2019. 08. 28 no. 1

Summary

- Adjoint rep.
- Proof of orthogonality
- Contragredient, tensori product reps
- Structure of characters

Recall : orthogonality of irred. chars.

$$(\chi_{\pi'}, \chi_\pi)_{L^2(G)} = \begin{cases} 1 & \pi \simeq \pi' \\ 0 & \pi \not\simeq \pi' \end{cases}$$

π, π' irred.

$\rightsquigarrow$ we want to understand $\chi_{\pi'}(g)\overline{\chi_\pi(g)}$

Def. (π, V), (π', V') reps (not irred.)

The <u>adjoint</u> <u>representation</u> is given by

- underlying sp. $\mathcal{L}(V, V') = \{T : V \to V' \text{ lin}\}$
- action of G: $Ad_g(T) = Ad_g^{\pi,\pi'}(T) = \pi'_g T \underbrace{\pi_g^{-1}}_{\pi_{g^{-1}}}$

$Ad_g Ad_h(T) = Ad_{gh}(T)$ from

$\pi'_g \pi'_h = \pi'_{gh}$, $\pi_h^{-1}\pi_g^{-1} = (\pi_g \pi_h)^{-1} = \pi_{gh}^{-1}$

Prop. $\chi_{Ad^{\pi,\pi'}}(g) = \chi_{\pi'}(g)\overline{\chi_\pi(g)}$

Proof : $(e_i)_{i=1}^m$ basis of V

$(f_j)_{j=1}^n$ $\sim$ V'

Step 1. $T^i_j : V \to V'$, $\sum_k \alpha_k e_k \mapsto \alpha_i f_j$

$\rightsquigarrow$ $(T^i_j)_{i,j}$ basis of $\mathcal{L}(V, V')$

$\because$ Up to $\mathcal{L}(V, V') \cong M_{n\times m}(\mathbb{C})$

$$T^i_j \longleftrightarrow \begin{array}{cc} \left[\begin{array}{ccc} & & \\ 1 & \cdots & \cdots \end{array}\right] & j \\ \begin{array}{c} \vdots \\ i \end{array} & \end{array} = E^{(ji)}$$

Step 2 If $(X^{(g^{-1})}_{ik})_{i,k}$, $(Y^{(g)})_{j,l}$

represent $\pi_{g^{-1}}$ and π'_g

$$Ad^{\pi,\pi'}_g(T^i_j) = Y^{(g)}_{jj} X^{(g^{-1})}_{ii} T^i_j + (\text{lin. comb. of the other } T^k_l)$$

$$\because \quad Y^{(g)} E^{(j,i)} X^{(g^{-1})} = \begin{array}{c} \begin{bmatrix} & \vdots & \\ \cdots & Y^{(g)}_{jj} X^{(g^{-1})}_{ii} & \cdots \\ & \vdots & \end{bmatrix} j \\ i \end{array}$$

Step 3 $Tr_{End(\mathcal{L}(V,V'))}(Ad^{\pi,\pi'}_g) = \chi_{\pi'}(g)\overline{\chi_\pi(g)}$

$\because$ Step 2 $\Rightarrow$ $Tr(Ad^{\pi,\pi'}_g) = \sum_{i,j} Y^{(g)}_{jj} X^{(g^{-1})}_{ii}$

$= \chi_{\pi'}(g)\chi_\pi(g^{-1})$

We know $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ (Rem. 08.27) $\boxtimes$

Proof of orthogonality.

Idea 1. $(\chi_{\pi'}, \chi_\pi)_{L^2(G)} = \frac{1}{|G|}\sum_g \chi_{\pi'}(g)\overline{\chi_\pi(g)}$

is the tr. of $\mathcal{L}(V,V') \to \mathcal{L}(V,V')$,

$T \mapsto \tilde{T}$ from 08.27

2. Schur's lem. implies what this map is.

(0 if $\pi \not\cong \pi'$, rank 1 prj. if $\pi = \pi'$)

Step 1 : point 1 above.

$\because$ By prop. $(\chi_{\pi'}, \chi_\pi)_{L^2(G)} = \frac{1}{|G|}\sum_g Tr(Ad_g)$

$= Tr\left(\frac{1}{|G|}\sum_g Ad_g\right)$

If $T \in \mathcal{L}(V,V')$ $\frac{1}{|G|}\sum_g Ad_g(T) = \frac{1}{|G|}\sum_g \pi'_g T \pi_{g^{-1}} = \tilde{T}$

Step 2 $\pi \not\cong \pi'$ (irred.) $\Rightarrow$ $(\chi_{\pi'}, \chi_\pi) = 0$

$\because$ $T \mapsto \tilde{T}$ is the zero map (08.27 Cor)

Step 3 $(\chi_\pi, \chi_\pi) = 1$. (π irred)

∵ $T \mapsto \tilde{T}$ is a projection to scalar maps (08.27. Cor) i.e. rk 1 prj.

⇒ has trace 1. □

Important ingredient : $(\mathrm{Ad}^{\pi,\pi'}, \mathcal{L}(V,V'))$

Def. Contragredient (or conjugate) rep. of (π, V) is $(\mathrm{Ad}, \mathcal{L}(V, \mathbb{C}) = V^*)$ with triv. rep. "$(1, \mathbb{C})$" on $\mathbb{C}$.

Notn (π^c, V^*), $(\bar{\pi}, \bar{V})$, etc.

Rem $(X^{(g)}_{i,j})_{i,j}$ rep mat for π_g

⇒ $Y^{(g)}_{i,j} = X^{(g^{-1})}_{j,i}$ repr. π^c_g

∵ (column vecs)* = (row vecs.)

Rec. tensor product $V \otimes V' = \{ \sum_{k=1}^{n} \alpha_k v_k \otimes w_k :$ $n \in \mathbb{N}$, $v_k \in V$, $w_k \in V'$; $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ etc. $\}$

$(e_i)_{i=1}^{m}$ basis of V, $(f_j)_{j=1}^{n}$ basis of V'

⇒ $(e_i \otimes f_j)_{i,j}$ basis of $V \otimes V'$.

Def. Tensor prod. of (π, V) & (π', V')

- underlying sp. $V \otimes V'$
- action $(\pi \otimes \pi')_g (v \otimes w) = \pi_g v \otimes \pi'_g w$

Rem. $\chi_{\pi \otimes \pi'}(g) = \chi_\pi(g) \chi_{\pi'}(g)$

$\mathrm{Ad}^{\pi,\pi'} \simeq \pi' \otimes \pi^c$ up to

$\mathcal{L}(V, V') \simeq V' \otimes V^*$

$(v \mapsto \varphi(v) w) \mapsfrom w \otimes \varphi$

Structure of representations

$(\pi_1, V_1), \dots, (\pi_n, V_n)$ the irred. reps of G.

$\pi_i \not\simeq \pi_j$ if $i \neq j$

Prop. If $\underbrace{\pi_1 \oplus \dots \oplus \pi_1}_{m_1 \times} \oplus \dots \oplus \underbrace{\pi_n \oplus \dots \oplus \pi_n}_{m_n}$

$\simeq \underbrace{\pi_1 \oplus \dots \oplus \pi_1}_{m'_1} \oplus \dots \oplus \underbrace{\pi_n \oplus \dots \oplus \pi_n}_{m'_n}$

then $m_k = m'_k$ for $k = 1, \dots, n$

Proof. Put π = above rep.

So $\chi_\pi = \underbrace{\chi_{\pi_1} + \dots + \chi_{\pi_1}}_{m_1 \times} + \dots + \underbrace{\chi_{\pi_n} + \dots + \chi_{\pi_n}}_{m_n}$

$= \sum_{k=1}^{n} m_k \chi_{\pi_k}$

By same arg. $\chi_\pi = \sum m'_k \chi_{\pi_k}$.

$(\chi_\pi, \chi_{\pi_i}) = \sum_k m_k (\chi_{\pi_k}, \chi_{\pi_i}) = m_i$

but this is also eq. to m'_i $\boxtimes$

Cor. $(\chi_\pi, \chi_\pi) = 1 \iff \pi$ is irred.

Proof In the above pres. $(\chi_\pi, \chi_\pi) = \sum_k m_k^2$

this is ≥ 1, $= 1$ only if $m_k = 1$ for just one k.

Exemples

1 $G = \mathbb{Z}/n\mathbb{Z}$, $G \overset{\pi}{\curvearrowright} V = \langle e_{[i]} : [i] \in G \rangle$.

$\pi_{[i]} e_{[j]} = e_{[i+j]}$.

We see $\pi \simeq \varphi^{(0)} \oplus \dots \oplus \varphi^{(n-1)}$ from

- $\chi_\pi([0]) = n$, $\chi_\pi([i]) = 0$ $i = 1, \dots, n-1$
  (fixed pt count).

$\Rightarrow (\chi_\pi, \varphi^{(k)}) = \frac{1}{n} \sum_{i=0}^{n} \chi_\pi([i]) \overline{\varphi^{(k)}_{[i]}} = \frac{1}{n} \times n = 1$.

2. $S_n \curvearrowright V = \langle e_i : i = 1, \dots, n \rangle$. $V_0 = \mathbb{C} v_0$ for $v_0 = \sum e_i$

$W = \{ \sum \alpha_i e_i : \sum \alpha_i = 0 \} \Rightarrow \pi|_W$ irr.