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Summary

- Characters
- Schur's lemma
- Orthogonality of irred. chars (s.t.)

Def. Character of (π, V) is

$\chi_\pi : G \to \mathbb{C}$, $\chi_\pi(g) = \mathrm{Tr}_{\mathrm{End}(V)}(\pi_g)$

i.e., $(e_i)_{i=1}^n$ basis of V,

$(X^{(g)}_{ij})_{i,j=1}^n$ repr. mat. of π_g

$\rightsquigarrow \chi_\pi(g) = \sum_{i=1}^n X^{(g)}_{ii}$

= sum of eigenvals of π_g.

Rem. • $\mathrm{Tr}_{\mathrm{End}(V)}(ABA^{-1}) = \mathrm{Tr}_{\mathrm{End}(V)}(B)$

for $A, B \in \mathrm{End}(V)$

$\Rightarrow \chi_\pi(ghg^{-1}) = \chi_\pi(h)$

χ_π is a class function on G.

const. on each conjugacy class.

• z : eigenval of π_g, $n \in \mathbb{N}$ s.t. $g^n = e$.

$\Rightarrow z^n$ eigenval of $\pi_g{}^n = \pi_e = \mathrm{Id}_V$, i.e.,

$z^n = 1$. (and $n \mid$ ord. of G)

$\Rightarrow \chi_\pi(g) \in \mathbb{Z}\left[e^{\frac{2\pi\sqrt{-1}}{|G|}}\right]$

Ex.)

Prop. $\chi_{\pi \oplus \pi'}(g) = \chi_\pi(g) + \chi_{\pi'}(g)$

∵ $(X^{(g)}_{ij})_{i,j}$ mat. rep of π_g

$(Y^{(g)}_{kl})_{k,l}$ ~ π'_g

$\Rightarrow \begin{bmatrix} X^{(g)} & 0 \\ 0 & Y^{(g)} \end{bmatrix}$ mat rep of $(\pi \oplus \pi')_g$

$\Rightarrow \mathrm{Tr}\,(\pi \oplus \pi')_g = \mathrm{Tr}\, X^{(g)} + \mathrm{Tr}\, Y^{(g)} = \mathrm{Tr}\,\pi_g + \mathrm{Tr}\,\pi'_g$

Examples

- 1-dim rep $G \xrightarrow{\varphi} GL_1(\mathbb{C})$ hom.

  $\rightarrow \chi_\varphi(g) = \varphi(g)$ Sometimes only this is char.

- $\pi : S_n \curvearrowright V = \langle e_1, \dots, e_n \rangle$ perm. rep.

  $\chi_\pi(\sigma) = \mathrm{Tr}_{M_n(\mathbb{C})}(\text{perm. mat for } \sigma)$

  $= \#$ of 1's in diag

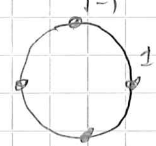
  $= \#\{ i : \sigma(i) = i \}$ fixed pts.

Ex. of poly.

$G = \mathbb{Z}/4\mathbb{Z}$. $\pi|_G \simeq \varphi^{(0)} \oplus \cdots \oplus \varphi^{(3)}$

up to $[i] \in G \longleftrightarrow \sigma^i$ for $\sigma = (1\ 2\ \cdots\ 4)$

$i = 1, 3 \Rightarrow \Big(\sigma^i$ no fixed pt.

$\sum_{k=0}^{3} \varphi^{(k)}_{[i]}$ = sum of  $= 0$

$i = 2 \Rightarrow \sigma^2 = (1\ 3)(2\ 4)$ still no fixed pt.

$\sum_{k=0}^{3} \varphi^{(k)}_i = 1 - 1 + 1 - 1 = 0$

Th'm (Schur's Lemma)

(π, V), (π', V') <u>irred.</u> rep of G.

$T : V \to V'$ intertwiner.

1) If $\pi \not\simeq \pi'$, then $T = 0$

2) If ($\pi \simeq \pi'$ and) $T \neq 0$, then any other intertwiner is sc. mult. of T.

Pf. Basic idea :

- T : intertw. $\Rightarrow$ $\mathrm{Im}\,T$, $\mathrm{Ker}\,T$: G-inv.
- π, π' irred. $\Rightarrow$ $\mathrm{Im}\,T$, $\mathrm{Ker}\,T$ have to be $0, V, V'$.

  $\Rightarrow T = 0$ or bijective

  $\pi \simeq \pi'$.

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Step 1. $\mathrm{Im}\,T$, $\mathrm{Ker}\,T$ G-inv.

$\because$ $v \in V$ $g \in G$.

$\pi'_g \underbrace{Tv}_{\in \mathrm{Im}\,T} = T\pi_g v \in \mathrm{Im}\,T$

If $v \in \mathrm{Ker}\,T$ $T\pi_g v = \pi_g \underbrace{Tv}_{0} = 0$

$\Rightarrow \pi_g v \in \mathrm{Ker}\,T$.

Step 2. $T \neq 0 \Rightarrow T$ inj. ([illegible] π' [illegible] 1)

$\because$ $\mathrm{Ker}\,T \neq V$ by assumption.

(π, V) irr. $\Rightarrow$ $\mathrm{Ker}\,T$ must be 0 or V.

$\Rightarrow$ $\mathrm{Ker}\,T = 0$, T inj.

Step 3 $T \neq 0 \Rightarrow T$ surj. ($\Rightarrow T$ bij.; proves 1.)

Otherwise $\mathrm{Im}\,T = 0$ by irr. of (π', V')

$\Rightarrow T = 0$.

Step 4. T bij $\Rightarrow \forall$ intertwiner $S : V \to V'$

$\exists\ \alpha \in \mathbb{C}$ s.t. $S = \alpha T$.

$\because$ $A = ST^{-1} : V' \to V'$, α : (any) eigenv. of A.

- A is an intertw.; $\pi'_g S T^{-1} = S\pi_g T^{-1} = ST^{-1}\pi'_g$

$A - \alpha$ has nontriv. ker. & (π', V') irr.

$\Rightarrow$ $A - \alpha = 0$. $\boxed{}$

Step 2.

Averaging maps to get intertwiners

Prop. $T : V \to V'$ linear map.

then $\tilde{T} = \frac{1}{|G|} \sum_{g \in G} \pi'_g T \pi_g^{-1}$ is an intertwiner.

Step 1. $S : V \to V'$ is an intertw. iff

$\pi'_h S \pi_h^{-1} = S$.

Step 2. $\pi'_h \tilde{T} \pi_h^{-1} = \tilde{T}$.

Cor. (π, V), (π', V') irr, $T: V \to V'$ lin.

$\tilde{T}$ as above.

1. $\pi \not\cong \pi' \Rightarrow \tilde{T} = 0$

2. $V = V'$, $\pi = \pi' \Rightarrow \tilde{T} = \dfrac{\mathrm{Tr}(T)}{\dim V} \mathrm{Id}_V$

$\because$ 1. is from Th'm. 1.

2. : $\mathrm{Tr}\,\tilde{T} = \dfrac{1}{|G|}\sum_g \mathrm{Tr}(\pi_g T \pi_g^{-1}) = \mathrm{Tr}\, T$

($\tilde{T}$ is scaler. mult. of Id.

Orthogonality of irred. characters

$\varphi, \psi : G \to \mathbb{C}$ maps $\Rightarrow$ Hermitian inn. prod. $(\varphi, \psi)_{L^2(G)} = \dfrac{1}{|G|}\sum_{g \in G} \varphi(g)\overline{\psi(g)}$.

Th'm. 1) π irred. rep $\Rightarrow (\chi_\pi, \chi_\pi)_{L^2(G)} = 1$

2) π' : another irred. rep, $\pi \not\cong \pi'$

$\Rightarrow (\chi_\pi, \chi_{\pi'})_{L^2(G)} = 0$

Example.

$G = \mathbb{Z}/n\mathbb{Z}$. $(\varphi^{(k)}, \varphi^{(l)})_{L^2(G)} = (\varphi^{(k-l)}, 1)_{L^2(G)}$

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi^{(k)}[i]\,\overline{\varphi^{(l)}[i]} = \frac{1}{n}\sum_i e^{\frac{2\pi\sqrt{-1}ki}{n}} e^{-\frac{2\pi\sqrt{-1}li}{n}}$$

$$= \frac{1}{n}\sum e^{\frac{2\pi\sqrt{-1}(k-l)i}{n}}.$$

$(\varphi^{(0)}, 1)_{L^2(G)} = 1$, $(\varphi^{(k)}, 1)_{L^2(G)} = 0$ $(k = 1, \ldots, n-1)$

$\frac{1}{n} \times n$

Why good? $(\chi_\pi, \chi_\pi)_{L^2(G)} = 1 \Rightarrow \pi$ irred.

e.g. $\pi \cong \pi_1 \oplus \pi_2$ π_1, π_2 irr, $\pi_1 \not\cong \pi_2$

$\Rightarrow (\chi_\pi, \chi_\pi) = \sum_{i,j} (\chi_{\pi_i}, \chi_{\pi_j}) = \sum_i (\chi_{\pi_i}, \chi_{\pi_i}) = 2$