· Summary
- Matrix groups
- Tangent vectors & vector fields

§2 Lie groups & Lie algebras

(Main ref. Fulton-Harris, Rep. theory)

Goal: understand ~~/~~ continuous groups

Pragmatically: (connected) subgroups of $GL_n(\mathbb{R})$ — matrix grp / lin grp

$U(n) = \{ X \in GL_n(\mathbb{C}) : \bar{U}^t = U^{-1} \} < GL_{2n}(\mathbb{R})$.

$SU(n) = U(n) \cap SL_n(\mathbb{C})$ $\quad (\det(U) = 1)$

$SO(n) = \{ X \in GL_n(\mathbb{R}) : X^t = X^{-1},\ \det X = 1 \}$

$\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ as

- subgroup of $GL_1(\mathbb{C})$
- $\left\{ \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix} : z \in \mathbb{T} \right\} < SU(2)$, ...

Abstractly (Lie groups) $\rightsquigarrow$ manifolds with (smooth) group structure

manifold: (top. sp.) neighborhood of any pt can be modelled by open set in $\mathbb{R}^n$. (n fixed)

smooth grp. str: prod & inverse maps cab be locally modelled by C^∞-func.

$G \times G \xrightarrow{m} G$ $\quad$ $g h$: has neigh $W \subset \mathbb{R}^n$

(g, h): has neigh $\sim U \times V \subset \mathbb{R}^n \times \mathbb{R}^n$

$\rightsquigarrow$ around (g, h) m is $(f_1, \dots, f_n)$ for $f_i(x_1, \dots, x_n, y_1, \dots, y_n)$.

$G \xrightarrow{S} G$, $g \mapsto g^{-1}$ similar

Ex. $GL_n(\mathbb{C})$: open set of $M_n(\mathbb{C}) \simeq \mathbb{R}^{2n^2}$

Rem. We'll stick to matrix groups
Which include

- compact Lie groups: $U(n)$, $SO(n)$, ...
- commutative Lie groups: $\mathbb{T}^n$, $\mathbb{R}^n$, ...
- algebraic groups.

"Only" sensible non-example: univ. cov. of $SL_2(\mathbb{R})$

How to understand cont. grps:

take "infinitesimal" model (Lie algebra)

rotation (by $\mathbb{T}$, $SO(n)$, ...) $\longrightarrow$ derivation.

$$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \xrightarrow[\text{diff.}]{} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

how to recover original rot:

× integration $\int_0^t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \partial x = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$

✓ exponential $\exp\left(t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$

$$= I_2 + t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{t^2}{2} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2}_{-I_2} + \frac{t^3}{3!} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^3 + \cdots$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = \cos(t) & -\sum \frac{(-1)^k}{(2k+1)!} t^{2k+1} = -\sin(t) \\ \sin t & \cos(t) \end{bmatrix}$$

Rem. We are computing the fundamental solution of diff. eq. $\frac{\partial v(t)}{\partial t} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v(t)$.

Tangent vectors, vec. fields.

M manifold, $p \in M$, U neigh. of p. $\quad U \overset{\varphi}{\leftarrow} V \subset \mathbb{R}^n$ open.

$\rightsquigarrow$ we can talk about smooth funcs around p,

$f : U \to \mathbb{R}$ ($f \circ \varphi(x_1, \ldots, x_n)$ is smooth as $V \to \mathbb{R}$.

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(directional derivative)

<u>tangent vector</u> at p :

D : {sooth func. around p} $\to \mathbb{R}$ lin. s.t.

$$D(f_1 f_2) = D(f_1) f_2(p) + f_1(p) D(f_2)$$

Leibniz rule at p.

($\rightsquigarrow$ will only "dep. on $\partial_i(f \circ \varphi)$ ")

$$F(x) = \underbrace{F(0)} + \sum_{i=1}^{n} (\partial_i F(0)) x_i + \underbrace{\sum h_i(x) x_i}_{\text{in } \operatorname{Ker}(D)}$$

smooth. $h_i(0)=0$

$p \Leftrightarrow 0 = x = (x_1, \dots, x_n)$

$\psi : M \to N \rightsquigarrow \psi_\# : T_p M \to T_{\psi(p)} N$

<u>vector field</u> : $D = (D_p)_{p \in M}$

- D_p : tangent vec. at p.
- varies smoothly in p

Ex. $M = \mathbb{R}$. $D_t(f) = t \dfrac{df}{dz}(t)$

Invariant vector fields on $GL_n(\mathbb{R})$

$$X \in M_n(\mathbb{R}) \rightsquigarrow e^{tX} = \exp(tX) = \sum \frac{t^k}{k!} X^k$$

$(X^0 = I_n.)$

$e^{sX} e^{tX} = e^{(s+t)X}$, $e^{0X} = I_n$, etc.

$g \in GL_n(\mathbb{R})$, f smooth func around g

$$\rightsquigarrow \tilde{X}^{(r)}_g (f) = \frac{d}{dt} f(g e^{tX}) \Big|_{t=0}$$

Prop. 1) $\tilde{X}^{(r)}_g$ is a tang. vec. at g

2) $f_{ij}(g) = g_{ij}$ ((i,j)-comp. of g)

$$\tilde{X}^{(r)}_g (f_{ij}) = \sum_{k=1}^{n} \underbrace{X_{kj}}_{\text{scalar}} \underbrace{f_{ik}}_{\text{func}}(g)$$

Proof 1) usual Leibniz rule for differentiation in t-variable.

2) $\tilde{X}^{(r)}_g (f_{ij}) = \frac{d}{dt} ((g e^{tX})_{ij}) \Big|_{t=0}$

$= \frac{d}{dt} (g_{ij} + t \sum_k g_{ik} X_{kj} + t^2 \cdots) \Big|_{t=0}$

$= \sum_k g_{ik} X_{kj} = \sum_k X_{kj} f_{ik}(g)$. $\boxed{}$

Cor. $\tilde{X}^{(r)} = (\tilde{X}^{(r)}_g)_g$ is a vec field on $GL_n(\mathbb{R})$, $\tilde{X}^{(r)}(f_{ij}) = \sum_k X_{kj} f_{ik}$