{Res}_t(df) = 0 \) for \( f \in K \).

(iv) \( \text{Res}_t(g^{-1} dg) = v(g) \) for all \( g \in K - k \).

Proof. These properties are easy to verify. (i) is clear. (ii) follows since all elements of \( A \) have zero residues by definition. For (iii), we write
\[
f = \sum_{i>0} a_{-i} t^{-i} + g
\]
where \( g \in A \). This gives that \( df = \sum_{i>0} (-i) a_{-i} t^{-i-1} dt + dg \) where \( dg \in \Omega_{A|k} \).

The coefficient of \( t^{-1} \) vanishes.

For (iv), write \( g = t^n \alpha \), where \( n = v(g) \) and \( \alpha \in A^\times \). Differentiation gives that \( dg = nt^{n-1} \alpha dr + t^n \alpha d \alpha \), which gives that
\[
g^{-1} dg = nt^{-1} dt + \alpha^{-1} d \alpha
\]
As $d\alpha \in \Omega_{A|k}$ and $\alpha$ is invertible in $A$, so $\alpha^{-1}d\alpha \in \Omega_{A|k}$ which gives the claim. 

### 19.1.3 Residues on $X$

Let now $X$ be a smooth projective curve over $k$, and let $\Omega_X = \Omega_{X|k}$ denote the sheaf of differentials. For each divisor $D$ we also have the invertible sheaf $\Omega_X(D) = \Omega_X \otimes \mathcal{O}_X(D)$ of differentials with poles at worst order $v_x(D)$ on $D$ for each $x \in X$.

We let $\mathcal{M} = \Omega_{K|k}$ denote the space of meromorphic differentials on $X$. Note that $\mathcal{M} \simeq Kdt$ is a 1-dimensional $K$-vector space. Given an element $\omega \in \mathcal{M}$, and a point $x \in X$, $K$ is the fraction field of $\mathcal{O}_{X,x}$ so we can use the residue map from the previous section. In particular, we get a map

$$\text{Res}_x : \mathcal{M} \to k.$$ 

**Proposition 19.12.** Let $X$ be a smooth projective curve over $k$. Then

(i) (Independence of uniformizer) $\text{Res}_x(\omega)$ does not depend on the choice of $t$.

(ii) (Residue theorem) For any $\omega \in \Omega_{K|k}$, $\text{Res}_x(\omega) = 0$ for all but finitely many $x$. Moreover, $\sum_{x \in X} \text{Res}_x \omega = 0$.

Over the field $k = \mathbb{C}$, these are indeed the same residues you know from complex analysis. Here the theorem is easy to prove: (i) follows since the residue is uniquely determined by

$$\text{Res}_x(\omega) = \int_{\sigma} \omega$$

for a small circle $\sigma$ around $x$. The point (ii) follows just as easily, by the Residue theorem. Indeed, if $D_i$ are small disks around each pole of $\omega$, $\omega$ is holomorphic in $U = X - \bigcup D_i$, and so $\int_{\partial U} \omega = 0$, which means that the residues sum to zero.

One can prove the above properties (i) and (ii) over any field, completely algebraically, but it is a bit more involved (especially in positive characteristic), so we will not do so here.

We now return to the proof of Serre duality. We recall the set of repartitions $R$, which form a vector space over $K$. (If $f \in K$ and $r$ is a repartition, the product $fr$ is given by $(fr)_x = fr_x$: Again this makes sense since a rational function is regular outside a finite number of points.)

**Definition 19.13.** Let $I(D) = R/(R(D) + K)$ and let

$$J(D) = \text{Hom}_k(I(D), k)$$

denote the space of $k$-linear functionals on $I(D)$. 

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We can (and will!) think of the elements of \( J(D) \) as the linear functionals on the huge ring of repartitions \( R \) that vanish on the subspace \( R(D) + K \). So we have an inclusion

\[
J(D) \subseteq \text{Hom}_k(R, k).
\]

If \( D' \geq D \), we saw before that \( R(D) \subseteq R(D') \), which implies that \( J(D') \subseteq J(D) \).

We define \( J = \bigcup_D J(D) \) where the sum is over all divisors on \( D \). This is yet another horribly big vector space over \( k \).

Actually, \( J \) also has the structure of a vector space over \( K \), via the product \( f\alpha(r) = \alpha(fr) \). Note that \( \alpha(fr) \) is a \( k \)-linear functional that vanishes on \( K \). If \( \alpha \) vanishes on \( R(D) \), and \( \text{div} f = D' \), then \( f\alpha \) vanishes on \( R(D - D') \): If \( r \in R(D - D') \), then \( fr \in R(D) \) because

\[
v_x(fr) = v_x(f) + v_x(r) \geq v_x(f) - (v_x(D) + v_x(f)) = -v_x(D)
\]

and hence \( \alpha(fr) = 0 \). Hence \( f\alpha \) belongs to \( J(D - D') \subseteq J \).

**Proposition 19.14.** We have \( \dim_K J \leq 1 \).

**Proof.** Assume the contrary and let \( \alpha, \beta \) be two linearly independent elements of \( J \). Let \( D \) be a divisor, such that \( \alpha \) and \( \beta \) both lie in \( J(D) \). Let \( d = \deg D \).

For each \( n \in \mathbb{Z} \) we pick a divisor \( D_n \) so that \( \deg D_n = n \) (for instance \( D_n = nP \) for a point \( P \) will do). If \( f \in H^0(X, \mathcal{O}_X(D_n)) \), then \( f\alpha \in J(D - D_n) \), by the argument above.

Define the map

\[
H^0(X, \mathcal{O}_X(D_n)) \otimes H^0(X, \mathcal{O}_X(D_n)) \to J(D - D_n)
\]

\[
(f, g) \mapsto f\alpha + g\beta
\]

This is injective since \( \alpha, \beta \) are assumed to be linearly independent over \( K \). In particular,

\[
\dim_k J(D - D_n) \geq 2h^0(X, \mathcal{O}_X(D_n))
\]

We will show that this gives a contradiction by estimating these dimensions using Easy Riemann–Roch.

On the left-hand side we get

\[
\dim_k J(D - D_n) = \dim_k I(D - D_n)
\]

\[
= h^1(X, \mathcal{O}_X(D - D_n))
\]

\[
= h^0(X, \mathcal{O}_X(D - D_n) - (d - n) + O(1)
\]

\[
= n + O(1)
\]
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for \( n \gg 0 \). Here \( O(1) \) means a constant independent of \( n \) and \( D_n \).

On the right-hand side we use the inequality \( h^0(D) \geq \deg D + 1 - g \):

\[
2h^0(X, \mathcal{O}_X(D_n)) \geq 2\deg(D_n) + O(1) = 2n + O(1)
\]

So if \( n \gg 0 \) we get a contradiction as the two sides cannot be equal. Hence \( \alpha, \beta \) cannot be linearly independent, which finishes the proof. \( \square \)

19.1.4 The final part of the proof

The statement of Serre duality is that two vectors are dual to each other. We have to define the pairing explicitly and check that it is indeed perfect\(^1\).

Let \( \omega \in \mathcal{M} \) be a meromorphic differential on \( X \). This defines a canonical divisor

\[
\text{div} (\omega) := \sum_{x \in X} v_x(\omega)x
\]

Note that this means that the invertible sheaf \( \Omega_X(-D) \) is the sheaf of differentials so that \( \text{div} \omega - D \geq 0 \).

**Definition 19.15.** We define the *Serre duality pairing* as follows:

\[
\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathbb{R} \to k
\]

\[
(\omega, r) \mapsto \langle \omega, r \rangle = \sum_{x \in X} \text{Res}_x(r_x \omega)
\]

This has the following properties:

1. \( \langle \omega, r \rangle = 0 \) if \( r \in K \)
   (by the Residue theorem)

2. \( \langle \omega, r \rangle = 0 \) if \( r \in R(D) \) and \( \omega \in \Omega_X(-D)(X) \).
   (The product \( r_x \omega \) cannot have a pole for any \( x \in X \), because the zeroes must at least cancel the poles by the assumptions on \( r \) and \( \omega \).)

3. \( \langle f\omega, r \rangle = \langle \omega, fr \rangle \) if \( f \in K \)
   (Since both pairings evaluate to the same sum of the residues over \( f\omega r \).)

Now, fixing a meromorphic differential \( \omega \in \Omega_X(-D)(X) \), gives a linear functional \( \theta(\omega) \) on \( R \). By properties (1) and (2), this descends to a functional on \( R/(R(D) + K) \). Hence we get a map

\[
\theta : \Omega_X(-D) \to J(D)
\]

\(^1\)Recall that a bilinear map \( M \times N \to A \) is a *perfect pairing* if the induced map \( M \to \text{Hom}_A(N, A) \) is an isomorphism

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(Recall that $J(D)$ is the dual of $R/(R(D) + K)$).

**Lemma 19.16.** Let $\omega \in \mathcal{M}$ such that $\theta(\omega) \in J(D)$. Then $\omega \in \Omega_X(-D)(X)$.

**Proof.** Suppose this is not true, so that $\omega \notin \Omega_X(-D)(X)$. Hence there is a point $x \in X$ so that $\omega$ has a pole in $x$ with a higher order than allowed by $D$: $v_x(\omega) < v_x(-D)$.

Pick the repartition $r \in R(D)$ by defining

$$r_q = \begin{cases} 0 & \text{for } q \neq x \\ t^{-v_x(\omega)-1} & \text{for } q = x \end{cases}$$

Then $v_x(r_x \omega) = -1$, and so

$$\langle \omega, r \rangle = \sum_{q \in X} \text{Res}(r_q \omega) = \text{Res}_x(r \omega) \neq 0$$

This means that $\theta(\omega) \neq 0$ on $R(D)$. However, $\theta(\omega)$ is supposed to vanish on all of $R(D) + K$, so we have a contradiction.

**Theorem 19.17 (Serre duality).** The map $\theta$ induces an isomorphism

$$\theta : H^0(X, \Omega_X(-D)) \rightarrow H^1(X, \mathcal{O}_X(D))^\vee$$

**Proof.** $\theta$ is injective. Let $\omega \in H^0(X, \Omega_X(-D))$ be so that $\theta(\omega) = 0$. Then by the previous lemma, we have that $\omega \in \Omega_X(-D')(X)$ for every divisor $D'$. This implies that $\omega = 0$, since it is impossible to have a non-zero $\omega \in \mathcal{M}$ so that $\text{div} \omega - D' \geq 0$ for every divisor $D'$ (just take $D'$ to have very high degree).

$\theta$ is surjective. $\theta$ is $K$-linear, and we can view it as a map from $\mathcal{M}$ to $J$. By definition, $\dim_K \mathcal{M} = 1$, and we also have $\dim_K J \leq 1$. An injection of finite dimensional vector spaces into a smaller dimensional vector space must be surjective. Hence if $\alpha \in J(D)$, we get a meromorphic differential $\omega \in \mathcal{M}$ so that $\theta(\omega) = \alpha$, and so the previous lemma shows that $\omega \in \Omega(-D)$. 

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Chapter 20

Applications of the
Riemann–Roch theorem

In this chapter we give a few of the (many) consequences of the Riemann–Roch formula. We start by translating the results of Chapter 16 into concrete numerical criteria for a divisor $D$ to be base-point free or very ample. Then we use these results to classify all curves of all genus $\leq 4$.

20.1 Very ampleness criteria

Recall the criterion of Theorem 16.18, that an invertible sheaf $L$ is very ample if and only if its linear system separates points and tangent vectors. Using Riemann–Roch we can translate that result into a very simple, numerical criterion for very ampleness on a curve:

**Theorem 20.1.** Let $X$ be a non-singular projective curve and let $D$ be a divisor on $X$. Then

(i) $|D|$ is base-point free if and only if

$$h^0(D - P) = h^0(D) - 1 \quad \text{for every point } P \in X. \quad (i)$$

(ii) $|D|$ is very ample if and only if

$$h^0(D - P - Q) = h^0(D) - 2 \quad \text{for every two points } P, Q \in X \quad (ii)$$

(including the case $P = Q$)
20.1. Very ampleness criteria

(iii) A divisor $D$ is ample iff $\deg D > 0$

Proof. (i) We take the cohomology of the following exact sequence

$$0 \to \mathcal{O}_X(D - P) \to \mathcal{O}_X(D) \to k(P) \to 0$$

and get

$$0 \to H^0(X, \mathcal{O}_X(D - P)) \to H^0(X, \mathcal{O}_X(D)) \to k$$

From this sequence, we get $h^0(D - 1) \leq h^0(D - P) \leq h^0(D)$.

The right-most map takes a global section of $\mathcal{O}_X(D)$ and evaluates it at $P$. To prove that $|D|$ is base point free, we must prove that there is a section $s \in \mathcal{O}_X(D)$ which does not vanish at $P$, or equivalently, that the map $H^0(X, \mathcal{O}_X(D)) \to k$ is surjective. But this happens if and only if $h^0(D - P) = h^0(D) - 1$.

(ii) If the above inequality is satisfied, we see in particular that $|D|$ is base-point free. So it determines a morphism $\phi : X \to \mathbb{P}^n$. We can use Theorem 16.18 that ensure that $\phi$ is an embedding. That is, $\phi$ separates (a) points and (b) tangent vectors.

For (a), we are assuming that $h^0(D - P - Q) = h^0(D) - 2$, so the divisor $D - P$ is effective and does not have Q as a base point (by (i)). But this means that there is a section of $H^0(X, \mathcal{O}_X(D - P))$ which doesn’t vanish at Q. We have $H^0(X, D - P) \subseteq H^0(X, D)$, so we get a section of $\mathcal{O}_X(D)$ which vanishes at $P$, but not at Q. Hence $|D|$ separates points.

For (b), we need to show that $|D|$ separates tangent vectors, i.e., the elements of $H^0(X, \mathcal{O}_X(D))$ should generate the $k$-vector space $m_p \mathcal{O}_X(D)/m_p^2 \mathcal{O}_X(D)$ at every point $P \in X$. This condition is equivalent to saying that there is a divisor $D' \in |D|$ with multiplicity 1 at $P$; Note that $\dim T_P(X) = 1$, $\dim T_P D' = 0$ if $P$ has multiplicity 1 in $D'$ and $\dim T_P(D') = 1$ if $P$ has higher multiplicity. But this is equivalent to $P$ not being a base-point of $D - P$. Applying (i) again, we see that $h^0(D - 2P) = h^0(D) - 2$, so we are done.

(iii) By definition, $D$ is ample if $mD$ is very ample for $m \gg 0$. So the result follows by the next result, since any divisor of degree $\geq 2g + 1$ is very ample. $\square$

Proposition 20.2. Let $X$ be a non-singular projective curve and let $D$ be a divisor on $X$. Then

1. If $\deg D \geq 2g$, then $|D|$ is base-point free.

2. If $\deg D \geq 2g + 1$, then $|D|$ is very ample.

Proof. By Serre duality, $h^1(D) = h^0(K - D) = 0$ because $\deg D > \deg K = 2g - 2$. Similarly, $h^1(D - P) = 0$.

(i) Applying Riemann–Roch, we find that $h^0(D - p) = h^0(D) - 1$ for any $P \in X$, so we are done by the above theorem.
(ii) In this case we also get $h^1(D - P - Q) = 0$, so Riemann–Roch shows that $h^0(D - P - Q) = h^0(D) - 2$, which is the conclusion we want. 

20.2 Curves on $\mathbb{P}^1 \times \mathbb{P}^1$

Let us consider one central example, namely curves on $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Recall, that $\text{Cl}(Q) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$ where $L_1 = [0 : 1] \times \mathbb{P}^1$ and $L_2 = \mathbb{P}^1 \times [0 : 1]$.

We can use this to prove that $Q$ contains non-singular curves of any genus $g \geq 0$. (This is in contrast with the case of $\mathbb{P}^2$, where only genera of the form $(d-1)^2$ were allowed).

To prove this, consider the divisor $D = aL_1 + bL_2$ where $a, b \geq 1$. $D$ is effective, so let $C \in |D|$ be a generic element.

**Lemma 20.3.** $C$ is non-singular.

To compute the genus of $C$, we use the formula $2g - 2 = \deg \Omega_C$. So we need to find $\Omega_C$ and compute its degree. This is best computed using the Adjunction formula of Proposition 11.43:

\[
\Omega_C = \omega_Q \otimes \mathcal{O}_Q(C)|_X = \mathcal{O}_Q(-2L_1 - 2L_2) \otimes \mathcal{O}(aL_1 + bL_2)|_C = \mathcal{O}_C((a-2)L_1 + (b-2)L_2)
\]

To compute the degree of this, we consider the degrees of $L_1|_C$ and $L_2|_C$ separately. Note that the degree $\deg L_1|_C$ is invariant under linear equivalence, so we can compute the degree of any $[s : t] \times \mathbb{P}^1$ for a general point $[s : t] \times \mathbb{P}^1$. The point is that as a Weil divisor, $L_1|_X$ is obtained by intersecting $[s : t] \times \mathbb{P}^1$ with $X$. When $[s : t] \in \mathbb{P}^1$ is a general point, the intersection $X \cap [s : t] \times \mathbb{P}^1$ is a reduced subscheme of $X$, consisting of $b$ points (as $C \subseteq Q = \mathbb{P}^1 \times \mathbb{P}^1$ is a divisor of type $aL_1 + bL_2$). Hence $\deg L_1|_C = b$ and $\deg L_2|_C = a$. It follows that

\[
2g - 2 = \deg \Omega_C = (a-2)b + (b-2)a = 2ab - 2a - 2b
\]

Solving for $g$ gives us the following theorem:

**Theorem 20.4.** Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Then a generic divisor $C$ in $|aL_1 + bL_2|$ is a smooth projective curve of genus

\[
g = (a-1)(b-1).
\]

In particular, $Q$ contains non-singular curves of any genus $g \geq 0$. 

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20.3 Curves of genus $0$

The results of the previous results are particularly strong when the genus is small. For instance, when $g = 0$, any divisor of positive degree is very ample! We can use this to classify all curves of genus $0$.

**Lemma 20.5.** Let $X$ be a non-singular curve. Then $X \simeq \mathbb{P}^1$ if and only if there exists a Cartier divisor $D$ such that $\deg D = 1$ and $h^0(X, \mathcal{O}_X(D)) \geq 2$. In this case, the divisor $D$ is even very ample.

**Proof.** This follows directly from Proposition 20.2. Still, let us give an alternative proof, which is perhaps a little bit more enlightening.

Let $g \in H^0(X, \mathcal{O}_X(D))$. Then $D' \sim \text{div } g + D \geq 0$, so we may assume that $D$ is effective. Since $\deg D = 1$, we must have $D = p$ for some point $p \in X$. Now take $f \in H^0(X, \mathcal{O}_X(D)) - k$. As above, $f$ induces a morphism $\phi : X \to \mathbb{P}^1$. This morphism has degree equal to 1, so it is birational, and hence $X$ is isomorphic to $\mathbb{P}^1$.

**Proposition 20.6.** A non-singular curve $X$ is isomorphic to $\mathbb{P}^1$ if and only if $\text{Pic}(X) \simeq \mathbb{Z}$.

**Proof.** We have seen that the Picard group of any $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}$.

Conversely, suppose $X$ is a curve with $\text{Pic}(X) \simeq \mathbb{Z}$. Let $p, q$ be two distinct points on $X$. By assumption, $\mathcal{O}_X(p) \simeq \mathcal{O}_X(q)$, so the linear system $|p|$ has at least two sections. Then $X \simeq \mathbb{P}^1_k$ by the previous lemma.

**Theorem 20.7.** Any curve of genus $0$ over an algebraically closed field is isomorphic to $\mathbb{P}^1$.

**Proof.** Let $p \in X$ be a point and consider the divisor $D = p$. If $X$ has genus $0$, then $1 = \deg D > 2g - 2 = -2$, so $H^1(X, \mathcal{O}_X(D)) = 0$. Then Riemann-Roch tells us that

$$\dim H^0(X, \mathcal{O}_X(D)) = 1 + 1 - 0 = 2$$

Hence $X \simeq \mathbb{P}^1_k$ by Lemma 20.5.

We conclude by yet another characterisation of $\mathbb{P}^1$:

**Lemma 20.8.** Let $C$ be a non-singular projective curve and $D$ any divisor of degree $d > 0$. Then

$$\dim |D| \leq \deg D$$

with equality if and only if $C \simeq \mathbb{P}^1$. 296
Proof. This is (IV, Ex. 1.5) in Hartshorne. Although one might guess that this lemma follows from Riemann-Roch this is not the case. Riemann-Roch gives a different sort of relationship between the dimension and degree of a divisor. We induct on $d$.

First suppose $d = 1$. There is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(P) \to k(P) \to 0.$$ 

Now $h^0(\mathcal{O}_C) = 1$ and $h^0(k(P)) = 1$ therefore $h^0(\mathcal{O}_C(P)) \leq 2$ so dim $|P| \leq 1$. If $\dim |P| = 1$ then $|P|$ has no base points so we obtain a morphism $C \to \mathbb{P}^1$ of degree $\deg P = 1$ which must be an isomorphism so $C$ is rational.

Next suppose $D = P_1 + \cdots + P_d$. Let $D' = P_1 + \cdots + P_{d-1}$. There is an exact sequence

$$0 \to \mathcal{O}_C(D') \to \mathcal{O}_C(D) \to k(P_d) \to 0.$$ 

Now $h^0(\mathcal{O}_C(D')) \leq d$ by induction and $h^0(k(P_d)) = 1$ so $h^0(\mathcal{O}_C(D)) \leq d+1$, therefore $\dim |D| \leq d$ with equality iff $h^0(\mathcal{O}_C(D')) = d$. By induction $h^0(\mathcal{O}_C(D')) = d$ iff $C$ is rational.

\[\square\]

### 20.4 Curves of genus 1

A curve in $\mathbb{P}^2_k$ of degree three has genus 1. This follows from our earlier work on the canonical bundle, which showed $\omega_X \simeq \mathcal{O}_{\mathbb{P}^2_k}(d-3)|_X \simeq \mathcal{O}_X$. In this section, we note that in fact every curve of genus 1 arises this way:

**Theorem 20.9.** Any projective curve of genus 1 can be embedded as a plane cubic curve in $\mathbb{P}^2_k$.

**Proof.** Pick a point $P \in X$ and consider the divisor $D = 3P$. $D$ has degree $3 \geq 2g + 1$, so it is very ample. Let $\phi : X \to \mathbb{P}^2_k$ denote the embedding. The image $\phi(X)$ is a smooth curve of degree equal to $\deg \phi^*\mathcal{O}_{\mathbb{P}^2}(1) = \deg D = 3$. \[\square\]

In contrast to the $g = 0$ case, there is however many non-isomorphic genus 1 curves. For instance, in the *Ledengre family* of curves in $X_\lambda \subseteq \mathbb{P}^2$ given by

$$y^2z = x(x - z)(x - \lambda z)$$

where $\lambda \in k$, each $X_\lambda$ is isomorphic to at most a finite number of other $X_\lambda$’s.

#### 20.4.1 Divisors on $X$

Let us study the divisors on $X$. To make the discussion a bit more concrete, let $X \subseteq \mathbb{P}^2$ be the curve given by $y^2z = x^3 - xz^2$. We claim that there is an exact
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sequence

\[ 0 \to X(k) \to \text{Cl}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0 \]

This means that the Class group \( \text{Cl}(X) \) is very big – its elements are in bijection with the \( k \)-points of \( X \), of which there might be uncountably many. In particular \( X \) cannot be isomorphic to \( \mathbb{P}^1 \).

If \( L \subseteq \mathbb{P}^2 \) is a line, we get a divisor \( L|_X \). That is, we take a section \( s \in \mathcal{O}_{\mathbb{P}^2}(1) \) defining \( L \) and restrict it to \( X \). The divisor of \( s \in \mathcal{O}_{X}(1) \) consists of three points \( P, Q, R \) (counted with multiplicity). In particular, since any two lines are linearly equivalent on \( \mathbb{P}^2 \), we get for every pair of lines \( L, L' \) and corresponding triples \( P, Q, R \), a relation

\[ P + Q + R \sim P' + Q' + R' \]

(where \( \sim \) denotes linear equivalence).

Let us consider the point \( O = [0,1,0] \) on \( X \). This is a special point on \( X \): it is an inflection point, in the sense that there is a line \( L = V(z) \subseteq \mathbb{P}^2 \) which has multiplicity three at \( O \), so that \( L \) restricts to \( 3O \) on \( X \). This has the property that any three collinear points \( P, Q, R \) in \( X \) satisfy

\[ P + Q + R \sim 3O \]

We will use these observations to define a group structure on the set of closed points \( X(k) \), using the point \( O \) as the identity. The group structure will be induced from that in \( \text{Cl}(X) \).

Consider the subgroup \( \text{Cl}^0(X) \subseteq \text{Cl}(X) \) consisting of degree 0. This fits into the exact sequence

\[ 0 \to \text{Cl}^0(X) \to \text{Cl}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0 \]

We now define a map

\[ \xi : X(k) \to \text{Cl}^0(X) \]

\[ P \mapsto [P - O] \]

**Lemma 20.10.** \( \xi \) is a bijection.

**Proof.** \( \xi \) is injective: \( \xi(P) = \xi(Q) \) implies that \( P \sim Q \). Then \( P = Q \) (otherwise \( X \) would be rational, by Proposition 20.6). (Alternatively, it follows because \( h^0(X, \mathcal{O}_X(P)) = 1 \).

\( \xi \) is surjective: Take a divisor \( D = \sum n_i P_i \in \text{Div}(X) \) of degree 0. Then \( D' = D + O \) has degree 1, so by Riemann–Roch, \( H^0(X, \mathcal{O}_C(D')) \) is 1-dimensional. Hence there exists an effective divisor of degree 1 in \( |D'| \), which must then be of the form \( D' = Q \). But that means that \( D + O \sim Q \), or, \( D \sim Q - O \), as

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desired.

Using this bijection, we can put a group structure on the set $X(k)$:

**Theorem 20.11.** The set of $k$-points $X(k)$ on a genus 1 form a group.

The group law has the following famous geometric interpretation. Given two points $p_1, p_2 \in X$, we draw the line $L$ they span (see the figure below). This intersects $X$ in one more point, say $p_3$. In the group $\text{Cl}^0(X)$ we have

\[ p_1 + p_2 + p_3 = 3O \]

To define the ‘sum’ $p_1 + p_2$ (which should be a new $k$-point of $X$), we then look for a point $p_4$ so that

\[ p_4 - O = (p_1 - O) + (p_2 - O) \]

or in other words, $p_4 + O = p_1 + p_2$. By the above, this becomes $p_4 + O = 3O - p_3$ or, $p_3 + p_4 + O = 3O$. This tells us that we should define $p_4$ as follows: We draw the line $L'$ from $O$ to $p_3$ (shown as the dotted line in the figure), and define $p_4$ to be the third intersection point of $L'$ with $X$. By construction, we get $(p_1 - O) + (p_2 - O) = (p_4 - O)$ in $\text{Cl}^0(X)$.

Given the equation of $X$ in $\mathbb{P}^2$, and coordinates for the points $p_1, p_2$, we can of course write down explicit formulas for the coordinates of $p_4$, and they are rational functions in the coordinates of $p_1, p_2$. This is almost enough to justify that $X$ is a *group variety*, i.e., it is an algebraic variety equipped with morphisms $m : X \times X \to X$ satisfying the usual group axioms.

### 20.5 Curves of genus 2

Let $X$ be a non-singular projective curve of genus 2. We saw one example of such a curve earlier in this chapter, namely the curve obtained by gluing two
copies of the affine curve $y^2 = p(x)$ where $p(x)$ is a polynomial of degree five. The condition that $X$ is non-singular implies that $p$ has distinct roots.

We already saw in Chapter XX that a genus 2 curve cannot be embedded in the projective plane $\mathbb{P}^2_k$ (since 2 is not a triagonal number). However, we show the following:

**Theorem 20.12.** Any curve of genus 2 is isomorphic to a hyperelliptic curve

Here, a curve $C$ is said to be hyperelliptic if there is a base point free linear system of degree 2 and dimension 1. Equivalently, there exists points $P, Q \in X$ so that the invertible sheaf $L = \mathcal{O}_X(P + Q)$ is globally generated and by two global sections.

It is classical notation that a base point free linear system of degree $d$ and dimension $r$ is called a $g^r_d$. So to say that a curve is hyperelliptic is to say that it has a $g^1_2$.

**Example 20.13.** If $g = 0$, then $X \cong \mathbb{P}^1$. Let $D = 2P$, then $H^0(D) = kx_0^2 + kx_0x_1 + x_1^2$, so $|D| \cong \mathbb{P}^2$ is identified with the space of quadratic polynomials up to scaling. If we take two quadratic polynomials $q_0, q_1$ with no common zeroes, we get a base-point free linear system $g^1_2 \subseteq |D|$.

**Example 20.14.** If $g = 1$ any divisor of degree 2 gives a $g^1_2$ by Riemann-Roch. Indeed, if $D$ has degree 2 then

$$h^0(D) - h^0(K - D) = 2 + 1 - g = 2$$

and $\deg(K - D) = -2$ so $h^0(D) = 2$ and hence $\dim |D| = 1$. This $D$ is base-point free, since $D - p$ has degree 1, and hence since $X$ is not rational, $h^0(D - p) = 1 = h^0(D) - 1$.

**Example 20.15.** Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth divisor of bidegree $(2, g + 1)$. Then $K_X \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, g - 1)$ and $X$ has genus $g$. Moreover, the projection $p_2 : X \to \mathbb{P}^1$ is finite of degree 2, which shows that $X$ is hyperelliptic.

The projections $p_1, p_2 : Q \to \mathbb{P}^1$ give rise to a degree 2 and a degree $g + 1$ morphism of $X$ to $\mathbb{P}^1$. Thus there exists a 2:1 morphism $f : X \to \mathbb{P}^1$. $f$ corresponds to a base point free linear system on $X$ of degree 2 and dimension 1. Thus $X$ is hyperelliptic.

In this example, $\Omega_X = \mathcal{O}_Q(X) \otimes \omega_Q|_X = \mathcal{O}_Q(2, g + 1) \otimes \mathcal{O}_Q(-2, -2) = \mathcal{O}_X(0, g - 1)$. The latter invertible sheaf has $g$ independent global sections so $X$ has genus $g$. Moreover $K_X$ is base point free, but not very ample, since the corresponding morphism $X \to \mathbb{P}^2$ is not an embedding (it maps $X$ onto a conic).

To prove the theorem, we must produce a degree two map $\phi : X \to \mathbb{P}^1$. We have a natural candidate: the canonical divisor $K_X$, which has degree $2g - 2 = 2$. We claim that $K_X$ is base point free.
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Note that we cannot apply Proposition 20.2 directly to prove this, since the degree is too small. However, we can use Riemann–Roch to check directly that the conditions in Theorem 20.1 apply. That is, we need to show that for every point $P \in X$, we have

$$h^0(X, K_X - P) = h^0(X, K_X) - 1 = 2 - 1 = 1$$

Applying Riemann–Roch to the divisor $D = P$, we also get $h^0(P) - h^0(K_X - P) = 1 + 1 - 2 = 0$. As $P$ is effective, and $X$ is not rational, we have $h^0(P) = 1$, and so also $h^0(X, K_X - P) = 1$, as we want.

20.6 Curves of genus 3

The case of curves of genus 3 is especially interesting. We have seen two examples of curves of genus 3 so far:

Example 20.16. A plane curve $X \subseteq \mathbb{P}^2$ of degree $d = 4$ has genus $\frac{1}{2}(d - 1)(d - 2) = 3$.

Notice that

$$\Omega_X = \mathcal{O}_{\mathbb{P}^2}(d - 3)|_X = \mathcal{O}_X(1)$$

so $\Omega_X$ is very ample, since it is the restriction of the very ample invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$. Hence $K_X$ is very ample, and the corresponding morphism is exactly the given embedding $X \hookrightarrow \mathbb{P}^2$.

Example 20.17. The curves in Section 18.5 on page 276 can be chosen to have genus $g = 3$. In this case, $X$ admits a 2:1 map to $\mathbb{P}^1$.

Example 20.18. A curve $X$ on the quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ of type $(2, 4)$ is hyperelliptic. It is a curve of degree 6 and genus 3.

Thus these examples are a bit different. The curves in the first example are ‘canonical’ whereas the curves in the other two are ‘hyperelliptic’. We show that this distinction is a general phenomenon for curves of genus three:

Proposition 20.19. Let $X$ be a curve of genus 3. Then there are two possibilities:

(i) $K_X$ is very ample. Then $X$ embeds as a plane curve of degree 4.

(ii) $K_X$ is not very ample. Then $X$ is a hyperelliptic curve, and it embeds as a $(2, 4)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $K_X \sim 2F$, where $F = L_1|_X$.

We will deduce this from a more general result:
Theorem 20.20. Let $X$ be a curve of genus $\geq 2$. Then $K$ is very ample if and only if $X$ is not hyperelliptic.

Proof. $K$ is very ample if and only if $h^0(K - P - Q) = h^0(K) - 2 = g - 2$ for every $P, Q \in X$. By Riemann–Roch, we compute

$$h^0(P + Q) - h^0(K - P - Q) = 2 + 1 - g = 3 - g$$

Hence $K$ is very ample if and only if $h^0(P + Q) = 1$ for every $P, Q$.

If $X$ is hyperelliptic, then there is a map $\phi : X \to \mathbb{P}^1$, so that $\phi^*([1 : 0]) = P + Q$ for some points $P, Q \in X$ (possibly equal). Here the linear system $|P + Q|$ is 1-dimensional, so $h^0(X, P + Q) = 2$, and hence $K_X$ is not very ample.

If $X$ is not hyperelliptic, we have $h^0(X, P + Q) = 1$ for any $P, Q$ (otherwise it is $\geq 2$, and $P + Q$ induces a map $X \to \mathbb{P}^1$ of degree two), and hence $K_X$ is very ample.

We still need to check the last part of the above theorem, namely that every hyperelliptic curve arises as a curve of type $(2,4)$ on $Q \subseteq \mathbb{P}^3$.

We proceed as follows. Let $D = P_1 + \cdots + P_4$ denote a generic degree 4 divisor on $X$ (so $P_1, \ldots, P_4$ are general points of $X$). By Riemann–Roch, we get

$$h^0(D) - h^0(K - D) = 4 + 1 - 3 = 2$$

We claim that $h^0(K - D) = 0$, so that $h^0(D) = 2$. Note that $K - D$ has degree $2g - 2 - 4 = 0$, so $K - D$ is a divisor of degree 0. This is effective if and only if $K \sim D$. However, there is a 4-dimensional family of divisors of the form $P_1 + \cdots + P_4$, whereas the space of effective canonical divisors has dimension $\dim |K| = 2$. Hence if the points $P_i$ are chosen generically, $K - D$ will not be effective, and hence the claim holds.

From this, we obtain a linear system $|D|$ of dimension 1. We claim that $D$ is base point free. We need to show that

$$h^0(D - P) = h^0(D) - 1 = \text{deg } D + 1 - 3) - 1 = 1$$

for every point $P$. Suppose not, and let $P$ be a base point of $D$. Since $D = P_1 + P_2 + P_3 + P_4$ we may suppose that $P = P_4$.

By Riemann–Roch, we are done if we can show $h^0(K - D + P) = 0$. However, $K - D + P = K - P_1 - P_2 - P_3$. There is a 3-dimensional space of effective divisors of the form $P_1 + P_2 + P_3$ for points $P_i \in X$, but only a 2-dimensional linear system of effective canonical divisors $|K|$. Hence $K - D + P$ is not effective.

We therefore have two morphisms from our hyperelliptic curve $X$; $f : X \to \mathbb{P}^1$ (induced by the $g^2_2$) and $g : X \to \mathbb{P}^1$ (induced by $D$). By the universal property
of the fiber product, this gives a morphism
\[ \phi = (f \times g) : X \to \mathbb{P}^1 \times_k \mathbb{P}^1 \]

We claim that this is a closed immersion. Let \( F = P + Q \in |g_2^1| \). The map \( D + F \) induces the map \( F : X \to \mathbb{P}^3 \), which coincides with \( j \circ \phi \) where \( j : \mathbb{P}^1 \times \mathbb{P}^1 \) is the Segre embedding. To prove the claim, it suffices to show that \( F \) is an embedding, or equivalently that \( D + F \) is very ample.

First claim that \( K \sim 2F \). Since both of these divisors have degree 4 it suffices to show that \( K - 2F \) is effective. Note that in any case \( h^0(X, 2F) \geq 3 \), since if \( H^0(X, F) = \langle x, y \rangle \), then \( x^2, xy, y^2 \) are linearly independent in \( H^0(X, 2F) \) (understand why!). Now applying Riemann–Roch to \( D = 2F \), we get
\[ h^0(2F) - h^0(K - 2F) = 4 + 1 - 3 = 2 \]
so \( h^0(K - 2F) \geq 1 \), and \( K \sim 2F \) as we want.

Now, to show that \( D + F \) is very ample, we need to show that
\[ h^0(X, D + F - p - q) = h^0(D + F) - 2 \]
for any pair of points \( p, q \in X \). By Riemann–Roch again, we can conclude if we know that \( h^0(K - D - F + p + q) = 0 \). But since \( K \sim 2F \), we have
\[ K - D - F + p + q \sim F - D + p + q \]
These are divisors of degree 0, so if this is effective, we must have \( D \sim F + p + q \). However, the space of effective divisors of the form \( D' + p + q \) with \( D' \sim F \) is 3-dimensional (since \( |F| \) has dimension 1, and \( p \) and \( q \) can be chosen freely on \( X \)). On the other hand, as we have seen, the space of divisors of the form \( D = P_1 + \cdots + P_4 \) is of dimension 4, so choosing \( D \) generically means that this \( F - D + p + q \) is not effective. It follows that \( h^1(D - p - q) = h^0(D + F) - 2 \) and hence \( D \) is very ample.

\[ \square \]

### 20.7 Curves of Genus 4

Recall that curves of genus \( g \geq 2 \) split up into two disjoint classes.

(I) Hyperelliptic curves: \( X \) admits a 2:1 to \( \mathbb{P}^1 \)

(II) Canonical curves: \( K_X \) is very ample

Here’s an example of a genus 4 curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \):

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Example 20.21. Consider a type \((2, 5)\) curve \(C\) on \(Q \subseteq \mathbb{P}^3\). Then \(C\) has degree \(7 = 2 + 5\) and \(C\) is hyperelliptic (because of the degree 2 map coming from projection onto the first fact \(p_1 : Q \to \mathbb{P}^1\)). A type \((3, 3)\) curve on \(Q\) is also of genus 4. It is a degree 6 complete intersection of \(Q\) and a cubic surface. Curves of type \((3, 3)\) have at least two \(g^1_3\)'s.

In fact, using the same strategy as for \(g = 3\), one can show that any hyperelliptic curve of genus 4 arises this way.

### 20.7.1 Classifying curves of genus 4

We start with an abstract curve \(X\) of genus 4. We may assume that \(X\) is not hyperelliptic (since in that case it embeds as a \((2, 5)\)-divisor on \(\mathbb{P}^1 \times \mathbb{P}^1\)). So we assume that \(K_X\) is very ample. Therefore we have the canonical embedding 

\[ X \hookrightarrow \mathbb{P}^{g-1} = \mathbb{P}^3. \]

The degree of the embedded curve is \(\deg \omega_X = 2g - 2 = 6\). Thus we can view \(X\) as a degree 6 genus 4 curve in \(\mathbb{P}^3\).

What are the equations of \(X\) in \(\mathbb{P}^3\)? To answer this question we use a very powerful technique in curve theory, namely we combine Riemann–Roch with the sheaf cohomology on \(\mathbb{P}^n\). Twisting the ideal sheaf sequence of \(X\) by \(\mathcal{O}_{\mathbb{P}^3}(2)\) and taking cohomology gives the exact sequence

\[ 0 \to H^0(\mathbb{P}^3, I_X(2)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(X, \mathcal{O}_X(2)) \to \cdots \]

Note that \(\mathcal{O}_{\mathbb{P}^3}(1)|_X = K_X\). Applying Riemann-Roch states to the divisor 
\(D = 2K_X\), we get

\[ h^0(\mathcal{O}_X(2)) = \deg 2K_X + 1 - g + h^1(\mathcal{O}_X(D)) = 12 + 1 - 4 + 0 = 9. \]

(Note that \(h^1(\mathcal{O}_X(2)) = h^0(K_X - 2K_X) = h^0(-K_X) = 0\) since \(K_X\) is effective). Since \(h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10\) it follows that the map \(H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2))\) must have a nontrivial kernel so \(h^0(\mathbb{P}^3, I_X(2)) > 0\).

The upshot of this is that we now know that \(X\) lies in some surface of degree 2. Since \(X\) is integral, this surface cannot be a union of hyperplanes. So \(X\) lies on either a singular quadric cone \(Q_0 = V(xy - z^2)\) or the nonsingular quadric surface \(Q = V(xy - zw)\).

If \(C\) lies on \(Q\) then it must have a type \((a, b)\) which must satisfy \(a + b = 6\) and \((a - 1)(b - 1) = 4\). The only solution is \(a = b = 3\). Since \(\mathcal{O}_Q(3, 3) \cong \mathcal{O}_{\mathbb{P}^3}(3)|_Q\), this implies that \(C\) is the restriction of a divisor on \(\mathbb{P}^3\), that is, \(C = Q \cap S\) for a degree 3 surface \(S\).

The other possibility is that \(C\) lies on \(Q_0\). Computing as before, we obtain

\[ 0 \to H^0(\mathcal{O}_X(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_X(3)) \to \cdots \]
As before one sees that $h^0(\mathcal{O}_X(3)) = 15$ and $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$. Thus $h^0(\mathcal{O}_C(3)) \geq 5$. Let $q \in H^0(\mathcal{O}_C(2))$ be the defining equation of $Q_0$. Then $xq, yq, zq, wq \in H^0(\mathcal{O}_C(3))$. But $h^0(\mathcal{O}_C(3)) \geq 5$ so there exists an $f \in H^0(\mathcal{O}_C(3))$ so that the global sections $xq, yq, zq, wq, f$ are independent. Thus there is an $f$ not in $(q)$. Since $f \notin (q)$ we see that $S = Z(f) \not\subseteq Q$ so $C' = S \cap Q$ is a degree 6 not necessarily nonsingular or irreducible curve. Since $C \subseteq S$ and $C \subseteq Q$ it follows that $C \subseteq C''$. Since these are both integral curves of the same degree $\deg C = 6 = \deg C'$, we must have $C = C'$. Thus in the case that $C$ lies on $Q_0$ we see that $C$ is also a complete intersection $C = Q_0 \cap S$ for some cubic surface $S$.

This proves the following theorem:

**Theorem 20.22.** Let $X$ be a non-singular curve of genus 4. Then either

(i) $X$ is hyperelliptic (in which case $X$ embeds as a $(2,5)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1$); or

(ii) $X = Q \cap S$ is the intersection of a quadric surface and a cubic surface in $\mathbb{P}^3$. 

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