§1 Linear representation of finite groups.

Main ref. Serre "Linear repr. of fin. groups."

Blanket assumption: work over fin. dim. cplx vec. sps. ($\simeq \mathbb{C}^n$ for some $n = 0, 1, \ldots$)

Notn $\mathbb{C} = \{ a + b\sqrt{-1} : a, b \text{ real} \}$

V : f.d. ($\mathbb{C}$-)vec sp ($\simeq \mathbb{C}^n$)

$\mathrm{End}(V) = \{ T : V \to V \text{ lin.} \} \simeq M_n(\mathbb{C})$

$GL(V) = \{ T \in \mathrm{End}(V) \text{ invertible} \} \simeq GL_n(\mathbb{C})$

$= \{ X \in M_n(\mathbb{C}) : \det X \neq 0 \}$

Id_V : identity map $\longleftrightarrow I_n$.

G: finite group.

linear rep. of G:

concrete) system of matrices $(X_g)_{g \in G}$

$\underset{\substack{\uparrow \\ \text{fixed}}}{X_g \in M_n(\mathbb{C})}$, $X_g X_h = X_{gh}$, $\underset{\substack{\uparrow \\ \text{neutral elem.}}}{X_e} = I_n$

abstract) V: f.d. vec. sp.

$\pi : G \to GL(V)$ group hom.

Convention: π_g instead of $\pi(g)$

so $\pi_g v \in V$ when $v \in V$.

Sometimes gv instead of $\pi_g v$

so $g(hv) = (gh)v$, $ev = v$.

First examples

1. $G = S_n = \{ \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \text{ bijective} \}$

$\{\sigma(1), \ldots, \sigma(n)\} = \{1, \ldots, n\}$.

$V = \langle e_i : i = 1, \ldots, n \rangle$. ($\simeq \mathbb{C}^n$)

$= \{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{C} \}$.

$\pi_\sigma e_i = e_{\sigma(i)}$ $\quad (\pi_\sigma \pi_\tau e_i = e_{\sigma\tau(i)} = \pi_{\sigma\tau} e_i)$

2. $G = \mathbb{Z}/n\mathbb{Z} = \{ [i] : i \in \mathbb{Z},\ [i] = [i+kn] \}$

1-dimensional representations of $\mathbb{Z}/n\mathbb{Z}$

$$\varphi^{(1)}_{[i]} = e^{\frac{2\pi\sqrt{-1}}{n} i} \in \mathbb{C} = M_1(\mathbb{C})$$

well-defied

$$\varphi^{(1)}_{[i]} \varphi^{(1)}_{[j]} = e^{\frac{2\pi\sqrt{-1}}{n}(i+j)} = \varphi^{(1)}_{i+j}.$$

More generally : $\varphi^{(k)}_{[i]} = e^{\frac{2\pi\sqrt{-1}}{n} k i}$

$\varphi^{(0)}, \cdots, \varphi^{(n-1)}$ are the only 1-dim reps

$\because$ $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times = GL_1(\mathbb{C})$ multiplicative group

- determined by $\psi_{[1]}$ $\left(\psi_{[i]} = \psi_{[1]}^{\,i} \right)$
- $\psi_{[1]}^{\,n} = \underbrace{\psi_{[1]} \cdots \psi_{[1]}}_{n\times} = \psi_{n[1]} = \psi_{[0]} = 1$

$\Rightarrow$ $\psi_{[1]}$ should be n-th root of unity.

Questions

- How do we compare different reps?
- Decomposition of reps
  $\rightsquigarrow$ indecomposable / irreducible reps
- Classification by <u>characters</u> $\chi_\pi(g) = \mathrm{Tr}(\pi_g)$.

Today's summary

- Direct sum of reps
- Comparison of reps

$\Big) \to$ Complete reducibility, irreducible reps

- Group ring

(- Tensor product)

Notn : G finite group

(π, V) linear rep of G $(\pi : G \to GL(V))$.

("V: "underlying space" of π)

Direct sum of reps.

(π, V), (π', V') rep. of G $\rightsquigarrow$ "$\pi \oplus \pi'$" on $V \oplus V'$

$V \oplus V' = \{ v \oplus w : v \in V, w \in V' \} \simeq V \times V'$

$(\pi \oplus \pi')_g (v \oplus w) = (\pi_g v) \oplus (\pi'_g w)$

Concrete) $(X_g)_{g \in G}$ $(X'_g)_{g \in G}$ $X_g \in M_m(\mathbb{C})$, $X'_g \in M_n(\mathbb{C})$

$\rightsquigarrow \begin{bmatrix} X_g & 0 \\ 0 & X'_g \end{bmatrix} \in M_{m+n}(\mathbb{C})$

Comparison of reps.

<u>Intertwiner</u> (or (G-)<u>homomorphism</u>) from (π, V) to (π', V')

$T : V \to V'$ linear map s.t. $\forall g \in G$ $T \pi_g = \pi'_g T$

Concrete) with above $(X_g)_g$, $(X'_g)_g$: $T \in M_{n \times m}(\mathbb{C})$

$\forall g$ $T X_g = X'_g T$

Notn : $\mathrm{Hom}_G(V, V') = \{ T : V \to V' \text{ intertwiner} \}$

$\mathrm{Hom}_G((\pi, V), (\pi', V'))$, $\mathrm{Hom}(\pi, \pi')$, (π, π'), ...

π and π' are <u>isomorphic</u> (or <u>equivalent</u>, <u>similar</u>)

if $\exists$ <u>bijective</u> intertwiner $T : V \to V'$.

Example $G = \mathbb{Z}/n\mathbb{Z}$

- $V = \langle e_{[i]} : [i] \in G \rangle \quad \pi_{[i]} e_{[j]} = e_{[i+j]}$

  (restriction of $S_n \curvearrowright V$ from yesterday
  $G \ni [1] \leftrightarrow$ cycl. perm $(1\ 2 \cdots n) \in S_n$)

- $\psi^{(0)} \oplus \cdots \oplus \psi^{(n-1)}$ on $\mathbb{C}^n$

  $\psi^{(k)}_{[i]} = e^{\frac{2\pi\sqrt{-1}}{n} ik}$, $f_k = 0 \oplus \cdots \oplus \underset{k\text{-th}}{1} \oplus \cdots \oplus 0$

$T : V \to \mathbb{C}^n$, $e_{[i]} \mapsto \sum_{k=0}^{n-1} e^{\frac{2\pi\sqrt{-1}}{n} ik} f_k$
$= 1 \oplus e^{\frac{2\pi\sqrt{-1}}{n} i} \oplus \cdots \oplus e^{\frac{2\pi\sqrt{-1}}{n} i(n-1)}$

Exercise : check $(\psi^{(0)} \oplus \cdots \oplus \psi^{(n-1)})_{[1]} T e_{[i]} = T \pi_{[1]} e_{[i]}$

What did we do ? :

$\pi_{[1]} \leftrightarrow$ "rotation" matrix $\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ & 1 & \ddots & \vdots \\ & & \ddots\ 0 & \\ & & 1 & 0 \end{bmatrix}$

relation between $T e_{[i]}$ and f_k

$\leftrightarrow$ rel. between $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} i$ & eigenvector

Complete reducibility

(π, V) rep of G. $W \subset V$ subspace is G-invariant
if $\forall w \in W$, $g \in G$ $\pi_g w \in W$.

$\rightsquigarrow ((\pi_g|_W)_g , W)$ is again rep of G.

subrep of (π, V)

Th'm. Under above setting $\exists W' \subset V$ s.t.

- W' is complement to W : $V = W \oplus W'$
- W' is G-invariant.

Rec ! (Hermitian) inner product (v, v') $v, v' \in V$

$(v, v') \in \mathbb{C}$, $\|v\|^2 = (v, v) \geq 0$, $\|v\| = 0 \Leftrightarrow v = 0$

$(v, v') = \overline{(v', v)}$, linear in v

Proof. Step 1 V has a G-invar. inner product.

$$(v, v') = (\pi_g v, \pi_g v')$$

$\because$ $(v, v')_0$: any inn. prod (from $V \cong \mathbb{C}^n$ say)

$$(v, v') = \sum_{h \in G} (\pi_h v, \pi_h v')_0$$

$$\leadsto (\pi_g v, \pi_g v') = \sum_h (\pi_h \pi_g v, \pi_h \pi_g v')_0$$

$$= \sum_h (\pi_{\underbrace{hg}_{\text{relabel as } h'}} v, \pi_{hg} v')_0 = (v, v')$$

Step 2 $W' = \{ w' \in V : \forall w \in W \quad (w, w') = 0 \}$

orthogonal complement.

is G-invar.

$$\because (w, \pi_g w') \underset{\text{Step 1}}{=} (\pi_{g^{-1}} w, \pi_{g^{-1}} \pi_g w') = (\underbrace{\pi_{g^{-1}} w}_{\in W}, w')$$

$$= 0 .$$

$\blacksquare$

Rem. Step 1 $\Rightarrow$ any rep of fin. grp. is <u>unitarizable</u>

unitary rep : $\pi : G \to U(H)$

H : Hilbert sp. (Herm. inn. prod. sp.)

$U(H)$: unitary group.

$= \{ T : H \to H, (T\xi, T\eta) = (\xi, \eta) \}$

(π, V) is <u>irreducible</u> if only subreps are the obvious ones $W = 0, V$ $0 \neq V$

Cor. Any rep is isom. to direct sum of irreducible ones

$\because$ V "irred" $\Rightarrow$ direct sum with one summand

$V = 0 \Rightarrow$ "zero summand"

not irred $\Rightarrow$ take $W \subset V$ $\underset{\text{Thm}}{\Rightarrow}$ $V \cong \underset{\text{subreps}}{W \oplus W'}$

keep decomposing W & W'.

Tensor product.

(π, V), (π', V') reps. $\rightsquigarrow$ <u>tensor product</u> rep.

$\pi \otimes \pi'$ on $V \otimes V'$.

$V \otimes V' = \{ \sum_{\alpha_i} \alpha_i v_i \otimes v'_i \quad v_i \in V,\ v'_i \in V'$
$\alpha_i \in \mathbb{C} \}$

$(v_1 + v_2) \otimes v' = v_1 \otimes v' + v_2 \otimes v'$
$(\alpha v) \otimes v' = \alpha (v \otimes v') = v \otimes \alpha v'$ etc.

$(\pi \otimes \pi')_g (v \otimes v') = \pi_g v \otimes \pi'_g v'$

Group algebra

$\mathbb{C}[G] = \{ \sum_{g \in G} \alpha_g \cdot g \ : \alpha_g \in \mathbb{C} \}$

$(\sum_g \alpha_g g)(\sum_h \beta_h h) = \sum \alpha_g \beta_h \underbrace{(gh)}_{\text{computed in } G}$

$= \sum_k \left(\sum_h \alpha_{kh^{-1}} \beta_h \right) k.$

Representations of G $\equiv$ modules over $\mathbb{C}[G]$
irred. reps $\equiv$ simple modules

Complete reducibility $\Leftrightarrow$ any $\mathbb{C}[G]$-module is a direct sum of simple ones

$\mathbb{C}[G]$ is <u>semisimple</u>

$\Longrightarrow$ $\mathbb{C}[G] \simeq \prod_{(\pi, V):\text{irred rep.}} \mathrm{End}(V)$

general theory of semisimple algs

Summary

- Characters
- Schur's lemma (intertwiners between irred. reps)
- Orthogonality of irred. chars.

G : finite group.

(π, V), (π', V'), $\cdots$: linear reps of G.

<u>Character</u> of (π, V) : $\chi_\pi : G \to \mathbb{C}$, $\chi_\pi(g) = \mathrm{Tr}_{\mathrm{End}(V)}(\pi(g))$

Rem

- $\chi_\pi(g)$ = sum of eigenvalues of $\pi(g)$

  $\pi(g)$: can be represented by a unitary mat.

  $\Rightarrow \chi_\pi(g) \in \mathbb{Z}[e^{\frac{2\pi\sqrt{-1}}{|G|}}]$
- $\mathrm{Tr}_{\mathrm{End}(V)}(ABA^{-1}) = \mathrm{Tr}_{\mathrm{End}(V)}(B)$

  $\Rightarrow \chi_\pi(ghg^{-1}) = \chi_\pi(h) \quad (g, h \in G)$.

  χ_π is a <u>class function</u> (const. on each conjugacy class).

Prop. $\chi_{\pi\oplus\pi'}(g) = \chi_\pi(g) + \chi_{\pi'}(g)$, $\chi_{\pi\otimes\pi'}(g) = \chi_\pi(g)\chi_{\pi'}(g)$

$\because$ $(e_i)_{i=1}^m$ basis of V, $(f_j)_{j=1}^n$ basis of V'

$\Rightarrow (e_1 \oplus 0, \cdots, e_m \oplus 0, 0 \oplus f_1, \cdots, 0 \oplus f_n)$: basis of $V \oplus V'$

$(e_1 \otimes f_1, \cdots, e_m \otimes f_1, e_1 \otimes f_2, \cdots, e_m \otimes f_n)$ basis of $V \otimes V'$

$\Rightarrow \mathrm{Tr}_{\mathrm{End}(V\oplus V')}(A \oplus B) = \mathrm{Tr}_{\mathrm{End}(V)}(A) + \mathrm{Tr}_{\mathrm{End}(V')}(B)$

$\mathrm{Tr}_{\mathrm{End}(V\otimes V')}(A \otimes B) = \mathrm{Tr}_{\mathrm{End}(V)}(A)\,\mathrm{Tr}_{\mathrm{End}(V')}(B)$

for $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(V')$

Use $A = \pi(g)$, $B = \pi'(g)$ ◻

Example ↗ ! sometimes "character" means this !

- 1-dim rep $\phi : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

$$\chi_\phi(g) = \phi(g)$$

- $\pi : S_n \curvearrowright V = \langle e_1, \cdots, e_n \rangle$

$$\chi_\pi(\sigma) = \mathrm{Tr}_{M_n(\mathbb{C})}(\text{permutation matrix for } \sigma)$$

$$= \text{number of 1's in the diagonal}$$

$$= \#\{ i : \sigma(i) = i \} \quad (\text{number of fixed points})$$

Schur's lemma) $(\pi, V), (\pi', V')$ irreducible reps of G. $T : V \to V'$ intertwiner.

1) π & π' are not isomorphic (write $\pi \not\cong \pi'$) : T must be 0

2) π & π' isomorphic : if $T \neq 0$, T is invertible & any other intertwiner is a scalar multiple of T.

Proof : Basic idea : $\mathrm{Im}\, T$, $\mathrm{Ker}\, T$ are G-invariant subspaces

$\because$ $\pi'(g) T v = T \pi(g) v \in \mathrm{Im}\, T$

$v \in \mathrm{Ker}\, T \Rightarrow T \pi(g) v = \pi'(g) T v = 0$

$\Rightarrow \pi(g) v \in \mathrm{Ker}\, T$.

Proof of 1). (proof by contradiction)

$$\left.\begin{array}{l} \text{if } T \neq 0 \quad \mathrm{Ker}\, T \neq 0 \\ \text{irreducibility of } \pi \end{array}\right) \Rightarrow \mathrm{Ker}\, T = 0$$

$\mathrm{Im}\, V \neq 0$ & irred'ty of π' $\Rightarrow \mathrm{Im}\, T = V'$

$\Rightarrow T$ is bijective. π & π' isom.

Proof of 2) By the same arg. $T \neq 0 \Rightarrow T$ invertible

$S : V \to V'$ another intertwiner

λ : any eigenvalue of $T^{-1}S \in \mathrm{End}(V)$

- $T^{-1}S - \lambda \mathrm{Id}_V$ is an intertwiner from π to π.
- $T^{-1}S - \lambda \mathrm{Id}_V$ has nontrivial kernel

$\Rightarrow$ $T^{-1}S - \lambda \mathrm{Id}_V = 0$ i.e. $S = \lambda T$.

Averaging maps into intertwiners.

Prop. (π, V), (π', V') reps.

$T : V \to V'$ linear map.

$\Rightarrow$ $\tilde{T} = \frac{1}{|G|} \sum_{h \in G} \pi'(h) T \pi(h)^{-1}$ is an intertwiner.

Proof Step 1 $\tilde{T}$ is intertwiner $\Leftrightarrow$ $\forall g \in G$ $\pi'(g) \tilde{T} \pi(g)^{-1} = \tilde{T}$

Step 2 $\pi'(g) \tilde{T} \pi(g)^{-1} = \frac{1}{|G|} \sum_h \pi(g)\pi(h) T \pi(h)^{-1} \pi(g)^{-1}$

$$= \frac{1}{|G|} \sum_h \pi(gh) T \pi(\underbrace{gh}_{\text{relabel as } h'})^{-1} = \tilde{T}$$

Cor. π, π' irred., $T : V \to V'$ linear, $\tilde{T}$ as above.

1) $\pi \not\cong \pi'$ $\Rightarrow$ $\tilde{T} = 0$

2) $V = V'$, $\pi = \pi'$ $\Rightarrow$ $\tilde{T} = \frac{\mathrm{Tr}(T)}{\dim V} \mathrm{Id}_V$

$\because$ for 2) $\cdot$ $\mathrm{Tr}(\tilde{T}) = \frac{1}{|G|} \sum_h \underbrace{\mathrm{Tr}(\pi(h) T \pi(h)^{-1})}_{= \mathrm{Tr}(T)} = \mathrm{Tr}(T)$

$\cdot$ $\tilde{T}$ is a scalar mult. of Id_V.

Orthogonality of irred. chars.

$\varphi, \psi : G \to \mathbb{C}$ maps $\Rightarrow$ Hermitian inner product

$$(\varphi, \psi)_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

Th'm 1) π irred. rep $\Leftrightarrow$ $(\chi_\pi, \chi_\pi)_{L^2(G)} = 1$.

2) π': another irred. rep, $\pi \not\cong \pi'$ $\Rightarrow$ $(\chi_\pi, \chi_{\pi'})_{L^2(G)} = 0$

Before proof,

Examples 1) $\varphi, \psi : G \to GL_1(\mathbb{C})$ 1-dim reps

$\varphi \simeq \psi$ as reps $\Leftrightarrow$ $\varphi = \psi$ as maps.

$(\varphi, \varphi')_{L^2(G)} = \frac{1}{|G|} \sum_{g \in G} \underbrace{|\varphi(g)|^2}_{\text{all } 1} = 1$.

$(\varphi, \psi)_{L^2(G)} = \frac{1}{|G|} \sum_g \varphi(g) \overline{\psi(g)} = 0$ if $\varphi \neq \psi$.

2) $S_n \curvearrowright V = \langle e_1, \cdots, e_n \rangle$.

$v_0 = e_1 + \cdots + e_n$ is fixed by any $\sigma \in S_n$

$\pi|_{\mathbb{C}v_0}$ is isom. to the trivial rep " "
$(\varphi(\sigma) = 1)$

W : S_n-invariant complement of $\mathbb{C}v_0$

$\Rightarrow V \simeq \mathbb{C}v_0 \oplus W$ as reps. $\pi' := \pi|_W$.

$\underbrace{\chi_\pi(\sigma)}_{\#\{i : \sigma(i) = i\}} = \underbrace{1}_{\text{char. of triv. rep}} + \chi_{\pi'}(\sigma')$

$(\chi_{\pi'}, \chi_{\pi'}) = \frac{1}{n!} \sum_{\sigma \in S_n} \left(\#\{i : \sigma(i) = i\} - 1\right)^2$

Fact : Right hand side is always 1.
($\Rightarrow \pi'$ is irreducible)

Check for $n = 3$

σ	e	$(1,2), (2,3), (3,1)$	$(1,2,3), (3,2,1)$
$\chi_\pi(\sigma)$	3	1	0
$\chi_{\pi'}(\sigma)$	2	0	-1.

Summary
- Contragredient & adjoint reps
- Proof of orthogonality
- Structure of characters
  - Regular representation

G : fin. grp. (π, V), (π', V') : lin. reps. of G

Rem $\pi \simeq \pi' \Rightarrow \chi_\pi = \chi_{\pi'}$ (we'll later see $\Leftarrow$).

$\because$ $T : V \to V'$ bijective intertwiner

$\mathrm{Tr}_{\mathrm{End}(V')}(TAT^{-1}) = \mathrm{Tr}_{\mathrm{End}(V)}(A)$ for $A \in \mathrm{End}(V)$

$A = \pi_g \Rightarrow TAT^{-1} = \pi'_g$

<u>Contragredient</u> <u>rep</u> :

$V^* = \{ \xi : V \to \mathbb{C} \text{ linear} \}$ dual space of V

$\rightsquigarrow$ rep π^c of G on V^* by $(\pi^c_g \xi)(v) = \xi(\pi_{g^{-1}} v)$

$$(\pi^c_g(\pi^c_h \xi))(v) = (\pi^c_h \xi)(\pi_{g^{-1}} v) = \xi(\pi_{h^{-1}} \pi_{g^{-1}} v)$$
$$= \xi(\pi_{(gh)^{-1}} v) = (\pi^c_{gh} \xi)(v)$$

Concretely, $(X_g)_{g \in G}$ matrices in $M_n(\mathbb{C})$ corresponding to π $\rightsquigarrow$ $X^c_g = (X_{g^{-1}})^t$ corresp. to π^c.

If X_g is unitary, X^c_g is componentwise conjugate

$\bar{X}_g := (\overline{(X_g)_{ij}})_{i,j}$.

<u>Adjoint</u> <u>rep</u> : $\mathrm{Hom}(V, V') = \{ T : V \to V' \text{ linear} \}$

(also write $\mathrm{Hom}_{\mathbb{C}}(V, V')$, $\mathrm{Lin}(V, V')$, $B(V, V')$, $\cdots$)

$\mathrm{Ad}_g(T) = \pi'_g T \pi_{g^{-1}}$ defs rep. of G on $\mathrm{Hom}(V, V')$

Rem $V' \otimes V^* \simeq \mathrm{Hom}(V, V')$ by $\sum v'_i \otimes \xi_i \mapsto (v \mapsto \sum \xi_i(v) v'_i)$

this is a intertwiner from $\pi' \otimes \pi^c$ to Ad.

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Translating Schur's lemma

Suppose $\sum v_i' \otimes \xi_i \in V' \otimes V^*$ corr. to $T \in \mathrm{Hom}(V, V')$

T is an intertwiner $\Leftrightarrow T = \pi_g' T \pi_g^{-1} \quad \forall g$

$\Leftrightarrow \sum v_i' \otimes \xi_i$ is fixed by $\pi_g' \otimes \pi_g^c$.

When π, π' irred

1) $\pi \not\simeq \pi'$ $\nexists$ G-invar. nonzero vec. in $V' \otimes V^*$

2) $\pi \simeq \pi'$ $\exists$ up to scalar unique G-inv. vec. in $V' \otimes V^*$.

Proof of orthogonality (almost)

Step 1) $(\chi_{\pi'}, \chi_\pi)_{L^2(G)} = \frac{1}{|G|} \sum_g \chi_{\pi'}(g) \overline{\chi_\pi(g)}$.

is the trace of $\frac{1}{|G|} \sum_g \pi_g' \otimes \pi_g^c$

$\because$ $\mathrm{Tr}_{\mathrm{End}(V' \otimes V^*)}(\pi_g' \otimes \pi_g^c) = \mathrm{Tr}_{\mathrm{End}(V')}(\pi_g') \mathrm{Tr}_{\mathrm{End}(V^*)}(\pi_g^c)$

So it's enough to have $\overline{\chi_\pi}(g) = \mathrm{Tr}_{\mathrm{End}(V^*)}(\pi_g^c)$

- $(X_g)_g$ corr to π $\Leftrightarrow$ $((X_{g^{-1}})^t)_g$ corr. to π^c

$\Rightarrow \mathrm{Tr}((X_{g^{-1}})^t) = \mathrm{Tr}(X_{g^{-1}}) = \mathrm{Tr}((X_g)^{-1})$

- λ : eigenval. of $X_g \Rightarrow |\lambda| = 1$.

$\Rightarrow \mathrm{Tr}((X_g)^{-1}) = \sum_{\substack{\lambda:\text{ e-val} \\ \text{of } X_g}} \lambda^{-1} = \sum_\lambda \bar{\lambda} = \overline{\mathrm{Tr}(X_g)}$

- Conceptually : using G-inv. inn. prod

$(\pi^c, V^*) \simeq (\bar{\pi}, \bar{V})$ $\quad \xi(v) = (v, w)$ for $\exists! w$

$\bar{V} = \{\bar{v} : v \in V, \ \lambda \bar{v} = \overline{\lambda v}\}$ conjug. sp.

$\bar{\pi}_g \bar{v} = \overline{\pi_g v}$ $\quad \mathrm{Tr}(\bar{\pi}_g) = \overline{\mathrm{Tr}(\pi_g)}$

Step 2) Set $\Phi = \frac{1}{|G|} \sum_g \pi_g' \otimes \pi_g^c$.

$\mathrm{Im}\, \Phi$ = G-invariant vectors in $V' \otimes V^*$

$\because$ $(\pi_h' \otimes \pi_h^c) \Phi = \frac{1}{|G|} \sum_g \pi'_{\underbrace{hg}_{\text{relabel as } g'}} \otimes \pi^c_{hg} = \Phi$.

Step 3) Proof of claim (2)

π, π' irred., $\pi \not\simeq \pi' \Rightarrow \Phi = 0$

$\because$ By Schur's lem. $\not\exists$ G-inv. vec. in $V' \otimes V^*$.

$\Rightarrow \operatorname{Im}\Phi = 0$

Step 4) Proof of "$\Rightarrow$" for claim (1)

π irred $\Rightarrow \operatorname{Tr}\Phi = 1$.

$\because$ • $\operatorname{Im}\Phi$ is 1-dim (span of $\sum_i e_i \otimes e^i$ corr. to Id_V ; $(e_i)_i$ basis of V, $(e^i)_i$ dual basis)

• $\Phi^2 = \Phi$ from $(\pi'_h \otimes \pi^c_h)\Phi = \Phi$. $\because$ average of left hand side is Φ^2 証

Structure of characters.

(π, V) rep of G. Suppose

$$(\pi, V) \simeq \underbrace{(\pi_1, V_1) \oplus \cdots \oplus (\pi_1, V_1)}_{n_1 \times} \oplus \underbrace{(\pi_2, V_2) \cdots \oplus (\pi_2, V_2)}_{n_2 \times} \oplus \cdots \oplus \underbrace{(\pi_k, V_k) \oplus \cdots \oplus (\pi_k, V_k)}_{n_k \times}$$

• each (π_i, V_i) irred.

• $i \neq j \Rightarrow \pi_i \not\simeq \pi_j$.

$$= \bigoplus_{i=1}^{k} (\pi_i, V_i)^{\oplus n_i}$$

Then $\chi_\pi = n_1 \chi_{\pi_1} + n_2 \chi_{\pi_2} + \cdots + n_k \chi_{\pi_k}$ and

• $(\chi_\pi, \chi_\pi) = n_1^2 + n_2^2 + \cdots + n_k^2$

• $n_i = (\chi_\pi, \chi_{\pi_i})$: "<u>multiplicity</u> of π_i in π"

$\because$ $(\chi_{\pi_i}, \chi_{\pi_j}) = \delta_{ij}$

So • $(\chi_\pi, \chi_\pi) = 1 \Rightarrow \pi$ is irred.

("$\Leftarrow$" for claim (1) of orthogonality)

• irred decomp of π is unique up to rearranging direct summands

(and replacing summands by isom. irreps.)

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<u>Regular representation</u> (λ, V) with

$V = \langle e_g : g \in G \rangle$. $\lambda_g e_h = e_{gh}$

(other form : underlying vec. sp $\mathbb{C}(G) = L^2(G)$

$(\lambda_g f)(h) = f(g^{-1} h)$

$V \ni e_g \leftrightarrow \delta_g \in L^2(G)$)

Concretely) $\lambda_g \leftrightarrow (X_{h,k})_{h,k\in G}$ with

$X_{h,k} = \begin{cases} 1 & h = gk \\ 0 & h \neq gk \end{cases}$

$\Rightarrow \chi_\lambda(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}$

Prop. $(\pi_1, V_1), \cdots, (\pi_k, V_k)$ all irred. reps of G (up to isom.)

$\pi_i \not\simeq \pi_j$ for $i \neq j$

Then $(\lambda, V) \simeq \bigoplus_{i=1}^{k} (\pi_i, V_i)^{\oplus \dim V_i}$

$\because$ mult. of π_i in λ $= (\chi_{\pi_i}, \chi_\lambda)$

$= \frac{1}{|G|} \sum_g \chi_{\pi_i}(g) \chi_\lambda(g) = \chi_{\pi_i}(e) = \mathrm{Tr}(\underbrace{\pi_i(e)}_{Id_{V_i}})$

$= \dim V_i$

Examples 1) $G = \mathbb{Z}/n\mathbb{Z}$. $V = \langle e_{[i]} : [i] \in \mathbb{Z}/n\mathbb{Z} \rangle$

$(\lambda, V) \simeq \bigoplus_{k=0}^{n-1} \varphi^{(k)}$ for $\varphi^{(k)}([i]) = e^{\frac{2\pi\sqrt{-1}}{n} ki}$

2) $G = S_3$. Irreducible reps :

two 1-dim : $\pi^{triv}(\sigma) = 1$. $\pi^{sig}(\sigma) = (-1)^{|\sigma|}$.

one 2-dim : π' : complement of π^{triv}.

for $S_3 \curvearrowright \langle e_1, e_2, e_3 \rangle$

$\underset{6\text{-dim}}{\lambda} \simeq \pi^{triv} \oplus \pi^{sig} \oplus \pi' \oplus \pi'$

Rem λ corresponds to $\mathbb{C}[G] \curvearrowright \mathbb{C}[G]$ left. mult.

We have $\mathbb{C}[G] \simeq \prod_{\pi_i : \text{irrep.}} \mathrm{End}(V_i) \simeq \prod_i V_i \otimes V_i^*$

left mult. becomes $\pi_i \otimes 1$ on $V_i \otimes \underbrace{V_i^*}_{\text{gives mult.}}$

Summary
- Counting irreducible representations $(= \#(\text{conjug. cls}))$
- Character table

· Counting irred. reps.

G : finite group $\quad (\pi, V)$ lin. rep. of G

$f : G \to \mathbb{C}$ map $\rightsquigarrow \pi_f \in \mathrm{End}(V)$ by

$$\pi_f(v) = \sum_{g \in G} f(g) \pi_g v.$$

Rem. this corresponds to $\mathbb{C}[G]$-module structure on V

f is a <u>class function</u> if $f(hgh^{-1}) = f(g) \quad \forall g, h \in G$

Ex. $\chi_{\pi'}$ is a class func.

Prop. 1) f class func. $\Rightarrow \pi_f \in \underset{\substack{\uparrow \\ \text{intertwiners from } \pi \text{ to } \pi}}{\mathrm{End}_G(V)}$

2) f class func. & π irred

$\Rightarrow \pi_f$ is the scalar mult. by $\dfrac{|G|}{\dim V} (f, \overline{\chi_\pi})_{L^2(G)}$

$$= \frac{1}{\dim V} \sum_{g \in G} f(g) \chi_\pi(g).$$

Proof 1) Want : $\pi_{h^{-1}} \pi_f \pi_h = \pi_f$

$$\pi_{h^{-1}} \pi_f \pi_h = \sum_{g \in G} f(g) \pi_{h^{-1}} \pi_g \pi_h = \sum_g f(g) \pi_{\underbrace{h^{-1}gh}_{\text{relabel as } g'}}$$

$$= \sum_{g'} f(hg'h^{-1}) \pi_{g'} \underset{f \text{ is a class func.}}{=} \pi_f.$$

2) π_f is scalar (say α) by 1) & Schur's lemma.

$$\alpha = \frac{1}{\dim V} \mathrm{Tr}_{\mathrm{End}(V)}(\pi_f) = \frac{1}{\dim} \sum_g f(g) \underbrace{\mathrm{Tr}_{\mathrm{End}(V)}(\pi_g)}_{\chi_\pi(g)}.$$

□

$Z(\mathbb{C}[G]) : \{ f : G \to \mathbb{C} \text{ class func.} \}$.

Thm The irreducible characters of G form an orthonormal basis of $Z(\mathbb{C}[G])$ (w.r.t. $(- , -)_{L^2(G)}$)

Proof. $\pi^1, \cdots, \pi^k$ all irred. reps. of G, $\chi_i = \chi_{\pi^i}$
$V_1 \quad V_k$ (up to isom, mutually nonisom)

We alredy know that $(\chi_i)_i$ is orthonormal

$\Rightarrow$ Enough to show $Z(\mathbb{C}[G]) \underset{\text{orth. compl}}{\ominus} \langle \chi_1, \cdots, \chi_k \rangle = 0$

i.e. $f \in Z(\mathbb{C}[G])$, $\forall i$ $(f, \chi_i)_{L^2(G)} = 0 \Rightarrow f = 0$

Step 1. f as above $\Rightarrow \pi^i_f = 0$

$\because$ $(\pi^i)^c$ is also irred. & $\chi_{(\pi^i)^c}(g) = \overline{\chi_i(g)}$

$\Rightarrow \overline{\chi_i} = \chi_{\bar{i}}$ for some $\bar{i}$ $\underset{\text{assumption.}}{\Rightarrow} (f, \overline{\chi}_i) = 0$

$\underset{\text{Prop. 2)}}{\Rightarrow} \pi^i_f = 0$

Step 2. $\pi^i_f = 0$ for $\forall i \Rightarrow \lambda_f = 0$ (λ : reg. rep.)

$\because$ $\lambda \simeq \bigoplus_{i=1}^{k} (\pi^i)^{\oplus \dim V_i}$ (see last time)

$\Rightarrow \lambda_f \simeq \bigoplus (\pi^i_f)^{\oplus \dim V_i} = 0$

Step 3. $\lambda_f = 0 \Leftrightarrow f = 0$

$\because$ $V = \langle \delta_g : g \in G \rangle$ underlying sp. of λ

$\lambda_f \delta_e = \sum_{g \in G} f(g) \lambda_g \delta_e = \sum_g f(g) \delta_g$

So $\lambda_f = 0 \Rightarrow \lambda_f \delta_e = 0 \Rightarrow f = 0$ $\square$

Abstract version)

f is a class function $\Leftrightarrow \sum_{g \in G} f(g) \cdot g$ is in the <u>center</u> of $\mathbb{C}[G]$.

(same computation as Prop 1))

We also know $\mathbb{C}[G] \simeq \prod_{i=1}^{k} \mathrm{End}(V_i)$.

Center of $\mathrm{End}(V_i) = \mathbb{C}\, Id_{V_i}$

$$\Rightarrow \underset{\text{center}}{Z}\left(\prod_{i=1}^{k} \mathrm{End}(V_i)\right) \simeq \mathbb{C}^k \Rightarrow \dim Z(\mathbb{C}[G]) = k.$$

Rem. $\dim Z(\mathbb{C}[G]) = \#(\text{conjugacy classes of } G)$

$\because$ $C_1, \cdots, C_{k'}$ conj. cls. $\Rightarrow$ $(\delta_{C_i})_{i=1}^{k'}$ basis of $Z(\mathbb{C}[G])$

Cor. $g \in G$ 1) $c(g) = |\{hgh^{-1} : h \in G\}|$

$$\sum_{i=1}^{k} |\chi_i(g)|^2 = \frac{|G|}{c(g)}$$

2) h <u>not</u> conjug. to g $\Rightarrow \sum_{i=1}^{k} \chi_i(h)\overline{\chi_i(g)} = 0$

Proof. Step 1. f class func $\Rightarrow f = \sum_{i=1}^{k} \alpha_i \chi_i$

$$\alpha_i = (f, \chi_i)_{L^2(G)}$$

$\because$ $(\chi_i)_i$ is ONB.

Step 2. $X = \{hgh^{-1} : h \in G\}$.

$$\Rightarrow \delta_X = \sum_{i=1}^{k} \frac{c(g)}{|G|}\overline{\chi_i(g)}\, \chi_i$$

$\because$ $(\delta_X, \chi_i) = \frac{1}{|G|} \sum_{g' \underset{\text{conj}}{\sim} g} \overline{\chi_i(g')}$ [δ_X is a clss. func.

Step 3 1) : $1 = \delta_X(g) \underset{\text{Step 2}}{=} \frac{c(g)}{|G|} \sum_i \overline{\chi_i(g)}\chi_i(g)$

2) : $h \not\sim g \Rightarrow 0 = \delta_X(h) = \frac{c(g)}{|G|} \sum_i \overline{\chi_i(g)}\chi_i(h)$

Character table.

Draw table with

- column : conjugacy classes (or representatives g)
- row : irreducible characters χ_i
- entries : $\chi_i(g)$

- Cyclic groups : $G = \mathbb{Z}/n\mathbb{Z}$

	$[0]$	$[1]$	$\cdots$	$[n-1]$
$\psi^{(0)}$	1	1	$\cdots$	1
$\psi^{(1)}$	1	$e^{\frac{2\pi\sqrt{-1}}{n}}$	$\cdots$	$e^{\frac{2\pi\sqrt{-1}}{n}(n-1)}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\psi^{(n-1)}$	1	$e^{\frac{2\pi\sqrt{-1}}{n}(n-1)}$	$\cdots$	$e^{\frac{2\pi\sqrt{-1}}{n}(n-1)^2}$

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- $G = S_3$: irred. chars : χ_{triv}, χ_{sig}, $\chi_{\pi'}$ (last time)

conjug. classes $\{e\}$, $\{(1\ 2), (2\ 3), (3\ 1)\}$, $\{(1\ 2\ 3), (3\ 2\ 1)\}$

	e	$(1\ 2)$	$(1\ 2\ 3)$
χ_{triv}	1	1	1
χ_{sig}	1	-1	1
$\chi_{\pi'}$	2	0	-1

($= \#$(fixed points) -1)

Ex. $c((1\ 2)) = 3$ $\quad |\chi_{triv}((1\ 2))|^2 + |\chi_{sig}((1\ 2))|^2 + |\chi_{\pi'}((1\ 2))|^2$

$$= 1 + 1 = \frac{6}{3} = \frac{|S_3|}{|c((1\ 2))|}$$

$$\chi_{triv}((1\ 2))\,\chi_{triv}((1\ 2\ 3)) + \chi_{sig}((1\ 2))\,\chi_{sig}((1\ 2\ 3)) + \chi_{\pi'}((1\ 2))\,\chi_{\pi'}((1\ 2\ 3))$$

$$= 1 \cdot 1 + (-1) \times 1 + 0 \times 1 = 0.$$

- $G = A_4 = \{\sigma \in S_4 : (-1)^{|\sigma|} = 1\}$

($= \{T \in SO(3) : T\Delta = \Delta\}$ $\quad \Delta$: reg. tetrahedron)

$|A_4| = \frac{|S_4|}{2} = 12$. $\quad A_4 = K_4 \rtimes (\mathbb{Z}/3\mathbb{Z})$

conjug. classes. $\{e\}$. $\{(i\ j)(k\ l) : i \sim l \text{ all diff}\}$ three elems.

$\{(1\ 2\ 3), (2\ 4\ 3), (2\ 1\ 4), (3\ 4\ 1)\}$

$\{(3\ 2\ 1), (3\ 4\ 2), (4\ 1\ 2), (1\ 4\ 3)\}$

irreps : three 1-dim (from $\mathbb{Z}/3\mathbb{Z}$)

one 3-dim. χ_π

	e	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(3\ 2\ 1)$
$\mu^{(0)}$	1	1	1	1
$\mu^{(1)}$	1	1	$e^{\frac{2\pi\sqrt{-1}}{3}}$	$e^{\frac{4\pi\sqrt{-1}}{3}}$
$\mu^{(2)}$	1	1	$e^{\frac{4\pi\sqrt{-1}}{3}}$	$e^{\frac{2\pi\sqrt{-1}}{3}}$
χ_π	3	-1	0	0

π : from $A_4 \to SO(3) \curvearrowright \mathbb{R}^3 \underset{sc.\,ext}{\rightsquigarrow} \mathbb{C}^3$

Summary
- Frobenius reciprocity
  - duality
- McKay correspondence
  (classification of finite subgroups of $SU(2)$)

Frobenius reciprocity

G finite group, (π, V), (π', V'), (π'', V'') : reps of G

$\dim \mathrm{Hom}_G(V, V' \otimes V'') = \dim \mathrm{Hom}_G(V \otimes \underbrace{(V'')^*}_{\text{contragred. rep}}, V')$

Rem. π irred, (ρ, W) another rep

$\Rightarrow \dim \mathrm{Hom}_G(V, W)$ is "the number of times" π happens in ρ.

$\rho = \pi_1 \oplus \cdots \oplus \pi_k$ irr. decomp.

$\Rightarrow \dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(V, \oplus V_i)$

$= \sum \underbrace{\dim \mathrm{Hom}_G(V, V_i)}_{1 \text{ if } \pi \simeq \pi_i,\ 0 \text{ otherwise}}$

Proof of reciprocity.

Step 1 : $(e_i)_i$ basis of V, $(e^i)_i$ dual basis of V^*

$\Rightarrow R_\pi : \mathbb{C} \to V^* \otimes V$, $\alpha \mapsto \alpha(\sum_i e^i \otimes e_i)$

$\bar{R}_\pi : \mathbb{C} \to V \otimes V^*$, $\alpha \mapsto \alpha(\sum_i e_i \otimes e^i)$

are intertwiners.

$\because$ $\sum e_i \otimes e^i \leftrightarrow \mathrm{Id}_V \in \mathrm{End}(V)$ fixed by $\mathrm{Ad}_g(T) = \pi_g T \pi_g^{-1}$

Step 2 With (invariant) inner prod. on V,

$(\bar{R}_\pi^* \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R_\pi) = \mathrm{Id}_V$

$(T : V \to V' \rightsquigarrow T^* : V' \to V$, $(Tv, v') = (v, T^* v'))$

$\because$ $(e_i)_i$: orthonormal basis $\Rightarrow R(\alpha) = \sum_i \bar{e}_i \otimes e_i$, etc.

$(\bar{R}^* \otimes \mathrm{Id})(\mathrm{Id} \otimes R) = (\sum e_i^* \otimes \bar{e}_i^* \otimes \mathrm{Id})(\sum \mathrm{Id} \otimes \bar{e}_j \otimes e_j)$

$= \sum_i e_i^* \otimes e_i = \mathrm{Id}.$

Step 3 $\mathrm{Hom}_G(V, V' \otimes V'') \to \mathrm{Hom}_G(V \otimes (V'')^*, V')$,

$$T \mapsto (\mathrm{Id}_{V'} \otimes \bar{R}^*_{\pi''})(T \otimes \mathrm{Id}_{(V'')^*})$$

$\mathrm{Hom}_G(V \otimes (V'')^*, V') \to \mathrm{Hom}_G(V, V' \otimes V'')$

$$S \mapsto (S \otimes \mathrm{Id}_{V''})(\mathrm{Id}_V \otimes R_{\pi''})$$

are inverse to each other.

$\therefore$ Want : $(\mathrm{Id} \otimes \bar{R}^*_{\pi''} \otimes \mathrm{Id})(T \otimes \mathrm{Id}_{(V'')^* \otimes V''})(\mathrm{Id}_V \otimes R_{\pi''})$
$= T$.

By $(T \otimes \mathrm{Id}_{(V'')^* \otimes V''})(\mathrm{Id}_V \otimes R_{\pi''}) = T \otimes R_{\pi''}$
$= (\mathrm{Id} \otimes R_{\pi''})T$

we can reduce this to step 2. $\square$

McKay correspondence

Finite subgroups of $SU(2)$ $\leftrightarrow$ finite groups with 2-dim <u>faithful</u> representation by "det = 1" matrices.

$SU(2) = \{ U \in M_2\mathbb{C} : U^* U = I_2,\ \det U = 1 \}$

$SO(3) = \{ X \in M_3\mathbb{R} : X^t X = I_3,\ \det X = 1 \}$.

Prop. $\exists\ \varphi : SU(2) \to SO(3)$, $\ker \varphi = \{\pm I_2\}$.

Proof. $V_0 = \{ X \in M_2\mathbb{C} : X^* = -X,\ \mathrm{Tr}\, X = 0 \}$

3-dim real sp. with basis $\begin{bmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{bmatrix}, \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}$

inner prod $(X, Y) = \mathrm{Tr}(X^* Y) = -\mathrm{Tr}(XY)$.

(above basis = $2\times$ orthonormal basis)

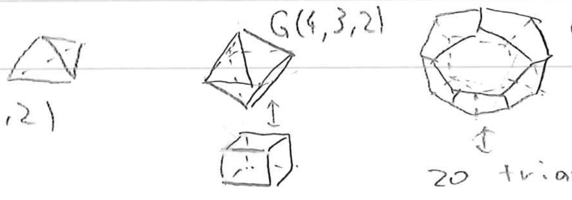
$SU(2)$ acts by $\mathrm{Ad}_U(X) = UXU^*$

preserving (X, Y) $\rightsquigarrow$ $SU(2) \to SO(3)$ $\square$

$G \subset SU(2)$ finite subgroup $\Rightarrow \varphi(G) \subset SO(3)$

Possibilities of $\varphi(G)$:

- cyclic groups $\mathbb{Z}/n\mathbb{Z}$: rotation around one axis
- dihedral groups $(\mathbb{Z}/n\mathbb{Z}) \rtimes_{\pm 1} (\mathbb{Z}/2\mathbb{Z})$: symmetry of regular polygon.
- symmetries of regular polyhedra



Possibilities of G:

- cyclic groups $\mathbb{Z}/n\mathbb{Z}$: $\left\langle \begin{bmatrix} e^{\frac{2\pi\sqrt{-1}}{n}} & 0 \\ 0 & e^{-\frac{2\pi\sqrt{-1}}{n}} \end{bmatrix} \right\rangle$
- inv. img of dihedr. grps $(\mathbb{Z}/2n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$
- inv. img of reg. polyhed. grps.

Drawing graph $\Gamma = (V, E)$ out of G

- vertex set $V = \{\pi_0, \ldots, \pi_k : \text{irred. reps of } G\}$
  π_0: trivial rep.
- edges $\pi_i \to \pi_j$: as many as $A_{ij} = \dim \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho)$

Prop. 1) Γ is unoriented $A_{ij} = A_{ji}$

2) $A_{ii} = 0$ unless G is trivial

3) $A_{ij} \leq 1$ $(i \neq j)$

Proof 1) ρ is self dual $(\rho, \mathbb{C}^2) \simeq (\rho^c, (\mathbb{C}^2)^*)$

isom : $e_1 \mapsto e_2$, $e_2 \mapsto -e_1$

i.e. $R : \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2$, $\alpha \mapsto \alpha(e_1 \otimes e_2 - e_2 \otimes e_1)$

$\bar{R} = -R$ give a model of $(\rho^c, \mathbb{C}^{2*})$

$A_{ji} = \dim \mathrm{Hom}_G(\pi_j, \pi_i \otimes \rho) \underset{\text{Frob.}}{=} \dim \mathrm{Hom}_G(\pi_j \otimes \rho^c, \pi_i)$

$\underset{\rho^c \cong \rho}{=} \dim \mathrm{Hom}_G(\pi_j \otimes \rho, \pi_i) = \dim \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho)$
$T \mapsto T^*$

2) $A_{ii} \leq 2$ by dim counting $(\dim(\pi_i \otimes \rho) = 2 \dim \pi_i)$

$A_{ii} = 1$ cannot happen.

$\because f : \mathrm{Hom}_G(\pi_i, \pi_i \otimes \rho) \to \mathrm{Hom}_G(\pi_i, \pi_i \otimes \rho)$
$T \mapsto (T^* \otimes \mathrm{Id})(\mathrm{Id} \otimes R)$

is conjug. lin & $f^2 = -\mathrm{Id}$.

But conj. lin $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is $\alpha \mapsto \bar{\alpha}\lambda$ $(\lambda = f(1))$

$\rightsquigarrow f^2(\alpha) = \alpha|\lambda|^2 \neq -\alpha$.

$A_{ii} = 2$ means $\pi_i \otimes \rho \cong \pi_i \oplus \pi_i$.

$\exists k : \lambda \subset \rho^{\otimes k} \Rightarrow \pi_i^{\oplus 2^k} \supset \lambda \Rightarrow G$ trivial.

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3) $\sum_{j=0}^{k} A_{ij}^2 = (\chi_{\rho\otimes\pi_i}, \chi_{\rho\otimes\pi_i})_{L^2(G)} = \frac{1}{|G|}\sum_g |\chi_\rho(g)|^2 |\chi_{\pi_i}(g)|^2$

$|\chi_\rho(g)| \leq 2$ and $\frac{1}{|G|}\sum_g |\chi_{\pi_i}(g)|^2 = 1$

$\Rightarrow \sum_j A_{ij}^2 \leq 4$.

Inequality happens $\Rightarrow$ $|A_{ij}| \leq 1$

Equality happens $\Rightarrow$ $\forall g$ $\underbrace{|\chi_\rho(g)| = 2}_{\text{only for } g = \pm I_2}$ or $|\chi_{\pi_i}(g)| = 0$

$A_{ij} = 2 \Rightarrow \rho\otimes\rho_i \simeq \rho_j \oplus \rho_j$, $\rho\otimes\rho_j \simeq \rho_i \oplus \rho_i$

$-I_2$ acts trivially on ρ_i or ρ_j (say ρ_i)

$\rightsquigarrow$ ρ_i is an irred. rep. of $H = G/G\cap\{\pm I_2\}$.

$\rightsquigarrow$ $\chi_{\rho_i}(h) = d_i \delta_e(h)$ on H ($d_i = \dim V_i$)

i.e. reg. character of H belongs to the lin. span of χ_{ρ_i} $\Rightarrow$ H trivial. $\Rightarrow$ ρ_i also

$\Rightarrow$ ρ trivial. $\Rightarrow$ $i = j$, G triv. ∎

Prop. $d_i = \dim V_i$

1) $2d_i = \sum A_{ij} d_j$

2) $\chi_\rho(g)\chi_{\pi_i}(g) = \sum A_{ij}\chi_{\pi_j}(g)$

Proof. 1) $2d_i = \dim(\rho\otimes\pi_i) = \dim \bigoplus \pi_j^{\oplus A_{ji}}$

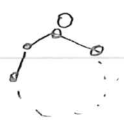
$= \sum \underbrace{A_{ji}}_{A_{ij}} d_j$

2) use same decomp, take character.

Recap: Γ has norm 2 $\|A\|_{B(L^2V)} = 2$

without loops around vertices

$G = \mathbb{Z}/n\mathbb{Z}$ $\rightsquigarrow$

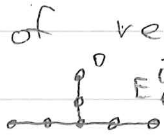


$A_n^{(1)}$ $n+1$ vertices.

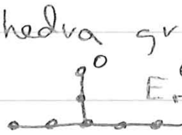
inv. img. of $(\mathbb{Z}/2(k-2)) \rtimes (\mathbb{Z}/2\mathbb{Z})$

$D_k^{(1)}$ $(k+1)$-vertices

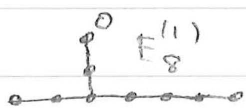
inv. img of reg. polyhedra grps



$E_6^{(1)}$



$E_7^{(1)}$



$E_8^{(1)}$

Summary
- Lie algebras
  - subalgs, ideal, center
- Representation of Lie algs
- Universal enveloping algs

∘ Lie algebra

K : field ($\mathbb{R}$ or $\mathbb{C}$ for us)

A <u>Lie alg.</u> over K is :
- K-vector space $\mathfrak{g}$ (fin. dim for us)
- bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$ s.t.
  "bracket"

  $[X, Y] = -[Y, X]$ (alternating)

  $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

  (Jacobi identity)

Rem : Lie alg $\equiv$ infinitesimal model of continuous grps (tomorrow)

Examples

1) $[X, Y] = 0$ for all X, Y. (<u>commutative</u>)

2) $V = K$-vector space, $\mathfrak{g} = \mathrm{End}(V)$ (as lin. sp.)
bracket $[X, Y]v = XYv - YXv$. $(v \in V)$
to emphasize this bracket we write $\mathfrak{gl}(V) = \mathfrak{g}$.
$\mathfrak{gl}_n(K) = \mathfrak{gl}(K^n)$ $(= M_n(K)$ as lin. sp.$)$

2') $\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) : \mathrm{Tr}(X) = 0\}$
$\mathfrak{sl}_n(K) = \mathfrak{sl}(K^n)$.

3) $\mathfrak{u}_n = \{X \in \mathfrak{gl}_n(\mathbb{C}) : X^* (= \bar{X}^t) = -X\}$ real Lie algs
$\mathfrak{su}_n = \{X \in \mathfrak{u}_n : \mathrm{Tr}\, X = 0\} = \{X \in \mathfrak{sl}_n(\mathbb{C}) : X^* = -X\}$

Rem. $\mathfrak{g}$ real Lie alg. $\rightsquigarrow$ $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \{\sum_{i=1}^{k} X_i \otimes \alpha_i : X_i \in \mathfrak{g}, \alpha_i \in \mathbb{C}\}$
cplx Lie alg.

Ex. $(\mathfrak{su}_n)_{\mathbb{C}} \simeq \mathfrak{sl}_n(\mathbb{C})$, $(\mathfrak{u}_n)_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$.
$\mathfrak{su}_2$ has basis $\begin{bmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{bmatrix}, \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}$.
(skew-Hermitian) Pauli matrices

these are basis of $\mathfrak{sl}_2(\mathbb{C})$: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = -\frac{1}{2}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \sqrt{-1}\begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}\right)$

<u>homomorphism</u> of Lie algs : $f : \mathfrak{g}_1 \to \mathfrak{g}_2$

linear map s.t. $[f(X), f(Y)] = f([X, Y])$.

<u>Lie subalgebra</u> of $\mathfrak{g}$: subspace $\mathfrak{h} \subset \mathfrak{g}$ s.t.

$X, Y \in \mathfrak{h} \Rightarrow [X, Y] \in \mathfrak{h}$.

<u>ideal</u> of $\mathfrak{g}$: $\mathfrak{h} \subset \mathfrak{g}$ subspace, $X \in \mathfrak{g}$, $Y \in \mathfrak{h} \Rightarrow [X, Y] \in \mathfrak{h}$

Examples.

1) $\mathfrak{sl}_n(\mathbb{K}) \underset{\text{ideal}}{\subset} \mathfrak{gl}_n(\mathbb{K}) \underset{\text{ideal}}{\supset} \mathbb{K}I_n$ $\left(\begin{array}{l} \mathrm{Tr}([X, Y]) = 0 \\ [X, I_n] = 0. \end{array} \right.$

$\Rightarrow \mathfrak{gl}_n(\mathbb{K}) = \mathfrak{sl}_n(\mathbb{K}) \oplus \mathbb{K}$

2) $\mathfrak{c} = \mathfrak{z}(\mathfrak{g}) = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ [X, Y] = 0 \}$

<u>center</u> of $\mathfrak{g}$: ideal.

$\mathfrak{g}$ is <u>simple</u> if 0 and $\mathfrak{g}$ are the only ideals.

· $\mathfrak{sl}_n\mathbb{K}$, $\mathfrak{su}_n$ simple

3) $\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} : a, b \in \mathbb{K}. \right\}$. $\left[\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}, \begin{bmatrix} a' & b' \\ 0 & -a' \end{bmatrix} \right] = \begin{bmatrix} 0 & 2(ab' - ba') \\ 0 & 0 \end{bmatrix}$.

$\mathfrak{h}_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} : a \in \mathbb{K} \right\}$ subalg but not ideal ($\mathrm{Char}\,\mathbb{K} \neq 2$)

$\mathfrak{h}_2 = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbb{K} \right\}$ ideal

4) $\mathfrak{so}_n(\mathbb{K})$

· Representation of Lie algebras

$\mathfrak{g}$: Lie alg over $\mathbb{K}$, V : vec. sp. over $\mathbb{K}$ write $\pi : \mathfrak{g} \curvearrowright V$

Linear <u>reprecention</u> of $\mathfrak{g}$ over V is given by :

linear map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ $X \mapsto \pi_X$ s.t.

$\pi_X \pi_Y v - \pi_Y \pi_X v = \pi_{[X, Y]} v$ $(v \in V)$

i.e. Lie alg hom $\mathfrak{g} \to \mathfrak{gl}(V)$

Rem. (π, V) rep of $\mathfrak{g}$ $\leftrightarrow$ Lie alg. structure on $\mathfrak{g} \oplus V$

$\supset \mathfrak{g}$ subalg

$\supset V$ ideal s.t.

· $[v, v'] = 0$

$[X \oplus v, X' \oplus v'] = [X, X'] \oplus (\pi_X v' - \pi_{X'} v)$

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Example: adjoint representation $ad: V = \mathfrak{g}$, $ad_X Y = [X, Y]$

Jacobi identity $\Leftrightarrow [ad_X, ad_Y] = ad_{[X,Y]}$.

center $\mathfrak{z}(\mathfrak{g}) \ni X \Leftrightarrow ad_X = 0$.

Fact: (Ado's theorem) $\dim_K \mathfrak{g} < \infty \Rightarrow \exists$ faithful rep. of $\mathfrak{g}$. ($X \neq 0 \Rightarrow \pi_X \neq 0$) (difficult)

Unitary representation: $\mathfrak{g}$: real Lie alg.

V: $\mathbb{C}$-vec. sp. with Hermitian inn. prod. ($\subset$ Hilbert sp.)

$\pi: \mathfrak{g} \curvearrowright V$ with $(\pi_X v, v') + (v, \pi_X v') = 0$

Ex. $\mathfrak{g} = \mathbb{R}$ (with zero bracket),

$V = \mathcal{S}(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{C}$ rapid decay $\forall m,n \ \sup_x |x^m \partial_x^n f(x)| < \infty\}$

$\pi_t f = t \partial_x f$ $\qquad (f_1, f_2) = \int f_1(x) \overline{f_2(x)}\, dx$

· Universal enveloping algebra

$\mathfrak{g}$: Lie alg. over K. $\mathcal{U}(\mathfrak{g})$: (unital) K-alg. with

· generators: $X \in \mathfrak{g}$

· relations: $\underbrace{X \cdot Y - Y \cdot X}_{\text{computation in } \mathcal{U}(\mathfrak{g})} = \underbrace{[X, Y]}_{\text{computation in } \mathfrak{g}}$

Formally: tensor algebra $T(\mathfrak{g}) = K \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots$

with prod. $(X_1 \otimes \cdots \otimes X_m) \cdot (Y_1 \otimes \cdots \otimes Y_n) = X_1 \otimes \cdots \otimes X_m \otimes Y_1 \otimes \cdots \otimes Y_n$

take bilateral ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$

$\rightsquigarrow \mathcal{U}(\mathfrak{g})$: quotient.

Rem. A K-alg. $\rightsquigarrow \mathfrak{gl}(A)$: A as vec sp. $[a,b] = ab - ba$.

Lie alg hom $\mathfrak{g} \to \mathfrak{gl}(A) \simeq$ alg hom $\mathcal{U}(\mathfrak{g}) \to A$.

i.e. left adjoint functor of $A \mapsto \mathfrak{gl}(A)$.

Rem. representation of $\mathfrak{g}$ = $\mathcal{U}(\mathfrak{g})$-module

$\pi: \mathfrak{g} \curvearrowright V \rightsquigarrow \pi: \mathcal{U}(\mathfrak{g}) \curvearrowright V$ by

$\pi_{X_1 \cdots X_m} v = \pi_{X_1} \cdots \pi_{X_m} v$

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Poincaré - Birkhoff - Witt theorem) $(X_i)_{i=1}^N$ basis of $\mathfrak{g}$ (N can be infinite) $\Rightarrow$ $\mathcal{U}(\mathfrak{g})$ has a basis $(X_1^{i_1} \cdots X_k^{i_k})_{k<\infty,\, k\le N,\, i_j\ge 0}$

Idea : any "noncomm. monomial" $X_{m_1} X_{m_2} \cdots X_{m_n}$ is a linear combination of above by induction on Σm_i E.g. :

$$X_3 X_1 = X_1 X_3 + \underbrace{[X_3, X_1]}_{\text{can be written as sum of } (X_i)_i}$$

Rem. $F_n = \langle Y_1 \cdots Y_m : m \le n,\ Y_i \in \mathfrak{g} \rangle_{K\text{-span}}$

increasing seq. of subspaces, $F_n F_{n'} \subset F_{n+n'}$

$\mathcal{U}(\mathfrak{g}) = \bigcup_{n=0}^{\infty} F_n \Rightarrow \mathcal{U}(\mathfrak{g})$ is a filtered alg

PBW $\Rightarrow$ assoc. graded $\bigoplus_{n=0}^{\infty} (F_n/F_{n-1}) \simeq \mathrm{Sym}(\mathfrak{g})$.

symmetric alg.

---

4) $\mathfrak{so}_n(K) = \{ X \in M_n(K) : X^t = -X,\ \mathrm{Tr}\, X = 0 \}$

$[X,Y]^t = (XY - YX)^t = [Y^t, X^t] = [Y, X] = -[X,Y]$

$\rightsquigarrow$ Lie subalg of $\mathfrak{sl}_n(K)$

Rem $\mathfrak{su}_2 \simeq \mathfrak{so}_3(\mathbb{R})$

$\mathfrak{so}_3(\mathbb{R})$ has basis $Y_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

compare with Pauli matrices & figure out rescaling factor

Summary

- Topological groups
  - Lie groups
- Lie groups to Lie algebras

- Topological groups.

Hausdorff top. sp. with group structure

$G \times G \to G \quad (g,h) \to gh, \qquad G \to G \quad g \mapsto g^{-1}$ cont.

Ex. discrete groups (discr. top.)

$\mathbb{R}, \ \mathbb{T} = \mathbb{R}/\mathbb{Z}, \cdots$

$\mathbb{Z}_p = \lim\limits_{n \to \infty} \mathbb{Z}/p^n\mathbb{Z} = \{ \sum_{k=0}^{\infty} a_k p^k : 0 \le a_k < p \}$

<u>Lie group</u> is a top. grp. s.t.

- $\exists$ neighborhood U of e, homeo. to an open set of $\mathbb{R}^n$ ($\varphi : U \hookrightarrow \mathbb{R}^n$ coord. func.)
- group laws are smooth around e

$U \times U \to G, \ (g,h) \mapsto gh^{-1}$ "looks like" a smooth map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ around (e,e)

Alternatively: top. group which is a C^∞-mfd.

(C^1-mfd is enough).

$G = \bigcup_{i \in I} U_i \qquad \varphi_i : U_i \hookrightarrow \mathbb{R}^n$ img open

$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ smooth

& smooth group law.

Examples

- $GL_n(\mathbb{R}), SL_n(\mathbb{R}), SO_n(\mathbb{R}), \cdots \qquad \mathbb{R}^n, \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, \cdots$
- $U_n \subset GL_n(\mathbb{C}) \subset GL_{2n}(\mathbb{R})$

$\uparrow \mathbb{C}^n \simeq \mathbb{R}^{2n}$ as real vec. sp.

- Lie groups to Lie algs

Tangent space at g = {"tangent vectors" at g}

$\rightsquigarrow$ formalize as "directional derivatives"

$g \in G$, U neigh. $U \xrightarrow{\varphi} \mathbb{R}^n$ homeo to open set

f: smooth func. around g $\rightsquigarrow$ $f \circ \varphi^{-1}$: smooth. around $\varphi(g) =: p$

$\rightsquigarrow$ partial derivatives $\partial_{x_i} f \circ \varphi^{-1}(p)$

$\alpha_1, \dots, \alpha_n \in \mathbb{R}$ $\rightsquigarrow$ $\xi_g(f) = \sum \alpha_i \partial_{x_i} f \circ \varphi^{-1}(p)$

such ξ_g's span an n-dim real vec. sp.

Intrinsically: $T_g G = \{ \xi : C^\infty(G) \to \mathbb{R}$
$\xi(f_1 f_2) = f_1(g)\xi(f_2) + \xi(f_1) f_2(p) \}$

is the <u>tangent space</u> at g. <u>Leibniz rule</u> at g

· invariant vector fields

G Lie group. $\xi \in T_e G$ $\rightsquigarrow$ two ways to get "translate" of ξ at $g \in G$:

ξ^l_g : $f \mapsto ((R_g)_\# \xi)(f) = \boxed{\xi(f \circ R_g)}$ $R_g : h \mapsto hg$

ξ^r_g : $f \mapsto ((L_g)_\# \xi)(f) = \boxed{\xi(f \circ L_g)}$ $L_g : h \mapsto gh$

satisfy the Leibniz rule at g

$f \in C^\infty(G) \rightsquigarrow (\xi^l_g(f))_{g \in G}$ is smooth in g.

$\xi^l = (\xi^l_g)$ is a <u>vector field</u> (the <u>right invariant</u> vec. field corresp. to ξ)

$\xi^l_{gh} = (R_g)_\# \xi^l_h$. right invariance

similarly $\xi^r = (\xi^r_g)_g$ <u>left</u> <u>invariant</u> vec. field

Prop. $G = GL_n(\mathbb{R})$ $f_{ij}(g) = g_{ij}$ (i,j)-entry of g

$F = (f_{ij})_{i,j=1}^n$ matrix of smooth funcs.

$X \in M_n(\mathbb{R}) = \mathfrak{gl}_n(\mathbb{R})$

0) $X(f) = \partial_t f(e^{tX})|_{t=0}$ $(f \in C^\infty(G))$

is a tangent vec. at $e = I_n$.

1) $X(f_{ij}) = X_{ij}$

2) $(X^l(f_{ij}))_{i,j} = X \cdot F$ matrix prod.

3) $(X^r(f_{ij}))_{i,j} = F \cdot X$

Proof. 1) $X(f_{ij}) = \partial_t\,((i,j)\text{-comp. of } I_n + tX + \frac{t^2}{2}X^2 + \cdots)$

$= (i,j)\text{-comp. of } \partial_t(I_n + tX + O(t^2))|_{t=0}$

$= X_{ij}$

2) $X^\ell(f_{ij})(g) = X(f_{ij} \circ R_g)$

$f_{ij} \circ R_g : h \mapsto (i,j)\text{-comp. of } hg$

$= \sum_k f_{ik}(h)\, g_{kj}$

$= \sum_{k=1}^n f_{ik}\, g_{kj}$

So $X(f_{ij} \circ R_g) = \sum X(f_{ik})\, g_{kj} = \sum X_{ik}\, g_{kj}$.

$\rightsquigarrow$ as a func. in g, eq. to XF.

3) similar

Lie alg structure on $\mathfrak{g} = T_eG$

$X, Y \in T_eG \rightsquigarrow X^\ell, Y^\ell$ right inv. vec. fields

$\Rightarrow [X^\ell, Y^\ell](f) = X^\ell(\underbrace{Y^\ell(f)}_{g \mapsto Y^\ell_g(f)}) - Y^\ell(X^\ell(f))$ "comm. bracket"

is still a right inv. vec. field

$\because [X^\ell, Y^\ell](f_1 f_2) = X^\ell(f_1 \cdot Y^\ell(f_2) + Y^\ell(f_1) \cdot f_2)$

$- Y^\ell(f_1 X^\ell(f_2) + X^\ell(f_1) f_2)$

$= f_1 X^\ell Y^\ell(f_2) + X^\ell Y^\ell(f_1) f_2 - f_1 Y^\ell X^\ell(f_2) - Y^\ell X^\ell(f_1) f_2$

$= f_1 [X^\ell, Y^\ell](f_2) + [X^\ell, Y^\ell](f_1) f_2$ Leibniz rule

right invariance from that of X^ℓ, Y^ℓ.

$\rightsquigarrow [X^\ell, Y^\ell] = Z^\ell$ for some $Z \in T_eG$

$[X, Y] := Z$.

Rem if we use X^r, Y^r, we get $(-1) \times$ above cf. Prop.

- Lie groups with same Lie algs

$\Gamma \triangleleft G$ (normal) G connected $\Rightarrow \Gamma < Z(G)$

discrete $\Rightarrow G/\Gamma$ has the "same" tang. sp at e as G.

$T_e(G/\Gamma) = T_eG$. brackets are also same

Universal cover (G: connected Lie grp)

concrete: $\tilde{G} = \{$ homotopy classes of paths from $e \}$

$= \{ \gamma : [0,1] \to G,\ \gamma(0) = e \} /$ homotopy

abstract: $\tilde{G} \xrightarrow{p} G$ $\tilde{G}$ simply connected

p local homeo.

Group structure (for concrete)

"concatenate" paths

$$\gamma' \cdot \gamma : \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1)\gamma(1) & \frac{1}{2} \le t \le 1 \end{cases}$$

path from e to $\gamma'(1) \cdot \gamma(1)$

neutral elem: const path at e.

$\rightsquigarrow \tilde{G}$ Lie grp, $\tilde{G} \to G$, $\gamma \mapsto \gamma(1)$ grp hom.

(local homeo)

Lie algs to Lie groups

· exponential map.

G : Lie group $\mathfrak{g} = T_eG$. tangent space at e

$X \in \mathfrak{g} \rightsquigarrow X^\ell$ right inv. vec. field $X^\ell_g(f) = X((R_g)^*(f))$

by general mfd. theory. $\exists\, \varepsilon > 0$, $\varphi_X(0) = e$

$\exists!\ \varphi_X : (-\varepsilon, \varepsilon) \to G$ s.t. $\varphi_X'(t) = X^\ell(\varphi_X(t))$.

i.e. $\partial_t f(\varphi_X(t))|_{t=t_0} = X^\ell_{\varphi_X(t_0)}(f)$

Prop. $\varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$ for $|s|, |t|, |s+t| < \varepsilon$

Proof. Fix t & consider deriv. at $s = s_0$

Left hand side $\partial_s f(\varphi_X(s)\varphi_X(t))|_{s=s_0}$

$= X^\ell_{\varphi(s_0)}(R_{\varphi_X(t)}{}^* f) = X^\ell_{\varphi_X(s_0)\varphi_X(t)}(f)$

Right hand side $\partial_s f(\varphi_X(s+t))|_{s=s_0} = X^\ell_{\varphi_X(s_0+t)}(f)$

$\Rightarrow$ as maps $(-\varepsilon', \varepsilon') \to G$ $\varepsilon' \ll 1$

$s \mapsto \varphi_X(s)\varphi_X(t)$ and $s \mapsto \varphi_X(s+t)$

are both $0 \mapsto \varphi_X(t)$, & has X^ℓ as "velocity"

$\underset{\text{uniqueness}}{\Rightarrow}$ they are same. ◻

Cor. φ_X extends to $\mathbb{R} \to G$ s.t. $\varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$

$\because$ Fix $t \in \mathbb{R}$, take $N \gg 1$ s.t. $|\frac{t}{N}| < \varepsilon$.

for ε in Prop. $\rightsquigarrow$ Put $\varphi_X(t) = \varphi_X(\frac{t}{N})^N$.

Prop $\Rightarrow$ this is well def'ed & satisfies claim.

<u>exponential</u> map : $\exp : \mathfrak{g} \to G$ $X \mapsto \varphi_X(1)$

Ex. $G = SU(n)$ $\mathfrak{g} = \mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) : X^* = -X\}$

$\varphi_X(t) = e^{tX} = I_n + tX + \frac{t^2}{2}X^2 + \cdots$

$\exp(X) = e^X = I_n + X + \cdots + \frac{1}{k!}X^k + \cdots$

$\therefore\ e^{sX}e^{tX} = e^{(s+t)X}$ from comparison of coeffs.

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$(\varphi_X(t) = \exp(tX))_{t\in\mathbb{R}}$: <u>one parameter group</u> generated by X (in G)

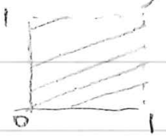
Lie subgroups.

N, M : manifolds, $j : N \to M$ inj. map (differentiable, ..) s.t. $j_\# : T_xN \to T_{f(x)}M$ is inj.

$X \mapsto (f \mapsto X(f\circ j))$

$\rightsquigarrow$ top of N does not have to be the same as the top. of img.

Ex. $\mathbb{R} \to \mathbb{T}^2$ (group hom!)



irrational slope

$\mathbb{T}^2$ has Lie alg $\mathbb{R}^2$, img of this emb. should corr. to $\{(t, \theta t) : t\in\mathbb{R}, \theta : \text{irr.}\}$ (subalg)

In the above situation, the img of j is called a <u>immersed</u> submfd.

Prop 2 G Lie group, $\mathfrak{g} = T_eG$ its Lie alg

$\mathfrak{h} \subset \mathfrak{g}$ Lie subalg

$\Rightarrow \exists$ immersed subgroup ($j : H \to G$ inj. hom) s.t. $\mathfrak{h} = T_eH$.

Proof : Step 1 H as a set $= \{h_1 \cdots h_k : h_i = \exp(X_i), X_i \in \mathfrak{h} \ \forall k\}$

this is a subgroup of G $(\exp(X)^{-1} = \exp(-X))$

Step 2 $\exists$ neigh Δ of $0\in\mathfrak{g}$ s.t. $G_0 = \exp(\Delta)$

$H_0 = \exp(\Delta\cap\mathfrak{h})$, satisfies $(H_0 \cdot H_0)\cap G_0 = H_0$

$H_0^{-1} = H_0$.

$\because$ We use <u>Ado's theorem</u> $\rightsquigarrow$ we may

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assume $G = GL_n(\mathbb{R})$

- $g \in G$ close to I_n ("$\| I_n - g \| < 1$")

$\Rightarrow \log g = \log(I + (g - I)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(g-I)^n}{n}$

is well-def'ed.

- $X, Y \in \mathfrak{h}$ small $\Rightarrow \exp(X)\exp(Y)$ close to I_n
- Baker-Camptell-Hausdorff formula

$$\log(\exp(X)\exp(Y)) = \sum_{n=0}^{\infty} \frac{1}{n} \sum_{\substack{r_1 + \cdots + s_1 + \cdots + s_{m-1} = n-1 \\ r_1 + s_1 \geq 1,\ r_2 + s_2 \geq 1 \cdots}} \frac{(-1)^{m-1}}{m} \left(\prod^{m-1} \frac{\mathrm{ad}_X^{r_i}}{r_i!} \frac{\mathrm{ad}_Y^{s_i}}{s_i!} \right) \frac{\mathrm{ad}_X^{r_m}}{r_m!} (Y)$$

$$+ (X \leftrightarrow Y)$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \cdots$$

$\|X\|, \|Y\|$ small $\Rightarrow$ this converges

cf. $-\log(2 - \exp(x+y)) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{r_i, s_i \\ r_i + s_i \geq 1}} \frac{x^{r_1}}{r_1!} \cdots \cdots \frac{y^{s_m}}{s_m!}$

Step 3. H becomes a Lie group by setting

$H_0 \underset{\log}{\simeq} \Delta \cap \mathfrak{h}$ open in $\mathfrak{h} \simeq \mathbb{R}^k$.

make it a (open) neighborhood of e in H. ☒

Rem In the above, G connected

$\mathfrak{h}$ ideal in $\mathfrak{g} \Leftrightarrow H$ normal in G.

Th'm. G conn. Lie group $\mathfrak{g}$ its Lie alg

K another Lie grp, $\mathfrak{k}$ its Lie alg

$\varphi : \mathfrak{g} \to \mathfrak{k}$ Lie alg hom

Then $\exists$ Lie grp hom $\tilde{G} \xrightarrow{f} K$ inducing φ

Conseq : $K = GL_n(\mathbb{R}) \rightsquigarrow \tilde{G} \to K$ rep. of $\tilde{G}$

$\mathfrak{g} \to \mathfrak{k} = \mathfrak{gl}_n(\mathbb{R})$ rep of $\mathfrak{g}$

so rep of $\tilde{G}$ $\equiv$ rep of $\mathfrak{g}$.

Proof of thm

Set $\mathfrak{h} = \{ (X, \varphi(X)) : X \in \mathfrak{g} \} \subset \mathfrak{g} \oplus \mathfrak{k}$.

(graph of φ).

Step 1 φ hom $(\Leftarrow)\Rightarrow$ $\mathfrak{h}$ is a Lie subalg

Step 2 Regard $\mathfrak{g} \oplus \mathfrak{k}$ as the Lie alg of $\tilde{G} \times K$, H immersed subgroup corr. to $\mathfrak{h}$. Then $H \to \tilde{G}$ prj. to second factor is homeo.

$\because$ $\mathfrak{h} \to \mathfrak{g}$ is iso $\Rightarrow$ $H \to \tilde{G}$ is local homeo. $\underset{\tilde{G}\text{ simply conn.}}{\Rightarrow}$ $H \to \tilde{G}$ is homeo.

Step 3. Up to $H \cong \tilde{G}$ of Step 2,

The prj. to first factor induces φ. $\boxtimes$

2018.09.19 no. 1

Summary
- algebraic groups
- ideals in Lie algs.
  - derived series, lower central series
  - solvable / nilpotent algs

- Algebraic groups

K : (comm.) field

<u>Algebraic group</u> over K is given by
- alg. variety G over K

(separ. scheme of finite type over K, no nilpot in $\mathcal{O}_{G,g}$)
- morphisms $\mathrm{Spec}\,K \xrightarrow{e} G$, $G \times G \xrightarrow{\mu} G$, $G \xrightarrow{\mathrm{inv}} G$.

satisfying the usual axioms of groups

Cartier) ($\mathrm{Char}\,K = 0$) <u>affine</u> alg. grps $\equiv$ fin. gen. comm. Hopf alg over K

i.e. comm Hopf $\Rightarrow$ no nilpotents.

Ex.
- Elliptic curves : not affine
- (split) torus : $(K^\times)^n$
- $GL_n(K)$: $\mathcal{O}_{GL_n(K)} = K[(x_{ij})_{i,j=1}^{n} ; \det(x_{ij}) \neq 0]$

$$= K[(x_{ij})_{i,j}, \tilde{d}] / (\tilde{d} \times \det((x_{ij})_{i,j}) - 1)$$

- $SL_n(K)$: $\mathcal{O}_{SL_n(K)} = K[(x_{ij})_{i,j=1}^{n}] / (\det((x_{ij})_{i,j}) - 1)$

why do we want to consider this?

Compact Lie group G $\rightsquigarrow$ alg. of "matrix coefficients" of G $\equiv \mathcal{O}_{G_{\mathbb{C}}}$ for the complexification $G_{\mathbb{C}}$ of G

Ex. $SU(n) \rightsquigarrow SL_n(\mathbb{C})$, $SO(n) \rightsquigarrow SO_n(\mathbb{C})$

2018.09.19 no. 2

- Ideals in Lie algs.

$\mathfrak{g}$: Lie alg.

<u>derived subalgebra</u> (commutator subalg)

$\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] =$ span of $[X, Y]$ $(X, Y \in \mathfrak{g})$

(also denoted by $\mathfrak{g}'$)

Prop 1 $\mathcal{D}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$

Proof) Almost by def of $\mathcal{D}(\mathfrak{g})$: $X, Y, Z \in \mathfrak{g}$

$\Rightarrow [X, [Y, Z]] \in \mathfrak{g}$ since $[Y, Z] \in \mathfrak{g}$ ∎

<u>Lower central series</u> $\mathcal{D}_n(\mathfrak{g})$: $\mathcal{D}_1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$

$\mathcal{D}_{n+1}(\mathfrak{g}) = [\mathcal{D}_n(\mathfrak{g}), \mathfrak{g}]$

<u>Derived series</u> $\mathcal{D}^n(\mathfrak{g})$: $\mathcal{D}^1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$,

$\mathcal{D}^{n+1}(\mathfrak{g}) = [\mathcal{D}^n(\mathfrak{g}), \mathcal{D}^n(\mathfrak{g})] = \mathcal{D}(\mathcal{D}^n(\mathfrak{g}))$, $(\subset \mathcal{D}^{n+1}(\mathfrak{g}))$

Prop.2 $\mathcal{D}_n(\mathfrak{g})$ and $\mathcal{D}^n(\mathfrak{g})$ are ideals of $\mathfrak{g}$

$(\Rightarrow \mathcal{D}_{n+1}(\mathfrak{g}) \subset \mathcal{D}_n(\mathfrak{g}),\ \mathcal{D}^{n+1}(\mathfrak{g}) \subset \mathcal{D}^n(\mathfrak{g}))$

Proof : By induction on n. (we'll check $\mathcal{D}_n(\mathfrak{g})$)

$n=1$) By Prop 1

general case) suppose $\mathcal{D}_n(\mathfrak{g}) \triangleleft \mathfrak{g}$,

$X, Z \in \mathfrak{g}$, $Y \in \mathcal{D}_n(\mathfrak{g})$

We want $[X, \underbrace{[Y, Z]}_{\mathcal{D}_{n+1}(\mathfrak{g})}] \in \mathcal{D}_{n+1}(\mathfrak{g})$

By Jacobi id. (& antisymm.)

$$[X,[Y,Z]] = \underbrace{[[Z,X],Y]}_{\text{in } [\mathfrak{g}, \mathcal{D}_n(\mathfrak{g})] = \mathcal{D}_{n+1}(\mathfrak{g})} + \underbrace{[\overbrace{[X,Y]}^{\mathcal{D}_n(\mathfrak{g}) \text{ by assumption}}, Z]}_{\text{in } \mathcal{D}_{n+1}(\mathfrak{g})}.$$

∎

Rem $\mathcal{D}_n(\mathfrak{g}) / \mathcal{D}_{n+1}(\mathfrak{g})$ is in the center of $\mathfrak{g} / \mathcal{D}_{n+1}(\mathfrak{g})$

$\mathcal{D}^n(\mathfrak{g}) / \mathcal{D}^{n+1}(\mathfrak{g})$ is commutative

2018.09.19 no.3

$\mathfrak{g}$ is <u>solvable</u> if $\mathcal{D}^k(\mathfrak{g}) = 0$ for $k \gg 1$

<u>nilpotent</u> if $\mathcal{D}_k(\mathfrak{g}) = 0$ for $k \gg 1$

$\mathcal{D}^k(\mathfrak{g}) \subset \mathcal{D}_k(\mathfrak{g}) \rightsquigarrow$ nilpot. $\Rightarrow$ solv.

Ex. "$ax+b$"-alg. $\left\{ \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} : a, b \in K \right\} = \mathfrak{g}$

$$\left[\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}, \begin{bmatrix} a' & b' \\ 0 & -a' \end{bmatrix} \right] = \begin{bmatrix} 0 & 2(ab'-ba') \\ 0 & 0 \end{bmatrix}$$

Char $K \neq 2 \Rightarrow \mathcal{D}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in K \right\}$

$\mathcal{D}^2(\mathfrak{g}) = 0$, $\mathcal{D}_2(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$ ($= \mathcal{D}_k(\mathfrak{g})$ $k \geq 1$)

so $\mathfrak{g}$ is solvable but not nilpotent

Ex. $\mathfrak{g}$ comm $\Leftrightarrow \mathcal{D}(\mathfrak{g}) = 0$ ($\Rightarrow$ nilpot.)

Solvable algs have "flexible" (and complicated) reps.

Ex. $\mathfrak{g} = K^2$ (comm.)

Rep of $\mathfrak{g}$ on $V \equiv$ commuting transforms

$T_1 = \pi_{(1,0)}$, $T_2 = \pi_{(0,1)} \in End(V)$

T_1 can be normalized using Jordan normal form

$\rightsquigarrow$ T_2 on each gen. eigenspace of T_1, with compatibility with "flag" structure...

Prop 3 $\mathfrak{g}$ is solvable iff $\exists\ \mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = 0$ s.t. $\mathfrak{g}_{n+1} \lhd \mathfrak{g}_n$, $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ comm.

Proof. $\Rightarrow$: we can take $\mathfrak{g}_n = \mathcal{D}^n(\mathfrak{g})$

$\Leftarrow$: $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ comm $\Leftrightarrow \mathfrak{g}_{n+1} \supset \mathcal{D}(\mathfrak{g}_n)$

$\rightsquigarrow$ By induction $\underbrace{\mathfrak{g}_n}_{0} \supset \mathcal{D}^n(\mathfrak{g})$ ∎

Cor. $\mathfrak{g} \rhd \mathfrak{h}$: The followings are equiv:

1) $\mathfrak{g}$ is solvable 2) $\mathfrak{h}$ & $\mathfrak{g}/\mathfrak{h}$ are solv.

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Proof $\Rightarrow$: $(\mathfrak{g}_n)_n$ as in Prop 3

$\rightsquigarrow$ $\mathfrak{h}_n = \mathfrak{h} \cap \mathfrak{g}_n$, $\bar{\mathfrak{g}}_n$ = img of $\mathfrak{g}_n$ in $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$

do the job.

$\Leftarrow$: $(\mathfrak{h}_n)_n$ for $\mathfrak{h}$, $(\bar{\mathfrak{g}}_m)_m$ for $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$, as in Prop 3, $\bar{\mathfrak{g}}_N = 0$

$\rightsquigarrow$ $\mathfrak{g}_m$: inv. img of $\bar{\mathfrak{g}}_m$ in $\mathfrak{g}$. $(m \leq N)$ so $\mathfrak{g}_N = \mathfrak{h}$.

$\mathfrak{g}_{N+k} = \mathfrak{h}_k$ do the job ☒

Thm There is a (unique) largest solvable ideal of $\mathfrak{g}$.

Proof. $\boxed{\text{Enough to prove}}$ $\mathfrak{a}$, $\mathfrak{b}$: solv. ideal of $\mathfrak{g}$ $\Rightarrow$ so is $\mathfrak{a}+\mathfrak{b}$ ($\Rightarrow$ span of all solv. ideals will be the max. one)

$\mathfrak{a} \lhd \mathfrak{a}+\mathfrak{b}$, $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} = \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$

$\rightsquigarrow$ Use Cor. to $\mathfrak{a} \to \mathfrak{a}+\mathfrak{b} \to \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$. ☒

$\mathrm{Rad}(\mathfrak{g})$, <u>solvable</u> <u>radical</u> of $\mathfrak{g}$: the largest solvable ideal of $\mathfrak{g}$.

$\mathfrak{g}$ is <u>semisimple</u> if there is no (nonzero) solvable ideal of $\mathfrak{g}$ ($\mathrm{Rad}(\mathfrak{g}) = 0$)

Rem. Char $K = 0$ $\Rightarrow$ $\forall$ semisimple $\mathfrak{g}$ is $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, $\mathfrak{g}_i$ simple.

2018.09.25 no.1

Summary
- Structure of solvable Lie algs
  - Engel's th'm & Lie's th'm.
- Killing form
  - Cartan's criterion

- Structure of solvable Lie algs

Motto : solvable Lie algs are like upper triangular matrices

Engel's theorem) V : vec. sp over K,
$\mathfrak{g} \subset \mathfrak{gl}(V)$ Lie subalg s.t. $\forall X \in \mathfrak{g}$ is a nilpotent endomor. of V ($X^n = 0$ $n \gg 1$)
Then $\exists\, 0 \neq v \in V$ s.t. $\forall X \in \mathfrak{g}$ $Xv = 0$

Cor. $\exists$ basis $(v_1, v_2, \cdots, v_n)$ of V s.t. $\forall X \in \mathfrak{g}$ is "strictly upper triangular"
i.e. $Xv_1 = 0$, $Xv_2 \in Kv_1$, $\cdots$, $Xv_k \in \langle v_1, \cdots, v_{k-1} \rangle$, $\cdots$

Ex. $\mathfrak{g} = \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in K \right\} \subset \mathfrak{gl}_3(K)$ $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Proof of Cor. Induction on $\dim V = n$.

. Set $v_1 = v$ from Th'm, $V' = V / Kv_1$
$\rightsquigarrow$ $\mathfrak{g}$ acts on V' $X[w + v_1] = [Xw + Xv_1] = [Xw]$

. Ind. hypo. $\Rightarrow$ V' has basis $[v_2], \cdots, [v_n]$ s.t.
$\forall X \in \mathfrak{g}$ is rep'ed by str. up. triang. mat.
$X[v_k] \in \langle [v_2], \cdots, [v_{k-1}] \rangle$ means $Xv_k \in \langle v_1, \cdots, v_{k-1} \rangle$
$v_1, \cdots, v_n$ basis of V.

Proof of Th'm

Step 1. X nilpot end. on V $\Rightarrow$ ad_X nilpot. end. on $\mathfrak{gl}(V)$
$\because$ $X^k = 0$ $\Rightarrow$ $\underbrace{\mathrm{ad}_X^{2k+1}(T)}_{\text{linear comb. of monoms } X^a T X^{2k-a}} = 0$

Step 2. Prove claim by ind. on $m = \dim \mathfrak{g}$.

Step 2-1 $\exists\, \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m - 1$
$\because$ Take any max. proper subalg as $\mathfrak{h}$

$\mathfrak{h}$ is invar. under ad_X for $X \in \mathfrak{h}$.
$\Rightarrow$ $\overline{ad} : \mathfrak{h} \curvearrowright \mathfrak{g}/\mathfrak{h}$. , ad_X nilp (Step 1) $\Rightarrow$ $\overline{ad}_X$ also.
Ind. hypo $\Rightarrow \exists 0 \neq \overline{Y} \in \mathfrak{g}/\mathfrak{h}$ s.t. $\overline{ad}_X(\overline{Y}) = 0$ $\forall X \in \mathfrak{h}$.
i.e. $\exists Y \in \mathfrak{g} \setminus \mathfrak{h}$ $\forall X \in \mathfrak{h}$ $[X, Y] \in \mathfrak{h}$
So $\mathfrak{h}' = \mathfrak{h} + K.Y$ is a subalg, $\mathfrak{h} \triangleleft \mathfrak{h}'$

maximality of $\mathfrak{h}$ $\Rightarrow$ $\mathfrak{h}' = \mathfrak{g}$.

Step 2-2 $\mathfrak{h}$ as in Step 2-1. , take $Y \in \mathfrak{g} \setminus \mathfrak{h}$.
Ind. hypo $\Rightarrow$ $W = \{ v \in V : \forall X \in \mathfrak{h} \ \ X.v = 0 \} \neq 0$
Enough to show $\exists v \in W : Yv = 0$
$\Leftarrow$ $YW \subset W$ $\because$ Y is a nilpotent end.

Step 2-3 $YW \subset W$

$\because$ $XYv = YXv + [X, Y]v$.
$X \in \mathfrak{h}$, $v \in W$ $\Rightarrow$ $YXv = 0$, $[X, Y] \in \mathfrak{h} \Rightarrow [X, Y]v = 0$
so $XYv = 0$ for $\forall X \in \mathfrak{h}$. $\boxtimes$

Lie's theorem. K alg. closed (like $\mathbb{C}$). V : K-vec. sp.
$\mathfrak{g} \subset \mathfrak{gl}(V)$ solvable subalg.
$\Rightarrow$ $\exists 0 \neq v \in V$ eigenvector for any $X \in \mathfrak{g}$.

Cor. $\exists$ basis $(v_1, \cdots, v_n)$ of V. s.t. $\forall X \in \mathfrak{g}$ is "upper triangular" $\begin{bmatrix} \lambda_1 & * & * \\ & \lambda_2 & * \\ 0 & & \ddots \end{bmatrix}$

Proof of Th'm. Ind. on $m = \dim \mathfrak{g}$.
Step 1 $\exists \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m - 1$
$\because$ $\mathcal{D}^k \mathfrak{g} \to 0$ by assumption. so $\mathcal{D}(\mathfrak{g}) \neq \mathfrak{g}$
$\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ comm. $\Rightarrow$ any subsp. is ideal.
$\rightsquigarrow$ we can take inv. img. of codim 1 subsp. in $\mathfrak{g}/\mathcal{D}(\mathfrak{g})$.
Step 2 Ind. hypo $\Rightarrow$ $\exists 0 \neq v \in V$ eigenvec. for $\forall X \in \mathfrak{h}$.
Set $Xv = \lambda(X) v$ $X \in \mathfrak{h}$ $\lambda : \mathfrak{h} \to K$ lin.
Step 3 $W = \{ v' \in V : \forall X \in \mathfrak{h} \ \ Xv' = \lambda(X) v' \}$
$Y \in \mathfrak{g} \setminus \mathfrak{h}$.

2018.09.25 no 3

5.3-1 Enough to show $YW \subset W$

$\because$ K alg. closed $\Rightarrow$ Y has eigenvec. in W.

Step 3-2 $YW \subset W \Leftrightarrow \lambda([X,Y]) = 0 \quad \forall X \in \mathfrak{h}$.

$\because$ Again $XYv' = YXv' + [X,Y]v'$

$X \in \mathfrak{h}$, $v' \in W \Rightarrow$ LHS is $\lambda(X)Yv' + \lambda([X,Y])v'$

so claim $\Leftrightarrow \lambda([X,Y]) = 0$

Step 4 $0 \neq w \in W \Rightarrow U_w = \langle w, Yw, Y^2 w, \cdots \rangle$

$X \in \mathfrak{h} \Rightarrow XY^k w \in \lambda(X)Y^k w + \langle w, \cdots, Y^{k-1} w \rangle$

$\because$ induction on k, $XY^k w = YXY^{k-1}w + \underbrace{[X,Y]}_{\text{in } \mathfrak{h}} Y^{k-1} w$

Step 5 $\lambda([X,Y]) = 0$

$\lambda([X,Y])$: "diagonal" entries of $[X,Y]|_{U_w}$.

$= \frac{1}{\dim U_w} \mathrm{Tr}([X,Y]|_{U_w})$.

But X, Y pres $U_w \Rightarrow \mathrm{Tr}([X,Y]|_{U_w}) = \mathrm{Tr}([X|_{U_w}, Y|_{U_w}]) = 0$

∎

∘ Killing form.

Recall K alg clos. V : K-vec. sp. $X \in \mathrm{End}(V)$

$\Rightarrow X = X_s + X_n$ $\quad X_s$: diagonalizable

X_n : nilpotent.

$X_s X_n = X_n X_s$. this decomp is unique.

(from Jordan normal form).

V : K-vec. sp. <u>Killing form</u> (assoc. to V) is

$B_V(X,Y) = \mathrm{Tr}(XY)$ $\quad X, Y \in \mathrm{End}(V)$.

$\mathfrak{g}$: Lie alg. the <u>Killing form</u> on $\mathfrak{g}$ is

$B(X,Y) = B_{\mathfrak{g}}(\mathrm{ad}_X, \mathrm{ad}_Y)$

(! careful when $\mathfrak{g} = \mathfrak{gl}(V)$)

$K \subset \mathbb{C}$ 2018.09.25 no 4

Cartan's criterion) $\mathfrak{g} < \mathfrak{gl}(V)$ $B_V(X,Y)=0$
for $X, Y \in \mathfrak{g}$ $\Rightarrow$ $\mathfrak{g}$ solvable

Idea : $\mathcal{D}(\mathfrak{g})$ nilpot $\Rightarrow$ $\mathfrak{g}$ solv.

Step 1 Engel's thm $\Rightarrow$ enough to check $\forall X \in \mathcal{D}(\mathfrak{g})$ is nilpot end. on V.
$\Leftrightarrow$ eigenvals (λ_i) of X are all 0

Step 2 $D = X_s$ claim $\Leftrightarrow$ $\underbrace{Tr(\bar{D}X)}_{=\sum|\lambda_i|^2} = 0$

$X = \sum[Y_i, Z_i]$ $\Rightarrow$ $\bar{D}X = \sum \bar{D}[Y_i, Z_i]$
$Tr(\bar{D}X) = \sum Tr([\bar{D}, Y_i]Z_i)$
$\Rightarrow$ Enough to have $[\bar{D}, Y_i] \in \mathfrak{g}$.

Step 3. $ad\,\bar{D}$ poly in $ad\,X$.
$\because$ $ad(X_s) = ad(X)_s$ &
$ad(\bar{D})$ poly of $ad(D)$

Cor. $B(\mathfrak{g},\mathfrak{g}) = 0$ $\Rightarrow$ $\mathfrak{g}$ solvable
$\because$ $\underset{\text{comm.}}{\mathfrak{z}(\mathfrak{g})} \to \mathfrak{g} \to \underset{\text{solvable from Cartan's criterion.}}{ad(\mathfrak{g}) = \text{img of } \mathfrak{g} \text{ in } End(\mathfrak{g})}$

Cor $\mathfrak{g}$ solvable $\Leftrightarrow$ $B(\mathfrak{g}, \mathcal{D}(\mathfrak{g})) = 0$ $(K = \mathbb{C})$
$\Rightarrow$: Lie's th'm $\Rightarrow$ $\exists$ basis of $\mathfrak{g}$ s.t. ad_X is uppertriangular. $\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_n \end{bmatrix}$

$\rightsquigarrow$ $\left[\begin{bmatrix} \lambda_1 & & * \\ 0 & \lambda_2 & \\ & & \ddots \end{bmatrix}, \begin{bmatrix} \mu_1 & & * \\ 0 & \mu_2 & \\ & & \ddots \end{bmatrix}\right] = \begin{bmatrix} 0 & & * \\ 0 & 0 & \\ & & \ddots \end{bmatrix}$

$\Rightarrow Tr(ad_X\, ad_{[Y,Z]}) = 0$.

$\Leftarrow$: $B(\mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathfrak{g})) = 0$ $\Rightarrow$ $\mathcal{D}(\mathfrak{g})$ solvable
$\mathcal{D}(\mathfrak{g}) \to \mathfrak{g} \to \underset{\text{comm.}}{\mathfrak{g}/\mathcal{D}(\mathfrak{g})}$

2018.09.26 no.1

Summary

- Semisimple Lie algebras
  - complete reducibility ; unitary trick
  - Nondegeneracy of Killing form.

∘ Semisimple Lie algs.

! Generally reps of Lie algs don't have complete reducibility. (and don't behave well with "$X = D + N$")

Ex $\mathfrak{g} = K \overset{\pi}{\curvearrowright} V = K^2$ by $\pi_t = \begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix}$ $(t \in K)$

$V' = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in K \right\}$ is the only nontriv. invar. subsp. $\Rightarrow$ doesn't have complement

also $\begin{bmatrix} t & t \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}}_{\text{diag.}} + \underbrace{\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}}_{\text{nilpot.}}$

is not of the form π_s

⇝ But all is good for semisimple algs

(Recall : semisimple $\equiv$ no solvable ideals)

$\mathfrak{g}$: Lie alg over K, $K \subset \mathbb{C}$.

Prop. $\mathfrak{g}$ is semisimple $\Leftrightarrow$ Killing form B is nondegenerate $(\forall X \neq 0\ \exists Y\ B(X,Y) \neq 0)$

Step 1 $\mathfrak{s} = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g}\ B(X,Y) = 0 \}$ is ideal of $\mathfrak{g}$

$\because$ $B([Z,X],Y) + B(X,[Z,Y]) = 0$ (invariance)

implies $X \in \mathfrak{s}, Z \in \mathfrak{g} \Rightarrow [Z,X] \in \mathfrak{s}$

Invariance : LHS $= \mathrm{Tr}(\mathrm{ad}_{[Z,X]} \mathrm{ad}_Y + \mathrm{ad}_X \mathrm{ad}_{[Z,Y]})$

$= \mathrm{Tr}\left((\mathrm{ad}_Z \mathrm{ad}_X - \cancel{\mathrm{ad}_X \mathrm{ad}_Z}) \mathrm{ad}_Y + \mathrm{ad}_X (\cancel{\mathrm{ad}_Z \mathrm{ad}_Y} - \mathrm{ad}_Y \mathrm{ad}_Z) \right)$

$= \mathrm{Tr}(\mathrm{ad}_Z \mathrm{ad}_X \mathrm{ad}_Y) - \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y \mathrm{ad}_Z) = 0$

(cf. "Step 2" of Cartan's criterion)

Step 2 "$\Rightarrow$" of the claim.

$\mathrm{ad}_{\mathfrak{s}}$: img of $\mathfrak{s}$ in $\mathrm{End}(\mathfrak{g})$. $B(\mathrm{ad}_{\mathfrak{s}}, \mathrm{ad}_{\mathfrak{s}}) = 0$

Cartan's criterion $\Rightarrow$ $\mathrm{ad}_{\mathfrak{s}}$ is solvable

$\underbrace{\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{s}}_{\text{comm.}} \to \mathfrak{s} \to \underbrace{\mathrm{ad}_{\mathfrak{s}}}_{\text{solv.}} \Rightarrow$ $\mathfrak{s}$ also soluable (ideal)

$\mathfrak{g}$ s.s. $\Rightarrow \mathfrak{s} = 0$ i.e. B is nondegen.

Step 3 $\mathfrak{h} \triangleleft \mathfrak{g} \Rightarrow \mathcal{D}(\mathfrak{h}) \triangleleft \mathfrak{g}$

$\because$ $[X,[Y,Z]] = [Z,[X,Y]] + [Y,[Z,X]]$

from (antisymm &) Jacobi identity

$Y, Z \in \mathfrak{h}$, $X \in \mathfrak{g}$ $\Rightarrow$ right hand side is in $\mathcal{D}(\mathfrak{h})$)

Step 4 $\mathrm{Rad}(\mathfrak{g}) \neq 0 \Rightarrow \mathfrak{g}$ contains a comm. ideal.

$\because$ $\mathrm{Rad}(\mathfrak{g})$ solvable $\Rightarrow \mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})) \searrow 0$.

if $\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})) \neq 0$ and $\mathcal{D}^{k+1}(\mathrm{Rad}(\mathfrak{g})) = 0$

$\mathfrak{a} = \mathcal{D}^k(\mathrm{Rad}(\mathfrak{g}))$ is comm. $\boxed{\mathcal{D}''(\mathcal{D}^k(\mathrm{Rad}(\mathfrak{g})))}$

Step 5 "$\Leftarrow$" of the claim. (works for any K)

Want : B nondeg $\Rightarrow$ $\mathfrak{g}$ has no nonzero comm. ideal.

suppose $\mathfrak{a} \triangleleft \mathfrak{g}$ comm. $X \in \mathfrak{a}$.

$Y \in \mathfrak{g} \Rightarrow \mathrm{ad}_X \mathrm{ad}_Y$ $\quad \mathfrak{g} \xrightarrow{\mathrm{ad}_X} \underbrace{\mathfrak{a}}_{\text{ideal}}$, $\underbrace{\mathfrak{a} \xrightarrow{\mathrm{ad}_X} 0}_{\text{comm.}}$

so matrix pres looks like

$$\begin{array}{cc} \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} & \begin{array}{l})\ \mathfrak{a} \\ )\ \text{complement} \end{array} \\ \quad\mathfrak{a} \quad \text{compl.} & \end{array}$$

$\Rightarrow \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y) = 0$

So $B(X,Y) = 0$ if $X \in \mathfrak{a}$ $\underset{\text{nondeg.}}{\Longrightarrow}$ $X = 0$. $\square$

Ex. $\mathfrak{g} = \mathfrak{sl}_2(K)$ $\quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

ad_E : $E \mapsto 0$, $F \mapsto H$, $H \mapsto -2E$

ad_F : $E \mapsto -H$, $F \mapsto 0$, $H \mapsto 2F$

ad_H : $E \mapsto 2E$, $F \mapsto -2F$, $H \mapsto 0$

2018.09.26 no.3

so $B(X,Y)$ is

$Y \backslash X$	E	F	H
E	0	4	0
F	4	0	0
H	0	0	8

Cor $\mathfrak{g}$ semisimple $\Leftrightarrow$ $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, $\mathfrak{g}_i$ simple

$\because$ $\Rightarrow$: $\mathfrak{h} \triangleleft \mathfrak{g}$, set $\mathfrak{h}^\perp = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{h} \ B(X,Y)=0\}$

$\mathfrak{h} \cap \mathfrak{h}^\perp$ solv. by Cartan's criterion

$\mathfrak{g}$ s.s. $\Rightarrow \mathfrak{h} \cap \mathfrak{h}^\perp = 0$

invar. of $B \Rightarrow \mathfrak{h}^\perp$ also ideal.

$\rightsquigarrow \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as Lie alg.

Keep decomposing.. ☑

∘ Complete reducibility of semisimple Lie algs

(π, V) : rep of $\mathfrak{g}$, $W \subset V$: $\mathfrak{g}$-invar. subsp.

$\mathfrak{g}$ semisimple $\Rightarrow \exists W' \subset V$ $\mathfrak{g}$-inv. complement

Analytic proof for $K = \mathbb{C}$ (Weyl's unitary trick)

Sketch for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

1 $(\mathfrak{su}_n)_\mathbb{C} = \mathfrak{sl}_n\mathbb{C}$ ($\mathfrak{sl}_n\mathbb{C}$ is the complexification of $\mathfrak{su}_n$)

so $\mathfrak{sl}_n\mathbb{C} \overset{\pi}{\curvearrowright} V \equiv \mathfrak{su}_n \overset{\pi}{\curvearrowright} V$
by cplx lin. transforms

2 $\mathfrak{su}_n \overset{\pi}{\curvearrowright} V$ by cplx lin. $\equiv SU(n) \overset{\tilde\pi}{\curvearrowright} V$ by cplx lin.
i.e. rep. of $SU(n)$ on V

3 $SU(n)$ is compact $\Rightarrow \exists$ $\tilde\pi$-invar. Herm. inn. prod. on V.

$\because$ Haar measure μ on $SU(n)$ $\int f(gh)\,d\mu(g) = \int f\,d\mu$

$(v, v')_0$ any inn. prod $\rightsquigarrow (v, v') = \int (\tilde\pi_g v, \tilde\pi_g v')_0\, d\mu(g)$
is invar.

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4. $W \subset V$ invar. $\Rightarrow W^\perp = \{ v \in V : \forall v' \in W \ (v, v') = 0 \}$ is $\tilde{\pi}$-invar.

5. $W^\perp$ is a rep of su_n. (by cplx lin. transf.)

6. $W^\perp$ is also a rep of $sl_n(\mathbb{C})$.

Algebraic proof (still requires $K \subset \mathbb{C}$) $\mathfrak{g} \curvearrowright V$

Step 0 we may assume $\mathfrak{g} \subset \mathfrak{gl}(V)$

(img of semisimple is still semisimple)

Step 1. B_V is nondeg. on $\mathfrak{g}$

$\because$ $\mathfrak{s} = \{ X \in \mathfrak{g} : \forall Y \in \mathfrak{g} \ B_V(X, Y) = 0 \}$ ideal

$\rightsquigarrow \mathfrak{s}$ solvable by Cartan's criterion

$\Rightarrow$ should be 0 by semisimplicity.

Step 2 $(X_i)_{i=1}^n$ basis of $\mathfrak{g}$, $(X^i)_{i=1}^n$ dual basis of $\mathfrak{g}$ rel. to B_V $B_V(X_i, X^j) = \delta_{ij}$

$C_V = \sum X_i X^i \in \mathrm{End}(V)$ <u>Casimir</u> operator

$\Rightarrow C_V$ is an intertwiner.

$\because$ $[Y, C_V] = \sum_i [Y, X_i] X^i + X_i [Y, X^i]$

$= \sum_{i,j} B_V([Y, X_i], X^j) X_j X^i + B_V(X_j, [Y, X^i]) X_i X^j$

invariance of B_V $\Rightarrow$ RHS is 0.

& relabel i & j

Step 3 Claim for $\dim W = \dim V - 1$, W irred.

· $C_V|_W$ is scalar (eigenspace would be subrep)

· $\mathfrak{g} \curvearrowright V/W$ is trivial ($\mathfrak{g} = D(\mathfrak{g})$ acts trivially)

· $\mathrm{Tr}(C_V) = \sum B_V(X_i, X^i) = \dim(\mathfrak{g})$

$\Rightarrow C_V|_W = \alpha \, \mathrm{Id}_W$, $\alpha \neq 0$, $V = W \oplus \mathrm{Ker}(C_V)$

Step 4 Claim for $\dim W = \dim V - 1$

Ind. on $\dim W$.

$0 \neq Z \subset W$ inv. $\underset{\text{ind.}}{\rightsquigarrow}$ $V/Z \simeq W/Z \oplus Y_{1\text{-}\dim}$.

$\tilde{Y}$: inv. img of Y splits as $Z \oplus Y_1$.
$\rightsquigarrow$ $V \simeq W \oplus Y_1$

Step 5 dim W general, W irred

$\mathrm{End}_{\mathfrak{g}}(W)$ is 1-dim., triv. rep. of $\mathfrak{g}$.

$$\begin{array}{ccc} \mathrm{Hom}_K(V, W) & \xrightarrow{\text{res.}} & \mathrm{End}_K(W) \\ \cup & & \cup \\ U & \longrightarrow & \mathrm{End}_{\mathfrak{g}}(W) \end{array}$$

surjective., $\mathfrak{g}$ acts by ad

surj.

inv. img of $\mathrm{End}_{\mathfrak{g}}(W)$

$\mathrm{Ker}(\mathrm{res}|_U)$ is codim 1. $\mathfrak{g}$-inv. subsp

$\Rightarrow$ $U \simeq \mathrm{Ker}(\mathrm{res}|_U) \oplus U_0$. U_0 : triv.

i.e. $\mathrm{Hom}_K(V, W)$ has $\mathfrak{g}$-inv. elem which restr. to Id_W.

this is $V \xrightarrow{p} W$ prj. intertwiner.

$V = W \oplus \mathrm{Ker}(p)$. as $\mathfrak{g}$-rep.

2018.10.02 no.1

Summary
- semisimple - nilpotent decomposition in semisimple Lie algs
- representation of $\mathfrak{sl}_2(\mathbb{C})$

∘ Semisimple - nilpot decomp.

K : alg - closed field (or perfect field ; Char $K = 0$, $\mathbb{F}_p, \cdots$)

V : fin-dim vec. sp. over K.

$T \in \mathrm{End}(V) \rightsquigarrow T = \underset{\text{diag'ble}}{T_s} + \underset{\text{nilpotent}}{T_n}$

~~T_s, T_n polynoms in T. ($\Rightarrow$ commute with each other)~~

~~$\mathfrak{g}$: Lie alg over K. $\mathfrak{g} \overset{\pi}{\curvearrowright} V$~~

! generally ~~$\nexists Z, Y \in \mathfrak{g}$ $\pi(X)_s = \pi(Y)$, $\pi(X)_n = \pi(Z)$.~~

Th'm $\mathfrak{g}$ semisimple, (π, V) rep of $\mathfrak{g}$, $X \in \mathfrak{g}$

$\Rightarrow \exists Y, Z \in \mathfrak{g}$ $\pi(X)_s = \pi(Y)$, $\pi(X)_n = \pi(Z)$

If π is faithful. Y & Z are indep. of π.

Key. Lem. $\mathfrak{g} \subset \mathfrak{gl}(V)$ semisimple $X \in \mathfrak{g}$

$\Rightarrow X_s, X_n \in \mathfrak{g}$

Proof of Lem

Step 1. For $W \subset V$ put $S_W = \{Y \in \mathfrak{gl}(V) : YW \subset W,\ \mathrm{Tr}(Y|_W) = 0\}$

~~and $\tilde{\mathfrak{g}} = \{Y \in \mathfrak{gl}(V) : [Y, \mathfrak{g}] \subset \mathfrak{g}\}$~~ (normalizer)

Then $\mathfrak{g} = \tilde{\mathfrak{g}} \cap \left(\bigcap_{W : \mathfrak{g}\text{-inv}} S_W\right)$

$\because$ $\subset$ is obvious from def. Put $\mathfrak{g}'$ = right hand side

$\mathfrak{g} \triangleleft \tilde{\mathfrak{g}} \Rightarrow \mathfrak{g} \triangleleft \mathfrak{g}'$. so $\mathfrak{g}$ acts on $\mathfrak{g}'$ by ad

Compl. reducibility $\Rightarrow \mathfrak{g}' = \mathfrak{g} \oplus U$ for ad - inv. U. We want $U = 0$.

Take $Y \in U$ Enough to show : $W \subset V$ irred subrep. of $\mathfrak{g}$ $\Rightarrow Y|_W = 0$

($\because$ $V = W_1 \oplus \cdots \oplus W_k$ irr. decomp)

$Y \in U \Rightarrow [\mathfrak{g}, Y] \in \mathfrak{g} \cap U = 0 \Rightarrow Y$ is intertwiner

Schur's lemm. $\Rightarrow Y|_W$ is scalar.

$Tr(Y|_W) = 0 \Rightarrow Y|_W = 0$.

ws. goal is to show $X_s \in \tilde{\mathfrak{g}}$, $X_s \in S_W$, etc.

Step 2 $W \subset V$ $\mathfrak{g}$-inv $\Rightarrow X_s, X_n \in S_W$

$\because$ $\mathfrak{g}$ semisimple $\Rightarrow \mathfrak{g} = \mathcal{D}(\mathfrak{g})$ (use $\mathfrak{g} = \oplus \mathfrak{g}_i$)

$\Rightarrow Tr(X|_W) = Tr(\Sigma [Y_i|_W, Z_i|_W]) = 0$

$X_n|_W$ is nilpot $\Rightarrow Tr(X_n|_W) = 0$

$X_s|_W = (X - X_n)|_W$ also 0-trace.

Step 3 $X_s, X_n \in \tilde{\mathfrak{g}}$

$\because$ $\left.\begin{matrix} ad_{X_s} \\ ad_{X_n} \end{matrix}\right)$ on $\tilde{\mathfrak{g}}$ is $\begin{pmatrix} \text{semisimple} \\ \text{nilpot.} \end{pmatrix}$ part of ad_X

$\because$ $ad_X \in End(\mathfrak{gl}(V))$ stabilizes $\mathfrak{g}$.

ad_{X_s} is diagonalizable ($v_1, v_2, \ldots$ eigenvec. of X)

$E_{ij} v_k = \delta_{jk} v_i \Rightarrow E_{ij}$ eigenvec. of ad_{X_s}

ad_{X_n} is nilpot. (see 09.25)

$ad_{X_s}\, ad_{X_n} = ad_{X_n}\, ad_{X_s}$, $ad_X = ad_{X_s} + ad_{X_n}$

$\boxed{ad_{X_s} \text{ stab } \mathfrak{g}}$

so ad_{X_s} is a polynom. of ad_X

$ad_X(\mathfrak{g}) \subset \mathfrak{g} \Rightarrow ad_{X_s}(\mathfrak{g}) \subset \mathfrak{g}$. $\boxed{2}$

Proof of thm. $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ $\mathfrak{g}' = Im\,\pi$.

Step 1. $\mathfrak{g} \simeq \ker \pi \oplus \mathfrak{g}'$ as Lie alg. ($\Rightarrow \mathfrak{g}'$ semisimpl)

$\because$ $\ker\pi$ ideal. $\mathfrak{g}' \simeq (\ker \pi)^\perp$ for Killing form (see 09.26)

Step 2. $\mathfrak{g} \xrightarrow{ad} End(\mathfrak{g})$ is faithful

$\Rightarrow \exists Y, Z$ $ad_Y = (ad_X)_s$, $ad_Z = (ad_Z)_n$

$\because$ Lem.

2018.10.02 no 3

Step 3 $\pi(Y) = \pi(X)_s$, $\pi(Z) = \pi(X)_n$

$\because$ $\mathrm{ad}\,\pi(Y)|_{\mathfrak{g}'}$ semisimple, $[\mathrm{ad}\,\pi(Y)|_{\mathfrak{g}'}, \mathrm{ad}\,\pi(Z)|_{\mathfrak{g}'}] = 0$
$\mathrm{ad}\,\pi(Z)|_{\mathfrak{g}'}$ nilpot.

$\mathrm{ad}\,\pi(X)_s$, $\mathrm{ad}\,\pi(X)_n$ have same prop.

Uniqueness of ss-nilpot. dec.

$\Rightarrow$ $\mathrm{ad}\,\pi(Y)|_{\mathfrak{g}'} = \mathrm{ad}\,\pi(X)_s|_{\mathfrak{g}'}$, etc.

so $\pi(Y) - \pi(X)_s \in \mathfrak{z}(\mathfrak{g}') = 0$. ⊠

Ex. $\mathfrak{sl}_2(\mathbb{C})$ $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ from deriv. of $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} = e^{tH}$

$t \in \sqrt{-1}\,\mathbb{R}$ $\Rightarrow$ e^{tH} unitary

(π, V) rep of V $\Rightarrow$ $\pi(e^{tH})$ diag'ble

$\Rightarrow$ $\pi(H)$ also diag'ble.

- Representation of $\mathfrak{sl}_2(\mathbb{C})$. (or $\mathfrak{sl}_2(K)$ Char $K = 0$)

$\mathfrak{sl}_2(\mathbb{C}) = \langle H, E, F \rangle$ $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. (simple Lie alg)

$\rightsquigarrow$ in the <u>defining rep</u> on $\mathbb{C}^2$

$H = H_s$, $E = E_n$, $F = F_n$.

$\Rightarrow$ $\forall$ rep (π, V) $\pi(H)$: diag'ble.

$\pi(E)$, $\pi(F)$: nilpotent.

Weight dec.

Prop. $\lambda \in \mathbb{C}$ $\pi(H)v = \lambda v$ $\Rightarrow$ $v' = \pi(E)v$, $v'' = \pi(F)v$
satisfy $\pi(H)v' = (\lambda+2)v'$, $\pi(H)v'' = (\lambda-2)v''$

Proof $[\pi(H), \pi(E)] = \pi([H, E]) = 2\pi(E)$

$\Rightarrow$ $\pi(H)\pi(E)v - \pi(E)\underbrace{\lambda v}_{\pi(H)v} = 2\pi(E)v$ ⊠

$v \in V$ is a <u>highest weight vector</u> (of wght λ) if

- it is an eigenvector of $\pi(H)$; $\pi(H)v = \lambda v$
- $\pi(E)v = 0$

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Prop. $v \in V$ h.w.vec. $\Rightarrow$ $W = \langle v, \pi(F)v, \pi(F)^2 v, \cdots \rangle$ is an $\mathfrak{sl}_2(\mathbb{C})$-inv. subsp.

Proof $\pi(F)$-inv : obvious

$\pi(H)$-invar : prev. Prop $\Rightarrow$ $\pi(F)^k v$ eigenvec

$\pi(E)$-invar : $[\pi(E), \pi(F)] = \pi([E,F]) = \pi(H)$

By induction $\pi(E)\pi(F)^{k+1} v \in \langle \pi(F)^k v \rangle$.

$k=0$: $\pi(E)\pi(F)v - \pi(F)\underbrace{\pi(E)v}_{0} = \lambda v$

gen. $\pi(E)\pi(F)^{k+1} v - \pi(F)\pi(F)\pi(F)^k v$

$= \pi(H)\pi(F)^k v$.

Weight decomposition.

$V = \bigoplus_{\alpha} V_\alpha$. $V_\alpha = \langle v \in V : \pi(H)v = \alpha v \rangle$.

h.w.v. of wght λ

Prop. v $\Rightarrow$ $\pi(F)^k v = 0$ for $k \gg 1$

only if $\lambda \in \mathbb{N}$

Proof. Claim $\pi(E)\pi(F)^k v = k(\lambda - k + 1)\pi(F)^{k-1} v$.

$\because$ induction on k, $[\pi(E), \pi(F)] = \pi(H)$.

2018.10.03 no. 1

Summary

- Representation of $\mathfrak{sl}_2(\mathbb{C})$, cont'd
  - irred reps
  - rep. of $SU(2)$
  - fusion rules.

- Repn of $\mathfrak{sl}_2\mathbb{C}$ cont'd.

$\mathfrak{sl}_2(\mathbb{C}) = \langle H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \rangle$

(π, V) : fin. dim rep of $\mathfrak{sl}_2(\mathbb{C})$

$\rightsquigarrow$ • $V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$, $V_\alpha = \{ v \in V : \pi(H) v = \alpha v \}$

weight sp. decomp. ($V_\alpha = 0$ for almost all α)

• $\pi(E) V_\alpha \subset V_{\alpha+2}$, $\pi(F) V_\alpha \subset V_{\alpha-2}$

highest weight vec v (of wght λ) : $0 \neq v \in V_\lambda \cap \operatorname{Ker} \pi(E)$.

$\rightsquigarrow$ $W = \langle \pi(F)^k v : k = 0, 1, 2, \cdots \rangle$ $\mathfrak{sl}_2(\mathbb{C})$-inv.

Prop. Under the above setting $\lambda \in \mathbb{N} = \{0, 1, 2, \cdots\}$

$\lambda = \max n$, $\pi(F)^n v \neq 0$

Proof. Step 1 $\pi(E)\pi(F)^k v = k(\lambda - k + 1)\pi(F)^{k-1} v$ for $k \geq 1$

$\because$ Induction on k (also makes sense for $k = 0$)

$k = 1$: $\pi(E)\pi(F)v - \pi(F)\underbrace{\pi(E)v}_{0} = \pi([E,F])v = \lambda v$, $[E,F] = H$

general : $\pi(E)\pi(F)^{k+1}v - \pi(F)\underbrace{\pi(E)\pi(F)^k v}_{k(\lambda-k+1)\pi(F)^{k-1}v \text{ by ind. hyp.}} = \pi(H)\pi(F)^k v$

$\Rightarrow \pi(E)\pi(F)^{k+1}v = (k(\lambda - k + 1) + (\lambda - 2k))\pi(F)^k v$

$= (k+1)(\lambda - k)\pi(F)^k v$. ∎

Step 2 take $k \geq 1$ s.t. $\pi(F)^k v = 0 \neq \pi(F)^{k-1} v$.

then $0 = \pi(E)\pi(F)^k v = k(\lambda - k + 1)\pi(F)^{k-1} v$

$\Rightarrow \lambda = k - 1$. ∎

Suppose (π, V) is irreducible. ($\Rightarrow W = V$).

$V = V_\lambda \oplus V_{\lambda-2} \oplus \cdots \oplus V_{-\lambda}$

$V_{\lambda - 2k} = \langle \pi(F)^k \cdot v \rangle$. 1-dim.

"Step 1" above determines how $\pi(E)$ should act. (and $\pi(H)$, $\pi(F)$ are already determined)

$\Rightarrow$ Irreducible rep. of $\mathfrak{sl}_2(\mathbb{C})$ are classified by the highest weight $\lambda = 0, 1, 2, \cdots$

(corresponding rep : $(\lambda+1)$-dimensional).

Concrete realization

$V^{(n)} = \langle x^n, x^{n-1}y, \cdots, y^n \rangle$ space of homogeneos polynoms in x & y. $(n+1)$-dim.

$\pi^{(n)}(E)f = y\partial_x f$, $\pi(F)f = x\partial_y f$ $\quad f \in V^{(n)}$

$\pi^{(n)}(H)f = y\partial_y f - x\partial_x f$

$\rightsquigarrow [\pi^{(n)}(E), \pi^{(n)}(F)] = \pi^{(n)}(H)$, $[\pi^{(n)}(H), \pi^{(n)}(E)] = 2\pi^{(n)}(E)$ etc.

$\pi^{(n)}(H)(x^{n-k}y^k) = 2k - n$.

$\Rightarrow y^n$: highest wght vec. of wght n.

Rep of $SU(2)$.

(cplx) rep. of $\mathfrak{sl}_2(\mathbb{C}) \leftrightarrow$ cplx rep. of $\mathfrak{su}_2$

$\leftrightarrow$ (f.d.) unitary rep. of $SU(2)$

$\rightsquigarrow V^{(n)}$ are the irred. unitary reps of $SU(2)$

generic elem of $SU(2)$: $\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix}$ $\quad |a|^2+|c|^2=1$

$$\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - \bar{c}y \\ cx + \bar{a}y \end{bmatrix}$$

invariant inn. prod. $(x^{n-k}y^k, x^{n-j}y^j) = \delta_{jk}\binom{n}{k}^{-1}$

$\sqrt{-1}H = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix} \in \mathfrak{su}_2$ $\quad (= \{X \in M_2(\mathbb{C}) : X^* = -X\})$

$\exp(2\pi t\sqrt{-1}H) = \begin{bmatrix} e^{2\pi t\sqrt{-1}} & 0 \\ 0 & e^{-2\pi t\sqrt{-1}} \end{bmatrix} \in SU(2)$

this is periodic. (same val $t + \mathbb{Z}$)

2018.10.03. no.3

$\rightsquigarrow$ any rep. π of $SU(2)$ should induce
$\pi(H)$ with integer eigenvals.
$\because$ $\pi(e^{2\pi\sqrt{-1}H}) = e^{2\pi\sqrt{-1}\pi(H)}$ should be
$\pi(I_2) = Id_V$.

Fusion rules of $\mathfrak{sl}_2(\mathbb{C})$.

(π, V), (π', V') rep of $\mathfrak{sl}_2(\mathbb{C})$
$\rightsquigarrow$ we want to understand $(\pi\otimes\pi', V\otimes V')$
(irred. decomp.)

We may assume π, π' are irred
($(\pi_1 \oplus \pi_2)\otimes\pi' \simeq (\pi_1\otimes\pi')\oplus(\pi_2\otimes\pi')$, etc.)

Task :

1 do weight decomp. of $V\otimes V'$

$$V\otimes V' \simeq \bigoplus_{m,n} \underbrace{V_m\otimes V'_n}_{\text{weight } (m+n)} = \bigoplus_k \bigoplus_m V_m \otimes V'_{k-n}$$

$$(\pi\otimes\pi')(X) = \pi(X)\otimes Id_{V'} + Id_V\otimes\pi'(X)$$

$$(\pi\otimes\pi')(H)(v\otimes v') = mv\otimes v' + v\otimes nv' = (m+n)v\otimes v'$$

Ex 1) $(\pi, V) = (\pi', V')$: def. rep on $\mathbb{C}^2$ $(= V^{(1)})$

$$V\otimes V = (V_1\otimes V_1)\oplus(V_{-1}\otimes V_1 \oplus V_1\otimes V_{-1})\oplus(V_{-1}\otimes V_{-1})$$

weight 2 0 -2

$v_1 \in V_1$ h.w. vec. $\Rightarrow$ $v_1\otimes v_1$ h.w. vec. in $V\otimes V$.

2 Find highest weigt vec. ξ in $V\otimes V'$

3 Take $W = \langle \xi, \pi(F)\xi, \pi(F)^2\xi, \cdots \rangle$

4 Take $W^\perp$, repeat from 2 (find $\xi' \in W^\perp, \cdots$)

$v_1\otimes v_1$, $(\pi\otimes\pi)(F)(v_1\otimes v_1) = v_1\otimes v_{-1} + v_{-1}\otimes v_1$
$(\pi\otimes\pi)(F)^2(v_1\otimes v_1) = v_{-1}\otimes v_{-1}$

span irred. subrep $W \simeq (\pi^{(2)}, V^{(2)})$

$\langle v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \rangle$ is the complement (triv. rep)

$\Rightarrow \pi^{(1)} \otimes \pi^{(1)} \simeq \pi^{(2)} \oplus \pi^{(0)}$

$\qquad\qquad Sym^2(V) \quad \Lambda^2(V)$

2) $\pi^{(1)} \otimes \pi^{(k)} \simeq \pi^{(k+1)} \oplus \pi^{(k-1)}$

again v_1, $v^{(k)}_k$ h.w. vecs in $V^{(1)}$, $V^{(k)}$

$\Rightarrow v_1 \otimes v^{(k)}_k$ h.w. vec of whght $k+1$

in $V^{(1)} \otimes V^{(k)} \rightsquigarrow$ get a irred. subrep. W

isom. to $V^{(k+1)}$

$(V^{(1)} \otimes V^{(k)})_{k-1}$ is 2-dim

$(V^{(1)}_1 \otimes V^{(k)}_{k-2}) \oplus (V^{(1)}_{-1} \otimes V^{(k)}_k)$

$\rightsquigarrow W^\perp$ has wght $(k-1)$-comp.

$\rightsquigarrow$ get a copy of $V^{(k-1)}$

$\dim V^{(k+1)} \oplus V^{(k-1)} = k+2+k$

$\dim V^{(1)} \otimes V^{(k)} = 2(k+1)$

$\Rightarrow$ We are done.

3) $\pi^{(m)} \otimes \pi^{(n)} \simeq \pi^{(m+n)} \oplus \pi^{(m+n-2)} \oplus \cdots \oplus \pi^{(|m-n|)}$

Rem. $U_n(x)$ Chebyshev polynom. of second kind

$U_0(x) = 1$, $U_1(x) = 2x$, $U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x)$

$$\sum U_n(x) t^n = \frac{1}{1-2tx+t^2}$$

$U_n(\frac{x}{2})$ basis of $\mathbb{Z}[x]$.

Rep. ring of $sl_2(\mathbb{C})$ $R(sl_2(\mathbb{C})) = \langle [\pi^{(n)}] : n \in \mathbb{N} \rangle$

with $\oplus$, $\otimes$ as ring structure

$\simeq \mathbb{Z}[x]$ by $[\pi^{(n)}] \longleftrightarrow U_n(\frac{x}{2})$

2018.10.09 no 1

Summary

- Representation of $\mathfrak{sl}_3(\mathbb{C})$.
  - Structure of $\mathfrak{sl}_3(\mathbb{C})$: Cartan subalg, root decomp.
  - eigenvalues, weight lattice, innerproduct

Structure of $\mathfrak{sl}_3(\mathbb{C}) = \{X \in M_3(\mathbb{C}) : \mathrm{Tr}\, X = 0\}$

Want : convenient "representatives" of diagonalizable / nilpotent elems.

Diagonalizable ones $\rightsquigarrow$ take diag matrices

Nilpotent ones $\rightsquigarrow$ take strictly upper / lower triangular mats

$\mathfrak{h} = \{X \in \mathfrak{sl}_3(\mathbb{C}) : \text{diag. matrix}\} = \langle H_1, H_2 \rangle$

$H_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}$.

Rem $\mathfrak{h}$ is maximal diagonalizable commutative "Cartan subalgebra"

$\because$ Y comm. with $\forall X \in \mathfrak{h}$ $\Rightarrow$ Y comm. with any diag.
extra : scalar mats
$\Rightarrow Y$ diag.

$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, ...

Rem $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus (\bigoplus_{i \neq j} \mathbb{C} E_{ij})$ "eigendecomp. for $\mathfrak{h}$"

- E_{ij} is a joint eigenvector for elems of $\mathfrak{h}$.

e.g. $[H_1, E_{12}] = \left[\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right] = 2E_{12}$

$[H_2, E_{12}] = \left[\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right] = -E_{12}$

- $\mathfrak{h}$ max comm. $\Leftrightarrow$ $\mathfrak{h}$ is the joint ker of $\mathrm{ad}_{\mathfrak{h}}$

- $\mathfrak{sl}_2$ - triples

Lem. $i \neq j \Rightarrow E_{ij}, E_{ji}, H_{ij} = [E_{ij}, E_{ji}]$ span a subalg isom to $\mathfrak{sl}_2$

Pf. $H_{ij} = E_{ii} - E_{jj}$. $[H_{ij}, E_{ij}] = 2E_{ij}$, $[H_{ij}, E_{ji}] = -2E_{ji}$

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from $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$

Notn. $s_{ij} = \langle E_{ij}, E_{ji}, H_{ij} \rangle$. $(= s_{ji})$.

Rem $H_1 = H_{12}$, $H_2 = H_{23}$

(π, V) : rep. of $\mathfrak{sl}_3(\mathbb{C})$ $\Rightarrow$ $\pi|_{s_{ij}} : \mathfrak{sl}_2(\mathbb{C}) \cong s_{ij} \curvearrowright V$ rep. of $\mathfrak{sl}_2(\mathbb{C})$.

- Irred. decomp of $\pi|_{s_{ij}}$ has $(\pi^{(n)}, V^{(n)})$ ($(n+1)$-dim irrep) as direct summands
- $\pi^{(n)}(H)$ has eigenvals $n, n-2, \ldots, -n$ $\in \mathbb{Z}$

$\Rightarrow$ $\pi(H_{ij})$ have integer eigenvalues.

$v \in V$ joint eigenvec. for $\mathfrak{h}$. $\rightsquigarrow \pi(X) v = \lambda(X) v$.

- $\lambda(X)$ is linear in X $\lambda \in \mathfrak{h}^*$ $(\simeq \mathbb{C}^2)$
- $\lambda(H_1), \lambda(H_2) \in \mathbb{Z}$. i.e. λ belongs to a subgrp $\simeq \mathbb{Z}^2$ (lattice in $\mathfrak{h}^*$)

$\Lambda_W = \{\lambda \in \mathfrak{h}^* : \lambda(H_1), \lambda(H_2) \in \mathbb{Z}\}$ weight lattice.

Example defining rep $V = \mathbb{C}^3$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ joint eigenvecs for $\mathfrak{h}$

$H_1 e_1 = e_1$, $H_2 e_1 = 0$, $H_1 e_2 = -e_2$, $H_2 e_2 = e_2$, ...

$\lambda_1(H_1) = 1$, $\lambda_1(H_2) = 0$ $\lambda_2(H_1) = -1$, $\lambda_2(H_2) = 1$, $\lambda_3(H_1) = 0$, $\lambda_3(H_2) = -1$

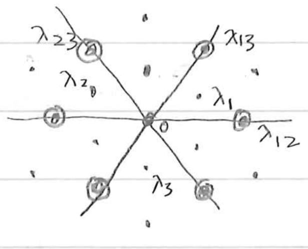
(so $\lambda_3 = -(\lambda_1 + \lambda_2)$, (λ_1, λ_2) basis of Λ_W, ...

Eigenvalues of $\mathrm{ad}\ \mathfrak{sl}_3(\mathbb{C}) \curvearrowright \mathfrak{sl}_3(\mathbb{C})$ span a sublattice of Λ_W root lattice Λ_R.

$[X, E_{ij}] = \lambda(X) E_{ij}$ $X \in \mathfrak{h}$

$\rightsquigarrow$ $\lambda_{12}(H_1) = 2$, $\lambda_{12}(H_2) = -1$, $\lambda_{23}(H_1) = -1$, $\lambda_{23}(H_2) = 2$, ...

$(\lambda_{12} = \lambda_1 - \lambda_2)$ $(\lambda_{23} = \lambda_1 + 2\lambda_2)$



$\cdot$: weight lattice

$\odot$: roots

$\Lambda_W / \Lambda_R \simeq \mathbb{Z}/3\mathbb{Z}$, ...

2018.10.09 no.3

- inner product

Killing form $B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$

$\rightsquigarrow$ restrict to $\mathfrak{h}$.

Prop. B is (symm.) pos. def. on $\langle H_1, H_2 \rangle_{\mathbb{R}}$.

Proof 1) $B(X,Y) = 6\,\mathrm{Tr}(XY)$ for $\mathfrak{sl}_3(\mathbb{C})$

H_1, H_2 are real symm $\Rightarrow$ pos. def.

Proof 2) $B(X,Y) = \sum_{\alpha:\text{root}} \alpha(X)\alpha(Y)$ $\quad X, Y \in \mathfrak{h}$

$= \sum_{i\neq j} \lambda_{ij}(X)\lambda_{ij}(Y)$

$\because$ ad_X is $\lambda_{ij}(X)$ on $\mathbb{C}E_{ij}$

0 on $\mathfrak{h}$

$\Rightarrow \mathrm{ad}_X \mathrm{ad}_Y$ is $\lambda_{ij}(X)\lambda_{ij}(Y)$ on $\mathbb{C}E_{ij}$

0 on $\mathfrak{h}$.

$\alpha(H_i) \in \mathbb{Z} \Rightarrow \alpha(X)^2 \geq 0. \Rightarrow B(X,X) \geq 0$

$B(X,X) = 0$ means $\alpha(X) = 0$ for $\alpha = \lambda_{ij}$

$\Rightarrow \mathrm{ad}_X = 0 \Rightarrow X \in \mathfrak{z}(\mathfrak{sl}_3(\mathbb{C})) = 0$ $\boxtimes$

$\langle H_1, H_2 \rangle_{\mathbb{R}}^* = \mathbb{R}\Lambda_W$ also has symm. pos. def inner prod.

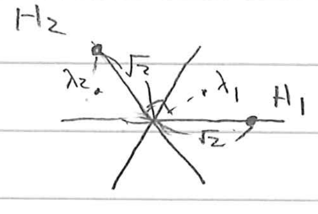
(E : Euclidean sp. $\rightsquigarrow E \overset{\phi}{\xrightarrow{\sim}} E^*$ by $\phi_v(w) = (v,w)$

$\rightsquigarrow E^*$ has inn. prod by $(\phi_v, \phi_w) = (v,w)$)

$(X,Y) = \mathrm{Tr}(XY) = \frac{1}{6}B(X,Y)$ has matrix

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ for basis (H_1, H_2)

$\rightsquigarrow$



2018.10.10 no1

Summary

- rep. of $\mathfrak{sl}_3(\mathbb{C})$, cont'd.
  - root vector action & weight decomp
  - positive roots, highest wght vecs
  - Weyl group

- Root vectors, wght decomp.

E_{ij} : matrix units $\rightsquigarrow$ λ_{ij} corr. roots.

(e.g. $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\lambda_{12}\Big(\underbrace{\begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}}_{H_1 = H_{12}}\Big) = 2$, $\lambda_{12}\Big(\underbrace{\begin{bmatrix} 0 & & \\ & 1 & \\ & & -1 \end{bmatrix}}_{H_2 = H_{23}}\Big) = -1$)

(π, V) : rep $\rightsquigarrow$ weight decomp. $V = \bigoplus_{\lambda \in \Lambda_W} V_\lambda$

$$V_\lambda = \{ v \in V : \pi(X) v = \lambda(X) v \quad (X \in \mathfrak{h}) \}$$

Lem. $\pi(E_{ij}) V_\lambda \subset V_{\lambda + \lambda_{ij}}$

Proof Take $v \in V_\lambda$.

Want : $\pi(X)\pi(E_{ij})v = (\lambda(X) + \lambda_{ij}(X))\pi(E_{ij})v$ for $X \in \mathfrak{h}$.

$$\begin{cases} \pi(X)\pi(E_{ij}) - \pi(E_{ij})\pi(X) = \pi([X, E_{ij}]) = \lambda_{ij}(X)\pi(E_{ij}) \\ \pi(X) v = \lambda(X) v \end{cases}$$

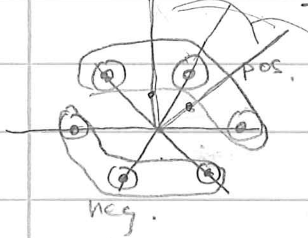
$\Rightarrow$ $\pi(X)\pi(E_{ij}) v = \lambda_{ij}(X)\pi(E_{ij}) v + \pi(E_{ij}) \underbrace{\pi(X) v}_{\lambda(X) v}$ $\square$

Def. $\lambda_{12}, \lambda_{23}, \lambda_{13}$: <u>positive roots</u>

$\lambda_{21}, \lambda_{32}, \lambda_{31}$: <u>negative</u> roots.

(π, V) rep, $v \in V_\lambda \cap \operatorname{Ker} \pi(E_{12}) \cap \operatorname{Ker} \pi(E_{23})$:

a <u>highest</u> <u>weigh</u> <u>vecto</u>



highest weight will be here.

Rem. $v \in \operatorname{Ker} \pi(E_{12}) \cap \operatorname{Ker} \pi(E_{23})$

$\Rightarrow$ $v \in \operatorname{Ker} \pi(E_{13})$ $\because$ $E_{13} = [E_{12}, E_{23}]$

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Prop. We can always find a h.w. vec.

Proof. $L(\lambda) = (\lambda, \lambda_{13})$ "length in 60° direction"

$V = \bigoplus_\lambda V_\lambda$ pick λ_0 with

$\cdot\ V_{\lambda_0} \neq 0$ $\quad \cdot\ L(\lambda_0)$ maximal among such.

$\Rightarrow L(\lambda + \lambda_{12}),\ L(\lambda + \lambda_{23}) > L(\lambda)$

$\because (\lambda_{12}, \lambda_{13}), (\lambda_{23}, \lambda_{13})$ are pos.

so $\pi(E_{12}) V_{\lambda_0} \subset V_{\lambda_0 + \lambda_{12}} = 0$, etc.

Thm. (π, V) rep., $v \in V_\lambda$ h.w. vec.

$W = \langle \pi(E_{21})^{a_1} \pi(E_{32})^{b_1} \cdots \pi(E_{21})^{a_k} \pi(E_{32})^{b_k} v$

$a_i, b_i \geq 0 \rangle$ is a subrep.

Proof Step 1 W is closed under $\pi(E_{ji})$ $i<j$

$\because$ obvious for E_{21}, E_{32}.

$\Rightarrow E_{31} = [E_{32}, E_{21}]$ also preserves W.

Step 2 W is closed under $\pi(H_1), \pi(H_2)$

$\because \pi(E_{21})^{a_1} \cdots \pi(E_{32})^{b_k} v \in V_{\lambda + (\sum a_i)\lambda_{21} + (\sum b_i)\lambda_{32}}$

by Lem $\Rightarrow \pi(H_i)$ is acting by scalar mult.

Step 3. W is closed under $\pi(E_{ij})$ $i<j$

$\because$ Induction on "word length" $\sum a_i + \sum b_i$

"length 0" : $\pi(E_{12}) v = 0 = \pi(E_{23}) v$

general : if $a_1 > 0$: $[E_{12}, E_{21}] = H_1 \Rightarrow$

$\pi(E_{12})\pi(E_{21}) = \pi(E_{21})\pi(E_{12}) + \pi(H_1)$

so $\pi(E_{12})\pi(E_{21})^{a_1} \cdots \pi(E_{32})^{b_k} v$

$= \pi(E_{21}) \underbrace{\pi(E_{12}) \underbrace{\pi(E_{21})^{a_1 - 1} \cdots \pi(E_{32})^{b_k} v}_{\text{len. 1 less}}}_{\text{in } W}$

$+ \underbrace{\pi(H_1)\pi(E_{21})^{a_1 - 1} \cdots \pi(E_{32})^{b_k} v}_{\text{in } W} \in W.$

$[E_{23}, E_{21}] = 0 \Rightarrow \pi(E_{32})\pi(E_{21})^{a_1} \cdots \pi(E_{32})^{b_k} v$

$= \pi(E_{21}) \underbrace{\pi(E_{32})\pi(E_{21})^{a_1 - 1}}_{\text{in } W} \cdots \pi(E_{32})^{b_k} v$

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If $a_1 = 0$: switch the role of E_{12} & E_{23} $\boxed{\Xi}$

Cor (π, V) irred $\Rightarrow$ V is generated by

- h.w. vec. $v \in V_\lambda$ · successive application of $\pi(E_{21})$, $\pi(E_{32})$.

(in fact: highest weight λ completely determines V cf. $\mathfrak{sl}_2$ case)

- Weyl group.

Recall rep of $\mathfrak{sl}_3(\mathbb{C}) \Longleftrightarrow$ rep of $SL_3(\mathbb{C})$

$\rightsquigarrow$ weight decomp of V

$\leftrightarrow$ decomp. as rep. of $H = \{ g \in SL_3(\mathbb{C}) : diag \}$

$\left(H = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & b^{-1} \end{bmatrix} ; a, b \in \mathbb{C}^\times \right\} \text{ cplx 2-torus} \right)$

<u>Normalizer</u> $N = N_{SL_3(\mathbb{C})}(H) = \{ g \in SL_3(\mathbb{C}) : gHg^{-1} = H \}$.

Ex. $g_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \in N$

1) $g_1 \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & b^{-1} \end{bmatrix} g_1^{-1} = \begin{bmatrix} a^{-1}b & & \\ & a & \\ & & b^{-1} \end{bmatrix}$, $g_2 \begin{bmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1} \end{bmatrix} g_2^{-1} = \begin{bmatrix} a & & \\ & b^{-1} & \\ & & a^{-1}b \end{bmatrix}$

2) $g_1^2 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$, $g_2^2 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \in H$.

$g_1 g_2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow (g_1 g_2)^3 = I_3$.

$\Rightarrow N/H \simeq S_3$

<u>Weyl group</u> of $\mathfrak{sl}_3(\mathbb{C})$: $W = N/H$; $s_i = [g_i]$ generators

$[g]h = ghg^{-1}$ $(g \in N, h \in H)$ defs

an action of W on H $([gh']h = gh'hh'^{-1}g^{-1}$
$= [g]h)$

$\Rightarrow$ induces an action on $\mathfrak{h}$ (= Lie alg of H).

$\begin{cases} s_1 H_1 = -H_1 \\ s_1 H_2 = H_1 + H_2 \end{cases} \leftrightarrow \begin{cases} s_1 \begin{bmatrix} a & & \\ & a^{-1} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} a^{-1} & & \\ & a & \\ & & 1 \end{bmatrix} \\ s_1 \begin{bmatrix} 1 & & \\ & b & \\ & & b^{-1} \end{bmatrix} = \begin{bmatrix} b & & \\ & 1 & \\ & & b^{-1} \end{bmatrix} \end{cases}$

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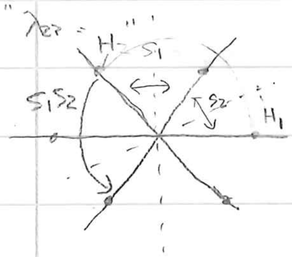
$s_2 H_1 = H_1 + H_2$, $s_2 H_2 = -H_2$



identifying $\langle H_1, H_2\rangle_{\mathbb{R}}$ with $\mathbb{R}\Lambda_W$ by inn. prd

so s_i acts by reflection along orthog. compl. of H_i

W acts on $\mathfrak{h}^*$, $\mathbb{R}\Lambda_W$, Λ_W, Λ_R in natural way $[g]\lambda = \lambda([g]^{-1}\sim)$.

Prop. (π, V) rep of $\mathfrak{sl}_3(\mathbb{C})$ $\mathrm{supp}(V) = \{\lambda \in \Lambda_W : V_\lambda \neq 0\}$

$\Rightarrow$ $\mathrm{supp}(V)$ is stable under W.

Proof. $\pi_{\mathfrak{g}}$ induces a rep of $SL_3(\mathbb{C})$.

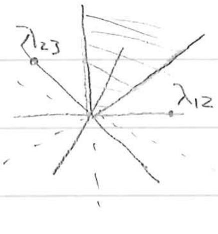
Claim: $\pi(g) V_\lambda = V_{[g]\lambda}$.

$\because$ $\pi(X)\pi(g)v = \pi(g)\underbrace{\pi(g^{-1})\pi(X)\pi(g)}_{\pi(g^{-1}Xg)} v$

$= \pi(g_i)\pi([g]^{-1}(X))v = \lambda([g]^{-1}(X))\pi(g_i)v$

$= ([g]\lambda)(X)\pi(g_i)v$. ☐

Fundamental dom of $W = S_3 \curvearrowright \mathbb{R}\Lambda_W$



highest weights belong here

summary
- Root system
  - axioms
  - canonical copies of $\mathfrak{sl}_2$.

- Root system.

Goal (after some time...)
complex simple Lie alg. $\mathfrak{g}$, $\mathfrak{h}$: Cartan subalg

$\rightsquigarrow$ roots (eigenvals of $\mathrm{ad}\,\mathfrak{h} \curvearrowright \mathfrak{g}$) ; Killing form on $\mathfrak{g}$ $\rightsquigarrow$ on $\mathfrak{h}$, $\mathfrak{h}^*$ $\rightsquigarrow$ root system

$\rightsquigarrow$ Dynkin diagram encoding $\left(\begin{matrix}\text{length}\\ \text{angle}\end{matrix}\right)$ of roots

$\rightsquigarrow$ classification of cplx simple Lie algs

Recap. $\mathfrak{g}$: simple Lie alg. over $\mathbb{C}$ (e.g. $\mathfrak{sl}_n(\mathbb{C})$)
$\mathfrak{h}$: Cartan subalg. of $\mathfrak{g}$ maximal diagonalizable
commutative ($X, Y \in \mathfrak{h} \Rightarrow [X,Y]=0$)
$\exists$ faithful rep / $\forall$ rep π $\forall X \in \mathfrak{h}$ $\pi(X)$ diag'ble
e.g. $\mathfrak{h} = \{X \in \mathfrak{sl}_n(\mathbb{C}) : \text{diag}\} < \mathfrak{sl}_n(\mathbb{C})$

$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$ root decomp
$\alpha \in \mathfrak{h}^*$ "root" $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{h}\ [Y, X] = \alpha(Y)X\}$

Fact 1. (for $\alpha \neq 0$) $\dim \mathfrak{g}_\alpha \leq 1$.
2. $\mathfrak{g}_\alpha \neq 0 \Rightarrow \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] =: S_\alpha$
is a subalg of $\mathfrak{g}$, isom to $\mathfrak{sl}_2(\mathbb{C})$
choose $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$, $H_\alpha := [E_\alpha, F_\alpha]$

(π, V) rep of $\mathfrak{g}$ $\Rightarrow$ $V = \bigoplus_\beta V_\beta$
$\beta \in \mathfrak{h}^*$ "weight" $V_\beta = \{v \in V : \forall Y \in \mathfrak{h}\ \pi_Y v = \beta(Y) v\}$
$\{0\} \cup$ roots $=$ weights of $\pi = \mathrm{ad}$

$\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$ & knowing weights of $\mathfrak{sl}_2$
$\Rightarrow \forall$ wght β, $\beta(H_\alpha) \in \mathbb{Z}$.

<u>wght lattice</u> $\Lambda_W = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z}\}$

<u>root lattice</u> Λ_R subgrp of Λ_W generated by roots.

$E = \mathbb{R} \cdot \Lambda_W$. (real) dual sp. of $\mathfrak{h}_0 = \langle H_\alpha : \alpha \text{ root} \rangle_{\mathbb{R}}$

Killing form $B_\mathfrak{g}(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \, \mathrm{ad}_Y)$

- nondeg. on $\mathfrak{g}$ ($\Leftarrow$ Cartan's criterion)
- invariance $B_\mathfrak{g}([Z,X],Y) + B_\mathfrak{g}(X,[Z,Y]) = 0$

$\Rightarrow$ $B_\mathfrak{g}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0$ iff $\alpha = -\beta$ (incl. $\alpha = 0$)
nondeg. on $\mathfrak{h}$ $(= \mathfrak{g}_0)$

Fact 4 $X \in \mathfrak{h}_0 \Rightarrow B_\mathfrak{g}(X,X) \geq 0$, $= 0$ only if $X = 0$

Write (v, w) $(v, w \in E)$ for the Euclidean inn. prod. on E.

Fact 3 α : root $\Rightarrow$ $T_\alpha = \dfrac{2}{B_\alpha(H_\alpha, H_\alpha)} H_\alpha$ is the elem s.t. $B_\mathfrak{g}(T_\alpha, H) = \alpha(H)$

(img of iso $\mathfrak{h}^* \to \mathfrak{h}$ corr. to $B_\mathfrak{g}$)

α, β roots $\Rightarrow$ $\dfrac{2B(\alpha,\beta)}{B(\alpha,\alpha)} = \alpha(H_\beta) \in \mathbb{Z}$.

Fact 5 α root. $k \in \mathbb{Z} \setminus \{\pm 1\} \Rightarrow k\alpha$ <u>not</u> root.

$s_\alpha(v) = v - \dfrac{2(\alpha, v)}{(\alpha,\alpha)} \alpha$ reflection along $\alpha^\perp$.

Rem. (π, V) rep. β weight $\Rightarrow \bigoplus_n V_{\beta + n\alpha}$ subrep of $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

$\Rightarrow$ $\mathrm{supp}(V) = \{\beta : \text{wght } V_\beta \neq 0\}$ is closed under s_α
$\because$ $\mathfrak{sl}_2(\mathbb{C})$ rep. has "symmetric" wghts.

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$\Rightarrow$ $R = \{\beta : \text{root}\} \subset E$ is closed under s_α.

$W = \langle s_\alpha : \alpha : \text{root} \rangle \subset O(E)$ <u>Weyl group</u>

Fact: R span E ($\mathbb{R}\Lambda_R = \mathbb{R}\Lambda_W$)

<u>Axiom</u> of root system.

E : Euclidean sp. ($\mathbb{R}$-vec. sp. & pos. def. inn. prd)

$R \subset E$ is a <u>root</u> <u>system</u> if

1) $|R| < \infty$, $\mathbb{R}$-span of R is E

2) $\alpha \in R$, $k \in \mathbb{Z}$ $\Rightarrow$ $k\alpha \in R$ iff $k = \pm 1$

3) $s_\alpha(v) = v - \dfrac{2(\alpha, v)}{(\alpha, \alpha)}\alpha$ maps R to R.

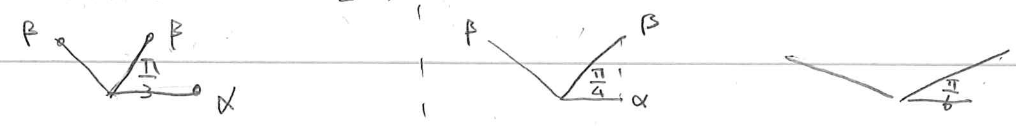
4) $n_{\alpha,\beta} = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Rem $n_{\alpha,\beta}\, n_{\beta,\alpha} = 4\cos^2\theta$ for θ : angle between α & β.

$4\cos^2\theta \in \mathbb{Z}$ $\Rightarrow$ $\cos\theta = \pm\dfrac{\sqrt{a}}{2}$ $a = 0, 1, 2, 3$.

$a = 1$ ($\cos\theta = \pm\frac{1}{2}$) $a = 2$ $a = 3$

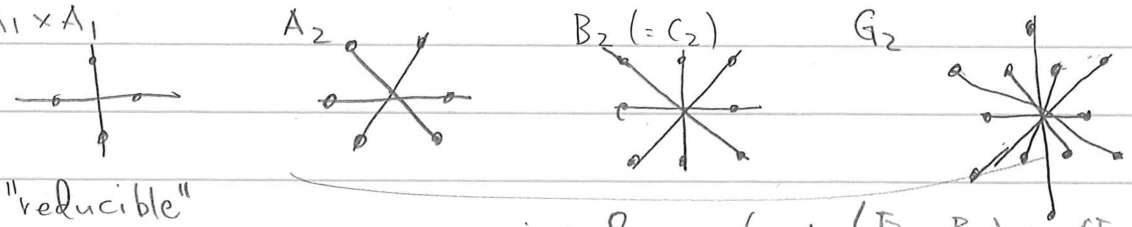
($\|\beta\| \geq \|\alpha\|$)



$\|\alpha\| = \|\beta\|$ $\|\beta\| = \sqrt{2}\,\|\alpha\|$ $\|\beta\| = \sqrt{3}\,\|\alpha\|$.

Ex. $\dim E = 2$

$A_1 \times A_1$ A_2 $B_2 (= C_2)$ G_2



"reducible"

irred. (not $(E_1, R_1) \times (E_2, R_2)$).

Motto : simple Lie alg $\rightsquigarrow$ irred root system,

"remembers everything"

Proof of Fact 6 $\quad \mathbb{R}\Lambda_R = \mathbb{R}\Lambda_W$

Equivalent to. $\mathbb{C}\Lambda_R = \mathfrak{h}^*$.

Take $X \in (\mathbb{C}\Lambda_R)^\perp \subset \mathfrak{h}$ (we want $X = 0$)

$[X, Y] = \alpha(X) Y = 0$ for all $Y \in \mathfrak{g}_\alpha$

$\Rightarrow [X, Y] = 0$ for all $Y \in \mathfrak{g}$. $\Rightarrow X \in \mathfrak{Z}(\mathfrak{g}) = 0$

Proof of Facts 1 & 2

Step 1 $\quad [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$

$\because$ Invariance $\Rightarrow B(X, [Y, Z]) = B([X, Y], Z)$

$= \alpha(X) B(Y, Z)$ for $X \in \mathfrak{h}, Y \in \mathfrak{g}_\alpha$

Cartan's criterion (& invar.) $\exists\, Z \in \mathfrak{g}_{-\alpha}$

$B(Y, Z) \neq 0 \rightsquigarrow$ for this Z $[Y, Z] \neq 0$

(If $T_\alpha \in \mathfrak{h}$ is s.t. $B(T_\alpha, X) = \alpha(X)$ $X \in \mathfrak{h}$

$[Y, Z] = B(Y, Z) T_\alpha$ for $Y \in \mathfrak{g}_\alpha, Z \in \mathfrak{g}_{-\alpha}$

Step 2 $\quad \alpha(T_\alpha) \neq 0$

$\because$ otherwise $s = \langle Y, Z, T_\alpha \rangle$ solvable

$\Rightarrow T_\alpha$ nilpot $\underset{\text{diag'ble}}{\Rightarrow} T_\alpha = 0$

Step 3 $\quad \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2(\mathbb{C})$

Step 4

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Summary

- canonical copies of $\mathfrak{sl}_2(\mathbb{C})$ in simple Lie algs
- Killing form

$\mathfrak{g}$ simple Lie alg over $\mathbb{C}$, $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha:\text{root}} \mathfrak{g}_\alpha)$, ...

Proof of (half of) "Fact 2":

$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ contains a subalg isom. to $\mathfrak{sl}_2(\mathbb{C})$.

$\because$ Take $0 \neq Y \in \mathfrak{g}_\alpha$ For $X \in \mathfrak{h}$

Step 1. $B(X, [Y, Z]) \underset{\substack{\text{invariance} \\ \text{\& antisymm.}}}{=} B([X, Y], Z) \underset{Y \in \mathfrak{g}_\alpha}{=} \alpha(X) B(Y, Z)$

Cartan's criterion $\Rightarrow \exists Z$ $B(Y, Z) \neq 0$. Invar. $\Rightarrow Z \in \mathfrak{g}_{-\alpha}$

If $T_\alpha \in \mathfrak{h}$ is the elem s.t. $B(T_\alpha, X) = \alpha(X)$ $(X \in \mathfrak{h})$

we have $[Y, Z] = B(Y, Z) T_\alpha$

Step 2 $\alpha(T_\alpha) \neq 0$

Otherwise $\mathfrak{s} = \langle Y, Z, T_\alpha \rangle$ has T_α in its center $\Rightarrow [\mathcal{D}(\mathfrak{s}), \mathfrak{s}] = 0$, $\mathfrak{s}$ solvable

$\Rightarrow \mathrm{ad}_{T_\alpha}$ on $\mathfrak{g}$ is nilpot. (Lie's thm) 09.25

But ad_{T_α} is diag'ble $\Rightarrow \mathrm{ad}_{T_\alpha} = 0$ contr.

Step 3 $H_\alpha = \dfrac{2}{\alpha(T_\alpha)} T_\alpha$, $E_\alpha = Y$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ s.t.

$$B(E_\alpha, F_\alpha) = \frac{2}{\alpha(T_\alpha)} \quad (\text{so } [E_\alpha, F_\alpha] = H_\alpha)$$

span a copy of $\mathfrak{sl}_2(\mathbb{C})$. $\because \alpha(H_\alpha) = 2$

Proof of "Fact 1" & "Fact 5".

$\dim \mathfrak{g}_{\pm 1} = 1$, $\dim \mathfrak{g}_{k\alpha} = 0$ $|k| \geq 2$

$\because$ $\mathfrak{s}_\alpha$: copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$

$V = \bigoplus_{k \in \mathbb{C}} \mathfrak{g}_{k\alpha}$ $(\mathfrak{g}_0 = \mathfrak{h})$

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Step 1 S_α acts triv. on $\ker(\alpha)$ $(\subset \mathfrak{h} \subset V)$

$\because$ $X \in \ker \alpha \Rightarrow [X, E_\alpha] = \alpha(X) E_\alpha = 0$ $(E_\alpha \in \mathfrak{g}_\alpha)$

$\Rightarrow [E_\alpha, X] = 0$. similarly $[F_\alpha, X] = 0$.

$[H_\alpha, X] = 0$ is from commutivity of $\mathfrak{h}$.

Step 2 "$k\alpha$" in V can only happen for $k \in \frac{1}{2}\mathbb{Z}$

$\because$ ad_{H_α} acts by $2k$ on $\mathfrak{g}_{k\alpha}$.

Step 3 "claim of Fact 5 & 1.

$\because$ 0-eigenspace for $\mathrm{ad}_{H_\alpha} \curvearrowright V$ is exhausted by $\ker(\alpha)$ and 0-wght sp. of $\mathfrak{s}_\alpha$

But $\mathfrak{g}_{k\alpha} \neq 0 \Rightarrow$ will give another 0-eig. vec. (We got Fact 5)

$\mathfrak{g}_\alpha \setminus S_\alpha$ will also create 0-wght sp.

$\Rightarrow \dim \mathfrak{g}_\alpha = 1$ $\boxtimes$

$\rightsquigarrow$ We also get "Fact 2" $(\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}])$

Proof of "Fact 3" $T_\alpha = \dfrac{2}{B(H_\alpha, H_\alpha)} H_\alpha$

$\because$ we already know $H_\alpha = \dfrac{2}{\alpha(T_\alpha)} T_\alpha$ and

$B(T_\alpha, X) = \alpha(X)$

$\Rightarrow B(H_\alpha, H_\alpha) = \dfrac{4}{\alpha(T_\alpha)}$

Proof of "Fact 4" $X \in \mathfrak{h}_0 = \langle H_\alpha : \alpha \text{ root} \rangle_{\mathbb{R}}$

$\Rightarrow B(X, X) \geq 0$, $= 0$ only when $X = 0$

$\because$ $\alpha(H_\beta) \in \mathbb{Z} \Rightarrow (X) \in \mathbb{R}$ for $X \in \mathfrak{h}_0$

$B(X, X) = \sum_{\alpha : \text{root}} \alpha(X) \alpha(X) \geq 0$.

$B(X, X) = 0$ means $\alpha(X) = 0$ $\forall \alpha$ $\Rightarrow X \in \mathfrak{z}(\mathfrak{g}) = 0$

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- Killing form

W : Weyl grp.

Thm (uniqueness of B) $\mathfrak{g}$: simple

1) $B_{\mathfrak{g}}$ is the unique (up to scalar) invar. bilin. form on $\mathfrak{g}$

2) induced inn. prod. on $\mathfrak{h}^*$ is the unique W-invar. inn. prod.

Proof 1) inv. bilin. form $\longleftrightarrow$ intertwiner $\mathfrak{g} \to \mathfrak{g}^*$

$\mathfrak{g}$ simple $\Rightarrow$ $ad : \mathfrak{g} \curvearrowright \mathfrak{g}$ is irred.

(subrep $\equiv$ ideal)

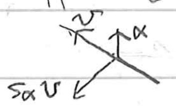
$\rightsquigarrow$ space of bilin. forms is 1-dim.

2) Enough to prove $W \curvearrowright \mathfrak{h}^*$ is irred.

Take $V \subset \mathfrak{h}^*$ W-invar.

Step 1. $\alpha \in \mathfrak{h}^* \setminus V$ root $\Rightarrow$ $\alpha \perp V$.

$\because$ if $\exists v \in V$, $v \not\perp \alpha$, $\Rightarrow$ $\alpha \in V$.



$\alpha \propto v - s_\alpha v \in V$

Step 2 $\mathfrak{g}' = \langle \mathfrak{g}_\alpha : \alpha$ root, $\alpha \in V \rangle_{\text{lin.sp.}} \underset{\text{ideal}}{\triangleleft} \mathfrak{g}$

$\because$ subalg : $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ $\alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

ideal : take $\beta \perp V$ root, $Y \in \mathfrak{g}_\beta$

$\alpha \in V$ root. $Z \in \mathfrak{g}_\alpha$.

$\Rightarrow$ $\alpha + \beta \notin V$, $\not\perp V$ $\Rightarrow$ $\alpha + \beta$ is not a root.

i.e. $\mathfrak{g}_{\alpha+\beta} = 0$ $\Rightarrow$ $[Y, Z] = 0$

$[Y, H_\alpha] = \underbrace{\beta(H_\alpha)}_{2(\beta,\alpha)/(\beta,\beta)} Y = 0$

$\Rightarrow$ $[Y, \mathfrak{g}'] = 0$. $X \in \mathfrak{h}$ $\Rightarrow$ $[X, \mathfrak{g}'] \subset \mathfrak{g}'$.

Step 3 claim.

$\mathfrak{g}' = 0$ or $\mathfrak{g}$ by Step 2. $\boxtimes$

$V = 0$ $V = \mathfrak{h}^*$.

Ex. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $\quad B_{\mathfrak{g}}(X,Y) = 2n\,\mathrm{Tr}(XY)$

- $\mathrm{Tr}(XY)$ has invariance. i.e.

$$\mathrm{Tr}([Z,X]Y) + \mathrm{Tr}(X[Z,Y]) = 0$$

- $H_1 = E_{11} - E_{22} \in \mathfrak{h}$ $\quad B(H_1, H_1) = 4n$, $\mathrm{Tr}(H_1 H_1) = 2$.

$$\mathrm{ad}_{H_1}(E_{ij}) = \begin{cases} 2E_{12}, & -2E_{21} \\ E_{1j}, & -E_{2j} \quad (j>2) \\ -E_{i1} & E_{i2} \quad (i>2) \end{cases} \quad \text{otherwise } 0$$

$\rightsquigarrow$ eigenvals $\quad 2 \quad -2 \quad 1 \quad -1$

mult $\quad 1 \quad 1 \quad 2(n-2) \quad 2(n-2)$

$$\Rightarrow \mathrm{Tr}((\mathrm{ad}_{H_1})^2) = 4 + 4 + 2(n-2) + 2(n-2) = 4n$$

Up to normalization, $H_i = E_{ii}$

$B(H_i, H_i) = 2$, $\quad B(H_i, H_{i\pm1}) = -1$, $\quad B(H_i, H_j) = 0$ $\ |i-j|>1$

$$\rightsquigarrow W = \langle (s_i)_{i=1}^{n-1} : s_i^2 = e,\ (s_i s_{i+1})^3 = e,\ s_i s_j = s_j s_i \rangle$$
$$\simeq S_n$$

Summary

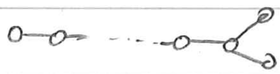
- Classification of root systems
  - Dynkin diagrams
  - positive roots
  - structure of irred. root system.

- Dynkin diagrams

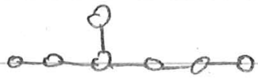
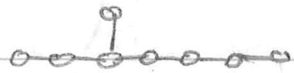
Goal : classify irred. root system (E, R)

(E : Euclidean sp. $R \subset E$ finite, ----)

by the <u>Dynkin diagrams</u>

- A_n 
- B_n 
- C_n 
- D_n 

} n : number of vertices

- E_6  E_7  E_8 
- F_4  , • G_2 

(Rem this will give a classification of simple Lie algs over $\mathbb{C}$

$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C})$, $B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C})$, $C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C})$

$D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C})$)

- Positive roots

$\mathfrak{g}$: simple Lie alg over $\mathbb{C}$, $\mathfrak{h}$: Cartan subalg

$E = \mathbb{R}\Lambda_W$, $R = \{\alpha : \text{root}\}$. $\left(\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha\right)\right)$

$E \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{h}^*$, ...

we want say which part of $\mathfrak{g}$ is "uppertriangular"

($\mathfrak{h}$ is "diagonal", ...)

concretely : take $\ell : E \to \mathbb{R}$ linear which is in "general position" : $\ker \ell \cap R = \emptyset$.

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and call $R^+ = \{\alpha \in R : \ell(\alpha) > 0\}$ <u>positive</u> roots

$R^- = \{\alpha \in R : \ell(\alpha) < 0\}$ <u>negative</u> roots

$\alpha \in R^+$ is <u>simple</u> if $\nexists\ \beta, \gamma \in R^+$ s.t. $\alpha = \beta + \gamma$.

Facts 1 any other $\ell' \rightsquigarrow R^{+'} = w(R^+)$ for $\exists\ w \in W$

2* simple pos. roots form a basis of Λ_R (& E)

(*prove later) 3* $(\alpha, \beta) \leq 0$ $\left(\Rightarrow\ = -\frac{\sqrt{\ell}}{2}, \ell = 0,1,2,3\right)$

Ex. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $\mathfrak{h} = \{X \in \mathfrak{sl}_n(\mathbb{C}) : \text{diag}\}$

$\mathfrak{h}_0 = \langle H_i = E_{ii} - E_{i+1, i+1} : i = 1, \cdots, n-1 \rangle_{\mathbb{R}}$

$E = \mathfrak{h}_0^*$ $\mathfrak{h}_0 \subset \tilde{\mathfrak{h}}_0 = \langle E_{ii} : i = 1, \cdots, n \rangle_{\mathbb{R}}$

$E \xleftarrow{\text{res.}} \tilde{E} = \tilde{\mathfrak{h}}_0^* = \langle L_i : L_i(E_j) = \delta_{ij} \rangle$

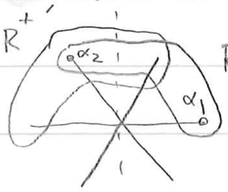
in E, we have $L_1 + \cdots + L_n = 0$ (corr. to trace)

$r_1 > r_2 > \cdots > r_{n-1} > 0 \Rightarrow \begin{cases} \ell(L_i) = r_i & (i < n) \\ \ell(L_n) = -(r_1 + \cdots + r_{n-1}) \end{cases}$

$R = \{L_i - L_j : i \neq j\}$ $L_i - L_j \leftrightarrow E_{ij}$

we want ℓ s.t. E_{ij} $(i<j)$ become pos.

Ex.



$s_{\alpha_1}(R^+) = R^{+'}$

generally $|R^{+'} \setminus R^+| = k \Rightarrow \exists\ w = s_{\alpha_1} \cdots s_{\alpha_k}$ st. $w(R^+) = R^{+'}$

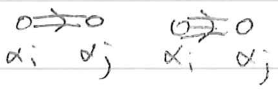
$L_i - L_{i+1}$ simple pos. root

$\leftrightarrow E_{i,i+1}$ "just above diagonal"

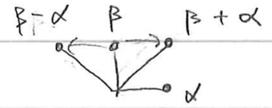
How to draw Dynkin diag.

- Vertices : simple pos. roots $\alpha_1, \cdots, \alpha_n$
- $\dfrac{(\alpha_i, \alpha_j)}{\sqrt{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}} = -\dfrac{\sqrt{\ell}}{2} \rightsquigarrow \ell$ edges between α_i & α_j

(Ex.  $\Rightarrow \dfrac{(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_i)} = -\frac{1}{2}$, $(\alpha_1, \alpha_3) = 0$

$\ell = 2, 3 \rightsquigarrow$  for $\|\alpha_i\| > \|\alpha_j\|$

Lem $\alpha, \beta \in R$ $\alpha = \pm \beta$ $p, q = \max$ s.t.
"α-string" $\beta - p\alpha, \beta - p\alpha + \alpha, \dots, \beta, \beta+\alpha, \dots, \beta + q\alpha$
are roots. Then $p + q \leq 3$, $p - q = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}$ $(\in \mathbb{Z})$
$(n_{\beta,\alpha})$

B_2 : 

Proof. $s_\alpha \in W$ reflects α-string. $\beta + q\alpha \longleftrightarrow \beta - p\alpha$.
Concretely $s_\alpha(\gamma) = \gamma - 2\frac{(\alpha, \gamma)}{(\alpha, \alpha)}\alpha$
so $s_\alpha(\beta + k\alpha) = \beta - (n_{\beta,\alpha} + k)\alpha$ $\qquad p = n_{\beta,\alpha} + q$

With $\beta' = \beta - p\alpha$: $3 \geq n_{\beta', \alpha} = p + q$

Prop $\alpha, \beta \in R$, $\alpha = \pm\beta$
1) $(\alpha, \beta) > 0 \Rightarrow \alpha - \beta$ is a root
$(\alpha, \beta) < 0 \Rightarrow \alpha + \beta$ is a root
2) $(\alpha, \beta) = 0$ either $\left(\begin{array}{l} \alpha \pm \beta \text{ are both roots} \\ \sim \text{ are both not roots} \end{array} \right.$

Proof 1) γ root $\Leftrightarrow -\gamma = s_\gamma(\gamma)$ root
enough to show $(\alpha, \beta) > 0 \Rightarrow \beta - \alpha$ root.
But $s_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$. $n_{\beta,\alpha} > 0$
"α-string" through β contains $\beta - \alpha$.
2) From Lem. $(\alpha, \beta) = 0$ means $p = q$.
$p = q = 0 \Rightarrow \beta \pm \alpha$ are not roots
$p = q > 0 \Rightarrow \beta \pm \alpha$ are roots. ∎

Cor α, β simple pos.
1) $\alpha - \beta$ $(\not\ni \beta - \alpha)$ not root
2) $(\alpha, \beta) \leq 0$

Proof 1) from def of simplicity
$\because$ $\alpha - \beta \in R \Rightarrow$ either $\alpha - \beta$ or $\beta - \alpha$ is pos

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2) Prop 1) and Cor 1).

Lem.. simple pos. are lin. indep.

$\because$ They $\begin{cases} \text{belong to } \ell(\gamma) > 0 \\ \text{have non-acute angles} \end{cases}$

Cor. simple pos. roots form a basis of E.

$\because$ gen. $\mathbb{N} \cdot R^+$ as semigrp.

$R = R^+ \cup (-R^+)$

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summary

- classification of Dynkin diag.
- Dynkin diag. to simple Lie alg.

∘ Class. Dynkin.

(E, R) root system $R = R^+ \amalg R^-$ choice of pos. roots.

$\rightsquigarrow$ $\Pi = \{\alpha_i : i = 1, \dots, n\}$ simple pos rts

Dynkin diag. Γ vertex set $= \Pi$

Goal. we want to say Γ is one of A_n, B_n, C_n, D_n, E_k $(6 \le k \le 8)$, F_4, G_2

Put $e_i = \frac{1}{\sqrt{(\alpha_i, \alpha_i)}} \alpha_i$ (rescaled unit vec)

$\rightsquigarrow (e_i, e_j) = -\frac{\sqrt{d}}{2}$ $d = 0, 1, 2, 3$.

Rem. $4(e_i, e_j)^2 = d$ = number of edges between α_i and α_j in Γ

Not. $i \sim j$ for $\exists$ edge α_i & α_j

We'll check Γ

P1 does not have loops

P2 any vertex is connected to < 4 edges.

P3 does not contain 

("the rest" are similar, FH §21.2)

P1) $J \subset \Pi$, $|J| = k$ $\Rightarrow \exists$ at most $k-1$ (unord.) pairs (j, j') with $j, j' \in J$ & $e_j, e_{j'}$ are conn.

$v = \sum_j e_j \Rightarrow (v, v) = \sum_j (e_j, e_j) + \sum_{j \sim j'} (e_j, e_{j'})$

$\le k + 2 \times (\#(j, j') \text{ as abv}) \times (-\frac{1}{2})$

this has to be pos. $\boxed{}$

P2) fix i. Want: $\sum_j \underbrace{\#(\text{edges between } \alpha_i \text{ \& } \alpha_j)}_{4(e_i,e_j)^2 \text{ by Rem.}} < 4$

Step 1. $i \sim j$ & $i \sim j' \Rightarrow j \not\sim j'$

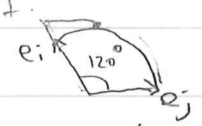
$\because$ P1

Step 2. estim. of claim.

$(e_j : i \sim j)$ are mut. orth., don't span e_i

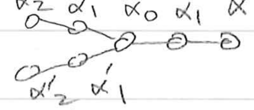
$\Rightarrow \sum (e_i, e_j)^2 < \|e_i\| = 1$ $\boxed{}$

P3) S1.



$\frac{2e_i + e_j}{\sqrt{3}}$ is a unit vec.

Step 2



contra. from

$v = \frac{2e_1 + e_2}{\sqrt{3}}$, v', v'' unit vecs.

- $(e_0, v) = \frac{2}{\sqrt{3}}(e_0, e_1) = -\frac{1}{\sqrt{3}}$

v, v', v'' mut orth, don't span e_0

$\Rightarrow \underbrace{(e_0, v)^2}_{\frac{1}{3}} + \underbrace{(e_0, v')^2}_{\frac{1}{3}} + \underbrace{(e_0, v'')^2}_{\frac{1}{3}} < \underbrace{(e_0, e_0)}_{1.}$ $\boxed{}$

Dynkin to $\mathfrak{g}$

Cartan matrix. $A = (a_{ij})_{i=1}^{n}$ $(n = \#\Pi)$

$a_{ij} = n_{\alpha_i, \alpha_j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}$ (another conv. n_{α_j, α_i})

Rem $a_{ii} = 2$, $i \neq j \Rightarrow a_{ij} \in \{0, -1, -2, -3\}$

$A_2 : \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B_2 : \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$, $G_3 : \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

2018.10.31. no3

Goal: $\mathfrak{g}$ cplx simple Lie, $\mathfrak{h} \subset \mathfrak{g}$ Cartan

R roots, $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in R^+$

$(R = R^+ \amalg R^-, \Pi)$ $H_\alpha \in \mathfrak{h}$ so $\langle E_\alpha, F_\alpha, H_\alpha \rangle \simeq \langle E, F, H \rangle$

Goal describe structure of $\mathfrak{g}$ interms of

- $E_\alpha, F_\alpha, H_\alpha$ for $\alpha \in \Pi$ as generators
- relation in terms of Cartan mat A (can read off from Dynkin diag)

$\rightsquigarrow$ which will give

- existence of $\mathfrak{g}$ for given Dynkin diag
- uniqueness $(\mathfrak{g}_i, \mathfrak{h}_i)$ $i = 1,2$ gave same Γ $\Rightarrow$ $\exists$ iso φ $\mathfrak{g}_1 \to \mathfrak{g}_2$ s.t. $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$

(we'll remove the choice of $\mathfrak{h}$ later)

"Obvious" relations

- $[E_\alpha, F_\alpha] = H_\alpha$ by def
- $[E_\alpha, F_\beta] = 0$ (would give elem in $\mathfrak{g}_{\alpha-\beta}$ but $\alpha - \beta$ is not a root)
- $[H_\alpha, H_\beta] = 0$
- $[H_{\alpha_i}, E_{\alpha_j}] = a_{ji} E_{\alpha_j}$, $[H_{\alpha_i}, F_{\alpha_j}] = -a_{ij} F_{\alpha_j}$

$\because$ equiv. to $a_{ji} = \alpha_j(H_{\alpha_i})$

$$\beta(H_{\alpha_i}) = \frac{2B(\beta, \alpha_i)}{B(\alpha_i, \alpha_i)}$$ for inv. bilin. form B (10.23)

Not-so-obvious relation (Serre relation)

$\mathrm{Ad}_{E_{\alpha_i}}^{1-a_{ji}}(E_{\alpha_j}) = 0$, $\mathrm{Ad}_{F_{\alpha_i}}^{1-a_{ji}}(F_{\alpha_j}) = 0$ $(i \neq j)$

Ex. $\mathfrak{sl}_3(\mathbb{C})$ claim is $[E_{12}, [E_{12}, E_{23}]] = 0$

$[E_{21}, [E_{21}, E_{32}]] = 0$

Pf of Serre rel. for E's

look at α_i - string through α_j

$\alpha_j - \alpha_i$ not root

$\rightsquigarrow \alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + (\underbrace{-n_{\alpha_j,\alpha_i}}_{-a_{ji}})\alpha_i$

Summary

- Construction of simple Lie algs from root sys.
  - free Lie algs
  - ideal from Serre's rel.
  - implementing Weyl group action

∘ Free Lie algs

Want : construct Lie algs from "generators & rels"

K : comm. field

Case 1 No rels.

S : set (of "generators")

$\rightsquigarrow M(S) = \coprod_{k=1}^{\infty} S_k$, $S_1 = S$, $S_n = \coprod_{p=1}^{n-1} S_p \times S_{n-p}$

"set of "words" with letters in S a, ab, $(ab)c$, ...

distinguish parenthesization. $(ab)c \neq a(bc)$

$$L_K(S) = \underbrace{K[M(S)]}_{\text{formal lin comb. of } M(S)} / \langle aa, (ab)c + (bc)a + (ca)b \rangle \quad a, b, c \in M(S)$$

nonassoc. alg

x_a : img of a in $L_K(S)$.

Lie bracket by $[x_a, x_b] = x_{ab}$

$L_K(S)$: free Lie alg generated by S

Case 2. put relations

R : set of "relations" $\subset K[M(S)]$

(like $a \cdot b - c \iff$ "$[x_a, x_b] = x_c$")

$$L_K(S; R) = L_K(S) / \langle \text{img of } r \in R \rangle$$

Lie alg gen. by S, with rel R.

(or, by $(x_t)_{t \in S}$)

2018.11.06 no.2

- Ideal from Serre's rel.

(E, R) root sys, $R = R^+ \amalg R^-$ pos - neg. ..

$\Pi = \{\alpha_1, \cdots, \alpha_n\}$ simple pos. roots.

$a_{ij} = n_{\alpha_i \alpha_j} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, $A = (a_{ij})_{i,j=1}^n$ Cartan mat

$\tilde{\mathfrak{g}}$: Lie alg with

generators : e_i, f_i, h_i $i = 1, \cdots, n$

$(S = \Pi \times \{-1, 0, 1\}$, $x(i,1) = e_i$, $x(i,-1) = f_i$, $x(i,0) = h_i)$

relations
- $[h_i, h_j] = 0$
- $[e_i, f_i] = h_i$
- $[e_i, f_j] = 0$ $(i \neq j)$
- $[h_i, e_j] = a_{ji} e_j$, $[h_i, f_j] = -a_{ji} f_j$

(all rels in the corresp. semisimple Lie alg <u>except</u> for the Serre rels)

$\tilde{\mathfrak{g}}_\pm \rightarrow$

Rem. $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(h_i) = -h_i$

is an aut. of $\tilde{\mathfrak{g}}$ ("$\theta(X) = -X^t$")

$x_{ij} = \mathrm{ad}_{e_i}^{1-a_{ji}}(e_j)$, $\mathfrak{b}_+ =$ ideal of $\tilde{\mathfrak{g}}_+$ gen'ed by $(x_{ij})_{i,j}$

$y_{ij} = \mathrm{ad}_{f_i}^{1-a_{ji}}(f_j)$, $\mathfrak{b}_- = \cdots\cdots \tilde{\mathfrak{g}}_-$

Prop. $[\tilde{\mathfrak{g}}_-, x_{ij}] = 0$ $([\tilde{\mathfrak{g}}_+, y_{ij}] = 0)$

Cor. $\mathfrak{b}_\pm \lhd \tilde{\mathfrak{g}}$ $(\mathfrak{b}_- = \theta(\mathfrak{b}_+))$

$\because$
- h_i stabilize $\mathfrak{b}_+$. ($[e_{k_1}, [\cdots, [e_{k_p}, x_{ij}]\cdots]$ eigenvec. of ad_{h_i})
- e_i obviously stab. $\mathfrak{b}_+$
- Prop $\Rightarrow$ f_i also stab $\mathfrak{b}_+$

2018.11.06 no 3

Proof of Prop.

Step 1. $[f_k, x_{ij}] = 0$ for $k \notin \{i, j\}$.

$\because$ $[f_k, e_i] = 0 = [f_k, e_j]$

Step 2 $[f_j, x_{ij}] = 0$

$\because$ f_j comm. with e_i $\Rightarrow$ ad_{f_j} comm. with ad_{e_i}

$\Rightarrow [f_j, x_{ij}] = ad_{e_i}^{1-a_{ji}} \underbrace{ad_{f_j}(e_j)}_{-h_j}$

$= ad_{e_i}^{-a_{ji}} (\underbrace{[e_i, h_j]}_{-a_{ij} e_i})$

$a_{ij} = 0 \Rightarrow$ this is zero

$a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0 \Rightarrow$ we'll have $ad_{e_i}(e_i) = 0$

Step 3 $[f_i, x_{ij}] = 0$

$\because$ use $[ad_{f_i}, ad_{e_i}] = -ad_{h_i}$, $[ad_{h_i}, ad_{e_i}] = 2 ad_{e_i}$

$\rightsquigarrow [ad_{f_i}, ad_{e_i}^k] = -k\, ad_{e_i}^{k-1}(ad_{h_i} + k - 1)$

(e.g. $[ad_{f_i}, ad_{e_i}^2] = \underbrace{[ad_{f_i}, ad_{e_i}] ad_{e_i}}_{-ad_{e_i} ad_{h_i} + 2 ad_{e_i}} + ad_{e_i}[ad_{f_i}, ad_{e_i}]$

Use this for $k = 1 - a_{ji}$, & $(ad_{h_i} - a_{ji}) e_j = 0$ $\square$

Goal: $\mathfrak{g}_A = \tilde{\mathfrak{g}} / (\mathfrak{b}_+ + \mathfrak{b}_-)$ is semisimple.

E_i = img of e_i, F_i, H_i similar.

- Implementing Weyl grp.

Want: impl. $W \curvearrowright \mathfrak{h}$ as "Adjoint by elems of G"

Derivation $D: \mathfrak{g} \to \mathfrak{g}$, $D([x,y]) = [x, D(y)] + [D(x), y]$

Ex. $D(x) = [z, x] = ad_z(x)$

2018. 11. 06. no. 4

If D is (locally) nilpotent $\forall x \in \mathfrak{g}\ \exists k\ D^k(x) = 0$

$e^D : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} D^k(x)$ makes sense. (Char $K = 0$)

Lem e^D is Lie a'g aut.

$\because$ By ind. $\frac{1}{k!} D^k([x,y]) = \sum_{n=0}^{k} \frac{1}{n!(k-n)!} [D^n(x), D^{k-n}(y)]$

For $\mathfrak{g} = \mathfrak{g}_A$ from before

ad_{E_i}, ad_{F_i} are (locally) nilpot.

$\theta_i = e^{ad_{E_i}} e^{-ad_{F_i}} e^{ad_{E_i}} \in Aut(\mathfrak{g})$.

Rem $e^{ad_X}(Y) = e^X Y e^{-X}$ for matrices

$$e^{ad_E} e^{-ad_F} e^{ad_E}(Y) = Ad(\underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{\begin{bmatrix}0&1\\-1&0\end{bmatrix}})(Y) \quad Y \in M_2(K)$$

Lem $\theta_i(H_j) = H_j - a_{ij} H_i$

($\rightsquigarrow \theta_i$ induces $s_{\alpha_i} \curvearrowright \mathfrak{h}^*$, $\mathfrak{h} = $ span of $(H_i)_i$)

$\because$ Direct computation.

- semi-simplicity of $\mathfrak{g}_A$

Th'm (E, R) (irred) root system

$\Rightarrow \mathfrak{g}_A$ (semi) simple.

Proof. Step 1 Enough to prove $\nexists$ nonzero comm. ideal.

$\because Rad(\mathfrak{g}) \neq 0 \Rightarrow \exists\, 0 \neq \mathfrak{b} \triangleleft \mathfrak{g}$, comm. (09.26)

Step 2 $\mathfrak{b} \triangleleft \mathfrak{g} \Rightarrow \mathfrak{b} = \mathfrak{b} \cap \mathfrak{h} \oplus (\bigoplus_{\alpha:\text{roots}} \mathfrak{b} \cap \mathfrak{g}_\alpha)$

$\because$ From ad_{h_i} (mutually commute, diag'ble)

$\mathfrak{g} = (\bigoplus_{\alpha} \mathfrak{g}_\alpha) \oplus \mathfrak{h}$ (fact 1)

Step 3. $\mathfrak{b} \triangleleft \mathfrak{g}$, $\mathfrak{b}$ comm $\Rightarrow$ $\mathfrak{b} \cap \mathfrak{g}_{\alpha_i} = 0$.

$\because$ $\dim \mathfrak{g}_{\alpha_i} = 1$ & $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] \neq 0$.

(Fact 2)

$\exists\, w \in W$ s.t. $w(\alpha_i) = -\alpha_i$ $\quad (w = s_{\alpha_i})$

$\mathfrak{b} \triangleleft \mathfrak{g}$, $\mathfrak{b} \cap \mathfrak{g}_{\alpha_i} \neq 0 \Rightarrow \mathfrak{b} \cap \mathfrak{g}_{-\alpha_i} \neq 0$

$\because$ action of $w = e^{\mathrm{Ad} x_1} \cdots e^{\mathrm{Ad} x_k}$.

But this will imply $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{b}$.

Step 4. $\mathfrak{b} \cap \mathfrak{h} \neq 0 \Rightarrow \exists\, i$ s.t. $E_i \in \mathfrak{b}$

($\Rightarrow \mathfrak{b}$ noncomm.)

$\because$ For simplicity suppose (E, R) irred

$\rightsquigarrow W \curvearrowright E$ is irreducible (cf. 10.24 Thm)

$\mathfrak{b} \cap \mathfrak{h}$ is W-invar $\Rightarrow \mathfrak{b} \cap \mathfrak{h} = \mathfrak{h}$.

then $2E_i = [H_i, E_i] \in \mathfrak{b}$. ☒

Complements

Fact 1 : $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\mu \in \Lambda_R} \mathfrak{g}_\mu \right)$ by const.
Λ_R : root lattice

$\rightsquigarrow$ want : $\mathfrak{g}_\mu = 0$ for $\mu \in \Lambda_R \setminus R$.

Step 1 μ not mult. of root

$\because$ $\exists\, w \in W$ s.t. $w\mu \notin \mathbb{N}R \cup (-\mathbb{N}R)$

$\Rightarrow \tilde{\mathfrak{g}}_{w\mu} = 0$ by const. $\Rightarrow \mathfrak{g}_{w\mu} = 0$

$\Rightarrow \mathfrak{g}_\mu = 0$

Step 2 $\mu = m\alpha$ $\quad m > 2$, $\alpha \in R$.

$\because$ $\tilde{\mathfrak{g}}_\mu = 0$ bc. $\tilde{\mathfrak{g}}_+ \simeq L_K((e_i)_{i=1}^n)$

Fact 2 : $\tilde{\mathfrak{g}}_+ = L_K((e_i)_{i=1}^n) \Rightarrow \dim \tilde{\mathfrak{g}}_{\alpha_i} = 1$

If $e_i \in \mathfrak{b}_+$ $\quad h_i = [e_i, f_i] \in \mathfrak{b}_+$ not possible. ☒

2018.11.07 no.1

Summary
- Uniqueness of Cartan subalg.
- Borel subgroups, (full) flag manifold
- compact form

- Uniqueness of Cartan subalg

$\mathfrak{g}$ complex (semi) simple Lie alg (like $\mathfrak{sl}_n(\mathbb{C})$)

Th'm $\mathfrak{h}, \mathfrak{h}'$ Cartan subalg
$\leadsto \exists g \in G$ s.t. $\mathrm{Ad}_g(\mathfrak{h}) = \mathfrak{h}'$.

G: complex (semi) simple Lie grp corr. to $\mathfrak{g}$
(like $SL_n(\mathbb{C})$)

Conseq. choice of a Cartan subalg $\mathfrak{h} \subset \mathfrak{g}$ is essentially unique:
- root system of $(\mathfrak{g}, \mathfrak{h})$ does not depend on this choice
- Dynkin diagrams classify $\mathfrak{g}$, not just $(\mathfrak{g}, \mathfrak{h})$
  (having $\mathfrak{g} \simeq \mathfrak{g}'$ is "same" as $(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{g}', \mathfrak{h}')$)

Outline for the proof of th'm.

Step 1. Find $Y \in \mathfrak{h}$ or $h \in H$ (subgrp of G corr. to $\mathfrak{h}$) s.t. $\mathfrak{h} = \begin{cases} \mathrm{Cent}_{\mathfrak{g}}(Y) = \{X : [X, Y] = 0\} \\ \mathrm{Cent}_{\mathfrak{g}}(h) = \{X : \mathrm{Ad}_g(X) = X\} \end{cases}$

$\because$ Y, h : any elem in "general position"
$\begin{cases} \alpha(Y) \neq 0 \text{ for any root } \alpha. \ (Y \text{ is regular}) \\ h \text{ generates a dense subgroup of } H \end{cases}$
(like $\begin{bmatrix} e^{2\pi\sqrt{-1}\theta_1} & & \\ & \ddots & \\ & & e^{2\pi\sqrt{-1}\theta_n} \end{bmatrix}$ $\theta_1, \dots, \theta_{n-1}$ lin. indep. $/\mathbb{R}$)

Step 2 Find $g \in G$ s.t. $\mathrm{Ad}_g(Y) \in \mathfrak{h}'$ or $\mathrm{Ad}_g(h) \in H'$
$\leadsto \mathfrak{h}' = \mathrm{Ad}_g(\mathfrak{h})$

2018.11.07 no.2

To carry out Step 2 (by Borel subgrp)

- Use quotient G/B' (full flag manifold) by a Borel subgroup, & Lefschetz fixed pt thm for action of h. $\rightsquigarrow$ $hg^{-1}B' = g^{-1}B'$ for some g^{-1}

$SL_2(\mathbb{C}) \rightsquigarrow SL_2(\mathbb{C}) / \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \right\} \simeq \mathbb{P}^1(\mathbb{C}) \simeq S^2$

- Use compact form $K \subset G$ & prove uniqueness (up to conjugation) for $T = K \cap H$
- any elem. of K is conj. to an elem. of T.

- $\mathfrak{h}_{reg} = \{ Y : \text{regular} \in \mathfrak{h} \}$

$F : G \times \mathfrak{h}_{reg} \to \mathfrak{g}$, $(g, Y) \mapsto \mathrm{Ad}_g(Y)$

img is Zariski open (..

$\rightsquigarrow$ $\mathrm{Ad}_{g_0}(Y) = \mathrm{Ad}_{g_0'}(Y')$ for some (g_0, Y) & (g_0', Y')

$\rightsquigarrow$ $g = g_0'^{-1} g_0$ conjugates Y into $\mathfrak{h}'$.

- Borel subgroups

$\mathfrak{g}$: (semi) simple / $\mathbb{C}$, $\mathfrak{h} \subset \mathfrak{g}$ Cartan

$R = R_+ \amalg R_-$ decomp of roots,

$(\Pi = \{ \alpha_1, \cdots, \alpha_n \}$ simple pos. roots$)$

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ $\quad \mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$

$\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. Borel subalgs ("upper/lower triang")

Prop. $\mathfrak{b}_\pm$ are maximal solvable subalgs containing $\mathfrak{h}$.

Pf. $\mathfrak{h} \subset \mathfrak{g}' \subset \mathfrak{g}$ subalgs $\Rightarrow$ $\mathfrak{g}' = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}' \cap \mathfrak{g}_\alpha \right)$

if $\mathfrak{b}_+ \subsetneq \mathfrak{g}'$ $\quad \exists \alpha \in R_-$ s.t. $\mathfrak{g}' \cap \mathfrak{g}_\alpha \neq 0$

$\mathfrak{g}'$ will contain $\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2(\mathbb{C})$ not solvable ⊡

2018. 11. 07 no. 3

Borel subalg : max. solvable subalg of $\mathfrak{g}$.

(all Borel subalgs are conjugate under $G \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g}$).

We'll look at B = subgroup of G for $\mathfrak{b}_+$.

$$SL_n(\mathbb{C}) \rightsquigarrow B = \left\{ \begin{bmatrix} * & * & \cdots & * \\ & * & & \vdots \\ 0 & & \ddots & \\ & & & * \end{bmatrix} \right\}.$$

G/B : full flagmanifold.

Facts : • $G/B = \coprod_{w \in W} B[w]$. ($G = \coprod_{w \in W} BwB$)

Bruhat decomposition.

each $B[w] \simeq \mathbb{C}^{\ell(w)}$ for "length" func.

$$\ell : W \to \mathbb{N}.$$

(• G/B is a projective variety).

$$\rightsquigarrow H^n(G/B; \mathbb{Q}) = \begin{cases} \mathbb{Q}^{\oplus \#\{w : 2\ell(w) = n\}} & n : \text{even} \\ 0 & n : \text{odd}. \end{cases}$$

Lefchetz fixed pt formula.

X : "nice" cpt. top. sp. (finite CW cplx. ...)

$f : X \to X$ ct map.

$$\rightsquigarrow \# \{ p \in X : f(p) = p \} \geq \sum (-1)^n \operatorname{Tr}(f_* |_{H^n(X;\mathbb{Q})}) =: \Lambda_f$$

Cor. $\forall h \in G$ $\exists g \in G$ s.t. $hg^{-1}B = g^{-1}B$.

$\because$ G/B is "nice", $h \curvearrowright G/B$ is homotop. to id. $\Rightarrow \Lambda_h = \sum \dim H^{2n}(X; \mathbb{Q}) = |W|$.

Application to uniqueness

$\mathfrak{h}, \mathfrak{h}'$ Cartan, $h \in H$ s.t. $\mathfrak{h} = \mathrm{Cent}_{\mathfrak{g}}(h)$

look at $h \curvearrowright G/B'$, find g as above.

$\rightsquigarrow \mathrm{Ad}_g(h) \in B'$ $\rightsquigarrow$ by semisimple - nilpot decomp

$$\mathrm{Ad}_g(h) \in H'$$

∘ Compact form.

Want : real Lie subalg $\mathfrak{g}_c \subset \mathfrak{g}$ s.t.

- $G_c \subset G$ is compact (small).
- $\mathfrak{g}_c \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$. (but recovers $\mathfrak{g}$)

How to do : if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we want

$\mathfrak{g}_c = \mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) : -\bar{X}^t = X\}$

$\sigma : X \mapsto -\bar{X}^t$ is conjug. lin. autmorphism

s.t. $\sigma^2 = \mathrm{Id}$.

(and root vec $E_{ij} \mapsto -E_{ji}$)

Generally : $(\mathfrak{g}, \mathfrak{h})$, $R = R_+ \sqcup R_-$.

$E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha \in \mathfrak{g}_{-\alpha}$ $H_\alpha = [E_\alpha, F_\alpha]$ for $\alpha \in R_+$

$\rightsquigarrow$ $\sigma(E_\alpha) = -F_\alpha$, $\sigma(F_\alpha) = -E_\alpha$, $\sigma(H_\alpha) = -H_\alpha$.

(enough to define on simple pos. roots)

and extend as conjugate linear.

Prop. 1) σ is an aut. of $\mathfrak{g}$

2) on $\mathfrak{g}_c = \{x \in \mathfrak{g} : \sigma(X) = X\}$, the Killing form is negative definite

3) $G_c < G$ for $\mathfrak{g}_c$ is compact

4) $\mathfrak{g}_c \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$.

Proof 2) $\mathfrak{h}_c = \mathfrak{h} \cap \mathfrak{g}_c$ is $\sqrt{-1}\,\mathfrak{h}_0$ ($\mathfrak{h}_0 = \langle H_i \rangle_{\text{real}}$)

$\rightsquigarrow$ $\forall \alpha \in R$ $\alpha : \mathfrak{h}_c \to \sqrt{-1}\,\mathbb{R}$

$\mathfrak{g}_c$ has basis $F_\alpha - E_\alpha$, $\sqrt{-1}(E_\alpha + F_\alpha)$ $(\alpha \in R_+)$

and $\sqrt{-1}\,H_\alpha$ $(\alpha \in \Pi)$

these are orth., $B(X, X) < 0$ by direct comp.

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$\left(F-E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sqrt{-1}(E+F) = \begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix} \right).$

Use

- $B_{\mathfrak{g}}(E_\alpha, F_\alpha) > 0$, $B(E_\alpha, E_\alpha) = 0$, etc.
- $B_{\mathfrak{g}}(\sqrt{-1}H_\alpha, \sqrt{-1}H_\alpha) = -\sum \beta(H_\alpha)^2$

3) $G_c \curvearrowright \mathfrak{g}_c$ img is in $O(\mathfrak{g}_c ; B)$ cpt. grp. $(\cong O(\mathbb{R}^{\dim \mathfrak{g}_c}))$

kernel : $\{g \in G_c : Ad_g = Id\} \subset \{g \in G : Ad_g = Id\}$ finite.

2018.11.13. no.1

Summary
- compact form, cont'd
  - uniqueness of maximal torus
- highest weight theory

- Compact form, continued
(Recall cpt invol & do Prop from last wk)

Rem. Good & bad of cpt frm

G: any $X \in \mathfrak{g}_c$ is semisimple (in $\mathfrak{g} = \mathfrak{g}_c \otimes \mathbb{C}$)
(cf. any $g \in SU(n)$ is diagonalizable)

B: we cannot take root (or weight) decomp. over $\mathbb{R}$. ($X \in \mathfrak{g}_c \rightsquigarrow \pi(X)$ have eigenvals λ in $\sqrt{-1}\mathbb{R}$; $e^{\lambda} \in \mathbb{T} = \{z : |z| = 1\}$)

Fact $\mathfrak{g}_c$ is unique up to conjugation.
i.e. $\mathfrak{g}_1 < \mathfrak{g}$ real Lie subalg generates a maximal cpt subgroup $G_1 < G$
$\Rightarrow \exists g \in G$ s.t. $Ad_g(\mathfrak{g}_1) = \mathfrak{g}_c$ (so $gG_1g^{-1} = G_c$)

Not'n $K = G_c$ $\mathfrak{h}_c = \mathfrak{h} \cap \mathfrak{g}_c$, $T < G$ corr. to $\mathfrak{h}_c$
$\Rightarrow \mathfrak{h}_c$ is a maximal comm. subalg of $\mathfrak{g}_c$
T is a maximal comm. subgrp of K
(maximal torus)

(Rem about $\Lambda_W = \hat{T}$.)

Th'm T' max. comm. $< K \Rightarrow \exists g \in K$ s.t. $gTg^{-1} = K'$
$\mathfrak{h}'_c$ ~ $< \mathfrak{g}_c \Rightarrow$ ~ $g\mathfrak{h}_c g^{-1} = \mathfrak{h}'_c$

Proof we'll do $gTg^{-1} = T'$ (but the proof does $Ad_g(\mathfrak{h}_c) = \mathfrak{h}'_c$)

Step 1 $\exists h$ s.t. $T = \mathrm{Cent}_K(h)$.

skip?

$\because$ take $h \in T \simeq \mathbb{R}^n / \mathbb{Z}^n$ s.t. $\{h^k : k \in \mathbb{Z}\}$ is dense in T

$\rightsquigarrow$ enough to find $g \in K$ s.t. $ghg^{-1} \in T'$.

Step 2 $\forall h \in K\ \exists g \in K$ s.t. $ghg^{-1} \in T'$

Step 2-1 $\mathfrak{g}_c \to K$, $X \mapsto e^X$ is surj.

Outline $\cdot$ K cpt $\overset{\text{Hopf-Rinow}}{\Longrightarrow}$ $\forall h \in K$ $\exists$ geodesic from e to h (for biinv. metric)

$\cdot$ geod. from e $\equiv$ exponential curve $(e^{tX})_t$ (for biinv. met)

Step 2-2 $\forall X \in \mathfrak{g}_c\ \exists g \in K$ s.t. $\mathrm{Ad}_g(X) \in \mathfrak{h}'_c$

(then $e^X = h \Rightarrow \mathrm{Ad}_g(h) \in T'$)

$\because$ (Y, Z) invariant inn. prod on $\mathfrak{g}_c$ $(-B_{\mathfrak{g}}(Y, Z))$

Take $Y' \in \mathfrak{h}'_c$ s.t. $\mathrm{Cent}_{\mathfrak{g}_c}(Y') = \mathfrak{h}'_c$.

(enough to take Y' s.t. $(e^{tY'})_t$ dense in T')

Put $A_X = \{\mathrm{Ad}_g(X) : g \in K\}$ cpt subset $\subset \mathfrak{g}_c$

$\rightsquigarrow$ $\exists X' \in A_X$ s.t. $X'' \mapsto |X'' - Y'|$ is min. at X'

Claim : $X' \in \mathfrak{h}'_c$.

$\because$ $|\mathrm{Ad}_{e^{tZ}}(X') - Y'|^2$ is smallest at $t=0$

$\Rightarrow$ deriv. is 0 at $t=0$

$\Rightarrow$ $(X' - Y', [Z, X']) = 0$ for all $Z \in \mathfrak{g}_c$

invar. for $\mathrm{ad}_{X'}$ $\Rightarrow$ $(\mathrm{ad}_{X'}(Y'), Z) = 0$

i.e. $\mathrm{ad}_{X'}(Y') = 0$ $\Rightarrow$ $X' \in \mathrm{Cent}_{\mathfrak{g}_c}(Y')$. $\square$

Rem. weight lattice Λ_W = Pontryagin dual of T

$= \{\varphi : T \to \mathbb{C}^\times \text{ cont. hom}\}$

(1-dim rep. of T)

2018.11.13 no.3

∘ Highest weight theory.

$\mathfrak{g}$ (semi) simple $\mathfrak{h}$, Λ_W, ...

(π, V) fin. dim rep $\rightsquigarrow V = \bigoplus_{\omega \in \Lambda_W} V_\omega$.

$V_\omega = \{ v \in V \ ; \ \pi_X v = \omega(X) v \ \ (X \in \mathfrak{h}) \}$
weight sp.

Want ∘ find "highest weight" of V.
∘ highest weight determines irred. rep

Fix pos/neg. dec. of roots $R = R_+ \amalg R_-$

$\rightsquigarrow$ simple pos. roots $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset R_+$

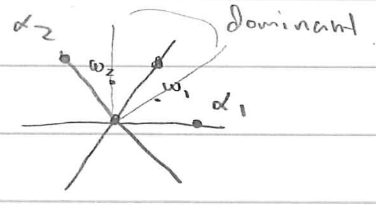
Def. · fundamental weights. $\omega_1, \dots, \omega_n$:

$\omega_i(H_{\alpha_j}) = \delta_{i,j} \iff \dfrac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{i,j}$.
(10/23) typo.

$\rightsquigarrow$ basis of Λ_W.

· $\omega \in \Lambda_W$ is dominant if $\omega(H_{\alpha_j}) \geq 0 \ \forall j$

i.e. $\omega = \sum m_i \omega_i \quad m_i \geq 0$ $\Updownarrow$ $\omega(R_+) \geq 0$

A_2 

$P_{++} = \{ \omega : \text{dominant} \}$.

Fact. $W \curvearrowright \Lambda_W$ P_{++} is a fundamental dom.
(any W-orbit intersects w/ P_{++} exactly once)

Recall $\pi(E_{\alpha_i})$ nilpot $\Rightarrow \exists$ highest weight vec.
i.e. $0 \neq v \in V_\omega \cap \left(\bigcap_{i=1}^n \operatorname{Ker} \pi(E_{\alpha_i}) \right)$ for some ω.

Th'm. $\omega \in P_{++} \Rightarrow \exists !$ irred. rep. (π, V) with highest wght. ω.

Ingredient. : Verma module $V(\omega)$

· infin dim $\mathfrak{g}$-module, dist. vec. $v_\omega \in V(\omega)$

2018.11.13. no4

- (π', V') has highest wght vec $v' \in V'_\omega$
  $\Rightarrow \exists!$ $\mathfrak{g}$-hom $V(\omega) \to V'$, $v_\omega \mapsto v'$
- $\exists!$ nontriv. max. $\mathfrak{g}$-inv. subsp. $M_\omega < V(\omega)$
  i.e. $V(\omega)$ has unique irred. quot. V^ω

Construction.

$U(\mathfrak{g})$, $U(\mathfrak{b}_+)$, $U(\mathfrak{h})$ univ. env. algs.

- $U(\mathfrak{b}_+) \subset U(\mathfrak{g})$
- $\mathfrak{b}_+ \to \mathfrak{h}$ prj. is Lie alg hom
  $\Rightarrow U(\mathfrak{b}_+) \to U(\mathfrak{h})$
- ω induces $U(\mathfrak{h}) \xrightarrow{\varphi_\omega} \mathbb{C}$. 1-dim rep

$\rightsquigarrow V(\omega) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} {}_{\varphi_\omega}\mathbb{C}$, $v_\omega = 1 \otimes 1$

any <u>proper</u> submodule doesn't contain v_ω.

$\Rightarrow M_\omega = \bigcup_{\substack{W \subset V(\omega) \\ \mathfrak{g}\text{-inv.}}} W$ is still proper submod.

2018.11.14. no. 1

Summary
- Highest weight theory (cont'd)
  - Verma module
- Categorical structure

° Highest weight theory

(Highest wght vecs, Verma modules)

(π^ω, V^ω) simple quot. of Verma mod.

Prop. $\omega \in P_{++} \Rightarrow \dim V^\omega < \infty$

Step 1 $V^\omega = \bigcup\limits_{\substack{\mathfrak{s}_\alpha\text{-inv} \\ \text{fin dim}}} W$ $\quad (\Rightarrow \pi^\omega_{E_\alpha}, \pi^\omega_{F_\alpha}$ loc. nilpot.)

∵ $\mathfrak{g} \otimes W \to V^\omega$ is $\mathfrak{s}_\alpha$-equivar. RHS is $\mathfrak{g}$-inv

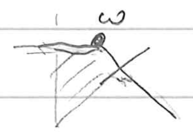
$\omega \in P_{++} \Rightarrow W = \langle F_\alpha^k v : k = 0, \dots, \omega(H_\alpha) \rangle$ is $\mathfrak{s}_\alpha$-inv.

Step 2 $\mathrm{Supp}(V^\omega) = \{\lambda : V^\omega_\lambda \neq 0\}$ is W-inv.

∵ s_α-inv. by Step 1

Step 3. $\mathrm{Supp}(V^\omega)$ is finite.

∵ $\mathrm{Supp}(V^\omega) \cap P_{++} \subset \{\omega - \sum_{i=1}^{n} k_i \alpha_i : k_i \geq 0\}$

E.g.  A_2 is $\begin{cases} \text{finite} \\ \text{fund. dom for } W. \end{cases}$

° Categorical structure

Overall motivation

G (cpt) grp ⇝ "monoidal category" & "fiber functor"
$\mathcal{C}$ F.

s.t. $\begin{cases} (\mathcal{C}, F) \text{ can recover } G \\ \text{abstract characterization of } (\mathcal{C}, F) \end{cases}$

2018.11.14 no.2

Ex. $G \rightsquigarrow \mathcal{C}$: category of finite G-sets $(G\text{-Sets}_f)$
$(X, \alpha : G \curvearrowright X)$, $(X \times Y, \alpha \times \beta)$
$F(X, \alpha) = X$.

Ex. $G \rightsquigarrow \mathcal{C}$: category of fin. dim unitary reps $(\mathrm{Rep}\,G)$
$(H, \pi : G \to U(H))$, $(H \otimes H', \pi \otimes \pi')$
$F(H, \pi) = H$.

How does "abstract characterization" work?

Ex. Galois theory
K field, $\bar{K}$ alg. closure of K.
$\mathcal{C}$ = category of finite étale algs $/K$.
(A étale $\Leftrightarrow$ $A \simeq L_1 \times \cdots \times L_m$, L_i fin. sep$/K$
$F(A) = \mathrm{Hom}_{K\text{-alg}}(A, \bar{K})$, $F(A \otimes B) = F(A) \times F(B)$
$F(A)$ has left action of $\mathrm{Gal}(\bar{K}/K) = \mathrm{Aut}_{K\text{-alg}}(\bar{K})$
in fact $\mathrm{Gal}(\bar{K}/K) = $ "$\mathrm{Aut}(F)$", ($\mathcal{C} \simeq \mathrm{Gal}$-Sets

Ex. 3-dim QFT $A \overset{\text{def}}{=} (A(O))_{O \subset \mathbb{R}^3}$ fam. of algs.
$\rightsquigarrow \mathcal{C}$: "rep. of A"
(minor cond. on A) $\rightsquigarrow$ $\mathcal{C} \simeq \mathrm{Rep}\,G$ for some G.
Doplicher-Roberts

Ex. étale fund. grp. of scheme (Grothendieck)
X : (conn, loc. Noetherian) $x : \mathrm{Spec}(\bar{K}) \to X$
$\mathcal{C}$: cat. of finite étale schemes $/X$.
$F(Y)$ = "fiber of $Y \to X$ over x"
$= \left\{ \begin{array}{c} \mathrm{Spec}(\bar{K}) \to Y \\ \searrow^{x} \quad \downarrow \\ \quad X \end{array} \right\}$

<u>Category</u>. $\mathcal{C}$ is given by

- objects $X, Y \cdots \in Ob(\mathcal{C})$
- morphisms $S, T \cdots \in Mor(X, Y)$ $(X, Y \in Ob(\mathcal{C}))$

$Id_X \in End(X) = Mor(X, X)$

$(S_1 S_2) S_3 = S_1 (S_2 S_3)$, $Id_Y S = S = S\, Id_X$.

Ex. $\mathcal{C} = (G\text{-Sets})$

$Mor((X, \alpha), (Y, \beta)) = \{ f : X \to Y,\ \beta_g f = f \alpha_g \}$

Ex. $\mathcal{C} = Rep\ G$

$Mor((H^1, \pi^1), (H^2, \pi^2)) = \{ T : H^1 \to H^2$ lin. $\pi^2_g T = T \pi^1_g \}$

<u>Functor</u> $F : \mathcal{C} \to \mathcal{C}'$ is given by

- maps $F : Ob(\mathcal{C}) \to Ob(\mathcal{C}')$

$Mor(X, Y) \to Mor(F(X), F(Y))$

s.t. $F(Id_X) = Id_{F(X)}$, $F(ST) = F(S)F(T)$

<u>Natural</u> <u>transform</u> $\phi : F \to F'$ is given by

morphisms $\phi_X : F(X) \to F'(X)$ s.t.

$F, F' : \mathcal{C} \to \mathcal{C}'$

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & F'(X) \\ {\scriptstyle F(T)}\downarrow & \circlearrowleft & \downarrow {\scriptstyle F'(T)} \\ F(Y) & \xrightarrow[\phi_Y]{} & F'(Y) \end{array} \quad \forall T \in Mor(X, Y)$$

Nat. iso. : all ϕ_X are invertible.

Ex.

$\mathcal{C} = (G\text{-Sets}_f)$, $\mathcal{C}' = (Sets_f)$ $F(X, \alpha) = X$.

$g \in G \rightsquigarrow \phi^g \in Aut(F)$ by $\phi^g_{(X,\alpha)} = \alpha_g$ as map $F(X, \alpha) \to F(X, \alpha)$

Ex. $\mathcal{C} = Vec_K = \mathcal{C}'$ $F(V) = V$, $F'(V) = V^{**}$

$\phi : F \to F'$ by $\phi_V(x)(\xi) = \xi(x)$ for $\xi \in V^*$

Summary

- Monoidal cat
- Tannaka-Krein duality

<u>Mon</u>. <u>cat</u>. $\mathcal{C}$ is given by

- cat $\mathcal{C}$
- functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ $\quad (X, Y) \mapsto X \otimes Y$.
- dist. ob $1 \in Ob(\mathcal{C})$ "mon unit"
- nat. isoms $1 \otimes X \overset{\lambda}{\simeq} X \simeq^{\rho} X \otimes 1$

$$\Phi : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

s.t.

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[rr] \arrow[d] & \arrow[d, phantom, "\circlearrowleft"] & (X \otimes (Y \otimes Z)) \otimes W \arrow[d] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[dr] & & X \otimes ((Y \otimes Z) \otimes W) \arrow[dl] \\
& X \otimes (Y \otimes (Z \otimes W)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr] \arrow[dr] & & X \otimes (1 \otimes Y) \arrow[dl] \\
& X \otimes Y &
\end{tikzcd}$$

etc. comm.

$\mathcal{C}, \mathcal{C}'$ mon cat $\rightarrow$

<u>Mon</u> <u>functor</u> $F : \mathcal{C} \to \mathcal{C}'$ is given by

- f'tr $F \ \mathcal{C} \to \mathcal{C}'$
- nat. iso. $F_2 : F(X) \otimes F(Y) \to F(X \otimes Y)$
- iso. $F_0 : 1_{\mathcal{C}'} \to F(1_{\mathcal{C}})$

with compat. for Φ & Φ', λ, λ', ...

Ex. (G-Sets) $\quad (X \times Y, (\alpha_g \times \beta_g)_g)$

RepG $\quad (H \otimes H', (\pi_g \otimes \pi'_g)_{g \in G})$

Fiber f'trs to Sets / $Hilb_f$ are mon. f'tr.

<u>Nat</u>. <u>trans</u> of mon. f'trs $F, F' : \mathcal{C} \to \mathcal{C}'$

nat trans $\phi_X : F(X) \to F'(X)$ compat with F_2 & F'_2.

Ex. $F : \mathrm{Rep}\, G \to \mathrm{Hilb}_f$

$g \in G \rightsquigarrow \phi^g : F \to F$ by $\phi^g_{(H,\pi)} = \pi_g : H \to H$

this is aut. as mon. ftr i.e.

$\phi^g_{(H \otimes H', \pi \otimes \pi')} = \pi_g \otimes \pi'_g = \phi^g_{(H,\pi)} \otimes \phi^g_{(H',\pi')}$

T-K duality.

Pt 1. ((Rep G, F) recovers G) $F : \mathrm{Rep}\, G \to \mathrm{Hilb}_f$

satisfies $G \simeq \mathrm{Aut}^{\otimes}(F) \leftarrow$ nat. unitary aut. of F.

Pt 2 (possibility of (Rep G, F))

$\mathcal{C}$: $\mathbb{C}$-lin. mon. cat with • duality, • invol. on mor $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(Y,X)$ $T \mapsto T^*$.

• symmetric br. $c : X \otimes Y \simeq Y \otimes X$

$F : \mathcal{C} \to \mathrm{Hilb}$ compat w/ duality, invol, br

$\Rightarrow \exists\, G$ s.t. $\mathcal{C} \overset{\otimes}{\simeq} \mathrm{Rep}\, G$, $F \leftrightarrow$ fiber ftr of G

Pt 1.

Step 1 $\mathrm{Aut}^{\otimes}(F)$ is cpt grp.

$\because$ $\mathrm{Aut}^{\otimes}(F) < \mathrm{Aut}(F) \simeq \prod_{(H_i,\pi_i) : \mathrm{Irrep}\, G} U(H_i)$

any nat iso $\phi : F \to F$ is det'ed by $\phi_{(H_i,\pi_i)}$ $H_i \to H_i$

Step 2. $G \to \mathrm{Aut}^{\otimes}(F)$ is cont inj.

(so $G < \mathrm{Aut}^{\otimes}(F)$)

S3.

Step 3 $\mathrm{Aut}^{\otimes}(F)/G = \{*\}$.

Facts • Stone-Weierstrass th'm.

X cpt, $\mathcal{A} \subset C(X)$ subalg $f \in \mathcal{A} \to \bar{f} \in \mathcal{A}$

sep. pts of X $\Rightarrow \forall f \in C(X), \varepsilon > 0 \cdots$

• Haar measure on G

S3. $\phi \in \mathrm{Aut}^{\otimes}(F) \rightsquigarrow \phi(\mathbb{C}, 1) = 1, \phi(\overline{H}, \overline{\pi}) = \overline{\phi(H,\pi)}$

$\phi(\mathbb{C},1) \otimes \phi(\mathbb{C},1)$ "=" $\phi(\mathbb{C},1)$

$(\phi(\overline{H},\overline{\pi}) \otimes \phi(H,\pi)) R = R \; \phi(\mathbb{C},1) = R.$

for $R : \mathbb{C} \to \overline{H} \otimes H, \; \lambda \mapsto \lambda \sum_{i=1}^{d} \overline{e}_i \otimes e_i$

$(e_i)_{i=1}^{d}$ ONB.

$(X_{ij})_{i,j=1}^{d} \leftrightarrow \phi(H,\pi), \quad (X'_{\bar{i},\bar{j}}) \leftrightarrow \phi(\overline{H},\overline{\pi})$ (unitarity)

$\Rightarrow \sum_j X'_{\bar{i}\bar{j}} X_{kj} = \delta_{ik} \Rightarrow X'_{\bar{i}\bar{j}} = \overline{X}_{ij}.$

S4. $\mathcal{O}(G)$: alg. of mat. coeffs.

$\mathcal{O}(G) \subset C(\mathrm{Aut}^{\otimes}(F))$ dense

$f^{\pi}_{\xi,\eta}(\phi) = (\phi(H,\pi)\eta, \xi), \quad \xi, \eta \in H$

- subalg : $f^{\pi}_{\xi,\eta} f^{\pi'}_{\xi',\eta'} = f^{\pi \otimes \pi'}_{\xi \otimes \xi', \eta \otimes \eta'}$
- closed under $f \rightsquigarrow \overline{f}$: $\overline{f^{\pi}_{\xi,\eta}} = f^{\overline{\pi}}_{\overline{\xi},\overline{\eta}}$

$\overline{(\phi(H,\pi)\eta, \xi)} = (\xi, \phi(H,\pi)\eta)$

$= (\overline{\phi(H,\pi)\eta}, \overline{\xi})$

$\overline{\phi(H,\pi)\eta} = \overline{\phi(H,\pi)}\,\overline{\eta} = \phi(\overline{H},\overline{\pi})\overline{\eta}$

- Sep. pts of $\mathrm{Aut}^{\otimes}(F)$. $\phi \neq \phi' \Rightarrow$ diff for some ξ, η.

S5. $f \to (\phi \mapsto \int f(\phi \cdot g)\, d\mu(g))$

is contr. $C(\mathrm{Aut}^{\otimes}(F)) \to C(\mathrm{Aut}^{\otimes}(F)/G)$

sends $\mathcal{O}(G)$ to $\mathbb{C}$.

$\Rightarrow C(\mathrm{Aut}^{\otimes}(F)/G)$ is 1-dim.