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Summary

Structure of nilp. / solv. Lie algs

- Engel's th'm nilpot : str. upper triang.
- Lie's th'm solv. : upper triangular.

Recall : $\mathfrak{g}$ solv. $\Leftrightarrow \exists\ \mathfrak{g} > \mathfrak{g}_1 > \mathfrak{g}_2 > \cdots > \mathfrak{g}_k = 0$

$\mathfrak{g}_{n+1} \triangleleft \mathfrak{g}_n$, $\mathfrak{g}_n / \mathfrak{g}_{n+1}$ comm.

Cor. if $\mathfrak{h} \triangleleft \mathfrak{g}$

$\mathfrak{h}, \mathfrak{g}/\mathfrak{h}$ solv. $\Leftrightarrow$ $\mathfrak{g}$ solv.

Proof : $\Leftarrow$: take $(\mathfrak{g}_n)_n$ as above

$\mathfrak{h}_n = \mathfrak{h} \cap \mathfrak{g}_n$, $(\mathfrak{g}/\mathfrak{h})_n$ = img of $\mathfrak{g}_n$.

do the job.

$\Rightarrow$: take $(\mathfrak{h}_n)_{n=1}^{k}$, $((\mathfrak{g}/\mathfrak{h})_n)_{n=1}^{l}$

set $\mathfrak{g}_n$ = inv. img of $(\mathfrak{g}/\mathfrak{h})_n$ $n < l$

$\mathfrak{g}_n = \mathfrak{h}_{n-l}$ $n \geq l$ $(\mathfrak{g}_l = \mathfrak{h})$

If $n < l-1$ $\mathfrak{g}_n / \mathfrak{g}_{n+1} \simeq (\mathfrak{g}/\mathfrak{h})_n / (\mathfrak{g}/\mathfrak{h})_{n+1}$

$n = l-1$ $\mathfrak{g}_n / \mathfrak{g}_{n+1} = \mathfrak{g}_{l-1}/\mathfrak{h} = (\mathfrak{g}/\mathfrak{h})_{l-1}$.

is comm. by $(\mathfrak{g}/\mathfrak{h})_l = 0$

$n \geq l$ $\mathfrak{g}_n / \mathfrak{g}_{n+1} \simeq \mathfrak{h}_{n-l} / \mathfrak{h}_{n-l+1}$. ☐

Engel's th'm. V : vec. sp.

$\mathfrak{g} < \mathfrak{gl}(V)$ suppose $\forall X \in \mathfrak{g}$ is a nilpot.

endom. of V. ($X^n = 0$ for $n \gg 1$)

Then $\exists\ 0 \neq v \in V$ $\forall X \in \mathfrak{g}$ $Xv = 0$

Cor. $\exists$ basis $(v_i)_{i=1}^{n}$ of V s.t. elems of $\mathfrak{g}$

have "strictly upper triangular" mat. pres.

Proof of Cor : Induction on $n = \dim V$

Set $v_1 = v$ from Thm, put $V' = V/\langle v_1 \rangle$.

$\mathfrak{g}$ acts on V' by nilpot. endos.

$\underset{\text{hyp.}}{\Rightarrow}$ $\exists$ basis $\bar{v}_2, \dots, \bar{v}_n$ of V'

s.t. $X \bar{v}_k \in \langle \bar{v}_2, \dots, \bar{v}_{k-1} \rangle$.

take inv. img. of $\bar{v}_k$: $v_k \in V$ ← ambiguity! $\langle v_1 \rangle$.

(• $(v_k)_{k=1}^n$ lin. indep.

• $X v_k \in \langle v_1, \dots, v_{k-1} \rangle$

Proof of Thm.

Step 1. X nilp. on $V \Rightarrow \mathrm{ad}_X$ nilp. on $\mathrm{End}(V)$

$\because$ $X^k = 0 \Rightarrow \mathrm{ad}_X^{2k}(T) = 0$

We will prove claim by ind. on $m = \dim \mathfrak{g}$

Step 2 $\exists\ \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m - 1$.

$\because$ Take any proper max subalg as $\mathfrak{h}$

$\overline{\mathrm{ad}} : \mathfrak{h} \curvearrowright \mathfrak{g}/\mathfrak{h}$ by nilp. maps (Step 1)

$\underset{\text{ind. hyp.}}{\Longrightarrow}$ $\exists\ 0 \neq \bar{Y}_1 \in \mathfrak{g}/\mathfrak{h}$ s.t. $\overline{\mathrm{ad}}_X(\bar{Y}) = 0$ for $X \in \mathfrak{h}$

i.e. $\exists Y \in \mathfrak{g} \setminus \mathfrak{h}$ $\forall X \in \mathfrak{h}$ $[X, Y] \in \mathfrak{h}$

so $\mathfrak{h}' = \mathfrak{h} + \langle Y \rangle$ is also subalg, $\mathfrak{h} \triangleleft \mathfrak{h}'$

$\underset{\text{maximality}}{\Longrightarrow}$ $\mathfrak{h}' = \mathfrak{g}$ i.e. $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$.

Step 3 Take $\mathfrak{h}$, $Y \in \mathfrak{g}/\mathfrak{h}$ as above

Put $W = \{ v \in V : \mathfrak{h} v = 0 \}$. $\underset{\text{hyp.}}{(\neq 0)}$

Enough to show $Y W \subset W$.

$\because$ Y is nilpot.

Step 4 $Y W \subset W$

$\because$ $X Y v = Y X v + \overbrace{[X, Y]}^{\ni \mathfrak{h}} v$ would be zero if $X \in \mathfrak{h}$.

⊡

Cor. $\mathfrak{g}$ nilpot $\Leftrightarrow$ ad : $\mathfrak{g} \curvearrowright \mathfrak{g}$ is by nilpot. maps.

$\Leftarrow$: Engel's th'm gives $\mathfrak{g}_k \searrow 0$,

$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}_k) \subset \mathfrak{g}_{k+1}$ (ideals).

$\rightsquigarrow$ $\mathcal{D}_k(\mathfrak{g}) \subset \mathfrak{g}_k$ goes to zero.

Lie's th'm) V : $\mathbb{C}$-vec. sp.

$\mathfrak{g} \subset \mathfrak{gl}(V)$ solvable.

$\Rightarrow \exists\ 0 \neq v \in V$ eigenvec. for all $X \in \mathfrak{g}$.

Cor. $\exists$ basis $(v_i)_{i=1}^n$ s.t. $\forall X \in \mathfrak{g}$ has upper triang. mat. pres.

Proof of Th'm. : ind. on $m = \dim \mathfrak{g}$.

Step 1 $\exists\ \mathfrak{h} \triangleleft \mathfrak{g}$ $\dim \mathfrak{h} = m-1$.

$\because$ By assumption $\mathcal{D}^k(\mathfrak{g}) \searrow 0 \Rightarrow \mathcal{D}(\mathfrak{g}) \neq \mathfrak{g}$

$\mathfrak{g}/\mathcal{D}(\mathfrak{g})$ is nonzero comm.

$\Rightarrow$ any subalg is ideal.

$\rightarrow$ take inv. img. of codim 1 subsp.

Step 2 Ind. hyp. $\Rightarrow \exists\ 0 \neq v$ eigenvec. for $\mathfrak{h}$

Set $Xv = \lambda(X)v$ ($\lambda : \mathfrak{h} \to \mathbb{C}$ linear)

$W = \{v' \in V : \forall X \in \mathfrak{h}\ \ Xv' = \lambda(X)v'\}$

$Y \in \mathfrak{g} \setminus \mathfrak{h}$ $\rightsquigarrow$ enough to show $YW \subset W$

$\because$ Y will have eigenvec. in W

Step 3. $YW \subset W \Leftrightarrow \lambda(\underbrace{[X,Y]}_{\in \mathfrak{h}}) = 0$ for $X \in \mathfrak{h}$

$\because$ Again $\underbrace{XYv'}_{} = \underbrace{YXv'}_{} + \underbrace{[X,Y]v'}_{} \to \lambda([X,Y])v'$

want these to be $\lambda(X)Yv'$

Step 4 For $w \in W$ put $U_w = \langle w, Yw, Y^2w, \dots \rangle$

$X \in \mathfrak{h} \Rightarrow XY^k w \in \lambda(X)Y^k w + \langle w, Yw, \dots, Y^{k-1}w \rangle$

$\because$ Induction on k ; $XY^k w = YXY^{k-1}w$

$+ \underbrace{[X, Y]}_{\in \mathfrak{h}} Y^{k-1} w$

Step 5 $\lambda([X, Y]) = 0$ for $X \in \mathfrak{h}$

$\because$ $\lambda([X, Y])$: diag. entries of

$[X, Y]|_{U_m}$ in terms of basis

$w, Yw, \cdots, Y^m w$. $\rightsquigarrow$ trace / $\dim U_m$

X, Y both pres. $U_m \Rightarrow [X, Y]|_{U_m} = [X|_{U_m}, Y|_{U_m}]$

$\rightarrow$ trace is zero $\boxtimes$