Summary

- Lin. reps. of Lie algs / grps.
- ideals in Lie algs & rep of $\mathfrak{sl}_2$
- derived / lower central series

G : Lie group (connected)

$\mathfrak{g} = T_e G$ Lie alg.

Def. A <u>finite dim</u> <u>rep</u>: of G (of $\mathfrak{g}$) is

- f.d. vec. sp. V ($\simeq \mathbb{R}^n$ or $\mathbb{C}^n$)
- Lie grp hom $G \xrightarrow{\pi} GL(V)$

(Lie alg hom $\mathfrak{g} \xrightarrow{\pi} \mathfrak{gl}(V) = \{T \in End(V)$
$[T, S] = TS - ST\}$)

$\{ Xv = \frac{d}{dt} e^{tX} v |_{t=0}$

Notn. gv , Xv for $v \in V$, $g \in G$, $X \in \mathfrak{g}$

$\pi_g v$ $\pi_X v$

Rem rep of $\mathfrak{g}$ $\equiv$ rep of univ. cov. $\tilde{G}$

Example. $G = SL_2(\mathbb{C})$ (or $SL_2(\mathbb{R})$)

$V_n = \langle x^n, x^{n-1}y, \dots, y^n \rangle_{\mathbb{C}\text{-sp.}}$ ($n+1$ -dim.

($V_n \subset \{ f : \mathbb{C}^2 \to \mathbb{C} \}$ as deg n homogen polys)

$\mathbb{C}^2 \curvearrowleft G$ right action $\pi^n : G \curvearrowright V_n$ by
"row vec"

by $(\pi^n_g)(f)(p) = f(pg)$ $p \in \mathbb{C}^2$

$\mathfrak{sl}_2(\mathbb{C})$: 3-dim cplx Lie alg, basis

$H = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$e^{tH} = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}$, $e^{tE} = \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$, $e^{tF} = \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix}$

$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \pi^n_g (x^k y^{n-k}) = (ax+cy)^k (bx+dy)^{n-k}$

$\pi^n_H : (2k-n) x^k y^{n-k}$

$x\partial_x - y\partial_y$

$\pi_E^n : (n-k)\, x^{k+1} y^{n-k-1}$ $\qquad$ $\pi_F^n : \; k\, x^{k-1} y^{n-k+1}$

$x\partial_y$ $\qquad$ $-y\partial_x$

Prop. These exhaust irred. fin dim reps of $\mathfrak{sl}_2\mathbb{C}$

(⇒ $SL_2\mathbb{C}$ is simply conn.)

Sketch (π, W) (irred.) f.d. rep of $\mathfrak{sl}_2$

$W_\lambda^{(k)} = \{ w \in W : (H-\lambda)^k w = 0 \}$ for k.

"subsp. of wght λ" $\quad$ $k=1$ will be enough

Step 1 $\quad E W_\lambda \subset W_{\lambda+2}$, $\quad F W_\lambda \subset W_{\lambda-2}$.

∵ $[H, E] = 2E$ (direct computation)

$\Rightarrow E(H-\lambda) = HE - 2E - \lambda E = (H-(\lambda+2))E$

So $(H-(\lambda+2))^k E = E(H-\lambda)^k$ by induction

Similar for F

Step 2. take $0 \neq v \in W_\lambda^{(1)} \cap \ker E$ $\quad$ (λ: highest wght)

then $\langle v, Fv, F^2 v, \cdots \rangle$ is $\mathfrak{sl}_2$-inv. $\quad$ ($\underset{(\pi, W)\text{ irr}}{\Rightarrow}$ everything)

∵ obviously F-inv.

H-inv. by $F^k v \in W_{\lambda-2k}^{(1)}$

E-inv: $[E, F] = H \rightsquigarrow EF^k = HF^{k-1} + FEF^{k-1}$

use induction.

Step 3 Such λ is int.

∵ $[E, F] = H \rightsquigarrow \mathrm{Tr}_V \pi_H = \mathrm{Tr}_V([\pi_E, \pi_F]) = 0$

if $F^k v \neq 0$ & $F^{k+1} v = 0$ (eventually happens)

$W = \langle v, Fv, \cdots, F^k v \rangle$

eval of π_H: $\lambda, \lambda-2, \cdots, \lambda-2k$

$\mathrm{Tr}\, \pi_H : \lambda + \lambda-2 + \cdots + \lambda-2k = (k+1)\lambda - k(k+1)$

so $\lambda = k$

Complete reducibility (Weyl's unitary trick)

$SL_2(\mathbb{C})$-rep $\equiv$ $SU(2)$-rep

(more like fin. grp.)

Rem. commutative Lie algs have **bad** rep. theory than $\mathfrak{sl}_2$

$\mathfrak{g} = \mathbb{R}$. rep of $\mathfrak{g}$: specifying $\pi_1 = T \in \mathrm{End}(V)$

$T \leftrightarrow$ Jordan normal form. $\begin{bmatrix} \lambda & 1 & & \\ & \lambda & & \\ & & \mu & 1 \\ & & & \mu \ \cdots \end{bmatrix}$

irrep: $\dim V = 1$, $\pi_1 = \lambda$ $\quad \lambda \in \mathbb{C}$.

So general rep might not be dir. sum of irreducibles

$\mathfrak{g} = \mathbb{R}^2$: rep of $\mathfrak{g}$: commuting ops S, T.

→ We want to understand

- dichotomy of Lie algs
  - semisimple (very noncomm, "rigid")
    $\mathfrak{sl}_n$, $\mathfrak{so}_n$, ...
  - solvable (close to comm, "soft")
    comm, $\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$, ...
- How they can "compose"

Def. $\mathfrak{g}$: Lie alg. An <u>ideal</u> of $\mathfrak{g}$ is $\mathfrak{h} \subset \mathfrak{g}$ subsp. s.t. $\forall X \in \mathfrak{g}, Y \in \mathfrak{h}$: $[X, Y] \in \mathfrak{h}$.
(stronger than Lie subalg) write $\mathfrak{h} \triangleleft \mathfrak{g}$

Def. The <u>derived</u> <u>subalg</u> (commutator subalg)
$\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] =$ span of $[X, Y]$ $(X, Y \in \mathfrak{g})$

Rem $\mathcal{D}(\mathfrak{g}) \triangleleft \mathfrak{g}$, $\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ comm.

Ex. $\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right\}$ Lie alg of $\left\{ \begin{bmatrix} e^a & b \\ 0 & 1 \end{bmatrix} \right\}$

$$\left[\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & ab' - ba' \\ 0 & 0 \end{bmatrix}$$

$\mathcal{D}(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right\}$.

Def. lower central series $\mathcal{D}_n(\mathfrak{g})$

$$\mathcal{D}_1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}), \quad \mathcal{D}_{n+1}(\mathfrak{g}) = [\mathcal{D}_n(\mathfrak{g}), \mathfrak{g}]$$

derived series $\mathcal{D}^n(\mathfrak{g}) \quad (\subset \mathcal{D}_n(\mathfrak{g}))$

$$\mathcal{D}^1(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}), \quad \mathcal{D}^{n+1}(\mathfrak{g}) = [\mathcal{D}^n(\mathfrak{g}), \mathcal{D}^n(\mathfrak{g})] = \mathcal{D}(\mathcal{D}^n(\mathfrak{g}))$$

Prop. $\mathcal{D}_n(\mathfrak{g}), \mathcal{D}^n(\mathfrak{g}) \triangleleft \mathfrak{g}$. ($\Rightarrow$ decreasing in n).

Proof for $\mathcal{D}_n(\mathfrak{g})$

Suppose we know $\mathcal{D}_n(\mathfrak{g}) \triangleleft \mathfrak{g}$, take $X, Z \in \mathfrak{g}$

$Y \in \mathcal{D}_n(\mathfrak{g})$ Want: $[X, \underbrace{[Y, Z]}_{\text{in } \mathcal{D}_n(\mathfrak{g})}] \in \mathcal{D}_{n+1}(\mathfrak{g})$

Jacobi & antisymm.

$$[X,[Y,Z]] = \underbrace{[\underbrace{[Z,X]}_{\text{in } \mathcal{D}_n}, Y]}_{\text{in } \mathcal{D}_{n+1}} + \underbrace{[\underbrace{[X,Y]}_{\text{in } \mathcal{D}_n}, Z]}_{\text{in } \mathcal{D}_{n+1}}$$

$\square$

Def. $\mathfrak{g}$ is solvable iff $\mathcal{D}^k(\mathfrak{g}) = 0 \quad k \gg 1$

nilpotent $\mathcal{D}_k(\mathfrak{g}) = 0 \quad k \gg 1$.

($\mathcal{D}^n \subset \mathcal{D}_n \rightsquigarrow$ nilpot $\Rightarrow$ solv.)

Ex. $\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$.

$\mathcal{D}_1(\mathfrak{g}) = \mathcal{D}^1(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right\}$

$\mathcal{D}^2(\mathfrak{g}) = 0, \quad \mathcal{D}_2(\mathfrak{g}) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right\}$ so

this is solvable but not nilpot.

Ex. $\mathcal{D}(\mathfrak{sl}_2) = \mathfrak{sl}_2$ so not solvable.

Prop. $\mathfrak{g}$ solvable iff $\exists$ dec. seq. $\mathfrak{g} > \mathfrak{g}_1 > \mathfrak{g}_2 > \cdots > \mathfrak{g}_k = 0$

s.t. $\mathfrak{g}_{n+1} \triangleleft \mathfrak{g}_n$, $\mathfrak{g}_n / \mathfrak{g}_{n+1}$ comm.

Proof $\Rightarrow$: take $\mathfrak{g}_n = \mathcal{D}^n(\mathfrak{g})$.

$\Leftarrow$: $\mathfrak{g}_n / \mathfrak{g}_{n+1}$ comm $\Rightarrow \mathcal{D}(\mathfrak{g}_n) \subset \mathfrak{g}_{n+1}$

By induction $\mathcal{D}^n(\mathfrak{g}) \subset \mathfrak{g}_n \rightsquigarrow 0$