MAT4215 - SYLLABUS 2026

A word about preparing for the exam: The exam will be oral, for about 30 minutes, so you should focus on concepts, definitions, examples and counterexamples in your preparation. You may also be asked for some basic computations, but nothing very complicated. Think about how the topics fit together, and why terms are defined in the way they are.

Important theorems for the oral exam

The following is a short list of some of the central theorems and results from the course. During the oral exam, you should expect to be asked to state and discuss one of these results. In particular, you should be prepared to explain the precise statement, the hypotheses, the main ideas of the proof, and examples or applications where appropriate.

(1) The construction of the structure sheaf on $\operatorname{Spec} A$.
(2) The natural bijection
$$\operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A) \simeq \operatorname{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathscr{O}_X))$$
for a scheme X and a ring A.
(3) The left exactness of the global section functor $\Gamma(U, -)$.
(4) For $X = \operatorname{Spec} A$, an A-module M, and an $\mathscr{O}_X$-module $\mathscr{F}$, the natural identification
$$\operatorname{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathscr{F}) \simeq \operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})).$$
(5) The equivalence between A-modules and quasi-coherent sheaves on $\operatorname{Spec} A$.
(6) The correspondence between closed subschemes and quasi-coherent ideal sheaves.
(7) The description of quasi-coherent sheaves on closed subschemes $X \subset \mathbf{P}^n_A$ in terms of graded modules.
(8) The description of morphisms to projective space in terms of invertible sheaves together with generating global sections.
(9) The cohomology of the sheaves $\mathscr{O}(m)$ on $\mathbf{P}^n_A$, especially their global sections.
(10) The finiteness theorem: if X is projective over a noetherian ring A and $\mathscr{F}$ is coherent, then the cohomology groups $H^i(X, \mathscr{F})$ are finitely generated A-modules.

The following is a list of the most important definitions and results in the course, in the order they appear in the current lecture notes. The points marked with * are especially important, and the points marked with ** are central results or constructions.

Chapter 2: The spectrum of a ring

* The Zariski topology on $\operatorname{Spec} A$.
* Closed sets $V(I)$ and distinguished open sets $D(f)$, and their basic properties.
* Maps of rings and the induced maps between prime spectra.
• When $\operatorname{Spec} A$ is irreducible or connected.
• Main examples: $\operatorname{Spec} \mathbb{Z}$, DVRs, polynomial rings, quotient rings and localizations.
• Fibers of maps between spectra.

Chapter 3: The structure sheaf

• Presheaves, sheaves and morphisms of sheaves.
* Stalks and germs.
• Sheaves defined on a basis.
• Pushforward sheaves.
** The construction of the structure sheaf on $\operatorname{Spec} A$.
• The sheaf associated to an A-module.

Chapter 4: Schemes

• Locally ringed spaces.
* The definition of a scheme.
• Morphisms of schemes, especially morphisms into an affine scheme.
** The category of affine schemes and the category of commutative rings.
• Schemes over a ring.
* Open embeddings and open subschemes.
* Closed embeddings and closed subschemes.
• Reduced schemes.
* Integral schemes and function fields.

- Points and residue fields.
- Affine varieties as schemes.

Chapter 5: Gluing

- Gluing sheaves.
* Gluing schemes.
- First examples of schemes obtained by gluing.

Chapter 6: Projective schemes

- Projective space: the underlying topological space, the structure sheaf and points.
* The sheaves $\mathscr{O}(m)$ on $\mathbf{P}^n_A$.
* The Proj construction.
- Basic examples of projective schemes.

Chapter 7: Fiber products and the functor of points

- Relative schemes.
- Definition of fiber products.
* Existence of fiber products for affine schemes.
- Superficial knowledge of the construction of fiber products in general.
- Base change.
* Scheme-theoretic fibers.
- The Segre embedding.
- R-valued points and the functor of points.

Chapter 8: Properties of schemes

- Covering intersections by distinguished opens.
- Finite type morphisms.
- Affine morphisms and finite morphisms.
* Noetherian schemes, including the theorem that $\operatorname{Spec} A$ is noetherian if and only if A is noetherian.
* Dimension.
- Codimension.

Chapter 9: More on sheaves

- Kernels, images, cokernels and quotient sheaves.
- When a map of sheaves is injective or surjective.
- Examples where the image or cokernel presheaf fails to be a sheaf.
* Exact sequences of sheaves.
* The left exactness of $\Gamma(U, -)$, and the failure of right exactness.
* Sheafification and its universal property.
- The inverse image sheaf.

Chapter 10: Quasi-coherent sheaves

- The basic constructions for $\mathscr{O}_X$-modules: direct sums, products, tensor products, Hom, kernels, images and cokernels.
* The $\sim$ functor and its basic properties.
** Quasi-coherent sheaves.
* Quasi-coherent sheaves on affine schemes.
** The equivalence between A-modules and quasi-coherent sheaves on $\operatorname{Spec} A$.
- Finiteness conditions for $\mathscr{O}_X$-modules, including coherent sheaves.
- Pushforwards and pullbacks of quasi-coherent sheaves.
- Relative Spec.
* Closed subschemes and quasi-coherent ideal sheaves. A sketch of the proof is enough.

Chapter 11: Locally free sheaves

- Locally free sheaves.
- Fibers and local freeness.
- Transition functions.
* Invertible sheaves and the Picard group.

Chapter 12: Sheaves on projective schemes

* The sheaf associated to a graded module and its basic properties.
• Closed subschemes of projective space.
• Superficial knowledge of the correspondence between closed subschemes of $\mathrm{Proj}(R)$ and saturated homogeneous ideals of R.
* Coherent sheaves on projective space and the role of graded modules.
• Pullbacks of invertible sheaves.
** Morphisms to projective space in terms of invertible sheaves and generating global sections.
• Application: automorphisms of projective space.

Chapter 13: Separated and proper morphisms

• Separated morphisms and separated schemes.
* Affine schemes are separated.
• Example of a non-separated scheme.
* Separatedness and intersections of affine opens.
• Graphs and equalizers; morphisms that agree over an open set.

Chapter 14: Varieties and rational maps

• Varieties (definition).

Chapter 15: First steps in sheaf cohomology

• Some basic homological algebra.
* Cech cohomology and its main properties.
• Long exact sequences in cohomology.
• Cech cohomology on schemes.
* Cohomology of quasi-coherent sheaves on affine schemes.
• Independence of the cover.
• Pushforwards and cohomology.
• Cohomology and dimension.
** Cohomology of the sheaves $\mathscr{O}(m)$ on $\mathbf{P}^n_A$, especially global sections.
** Finiteness of cohomology for coherent sheaves on projective schemes over a noetherian ring.
• Euler characteristic and arithmetic genus.
• Being comfortable with calculations involving cohomology, including examples with plane curves, the twisted cubic, hyperelliptic curves and complete intersections.