MAT4270: ADVANCED QUESTIONS

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These questions are about things slightly (or greatly) beyond what I can cover during the lecture, but I encourage you to think about them over the course. Some of these will be used as assignments for evaluation.

1. Finite groups

Notation. G: finite group, K: field of characteristic 0 (can be $\mathbb{C}$), X: finite G-set

Problem 1 (double transitivity)**.** Suppose that G acts *transitively* on X. Let $C(X) = \langle \delta_x \mid x \in X \rangle$ be the space of complex functions on X, and consider the induced action $G \curvearrowright C(X)$. Moreover, let V be the invariant complement of the trivial subrepresentation $\mathbb{C}1_X \subset C(X)$. Explain that the the action of G on V is irreducible if and only if $G \curvearrowright X$ is *doubly transitive* in the sense that

$$\forall x \neq x', y \neq y' \exists g \colon gx = y, gx' = y'.$$

(That is, G acts transitively on $X \times X \setminus \{(x, x) \mid x \in X\}$.)

Reference. [Ser77, Section 2.3, Exercise 2.6]

Problem 2 (integrality of characters)**.** Let G be a finite group, and denote by R the set of class functions on G which take values in algebraic integers. Explain the *integrality* of R and its application to an estimate of dimension of irreducible representations.

Reference. [Ser77, Section 6.5]

Problem 3 (group cohomology of finite groups)**.** Why is $H^n(G; M)$ trivial for $n > 0$ if M is a representation of G over K?

- give a conceptual reasoning using $H^n(G; M) = \mathrm{Ext}^n_{K[G]}(K, M)$ and a categorical property of representations of G over K.
- give a concrete reasoning for $Z^n(G; M) = B^n(G; M)$ by "averaging" argument. Try $n = 1, 2$ first to get the hang of it, then general n if you have time.

Reference. [Bro94, Wei94]

Problem 4 (fixed point counting and positivity)**.** For $g \in G$, let $f(g)$ be the number of fixed points,

$$f(g) = |\{x \in X \mid gx = x\}|.$$

Show that f is *positive definite*: that is, if $(g_i)_{i \in I}$ is a family of elements of G and $(c_i)_{i \in I}$ is a family of complex numbers on the same index set, we have

$$\sum_{i,j \in I} c_i \bar{c}_j f(g_j^{-1} g_i) \geq 0.$$

Hint: express $f(g)$ as the trace of some matrix, and use $\mathrm{Tr}(A^* A) \geq 0$ if A^* is the conjugate transpose of A.

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Also: $f'(g) = |X| - f(g)$ is *conditionally negative definite*: if the coefficients $(c_i)_i$ satisfy $\sum_i c_i = 0$, then

$$\sum_{i,j \in I} c_i \bar{c}_j f'(g_j^{-1} g_i) \leq 0.$$

This implies that $\exp(-f'(g))$ is positive definite.

Reference. [BO08]

Problem 5 (Hopf algebras)**.** What are the Hopf algebras associated with G, and how are they related?

- explain the Hopf algebra structure on the group algebra $\mathbb{C}[G]$.
- do the same for the function algebra $\mathcal{O}(G)$.
- explain the duality between $\mathbb{C}[G]$ and $\mathcal{O}(G)$.
- formulate the notion of representation of G in terms of $\mathcal{O}(G)$.

Reference. [Car07]

2. Lie algebras

Notation. $\mathfrak{g}$: Lie algebra (say over $\mathbb{R}$) with bracket $[x, y]$

Problem 6 (deformation)**.** Suppose that we are given an infinitesimal deformation of $\mathfrak{g}$. Concretely, that means we are given another map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]'$ such that $[x, y] + \epsilon[x, y]'$ defines a Lie algebra over $\mathbb{R}[\epsilon]/(\epsilon^2)$ when interpreted as the natural ϵ-linear extension on $\mathfrak{g}[\epsilon]/(\epsilon^2) = \{x_1 + \epsilon x_2 \mid x_1, x_2 \in \mathfrak{g}\}$. Here ϵ is an 'infinitesimal' such that $\epsilon^2 = 0$.

- what is the consistency condition between $[x, y]$ and $[x, y]'$?
- describe the above relation in terms of Lie algebra cohomology; interpret $[x, y]'$ as a cochain in the Chevalley–Eilenberg cochain complex of $\mathfrak{g}$ with coefficient $\mathfrak{g}$ itself. Which $[x, y]'$ correspond to the coboundaries?
- present the Lie algebra of $ax + b$ group as a deformation of the commutative Lie algebra $\mathbb{R}^2$.
- what does Whitehead's lemma imply for semisimple Lie algebras?

Reference. [Kna88, Wei94]

Problem 7 (Levi's theorem)**.** Let K be a field of characteristic 0 (let's further assume $K \subset \mathbb{C}$, although this is avoibable), and $\mathfrak{g}$ be a finite dimensional Lie algebra over K. Explain that there is a semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} + \mathrm{Rad}(\mathfrak{g})$. Also, give an nontrivial example of this decomposition ($\mathfrak{s} \neq 0 \neq \mathrm{Rad}(\mathfrak{g})$).

Reference. [FH91, Section E.1]

Problem 8 (Ado's theorem)**.** Let K be a field of characteristic 0, and let $\mathfrak{g}$ be a finite dimensional Lie algebra over K. Show that there is an integer $n > 0$ and an injective Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}_n(K)$.

Bonus: this is still true for positive characteristic, and the proof is shorter.

Reference. [FH91, Section E.2] (characteristic 0), [Bou07, Section 1.7, Exercise] (positive characteristic)

Problem 9 (Casimir operator of $\mathfrak{sl}_2$)**.** Let K be a field of characteristic 0. Consider the element

$$C = \frac{1}{8}H^2 + \frac{1}{4}H + \frac{1}{2}FE$$

in the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(K))$.

- Explain the reason that if (π, V) is a representation of $\mathfrak{sl}_2(K)$, the endomorphism $\pi(C) \in \mathrm{End}(V)$ is a scalar multiple of the identity map.
- Compute the corresponding scalar for the natural representation of $\mathfrak{sl}_2(K)$ on the space homogeneous polynomials of degree n in two variables, $\langle x^n, x^{n-1}y, \dots, y^n \rangle$.
- Express C using the standard basis of $\mathfrak{su}_2$.
- Use the identification $\mathfrak{su}_2 \simeq \mathfrak{so}_3$, express C as a differential operator on the unit sphere $S^2 \subset \mathbb{R}^3$.

Problem 10 (geometric realization of representations)**.** Explain geometric realization of the irreducible representations of SL_2 on the complex projective space $\mathbb{P}^1(\mathbb{C})$.

- find a representation of $\mathfrak{sl}_2(\mathbb{C})$ as algebraic vector fields on $\mathbb{P}^1(\mathbb{C})$.
- what are the holomorphic line bundles on $\mathbb{P}^1(\mathbb{C})$ (usually denoted $\mathcal{O}(n)$ for $n \in \mathbb{Z}$), and what are the space of holomorphic sections $H^0(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n)) = \Gamma(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n))$?
- describe the first cohomology $H^1(\mathbb{P}^1(\mathbb{C}), \mathcal{O}(n))$, either through holomorphic differential forms in these coefficients, or through the Čech cohomology for the decomposition $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup_{\mathbb{C}^\times} \mathbb{C}$.
- identify these with the spaces of homogeneous polynomials in two variables, and compare the actions of $\mathfrak{sl}_2$ (or SL_2).
- interpret this construction as the Borel–Weil–Bott construction.

Reference. [Sep07]

3. More specialized

Problem 11 (Peter–Weyl theorem)**.** Let G be a compact group.

(i) Show that the space $\mathcal{O}(G)$ of matrix coefficients of finite representations form a subalgebra of $C(G)$.
(ii) Use the Stone–Weierstrass theorem to check that $\mathcal{O}(G)$ is a dense subalgebra of $C(G)$.
(iii) What does the Peter–Weyl theorem say?

Problem 12 (oadjoint action)**.** $\mathfrak{g}$: Lie algebra, $\mathfrak{g}^*$: the linear dual of $\mathfrak{g}$, $\mathrm{Sym}(\mathfrak{g})$: the symmetric algebra

$$\mathrm{Sym}(\mathfrak{g}) = \left(\bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \right) / \langle x \otimes y - y \otimes x \mid x, y \in \mathfrak{g} \rangle .$$

What is the Kirillov bracket on $\mathfrak{g}^*$?

- interpret the elements of the symmetric algebra $\mathrm{Sym}(\mathfrak{g})$ as functions on $\mathfrak{g}^*$.
- extend the bracket $[x, y]$ on $\mathfrak{g}$ to a Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$.
- explain the relation between the universal enveloping algebras of $\mathfrak{g}$ with rescaled brackets the deformation quantization of this Poisson bracket.

Problem 13 (loop groups and affine Kac–Moody algebras)**.** G: simple compact Lie group
What is a loop group LG, and what is its Lie algebra?

- how do you realize these objects in the following settings? (and what are the merits?)
  • rational • smooth • continuous
- what is the affine Lie algebra $\hat{\mathfrak{g}}$, and what is the affine Kac–Moody algebra $\tilde{\mathfrak{g}}$? (the terminology varies among different authors)
- describe its Cartan matrix and Dynkin diagram of $\tilde{\mathfrak{g}}$.

Reference. [Kac90]

Problem 14 (deformation quantization)**.** M: C^∞-manifold
Explain the Lie algebraic concepts in the problem of Poisson deformation quantization.

- what is a polyvector field on M? Explain the graded Lie algebra structure (Schouten–Nijenhuis bracket) on the space of polyvector fields $\Gamma(\bigwedge^* TM)$.
- what is a polydifferential operator on M? Explain the differential graded Lie algebra structure on the space of polydifferential operators $D^*(M)$.
- how are they related? What are the Maurer–Cartan elements in these differential graded Lie algebras?
- explain the Poisson deformation quantization problem in terms of the above objects.

Problem 15 (Drinfeld–Kohno Lie algebra)**.** $\mathfrak{t}_n$: the nth Drinfeld–Kohno Lie algebra, G: simple compact Lie group

- describe the generators and relations of $\mathfrak{t}_n$.
- what is the relation between $\mathfrak{t}_n$ and the pure braid group PB_n?
- how are they related to the configuraton space of n distinct complex numbers?
- when (π, V) is a representation of G, explain that there is an induced representation of $\mathfrak{t}_n$ on $V^{\otimes n}$.

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