Summary

- Exponential map (cont.)
- Lie subalgebras & immersed subgrps
- Lie algs to Lie grps

Recall. $X \in T_eG \rightsquigarrow \tilde{X}^{(r)} = ((L_g)_\# X)_{g\in G}$ left inv. vec. field

$\exists!$ $\varphi_X : (-\varepsilon, \varepsilon) \to G$ $\quad \varphi_X'(t) = \tilde{X}^{(r)}_{\varphi_X(t)}$ integrating curve

Prop. $\varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$ for $|s|, |t|, |s+t| < \varepsilon$

Proof. Fix s & consider deriv. at $t = t_0$

Left hand side : $\frac{d}{dt} f(\varphi_X(s)\varphi_X(t))|_{t=t_0}$

$= (L_{\varphi_X(s)})_\# \tilde{X}^r_{\varphi_X(t_0)}(f) \underset{\text{invariance}}{=} \tilde{X}^r_{\varphi_X(s)\varphi_X(t_0)}(f)$

Right hand side : $\frac{d}{dt} f(\varphi_X(s+t)) = \tilde{X}^r_{\varphi_X(s+t_0)}(f)$

i.e. both rep. curve w/ velocity $\tilde{X}^{(r)}$

$\underset{\text{uniqueness}}{\Rightarrow}$ they should agree ∎

Cor φ_X extends to $\mathbb{R} \to G$ s.t. $\varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$

$\because$ For $t \in \mathbb{R}$ put $\varphi_X(t) = \varphi_X(\frac{t}{N})^N$ $\quad |\frac{t}{N}| < \varepsilon$

right hand side is well-def'd by Prop.

Prop $\Rightarrow \varphi_X(s)\varphi_X(t) = \varphi_X(s+t)$ in general.

(If $N \gg 1$ $\quad \varphi_X(s)\varphi_X(t) = \varphi_X(\frac{s}{N})^N \varphi_X(\frac{t}{N})^N$

$= (\varphi_X(\frac{s}{N})\varphi_X(\frac{t}{N}))^N = \varphi_X(\frac{s+t}{N})^N = \varphi_X(s+t)$)

Exponential map $\mathfrak{g} \to G$, $X \mapsto e^X = \varphi_X(1)$.

Rem $\varphi_X(t) = \varphi_{tX}(1) = e^{tX}$ $\quad$ one parameter group

Ex. $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$

$\rightsquigarrow \ e^{tX} = \sum \frac{1}{k!}(tX)^k = I_n + tX + \frac{t^2}{2}X^2 + \cdots$

Lie subgroups

Goal: understand "subgroups" (of G) corresponding to Lie subalgs of $\mathfrak{g} = T_eG$

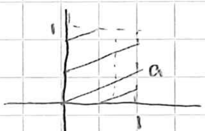
Warning: $H < G$ for $\mathfrak{h} < \mathfrak{g}$ might not be "nice" subset. ex:

$G = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. $\mathfrak{g} = \mathbb{R}^2$ comm. bracket

Any subsp. $\mathfrak{h} \subset \mathbb{R}^2$ is a Lie subalg

nontriv: 1-dim subsp. $\{(x, ax) : x \in \mathbb{R}\}$

Corresponding subgroup. $H_a = \{[(x, ax)] : x \in \mathbb{R}\}$



$\rightsquigarrow$ dense subgroup if (and only if) a is irrational.

$\rightsquigarrow$ not manifold by induced topology (but an <u>immersed</u> <u>submanifold</u>)

Prop. $\mathfrak{h} \subset \mathfrak{g} = T_eG$ Lie subalg.

then $\exists$ Lie group H, inj. hom $j : H \to G$

s.t. $j_*(T_eH) = \mathfrak{h}$. (H is an <u>immersed</u> subgroup)

Proof. Step 1. Define H as a set by

$H = \{e^{X_1}e^{X_2}\cdots e^{X_n} : X_i \in \mathfrak{h}\}$

closed under prod. obvious

inverse $(e^X)^{-1} = e^{-X}$.

Step 2 $\exists$ neighborhood $0 \in \Delta \subset \mathfrak{g}$ s.t.

$G_0 = e^{\Delta} \subset G$, $H_0 = e^{\Delta \cap \mathfrak{h}} \subset H$ satisfy

$(H_0 \cdot H_0) \cap G_0 = H_0$, $H_0^{-1} = H_0$.

$\because$ We use

- Ado's theorem : $\mathfrak{g}$ has a faithful hom $\mathfrak{g} \to \mathfrak{gl}_n(\mathbb{R})$ ($\Rightarrow$ we may assume $G = GL_n(\mathbb{R})$)
- Baker-Campbell-Hausdorff formula

$$\log(e^X e^Y) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{r_1+s_1+r_2+\cdots+s_{m-1}=n-1 \\ r_1+s_1 \geq 1,\ r_2+s_2 \geq 1, \ldots}} \frac{(-1)^m}{m} \left(\prod \frac{ad_X^{r_i}}{r_i!} \frac{ad_Y^{s_i}}{s_i!} \right) \frac{ad_X^{r_m}}{r_m!}(Y)$$

$$+ (X \leftrightarrow Y).$$

with $ad_X(Y) = [X, Y]$.

$\|X\|, \|Y\|$ small (as matrices in $M_n(\mathbb{R})$)

$\rightsquigarrow$ above sum converges.

$X, Y \in \mathfrak{h} \Rightarrow$ right hand side stays in $\mathfrak{h}$.

Step 3. H becomes a Lie group.

$\because$ $H_0 \supseteq \Delta \cap \mathfrak{h}$. open in $\mathfrak{h}$ (vec. sp.)

use translation by $h \in H$ to define coordinates elsewhere. $\boxtimes$

Th'm G (connected Lie group $\mathfrak{g} = T_e G$

K another Lie group $\mathfrak{k} = T_e K$.

$\varphi : \mathfrak{g} \to \mathfrak{k}$ Lie alg hom.

Then $\exists$ Lie grp hom $\underset{\text{univ. cover}}{\widetilde{G}} \xrightarrow{f} K$ s.t. $f_* = \varphi$

Conseq. $K = GL_n(\mathbb{R}) \rightsquigarrow \mathfrak{k} = \mathfrak{gl}_n(\mathbb{R})$

$\mathfrak{g} \to \mathfrak{gl}_n(\mathbb{R})$ "rep of $\mathfrak{g}$"

$\equiv \widetilde{G} \to GL_n(\mathbb{R})$ rep of $\widetilde{G}$.

Proof of Th'm.

Set $\mathfrak{h} = \{ (X, \varphi(X)) : X \in \mathfrak{g} \}$ graph of φ

Step 1 φ is a Lie alg hom

$\Leftrightarrow$ $\mathfrak{h}$ is a Lie subalg of $\mathfrak{g} \oplus \mathfrak{k}$.

cf. for groups. $\{ (g, f(g)) : g \in G \}$

is a subgrp of $G \times K$

$\Leftrightarrow$ $(g, f(g))(h, f(h))$ is in this.

but $(gh, f(gh))$ is the only possibility.

Step 2 $\mathfrak{g} \oplus \mathfrak{k}$ is the Lie alg of $\tilde{G} \times K$.

$\exists$ $H \subset \tilde{G} \times K$ immersed subgrp.

s.t. $T_e H = \mathfrak{h}$ by Prop.

Step 3 first proj. $H \to \tilde{G}$ is homeo.

$\because$ locally homeo & $\tilde{G}$ is simply conn.

$\Rightarrow$ $\exists$ lift $\tilde{G} \to H$

Step 4 Up to $\tilde{G} \cong H$, second prj

$H \to K$ gives a map $\tilde{G} \to K$.