Summary

- Direct sum of reps.
- Comparison of reps.
- Complete reducibility.
- Group algebra.

Notn. G fin. grp.

(π, V), (π', V') : reps of G

How to combine two reps : <u>direct sum</u>

((π, V), (π', V') rep of G. $\rightsquigarrow$)

- $V \oplus V' = \{ v \oplus w : v \in V, w \in V' \} \simeq V \times V'$
- $(\pi \oplus \pi')_g \in GL(V \oplus V')$ by

$$(\pi \oplus \pi')_g (v \oplus w) = \pi_g v \oplus \pi'_g w.$$

Concretely : $X^{(g)} \in GL_m(\mathbb{C})$, $Y^{(g)} \in GL_n(\mathbb{C})$

$$\rightsquigarrow \begin{bmatrix} X^{(g)} & 0 \\ 0 & Y^{(g)} \end{bmatrix} \in GL_{m+n}(\mathbb{C})$$

How to compare two reps :

Def. <u>intertwiner</u> (or <u>G-homomorphism</u>)

from (π, V) to (π', V') :

lin. map $T : V \to V'$ s.t. $T \pi_g = \pi'_g T$ $\forall g \in G$.

Notn. $\mathrm{Hom}_G((\pi, V), (\pi', V')) = \{ T : V \to V' \text{ intertw.} \}$

$\mathrm{Hom}_G(V, V')$, $\mathrm{Hom}(\pi, \pi')$, (π, π'), ---

(π, V) and (π', V') are <u>isomorphic</u>

(<u>equivalent</u>, <u>similar</u>) if $\exists$ <u>bijective</u>

intertw. $T : V \to V'$

Write $(\pi, V) \simeq (\pi', V')$ or $V \simeq V'$

Ex. $G = \mathbb{Z}/n\mathbb{Z}$.

- $V = \langle e_{[i]} : [i] \in G \rangle$, $\pi_{[i]} e_{[j]} = e_{[i+j]}$.

  restr. of $S_n \overset{\pi}{\curvearrowright} V$ yesterday

  $[1] \in G$ reps cycl. perm $(1, 2, \dots, n) \in S_n$.

- $\varphi^{(0)} \oplus \cdots \oplus \varphi^{(n-1)}$ on $\mathbb{C}^n$

  $\varphi^{(k)}_{[i]} = e^{\frac{2\pi\sqrt{-1}ki}{n}}$.

Prop. these are isomorphic reps.

Idea. define $T : V \to \mathbb{C}^n$ by

$$T(e_{[i]}) = 1 \oplus e^{\frac{2\pi\sqrt{-1}i}{n}} \oplus e^{\frac{2\pi\sqrt{-1}i2}{n}} \oplus \cdots \oplus e^{\frac{2\pi\sqrt{-1}i(n-1)}{n}}$$

$$=: \xi_{[i]}$$

Need to check:

1. $(\varphi^{(0)}_{[i]} \oplus \cdots \oplus \varphi^{(n-1)}_{[i]}) \xi_{[j]} = \xi_{[i+j]}$

   $= T(\pi_{[i]} e_{[j]})$

2. $(\xi_{[i]})_{i=0}^{n-1}$ is a basis of $\mathbb{C}^n$

   $(\Leftrightarrow T$ bij.$)$

   $\Leftarrow$ they are <u>orthogonal</u> for Herm. inn. prod.

Complete reducibility

"Any rep is a combination of simple ones".

Def. <u>G-invariant</u> <u>subspace</u> of (π, V):

$W \subset V$ subsp. s.t. $w \in W$, $g \in G$

$\Rightarrow \pi_g w \in W$.

(π, V) is <u>simple</u> if $0, V$ are the only inv. subsp. (<u>irreducible</u>)

Th'm. (π, V) (f.d.) rep. of G.
$W \subset V$ G-inv.
Then $\exists\, W' \subset V$ G-invar. s.t.
$V \simeq W \oplus W'$

Moral : we can "split off" inv. subsp. as a direct summand (as rep.)

For proof : we need <u>Hermitian</u> <u>inn. prod</u>
(v, v') for $v, v' \in V$
- $(v', v) = \overline{(v, v')}$
- $(v, v) \geq 0$, $= 0$ only if $v = 0$
- lin. in v, conj. lin in v'

Lem. V has a G-invariant inner prod.
$$(gv, gv') = (v, v')$$

Pf. Take any inn. prod $(v, v')_0$
and put $(v, v') = \frac{1}{|G|}\sum_{h \in G} (hv, hv')_0$ averaging.
Then $(gv, gv') = \frac{1}{|G|}\sum_h (hgv, hgv')_0$
$$= \underset{h' = hg}{} \frac{1}{|G|}\sum_{h' \in G} (h'v, h'v')_0 = (v, v')$$ $\square$

Pf of Th'm. : Take G-inv. inn. prod.
Step 1 $W' = \{ w' \in V : \forall w \in W \ (w, w') = 0 \}$
orthog. compl.
$\rightarrow$ W' is G-inv.
$\because$ Check $(w, gw') = 0$
use $(w, gw') \underset{\text{inv.}}{=} (\underbrace{g^{-1}w}_{\text{in } W}, \underbrace{g^{-1}gw'}_{w'}) \underset{\text{def. cond.}}{=} 0$

Step 2 $V \simeq W \oplus W'$ $\square$

Rem. Lem says $\forall$ f.d. rep. of G is <u>unitarizable</u> (equiv. to $G \to U_n$.)

Cor. any rep. is isom to dir. sum of irred. reps.

$\because$ Repeat: take $0 \subsetneq W \subsetneq V$ G-inv. (if any), write $V \simeq W \oplus W'$

(next step: take $0 \subsetneq W_1 \subsetneq W$, etc. $\cdots$)

until all summands are irred.

Group alg.

$$\mathbb{C}[G] = \{ \sum_g \alpha_g \cdot g : \alpha_g \in \mathbb{C} \}.$$

define prod. by

$$\Big(\sum_g \alpha_g \cdot g \Big) \Big(\sum_h \beta_h \cdot h \Big) = \sum_{g,h} \alpha_g \beta_h \cdot \underbrace{(gh)}_{\text{comp. in } G}$$

$$= \sum_{k \in G} \Big(\underbrace{\sum_{h \in G} \alpha_{kh^{-1}} \beta_h}_{} \Big) k$$

"convolution prod" of α & β at k.

this is assoc. prod, $1 \cdot e$ unit.

Rep of G $\equiv$ modules over $\mathbb{C}[G]$

$\because$ $\big(\sum_g \alpha_g \cdot g \big) v = \sum \alpha_g \, g v$ makes sense

Irred. reps $\equiv$ simple modules.

Compl. red'bility $\equiv$ any $\mathbb{C}[G]$-mod is a dir. sum of simple modules.

($\mathbb{C}[G]$ is <u>semisimple</u> algebra)

$$\underset{\text{genera theory}}{\Longrightarrow} \quad \mathbb{C}[G] \simeq \prod_{(\pi, V) \text{ irr}} \mathrm{End}(V).$$