2019.09.03

Summary

- regular representation
- counting irred. reps ($\to$ # conjug classes)
- character table.

Def. Regular representation of G is

- underlying sp. $V = \langle e_g : g \in G \rangle$
- rep. $\lambda_g e_h = e_{gh}$

Rem this is rank 1 free mod. of group alg. $\mathbb{C}[G]$.

$\rightsquigarrow$ λ_g is rep'ed by mat. $(x^{(g)}_{h,k})_{h,k} \in M_G(\mathbb{C})$

$$x^{(g)}_{h,k} = \begin{cases} 1 & h = gk \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $$\chi_\lambda(g) = \begin{cases} |G| & g = e \\ 0 & \text{otherwise} \end{cases}$$

Thm. $(\pi_1, V_1), \dots, (\pi_n, V_n)$ all irreps of G

$(\pi_i \not\simeq \pi_j \text{ if } i \neq j)$

Then $$(\lambda, V) \simeq \bigoplus_{i=1}^{n} (\pi_i, V_i)^{\oplus \dim V_i}$$

$$\Big(= \underbrace{(\pi_1, V_1) \oplus \cdots \oplus (\pi_1, V_1)}_{\dim V_1 \times} \oplus \underbrace{(\pi_2, V_2) \oplus \cdots}_{\dim V_2 \times} \Big)$$

Proof. By Prop from 08.28, π_i appears $(\chi_\lambda, \chi_{\pi_i})$ times.

$$(\chi_\lambda, \chi_{\pi_i}) = \frac{1}{|G|} \sum_{g \in G} \chi_\lambda(g) \overline{\chi_{\pi_i}(g)} = \overline{\chi_{\pi_i}(e)} = \overline{\mathrm{Tr}(\mathrm{Id}_{V_i})}$$
$$= \dim V_i$$ ∎

Examples 1. $G = \mathbb{Z}/n\mathbb{Z}$.

λ is $G \curvearrowright V = \langle e_{[i]} : i = 0, \dots, n-1 \rangle$ $\quad \lambda_{[i]} e_{[j]} = e_{[i+j]}$

$\lambda \simeq \varphi^{(0)} \oplus \cdots \oplus \varphi^{(n-1)}$

2. $G = S_3$ irreps :

1-dim : $\left(\begin{array}{l} \pi^{triv}_\sigma = 1 \\ \pi^{sig}_\sigma = (-1)^{|\sigma|} \end{array} \right.$

2-dim π' : compl. of π^{triv} for $\curvearrowright$

$\pi : S_3 \curvearrowright V = \langle e_1, e_2, e_3 \rangle \quad \pi_\sigma e_i = e_{\sigma(i)}$

$\underset{6\text{ dim}}{\lambda} \cong \pi^{triv} \oplus \pi^{sig} \oplus \pi'^{\oplus 2}$

Rem. $\mathbb{C}[G] \simeq \prod_{i=1}^{n} End(V_i)$ as alg.

$\pi_i \longleftrightarrow End(V_i) \curvearrowright V_i$

$\pi_i^{\oplus \dim V_i} \longleftrightarrow End(V_i) \curvearrowright V_i^{\oplus \dim V_i} \longleftrightarrow End(V_i) \circlearrowleft$

Counting irred. reps.

conjug. rel (on G) : $g \sim h$ iff $g = khk^{-1}$ for some k

conjug class : equiv. class

Thm. Number of irred. reps (up to isomorphisms)
= number of conjug classes

Idea : 1. # conjug classes = dim (class funcs)
$f(khk^{-1}) = f(h)$

2. (class funcs) = $\langle \chi_\pi : \pi \text{ rep.} \rangle_{span}$

3. $\pi_1, \cdots, \pi_n$ irreps $\Rightarrow (\chi_{\pi_i})$: basis of above.

Proof (partial proof, for "$\leq$")

Step 1. (point 1 above)

$\because$ $G/\sim$ quot. set by rel $\sim$, $G \xrightarrow{p} G/\sim$

$|G/\sim| = \dim(\text{func } G/\sim \to \mathbb{C})$

$f : G/\sim \to \mathbb{C} \rightsquigarrow f \circ p : G \to \mathbb{C}$ is a class func

Step 2 (point 3; $(\chi_{\pi_i})_i$ form a basis of $V = \langle \chi_\pi : \pi \text{ rep} \rangle_{\mathbb{C}\text{-sp.}}$

$\because$ $(\chi_{\pi_i})_i$ generates V:

enough to check $\forall \chi_\pi$ is a lin comb. of $(\chi_{\pi_i})_i$ $\to$ OK by irr. decomp

$\pi \simeq \bigoplus_{i=1}^{n} \pi_i^{\oplus n_i} \to \chi_\pi = \sum n_i \chi_{\pi_i}$

$(\chi_{\pi_i})_i$ lin indep. : from orthogonality.

Step 3. # conj. classes $\geq n$

$\because$ $V \subset$ (class funcs)

$\downarrow$ step 2 $\quad$ $\downarrow$ step 1 $\quad$ take dim

n $\quad$ # conj. class $\quad$ $\square$

We'll prove "$\geq$" later.

Character table

Draw a table with

- columns : conjug classes c (or representatives $g \in c$)
- rows : irred. reps π_i (or chars χ_{π_i})
- entries : values $\chi_{\pi_i}(g)$

Ex. 1. Cyclic group $G = \mathbb{Z}/n\mathbb{Z}$

	$[0]$	$[1]$	$\cdots$	$[n-1]$
$\varphi^{(0)}$	1	1	$\cdots$	1
$\varphi^{(1)}$	1	$e^{\frac{2\pi\sqrt{-1}}{n}}$	$\cdots$	$e^{2\pi\sqrt{-1}\frac{n-1}{n}}$
$\vdots$	$\vdots$			
$\varphi^{(n-1)}$	1	$e^{2\pi\frac{n-1}{n}}$	$\cdots$	$e^{2\pi\sqrt{-1}\frac{(n-1)^2}{n}}$

2. $G = S_3$ irred. chars $\chi_{triv}, \chi_{sig}, \chi_{\pi'}$

	e	$(1\,2)$	$(1\,2\,3)$
χ_{triv}	1	1	1
χ_{sig}	1	-1	1
$\chi_{\pi'}$	2	0	-1

3. $G = A_4 = \{\sigma \in S_4 : (-1)^{|\sigma|} = 1\}$

$(\simeq \{T \in SO(3) : T\Delta = \Delta\})$

regular tetrahedron centered at 0

$|A_4| = \frac{|S_4|}{2} = 12$, conjug. classes:

$\{e\}$, $\underbrace{\{(i\,j)(k\,l) : i \sim l \text{ different}\}}_{3 \text{ elems}}$

$\{(1\,2\,3), (2\,4\,3), (2\,1\,4), (3\,4\,1)\}$

$\{(3\,2\,1), (3\,4\,2), (4\,1\,2), (1\,4\,3)\}$

union of first two classes

$\simeq \{e, a, b, c : a^2 = b^2 = c^2 = e,\ ab = c\}$

$\simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad (= K_4)$

$A_4 \simeq K_4 \rtimes (\mathbb{Z}/3\mathbb{Z})$

1-dim rep $\psi^{(0)}, \psi^{(1)}, \psi^{(2)}$

from quotient to $\mathbb{Z}/3\mathbb{Z}$

3-dim rep π'

	e	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(3\,2\,1)$
$\psi^{(0)} = \chi_{triv}$	1	1	1	1
$\psi^{(1)}$	1	1	$e^{\frac{2\pi\sqrt{-1}}{3}}$	$e^{-\frac{2\pi\sqrt{-1}}{3}}$
$\psi^{(2)}$	1	1	$e^{-2\pi\sqrt{-1}}$	$e^{2\pi\sqrt{-1}}$
$\chi_{\pi'}$	3	-1	0	0

$\dim \pi' = 3$ from $\underset{12}{\chi_\lambda(e)} = \underset{1}{\chi_{triv}(e)} + \underset{1}{\psi^{(1)}(e)} + \underset{1}{\psi^{(2)}(e)} + \chi_{\pi'}(e)^2$

other $\chi_{\pi'}(g)$ from $\chi_\lambda = \chi_{triv} + \psi^{(1)} + \psi^{(2)} + 3\chi_{\pi'}$

$\pi' \simeq A_4 \curvearrowright \mathbb{C}^3$ extending $A_4 < SO(3) \underset{rot}{\curvearrowright} \mathbb{R}^3$