Summary

- Lie algebras
- Lie groups to Lie algebras
- Exponential map

Recap.

$X \in M_n(\mathbb{R})$: $\rightsquigarrow$ $\tilde{X}^{(r)}_g(f) = \frac{d}{dt} f(g e^{tX})\big|_{t=0}$

defs $\tilde{X}^{(r)}_g \in T_g GL_n(\mathbb{R})$

Prop $\tilde{X}^{(r)} = (\tilde{X}^{(r)}_g)_{g \in GL_n(\mathbb{R})}$ is left invariant

$L_h : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $g \mapsto hg$

then $(L_h)_\# : T_g GL_n\mathbb{R} \to T_{hg} GL_n\mathbb{R}$ maps $\tilde{X}^{(r)}_g$ to $\tilde{X}^{(r)}_{hg}$

Proof. $(L_h)_\# (\underset{T_g GL_n\mathbb{R}}{D})(f) = D(f \circ L_h)$ ← overall Leibniz rule at gh.

Leibniz rule at g

so $(L_h)_\# (\tilde{X}^{(r)}_g)(f) = \frac{d}{dt} f(hg e^{tX})\big|_{t=0}$

$= \tilde{X}^{(r)}_{hg}$ ∎

What is the correct alg str. on vec. fields?

→ M : mfd. $X = (X_p)_{p \in M}$, $Y = (Y_p)_p$

$\rightsquigarrow$ $X_p(Y(f))$ makes sense at each p

$Y(f)$: func $q \mapsto Y_q(f)$.

but $f \mapsto X_p(Y(f))$ does not satisfy

Leibniz rule at p.

$X_p(Y(f_1 f_2)) = X_p(Y(f_1)f_2 + f_1 Y(f_2))$

$= X_p(Y(f_1)) f_2(p) + Y_p(f_1) X_p(f_2) + X_p(f_1) Y_p(f_2) + f_1(p) \cdots$

Remedy : $[X,Y](f) = X(Y(f)) - Y(X(f))$

Ex. Leibniz rule from above computation

Rem. $M = GL_n(\mathbb{R})$ $X, Y \in M_n(\mathbb{R})$

$\rightsquigarrow [\tilde{X}^{(r)}, \tilde{Y}^{(r)}] = \widetilde{(XY - YX)}^{(r)}$

$\because$ Claim $\tilde{X}^{(r)}(\tilde{Y}^{(r)}(f_{ij})) = \sum_k (XY)_{kj} f_{ik}$

$\because$ follows from $\tilde{Y}(f_{ij}) = \sum_l Y_{lj} f_{il}$

$\rightsquigarrow [\tilde{X}^{(r)}, \tilde{Y}^{(r)}]$ and $\widetilde{(XY - YX)}^{(r)}$ agrees

on $\{f_{ij}\}_{ij}$.

Any func. on $GL_n(\mathbb{R})$ is (approxed by) polynom. of f_{ij}.

Def. Lie algebra is given by

- vector space $\mathfrak{g}$
- bilin. map (Lie bracket) $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $[X,Y]$

s.t. • $[X,Y] = -[Y,X]$

• $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Jacobi identity.

Ex. • $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$, $\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$

$[X,Y] = XY - YX$ comm. br.

• $\mathfrak{X}(M) := \{ D = (D_p) :$ smooth vec. field on $M \}$

Def. Homomorphism of Lie a's is

lin. map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ s.t.

$[f(X), f(Y)] = f([X,Y])$

Ex. $M_n(\mathbb{R}) \to \mathfrak{X}(GL_n(\mathbb{R}))$, $X \mapsto \tilde{X}^{(r)}$

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Lie groups to Lie algebras

G : Lie group (enough to think of mat. grp)

We want to make $\mathfrak{g} = T_e G$ a Lie al's.

- $T_e G$ real vec. sp, dim = dim of G.

- $X \in T_e G \rightsquigarrow (L_g)_\#(X) \in T_g X$.

$\rightarrow \tilde{X}^{(r)} = (L_g X)_{g \in G}$ left inv. vec. field

$\Rightarrow [X, Y] = [\tilde{X}^{(r)}, \tilde{Y}^{(r)}]_e \in T_e X$ makes sense

Obs. $T_e G \simeq \underline{\text{left inv. vec. fields}}$ as lin. sp.

Lie alg str $\leftarrow$ has Lie alg str.

$X \to \tilde{X}^{(r)}$, $D_e \leftarrow D$.

Ex. $T_e GL_n(\mathbb{R}) \simeq (M_n(\mathbb{R}), \text{comm. br.})$

Functoriality $G \to H$ hom $\overset{\mathfrak{g}}{T_e G} \to \overset{\mathfrak{h}}{T_e H}$

What do we lose by $G \rightsquigarrow \mathfrak{g}$?

Ex. $\mathbb{R}$ and $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ have the same Lie alg. $(\simeq \mathbb{R}, [X, Y] = 0)$ commutative bra

Ex. $SU(2)$ and $SO(3)$ also.

(And $\mathfrak{g}$ only depends on the conn. comp. of $e \in G$).

G conn, $\Gamma \triangleleft G$ discrete subgrp (ind. top)

$\Rightarrow \Gamma < Z(G)$, G/Γ has the "same" tangent space as G at e.

$G \to G/\Gamma$ local homeo.

$\Rightarrow T_e(G/\Gamma) \simeq T_e G$ as Lie al's.

Fact G_1, G_2 conn. Lie grps

$\mathfrak{g}_1 \simeq \mathfrak{g}_2 \iff$ univ cov. $\tilde{G}_1 \simeq \tilde{G}_2$

Nontriv part $(\Rightarrow)$: $\mathfrak{g}_1 \xrightarrow{f} \mathfrak{g}_2$ induces

$\hat{G}_1 \xrightarrow{\tilde{f}} G_2$ s.t. $(\tilde{f})_\# = f$ on $T_e\hat{G}_1 = \mathfrak{g}_1$

(will see in next part)

Exponential map.

(first part of Lie alg $\rightsquigarrow$ Lie grp)

We want to make sense of "e^{tX}" in general

$\rightarrow$ for matrix group $G < GL_n(\mathbb{R})$

$\mathfrak{g} = T_eG$ makes sense as a subsp. of $T_eGL_n\mathbb{R}$
$= M_n\mathbb{R}$

(as img of $j_\# : T_eG \to T_eGL_n\mathbb{R}$ for $G \overset{j}{\hookrightarrow} GL_n\mathbb{R}$)

Q. ~~if~~ Given $X \in \mathfrak{g}$ if we compute e^{tX} in $GL_n(\mathbb{R})$, do we stay in G?

A. Yes but we need some elaboration

Ex. $SO(n) \rightsquigarrow so_n = \{X \in M_n(\mathbb{R}) : X^t = -X\}$

$$(e^{sX})^t = \left(\sum_{k=0}^{\infty} \frac{s^k}{k!} X^k\right)^t = \sum_k \frac{s^k}{k!} (X^t)^k$$
$$(X^0 = I_n)$$
$$= e^{-sX} = (e^{sX})^{-1}.$$

$$\det(e^{sX}) = e^{s\operatorname{Tr} X} \quad \text{(prod. of eigenvals)}$$
$$= e^0 = 1$$

Generally : $\tilde{X}^{(r)} = (L_g X)_{g \in G}$ vec. field

$\Longrightarrow$ (mfd. thry (solve ODE)) $\exists\, \varepsilon > 0, \exists! \varphi_X : (-\varepsilon, \varepsilon) \to G$ s.t.

$\varphi_X'(t) = \tilde{X}^{(r)}_{\varphi_X(t)}$, $\varphi_X(0) = e$.

Mat exponential also has this prop.

$\rightarrow$ e^{tX} should rep φ_X, stay in G