MANDATORY ASSIGNMENT MAT4215 – SPRING 2022

The assignment must be submitted electronically by **14:30, Thursday May 12th**.
You need at least 50% to pass. If you have any questions or comments about the problems, feel free to email me at johnco@math.uio.no.
Feel free to use whatever results from Commutative Algebra you want.

Problem 1. [10%]
Give an example of a scheme X, a field K and a morphism of ringed spaces $\operatorname{Spec} K \to X$ which is *not* a morphism of schemes.

Problem 2. [10%] For each of the following rings A, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type:

(1) $\mathbb{Z}[i]$
(2) $\mathbb{Z}[\frac{1}{p}]$
(3) $\mathbb{Z}_{(p)}$
(4) $\mathbb{Z} \times \mathbb{Z}$
(5) $\mathbb{Z}[x]$

Problem 3. [25%]
Describe the scheme theoretic fibers in all points of the following morphisms.

(1) $f : \operatorname{Spec} \mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec} \mathbb{C}[x]$
(2) $f : \operatorname{Spec} \mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec} \mathbb{C}[x]$
(3) $f : \operatorname{Spec} \mathbb{C}[x,y]/(xy) \to \operatorname{Spec} \mathbb{C}[x]$
(4) $f : \operatorname{Spec} \mathbb{Z}[x,y]/(xy^2-m) \to \operatorname{Spec} \mathbb{Z}$, where m is a non-zero integer.

Which fibers are irreducible? Which are reduced?

Problem 4. [20%]
Show the following:
(i) The skyscraper sheaf of k on $\mathbb{A}^1_k$ at the origin 0 is quasi-coherent.
(ii) The skyscraper sheaf of $k(T)$ on $\mathbb{A}^1_k$ at the origin 0 is *not* quasi-coherent.
(iii) If X is variety over an algebraically closed field k, then the skyscraper sheaf of k at any closed point $x \in X$ is quasi-coherent.

Problem 5. [35%]
Let $f : X \to Y$ be a morphism of schemes and let $x \in X$ be a point. We say that:

- A quasi-coherent sheaf $\mathcal{F}$ on X is *flat over* Y at x if $\mathcal{F}_x$ is flat as a $\mathscr{O}_{Y,f(x)}$-module (where $\mathcal{F}_x$ is considered as a $\mathscr{O}_{Y,f(x)}$-module via the natural map $f_x^{\#} : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,f(x)}$).
- $\mathcal{F}$ is *flat* if it is flat at every point in X.
- f is flat if $\mathscr{O}_X$ is flat over Y

(i) Show that open embeddings are flat. What about closed immersions?
(ii) Show that a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if the map of rings $A \to B$ is flat. More generally, a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Spec} B$ is flat over $\operatorname{Spec} A$ if and only if M is flat as an A-module.
(iii) Which of the morphisms in Problem 2 are flat?
(iv)* Prove that the blow-up morphism $\pi : Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ is not flat.

Date: April 13, 2022.