Summary

- Spectral property of fusion graphs (McKay corresp.)
- Recap(?) of fin. grp reps.

Fusion graph & its sp. prop.

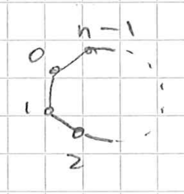
$G < SU(2)$ (fin) subgrp. $\rho : G \hookrightarrow SU(2)$

fusion graph Γ (of ρ)

- vertices : irreps. $(\pi_i)_{i=1}^{n}$ of G
- # edges $(\pi_i \to \pi_j)$: $\dim \mathrm{Hom}_G(\pi_i \otimes \rho, \pi_j) = A_{ij}$

A_{ij} : adjacency matrix

Ex. $\mathbb{Z}/n\mathbb{Z} < SU(2)$ as diag. matrices

$\to$ 

Thm. 1) $A_{ij} = A_{ji}$ (Γ is unoriented)

2) $A_{ii} = 0$ unless $G = \{e\}$ (no loops)

3) $A_{ij} \leq 1$ unless $G = \{\pm I_2\}$ (no par. edges)

Proof Step 1 ρ is self dual $\rho^c \simeq \rho$.

$\because$ lin bij $\mathbb{C}^2 = \langle e_1, e_2 \rangle \xrightarrow{T} (\mathbb{C}^2)^* = \langle \varphi^1, \varphi^2 \rangle$

by $T(e_1) = \varphi^2$, $T(e_2) = -\varphi^1$ $\boxed{\text{Want: intertw.}}$

$X^{(g)} = (X^{(g)}_{ij})_{i,j=1}^{2}$ rep mat of ρ_g (is in $SU(2)$)

$Y^{(g)} = (Y^{(g)}_{ij})_{i,j=1}^{2}$ rep. mat of ρ^c_g satisfies

$Y^{(g)}_{ij} = X^{(g^{-1})}_{ji} \overset{\uparrow}{=} \overline{X}^{(g)}_{ij}$

$X^{(g^{-1})} = (X^{(g)})^{-1} = \overline{X}^{(g)t}$

$T(\rho_g e_1) = T(X^{(g)}_{11} e_1 + X^{(g)}_{21} e_2) = X^{(g)}_{11} \varphi^2 - X^{(g)}_{21} \varphi^1$

$\rho^c_g T(e_1) = \rho^c_g \varphi^2 = Y^{(g)}_{12} \varphi^1 + Y^{(g)}_{22} \varphi^2 = \overline{X}^{(g)}_{12} \varphi^1 + \overline{X}^{(g)}_{22} \varphi^2$

$X \in SU(2) \Leftrightarrow X = \begin{bmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{bmatrix}$

Step 2 (Claim 1)

∵ $A_{ij} = \dim \mathrm{Hom}_G(\pi_i \otimes \rho, \pi_j) \underset{\text{Frob.}}{=} \dim \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho^c)$

$\underset{\text{Step 1}}{=} \dim \mathrm{Hom}_G(\pi_i, \pi_j \otimes \rho) \underset{\text{(mult. count)}}{=} \dim \mathrm{Hom}_G(\pi_j \otimes \rho, \pi_i)$

Step 3 $A_{ii} \le 1$ unless $G = \{e\}$

∵ $A_{ii} \le 2$ by dim counting

$\pi_i \oplus \pi_i$ has same dim as $\pi_i \otimes \rho$.

$A_{ii} = 2$ means $\pi_i \otimes \rho \simeq \pi_i \oplus \pi_i$

(ρ faithful $\Rightarrow \exists k$ s.t. $\lambda \subset \rho^{\otimes k}$,

$\pi \otimes \lambda \simeq \lambda^{\oplus \dim \pi}$

$\Rightarrow \lambda \simeq \pi_i^{\oplus 2^m}$ $\Rightarrow$ π_i is the only irr. rep. $\Rightarrow G = \{e\}$

Step 4 (Claim 2)

∵ Make π_i unitary by some Herm. inn. prod. on V_i

$\Rightarrow \forall T \in \mathrm{Hom}_G(\pi_i \otimes \rho, \pi_i)$: T^* is an intertw.

($T^* : V_i \to V_i \otimes \mathbb{C}^2$, $(Tv, w)_{V_i} = (v, T^* w)_{V_i \otimes \mathbb{C}^2}$

$R : \mathbb{C} \to \mathbb{C}^2 \otimes \mathbb{C}^2$, $\alpha \mapsto \alpha(e_1 \otimes e_2 - e_2 \otimes e_1)$

intertwiner (cf. Step 1)

So $(T^* \otimes \mathrm{Id})(\mathrm{Id} \otimes R)$ is again in $\mathrm{Hom}_G(\pi_i \otimes \rho, \pi_i)$

$(R^* \otimes \mathrm{Id})(\mathrm{Id} \otimes R) = -1$

$f : T \mapsto T^*$ is a conjug. lin map, $f^2 = -1$

If $\dim \mathrm{Hom}_G(\pi_i \otimes \rho, \pi_i) = 1$, f should look like $\alpha \mapsto \beta\bar{\alpha}$, $\mathbb{C} \to \mathbb{C}$.

But

Step 5 $\forall i \quad \sum_j A_{ij}^2 \leq 4$

$\because \quad \chi_{\pi_i} \chi_\rho = \sum_j A_{ij} \chi_{\pi_j}$

So $\sum A_{ij}^2 = (\chi_{\pi_i}\chi_\rho, \chi_{\pi_i}\chi_\rho)_{L^2(G)}$

$= \frac{1}{|G|} \sum_g |\chi_{\pi_i}(g)|^2 |\chi_\rho(g)|^2 \quad \cdots (*)$

We have $|\chi_\rho(g)| \leq 2 \quad (\rho_g \in U(2))$

and $\frac{1}{|G|} \sum_g |\chi_{\pi_i}(g)|^2 = (\chi_{\pi_i}, \chi_{\pi_i}) = 1$.

$\Rightarrow (*) \leq 4$

Step 6 Claim 3

$\because$ If $\sum_j A_{ij}^2 < 4 \qquad A_{ij} \leq 1$ for $\forall j$

If $\sum_j A_{ij}^2 = 4 \qquad |\chi_\rho(g)| = 2$ for all g

with $\chi_{\pi_i}(g) \neq 0$

$\rho_g \in SU(2)$ & $|\mathrm{Tr}\, \rho_g| = 2 \Rightarrow \rho_g = \pm I_2$

Suppose $A_{ij} = 2$ happens for some j

(otherwise we have Claim 3)

Want : $G < \{\pm I_2\}$ will show π_i, π_j 1-dim s.a.

$\left(\begin{array}{l} A_{ik} = 0 \text{ for } k \neq j \Rightarrow \pi_i \otimes \rho \simeq \pi_j \oplus \pi_j \\ A_{jk} = 0 \text{ for } k \neq i \Rightarrow \pi_j \otimes \rho \simeq \pi_i \oplus \pi_i \end{array} \right.$

$\Rightarrow g = -I_2$ must act trivially on π_i or π_j (

(say π_j)

$\Rightarrow \pi_j$ is a rep of $G / G \cap \{\pm I_2\} = H$

$\chi_{\pi_j}(g) = \frac{1}{2} \chi_{\pi_i}(g) \chi_\rho(g) = \begin{cases} 1 & g \in \{\pm I_2\} \\ 0 & \text{otherwise} \end{cases}$

triv. char.

$\Rightarrow \rho \simeq \pi_i \oplus \pi_i \qquad \pi_i \simeq \pi_i^c$ 1-dim $(\Rightarrow \rho$ triv$)$

$\square$

Prop. $d_i = \dim |\pi_i| \rightsquigarrow 2d_i = \sum_j A_{ij} d_j$

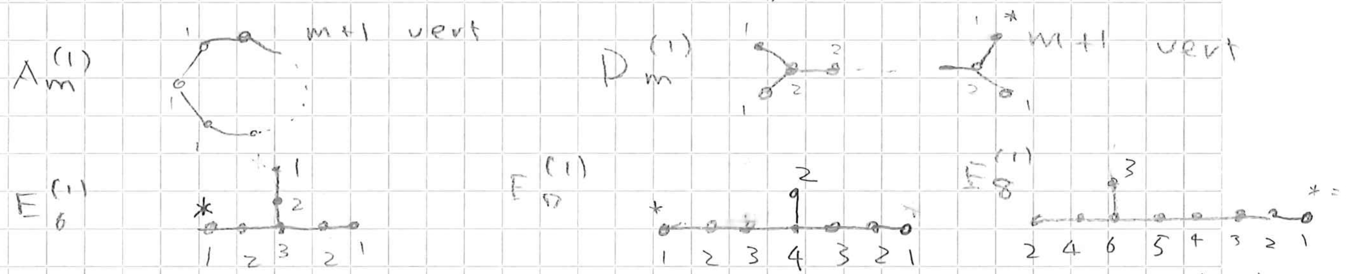
Proof $\chi_{\pi_i}\chi_\rho = \sum A_{ij} \chi_{\pi_j}$ take val at $g = e$

Rem. $(\chi_{\pi_j}(g))_{j=1}^{n}$ become eigenvec. of eigenval $\chi_p(g)$

Recap. $\|\Gamma\| = \max\{|\alpha| : \alpha : \text{eigenval of } A\} = 2$

Unless $G = \{e\}$ ($\Rightarrow \Gamma$:) Γ has no loops, no par. edges, is unoriented.

Fact (or exercise) such graphs are extended ADE Dynkin diagrams



$A^{(1)}_{n-1} \leftrightarrow \mathbb{Z}/n\mathbb{Z}$ $\qquad$ $D^{(1)}_{n+2} \leftrightarrow \widetilde{(\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})}$ (inv. img) ± 1

$E_6^{(1)} \leftrightarrow \widetilde{A}_4$ (Sym ( )) $\qquad$ $E_7^{(1)} \leftrightarrow$ Sym ( ) $\simeq \widetilde{S}_4$

$E_8^{(1)} \leftrightarrow \widetilde{\mathrm{Sym}(\text{icosahedron})} = \widetilde{A}_5$