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Weekly Exercises

1.1 Week 1

Exercise 1.1.1. Let $\{A_i\}_{i\in I}$ be an infinite sequence of non-trivial rings, and let X be the disjoint union of the spectra $\operatorname{Spec} A_i$. Show that X is not homeomorphic to a spectrum of a ring.

Exercise 1.1.2. Consider the ring map

$$\phi\colon \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y,z]/(xz-y)$$

which induces $f\colon \operatorname{Spec}\mathbb{C}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbb{C}}$. Show that the map f on the level of closed points sends (a, ab, b) to (a, ab), and the generic point to the generic point. Show that in this example, the image is neither open nor closed: it equals $D(x) \cup V(x,y)$.

Exercise 1.1.3. Show that $\operatorname{Spec}\overline{\mathbb{Q}}[x]$ is homeomorphic to $\operatorname{Spec}\mathbb{Z}$.

Exercise 1.1.4. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a map of spectra induced by a ring map $\phi\colon A \to B$.

a) Show that if ϕ is injective, then the image of f is dense in $\operatorname{Spec} A$.
b) More generally, show that $f(\operatorname{Spec} B)$ is dense if and only if $\operatorname{Ker}\phi \subset \sqrt{(0)}$.
c) Define $\phi : \mathbb{Q}[x] \to \mathbb{C}$ by sending x to π. Show that ϕ is injective. Describe the morphism $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{A}^1_{\mathbb{Q}}$.

Exercise 1.1.5. Let $A = \mathbb{R}[x,y]/(x^2+y^2+1)$.

a) Show that for each $a, b, c \in \mathbb{R}$ with $(a,b) \neq (0,0)$, the ideal $\mathfrak{m} = (ax+by+c)$ is a maximal ideal of A.
b) Show that every non-zero prime ideal is of this form. Use this to describe $\operatorname{Spec} A$. HINT: $A/\mathfrak{m}$ is a finite extension of $\mathbb{R}$. Show that $1, x, x^2$ are linearly dependent, and likewise $1, y, y^2$.

Exercise 1.1.6 (The sheaf of homomorphisms)**.** Given two presheaves $\mathcal{F}$ and $\mathcal{G}$, we may form a presheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$ by letting the sections over an open U be given by

$$\mathcal{H}om(\mathcal{F},\mathcal{G})(U) = \operatorname{Hom}_{\mathsf{Ab}(X)}(\mathcal{F}|_U, \mathcal{G}|_U). \tag{1.1}$$

If $V \subset U$, the restriction map sends $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ to the restriction $\phi|_V\colon \mathcal{F}|_V \to \mathcal{G}|_V$. Show that $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is a sheaf whenever $\mathcal{G}$ is a sheaf.

Exercise 1.1.7 (The restriction of a sheaf)**.** Let $U \subset X$ be an open subset of a topological space X and let $\mathcal{F}$ be a sheaf on X. Define $\mathcal{F}|_U$ by the assignment $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subset U$.

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a) Show that $\mathcal{F}|_U$ defines a sheaf on U, and that the stalks at points in U are equal to the corresponding stalks of $\mathcal{F}$.
b) Let $\mathcal{G}$ be a sheaf on U. Let $\iota\colon U \to X$ be the inclusion map. Show that $\iota_*(\mathcal{G}|_U) \simeq \mathcal{G}$.

Exercise 1.1.8. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. We say that ϕ is *injective* (resp. surjective) if $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective) for every point $x \in X$.

a) Show that ϕ is injective if and only if $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U.
b) Give an example of a map of sheaves ϕ so that ϕ_x is surjective for all $x \in X$, but $\phi_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective. HINT: Example **??**.

1.2 Week 2

Exercise 1.2.1. Let $A = k[x,y,z]/(xyz)$ and $X = \operatorname{Spec} A$. Compute $\mathcal{O}_{X,p}$ where x corresponds to the prime ideal $\mathfrak{p} = (x-1, y, z)$. Show that $yz \neq 0$ in $\mathcal{O}_{X,p}$, but takes the value 0 for all points in a neighbourhood of p.

Exercise 1.2.2. Show that if $f\colon X \to Y$ is a morphism of locally ringed spaces, the stalk maps $f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induce maps between the residue fields $\kappa(f(x))$ and $\kappa(x)$. What happens when X and Y are affine varieties?

Exercise 1.2.3. Is $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ a closed embedding?

Exercise 1.2.4. Let X be a scheme and let $x \in X$ be a point.

a) Show that there is a canonical morphism
$$f\colon \operatorname{Spec}\mathcal{O}_{X,x} \longrightarrow X$$
b) Show that f induces a homeomorphism between $\operatorname{Spec}(\mathcal{O}_{X,x})$ and the subset $W \subset X$ of points $w \in X$ such that $w \in \overline{x}$.
c) Show that the map $\iota_x\colon \operatorname{Spec}\kappa(x) \to X$ defined in the text factors via f.
d) Show that on the level of topological spaces, the image of f is the intersection of all open neighbourhoods containing x.
e) Compute the image of f when:
   (i) x is the generic point of an irreducible scheme.
   (ii) x is a closed point of $\mathbb{A}^2_{\mathbb{C}}$.

Exercise 1.2.5. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. Prove that if both f and g are closed embeddings, then $g \circ f$ is one as well.

Exercise 1.2.6 (The sheaf of units)**.** Let X be a scheme with structure sheaf $\mathcal{O}_X$. We say that $s \in \mathcal{O}_X(U)$ is a *unit* if there exists a multiplicative inverse $s^{-1} \in \mathcal{O}_X(U)$.

a) Show that $s \in \mathcal{O}_X(U)$ is a unit if and only if for all $x \in U$, the germ s_x is a unit in the ring $\mathcal{O}_{X,x}$; that is, if and only if s_x does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.
b) We let $\mathcal{O}_X^\times(U)$ denote the subgroup of units in $\mathcal{O}_X(U)$. Show that $\mathcal{O}_X^\times(U)$ is a sub*sheaf* of $\mathcal{O}_X$.

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Exercise 1.2.7 (The Frobenius morphism)**.** Let p be a prime number and let A be a ring of characteristic p. The ring map $F_A : A \to A$ given by $a \mapsto a^p$ is called the *Frobenius map* on A.

a) Show that F_A induces the identity map on $\operatorname{Spec} A$.
b) Show that if A is local, then F_A is a map of local rings.
c) For a scheme X over $\mathbb{F}_p$, define the *Frobenius morphism* $F_X \colon X \to X$ by the identity on the underlying topological space and with $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$. Show that F_X is a morphism of schemes.
d) Show that F_X is natural in the sense that if $f : X \to Y$ is a morphism of schemes over $\mathbb{F}_p$, we have $f \circ F_X = F_Y \circ f$.

In particular, this exercise shows that for a morphism of schemes $f \colon X \to Y$, in order to check that f is an isomorphism, is not enough to check that f is a homeomorphism; also the map $f^\sharp$ must be an isomorphism.

Exercise 1.2.8. Let $X = \operatorname{Spec} k[x, y, z, w]/(xw - yz)$ and consider the open set $U = X - V(x, y)$. Use the above strategy as in Example **??** to compute $\mathcal{O}_X(U)$. Conclude that U is not affine.

Exercise 1.2.9. Let X be a scheme so that the underlying topological space is finite and discrete. Show that X is an affine scheme. HINT: Consider the case where X has one point first.

1.3 Week 3

Exercise 1.3.1 (The sheaf of units)**.** Let X be a scheme with structure sheaf $\mathcal{O}_X$. We say that $s \in \mathcal{O}_X(U)$ is a *unit* if there exists a multiplicative inverse $s^{-1} \in \mathcal{O}_X(U)$.

a) Show that $s \in \mathcal{O}_X(U)$ is a unit if and only if for all $x \in U$, the germ s_x is a unit in the ring $\mathcal{O}_{X,x}$; that is, if and only if s_x does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.
b) We let $\mathcal{O}_X^\times(U)$ denote the subgroup of units in $\mathcal{O}_X(U)$. Show that $\mathcal{O}_X^\times(U)$ is a sub*sheaf* of $\mathcal{O}_X$.

Exercise 1.3.2. Describe the following schemes and the structure sheaf on them.

a) $\operatorname{Spec} \mathbb{C}[t]/(t^2 + 1)$
b) $\operatorname{Spec} \mathbb{R}[t]/(t^3 - t^2)$
c) $\operatorname{Spec} \mathbb{F}_3[t]/(t^3 - 1)$.

Exercise 1.3.3. Let X be a scheme whose underlying topological space is a finite and discrete set. Show that X is affine.

Exercise 1.3.4. Show that $\operatorname{Proj} k[x_0, x_1]/(x_0^2, x_1^2) = \varnothing$.

1.4 Week 4

Exercise 1.4.1. Describe $X = \operatorname{Spec} \mathbb{Z}[x]/(5x - 15)$. Is X irreducible? Reduced? What are the fibers of the canonical map $X \to \operatorname{Spec} \mathbb{Z}$?

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Exercise 1.4.2. a) Find an open subset of an affine scheme which is not quasi-compact.
b) Let U, V be quasi-compact open subsets of $\operatorname{Spec} A$ for some ring A. Show that $U \cap V$ is also quasi-compact.

Exercise 1.4.3. For each of the following rings A, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type.

$$\mathbb{Z}[i],\ \mathbb{Z}[1/p],\ \mathbb{Z}_{(p)},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}[x].$$

Exercise 1.4.4. Consider the map

$$\begin{aligned} \phi : \mathbb{A}^1_k &\longrightarrow \mathbb{P}^1_k \\ u &\mapsto (u : 1 - u^2). \end{aligned}$$

Show that ϕ is finite and surjective.

1.5 Week 5

Exercise 1.5.1. Let p and q be different prime numbers. Show the following identities:
a) $\operatorname{Spec} \mathbb{F}_p \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_q = \varnothing$.
b) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)} = \operatorname{Spec} \mathbb{Z}_{(p)}$.
c) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(q)} = \operatorname{Spec} \mathbb{Q}$.

Exercise 1.5.2. Let $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2)$.
a) Show that X is irreducible.
b) Show that $X \times_{\mathbb{R}} \mathbb{C}$ is not irreducible, and describe the irreducible components.
c) Compute $X(\mathbb{R})$ and $X(\mathbb{C})$.

Exercise 1.5.3. a) Show that the base change of a surjective morphism is surjective.
b) Show that the base change of an injective morphism need not be injective.
HINT: Consider $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

Exercise 1.5.4. Show that if B is an A-algebra, then $\mathbb{A}^n_B \simeq \mathbb{A}^n_A \times_A \operatorname{Spec} B$ and that $\mathbb{P}^n_B \simeq \mathbb{P}^n_A \times_A \operatorname{Spec} B$.

1.6 Week 6

Exercise 1.6.1. Let X be the gluing of two copies of $\operatorname{Spec} \mathbb{Z}$ along the open set $U = \operatorname{Spec} \mathbb{Z} - \{(2)\}$. Show that X is not separated and describe the diagonal map.

Exercise 1.6.2. Let X be a separated scheme. Show that, for any affine opens $U_1, \ldots, U_m \subseteq X$, $U_1 \cap \cdots \cap U_m$ is affine.

Exercise 1.6.3 (Monomorphisms)**.** Let $f\colon X \to Y$ be a morphism of schemes. We say that f is a *monomorphism* if the composition function

$$\operatorname{Hom}(T, X) \longrightarrow \operatorname{Hom}(T, Y)$$

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is injective defined by $\alpha \mapsto f \circ \alpha$ is injective for any scheme T. That is, if $g_1 : T \to X$ and $g_2 : T \to X$ are morphisms such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

a) Prove that if f is a closed embedding or an open embedding then it is a monomorphism.
b) Prove that f is a monomorphism if and only if the diagonal $\Delta : X \to X \times_Y X$ of the morphism f is an isomorphism.
c) Prove that if f is a monomorphism, then f is separated.

Exercise 1.6.4. Let $\mathcal{F}$ be a presheaf on a scheme X and let $\mathcal{F}^+$ be the sheaf associated to $\mathcal{F}$. Show that there is a natural map $\mathcal{F}(X) \to \mathcal{F}^+(X)$. Find examples where this fails to be injective or surjective.

Exercise 1.6.5. Let $\mathcal{F}$ be a presheaf of rings on X. Show that the associated sheaf $\mathcal{F}^+$ is a sheaf of rings.

1.7 Week 7

Exercise 1.7.1. For each of the schemes below, describe the $\mathcal{O}_X$-modules on X.

a) X is the scheme obtained by gluing $\operatorname{Spec} \mathbb{Z}_{(2)}$ and $\operatorname{Spec} \mathbb{Z}_{(3)}$ along their common open subscheme $\operatorname{Spec} \mathbb{Q}$.
b) X is the scheme obtained by gluing two copies of $\operatorname{Spec} \mathbb{Z}_{(2)}$ along $\operatorname{Spec} \mathbb{Q}$.
c) Let X be the scheme obtained by gluing the schemes $X_i = \operatorname{Spec} \mathbb{Z}_{(p_i)}$ together along their common open subschemes $\operatorname{Spec} \mathbb{Q}$. Describe the $\mathcal{O}_X$-modules on X.

Exercise 1.7.2. Suppose that $\mathcal{F}$ is a *presheaf* of $\mathcal{O}_X$-modules (i.e. a presheaf satisfying the usual $\mathcal{O}_X$-module axioms). Show that the associated sheaf $\mathcal{F}^+$ is an $\mathcal{O}_X$-module in a natural way. HINT: One can use the universal properties of sheafification, or the explicit description of $\mathcal{F}^+$.

Exercise 1.7.3. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Show that $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)$.

Exercise 1.7.4. Show in detail that the kernel, cokernel and image of a map of $\mathcal{O}_X$-modules indeed are $\mathcal{O}_X$-modules.

Exercise 1.7.5 (Godement sheaves)**.** Given any collection of abelian groups $\{A_x\}_{x \in X}$ indexed by the points x of X, we can define a sheaf $\mathcal{A}$ by

$$\mathcal{A}(U) = \prod_{x \in U} A_x,$$

and whose restriction maps to smaller open subsets are just the projections onto the corresponding smaller products.

a) Show that $\mathcal{A}$ is a sheaf.
b) If we assume that each A_x be a module over the local ring $\mathcal{O}_{X,x}$, show that sheaf $\mathcal{A}$ becomes an $\mathcal{O}_X$-module.

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